HEREDITARY GROUP RINGS

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Summary

The purpose of this note is to describe those group rings that are right hereditary. The characterization necessarily involves a number of concepts from ring theory and from group theory, and we briefly review these for the benefit of the reader. From ring theory we need the following definitions. A ring R is said to be:

right hereditary if every right ideal is projective (as right R-module);

completely reducible if R is a finite direct product of full matrix rings over skew fields (or equivalently, R is nonzero and every right ideal of R is a direct summand of R);

von Neumann regular if every right R-module is flat (or equivalently, for each element r of R there exists an element x of R such that $rxr = r$);

right $\aleph_0$*-Noetherian* if every right ideal of R is countably generated.

From group theory we require the concept of the fundamental group of a connected graph of groups, and the definition of this will be recalled in Section 1, below. Finally, for any ring R, let us call a group G an R^{-1}*-group* if the order of every finite subgroup of G is invertible in R. (The usual terminology for this property is "G has no R-torsion", but "R^{-1}-group" has the advantage of brevity, and is descriptive in that one can think of the orders of the elements of G as being of the form r^{-1}, $4 \in R$, where either r is the inverse in R of a natural number, or $r = 0$ and 0^{-1} is taken to be ∞.)

Throughout, we fix a nonzero ring R and a group G, and denote the corresponding group ring RG.

THEOREM 1. *The group ring RG is right hereditary if and only if one* (*or more*) *of the following holds:*

(H1) *R is completely reducible and G is the fundamental group of a connected graph of finite R^{-1}-groups;*

(H2) *R is right $\aleph_0$-Noetherian, von Neumann regular, and G is a countable locally finite R^{-1}-group;*

(H3) *R is right hereditary and G is a finite R^{-1}-group.*

The proof could be left to any reader familiar with **[1]**, **[4]**, **[5]**, [7] and **[12]**; but a proof that consists largely of references to the literature is rather time-consuming for a reader unfamiliar with some of the references. For this reason, an effort has been made to give an almost self-contained proof here, dependent only on **[5]** and fundamental facts about projective modules.

Received 18 September, 1978; revised 15 December, 1978.

[J. LONDON MATH. SOC. (2), 20 (1979), 27–38]

I am indebted to W. A. Hodges, P. Menal and W. Stephenson for helpful and stimulating conversations on this subject.

Remark. Since right hereditary means that the right global dimension is at most 1, a natural thing to ask for after Theorem 1 is a *formula* for the right global dimension of a group ring. An immediate conjecture is the bound

$$\text{r.gl.dim}\, RG \leqslant \text{r.gl.dim}\, R + \text{cd}_R G, \tag{1}$$

where $\text{cd}_R G$ denotes the projective RG-dimension of R, viewed as right RG-module with trivial G-action. In the case where R is a field, (1) follows from [**2**; XVI, 7(6)] (*cf.* [**6**], [**10**; Theorem 10.3.6]) and the basic idea of this case can be used to show that for an arbitrary ring R

$$\text{r.gl.dim}\, RG \leqslant \text{r.gl.dim}\, R + \text{cd}_K G, \tag{1$'$}$$

where K is the centre of R. (Briefly, the proof runs as follows. Let M be a right RG-module, and take a projective RG-resolution of it, say $\ldots \to Q_n \to \ldots \to Q_0 \to M \to 0$. If $r = \text{r.gl.dim}\, RG$ then the kernel Q of $Q_{r-1} \to Q_{r-2}$ is R-projective. Now consider a projective KG-resolution of K, say $\mathscr{P} \to K \to 0$. Then Q has a projective RG-resolution, $\mathscr{P} \otimes_K Q \to Q \to 0$, where G acts on $\mathscr{P} \otimes_K Q$ by the diagonal action, *cf.* [**10**; Lemma 10.3.5].) Now (1) would follow from (1′) if we knew that $\text{cd}_K G = \text{cd}_R G$, but this also is not known in general. There is however a result of Swan [**12**] (and recalled in Lemma 7, below) that says that $\text{cd}_K G = \text{cd}_R G$ if K has a subring k such that R/k is right k-projective; for example, if R is an algebra over a field k.

Let us record the form taken by (1′) (and (1)) in each of the three cases of Theorem 1: (H1) $1 \leqslant 0 + 1$; (H2) $1 \leqslant 1 + 1$; (H3) $1 \leqslant 1 + 0$. (Here "1" is to be read as "1 or less".) Indeed, the sufficiency of (H1), (H3) could be proved via these inequalities. (But we shall not be doing so, since in each of these cases, r.gl.dim R is not substantially harder to compute than $\text{cd}_R G$, and the direct ring-theoretic proofs seem more illuminating.)

1. *The fundamental group of a connected graph of groups*

By a *graph*, X, we understand a system consisting of: a nonempty set, $V(X)$, whose elements are called the *vertices* of X, a set, $E(X)$, whose elements are called the *edges* of X, and an incidence map $(\iota, \tau): E(X) \to V(X) \times V(X)$. For any edge e of X, ιe, τe are called the *initial* and *terminal* vertices of e, respectively.

The reader should have no difficulty in translating to this setting the definitions of the familiar graph-theoretic concepts that we shall be using.

Let us fix a connected graph X. We may view X as a small category, whose object set is $\text{ob}(X) = E(X) \cup V(X)$, and whose nonidentity morphisms are $\iota_e : e \to \iota e$, $\tau_e : e \to \tau e (e \in E(X))$. Then by a connected graph of groups, $\mathscr{G}$, we understand a functor $\mathscr{G} : X \to \mathscr{G}\!\mathit{roups}$. For vertices v of X, the $\mathscr{G}(v)$ will be called the *vertex groups* of $\mathscr{G}$, and for edges e of X, the $\mathscr{G}(e)$ will be called the *edge groups* of $\mathscr{G}$. The homomorphisms $\mathscr{G}(\iota_e) : \mathscr{G}(e) \to \mathscr{G}(\iota e)$ will be denoted $g \mapsto g^{\iota_e}$, and similarly for τ.

Since X is connected we can find a *spanning tree*, that is, a subgraph with the same vertex set and with a minimal edge set so that the subgraph is still connected. For any spanning tree T of X, and any graph of groups $\mathscr{G} : X \to \mathscr{G}\!\mathit{roups}$, the *fundamental group* *of* $\mathcal{G}$ *with respect to* T, $\pi(\mathcal{G}, T)$, is the group universal with the following properties: for each vertex v of X there is given a group homomorphism $\mathcal{G}(v) \to \pi(\mathcal{G}, T)$; for each edge e of X there is given an element $q(e)$ of $\pi(\mathcal{G}, T)$ such that $q(e)^{-1} . g^{\iota_e} . q(e) = g^{\tau_e}$ for all $g \in \mathcal{G}(e)$, and if e is an edge of T then $q(e) = 1$. One can show that the isomorphism class of $\pi(\mathcal{G}, T)$ is independent of the choice of T, *cf.* [**11**; Proposition 20]; for this reason one usually speaks of the *fundamental group* of $\mathcal{G}$, without reference to a spanning tree.

Since the fundamental group of a certain type of connected graph of groups is used in the characterization given in Theorem 1, it is illuminating to know the simplest examples of groups that arise in this way, and we shall now describe these. This is a special case of a result of Bass [**1**; (6.5)] for which we give a correspondingly simpler argument.

THEOREM 2. *The following are equivalent for a group G:*

(i) *G is the fundamental group of a connected graph of finite groups and G does not have a free subgroup of rank* 2;

(ii) *one of the following holds:*

(a) *G is countable and locally finite;*

(b) *G is finite-by-*$\mathbb{Z}$;

(c) $G \simeq A \coprod_V B$ *where A and B are finite and* $(A : V) = (B : V) = 2$.

Proof. (i) $\Rightarrow$ (ii). Let G be a group with no free subgroups of rank 2, and suppose that G is the fundamental group of a connected graph of finite groups, $\mathcal{G} : X \to \mathcal{G}roups$. By replacing the vertex groups of $\mathcal{G}$ with their images in G, we may assume that all the maps $\mathcal{G}(v) \to G$ are inclusions. On factoring out the normal subgroup of G generated by the vertex groups, we see that the elements $q(e)$, as e ranges over the edges of X not in the given spanning tree T, freely generate a free subgroup of G. Hence X has at most one edge not in T.

Let us consider first the case where there is one edge, e say, not in T. Let H denote the subgroup of G generated by the vertex groups. Then there are homomorphisms $\mathcal{G}(\iota_e), \mathcal{G}(\tau_e) : \mathcal{G}(e) \to H$, and G is the associated HNN extension. Consider any $h \in H$. By hypothesis, $h . q(e) . h . q(e)^{-1}$ and $h . q(e)^{-1} . h . q(e)$ do not freely generate a free subgroup of G, and by considering normal forms in HNN extensions we see that this can only happen if h lies in the image of $\mathcal{G}(\iota_e)$ or of $\mathcal{G}(\tau_e)$. Since a group H cannot be the union of two proper subgroups, it follows that $\mathcal{G}(\iota_e)$ or $\mathcal{G}(\tau_e)$ is surjective. Without loss of generality, $\mathcal{G}(\iota_e)$ and $\mathcal{G}(\tau_e)$ are both injective and so have images of the same order. Hence $\mathcal{G}(\iota_e)$ and $\mathcal{G}(\tau_e)$ are *both* surjective. Now H, as image of $\mathcal{G}(e)$, is finite, and further, is a normal subgroup of G such that G/H is the infinite cyclic group generated by the image of $q(e)$. Thus we are in case (ii)(b).

We may now suppose that $X = T$, so X is a tree. Here G is the colimit of the vertex groups amalgamating the edge groups. Consider any edge e of X. If e is deleted from X we are left with two connected components X_0, X_1 say. Let G_0, G_1 denote the subgroups of G generated by the vertex groups as the vertices range over X_0, X_1 respectively. Then it is clear that $G \simeq G_0 \coprod_{\mathcal{G}(e)} G_1$. On replacing $\mathcal{G}(e)$ by its image in G, we may view $\mathcal{G}(e)$ as a subgroup of G_0, G_1. Consider any $x \in G_0$, and any $y, z \in G_1$. By assumption xy and xz do not freely generate a free subgroup of G, and by the normal form for coproducts with amalgamation, this forces one of $x, y, z, y^{-1}z$ to lie in $\mathscr{G}(e)$. This proves that either $G_0 = \mathscr{G}(e)$ or $(G_1 : \mathscr{G}(e)) \leqslant 2$. By symmetry, there are then three possibilities: $(G_0 : \mathscr{G}(e)) = (G_1 : \mathscr{G}(e)) = 2$ or $G_0 = \mathscr{G}(e)$ or $G_1 = \mathscr{G}(e)$. If the first possibility holds for any edge e then we are in case (ii)(c), so we may assume that the second or third possibilities hold for each edge e. If $\mathscr{G}(\iota e) = \mathscr{G}(e) = \mathscr{G}(\tau e)$ then e may be replaced by a vertex in the representation of G as the fundamental group of a graph of finite groups, so we may assume that there are no such edges for $\mathscr{G}$. Let v_0, v_1 denote the vertices of e in X_0, X_1 respectively. Then either

$$G_0 = \mathscr{G}(v_0) = \mathscr{G}(e) \subset \mathscr{G}(v_1) \subseteq G_1$$

or

$$G_0 \supseteq \mathscr{G}(v_0) \supset \mathscr{G}(e) = \mathscr{G}(v_1) = G_1.$$

We summarise this as follows.

(2) For any vertex v_0 of any edge e, if $\mathscr{G}(v_0) = \mathscr{G}(e)$ then $\mathscr{G}(v_0) \supseteq \mathscr{G}(v)$ for all vertices v of T for which the geodesic from v to v_0 does not traverse e (that is, for all $v \in V(X_0)$).

(3) For any vertex v_0 of any edge e, if $\mathscr{G}(v_0) \supset \mathscr{G}(e)$ then $\mathscr{G}(v_0) \supseteq \mathscr{G}(v)$ for all vertices v for which the geodesic from v to v_0 traverses e (that is, for all $v \in V(X_1)$).

Now choose a sequence, possibly infinite, $v_0, e_0, v_1, e_1, v_2, \ldots$ such that the vertices of e_i are v_i and v_{i+1}, and $\mathscr{G}(v_i) = \mathscr{G}(e_i) \subset \mathscr{G}(v_{i+1})$, for $i = 0, 1, \ldots$. If this sequence is finite and cannot be extended then it stops at some v_n with $\mathscr{G}(v_n) \supset \mathscr{G}(e)$ for every edge e which has v_n as a vertex. Then by (3), $\mathscr{G}(v_n) \supseteq \mathscr{G}(v)$ for all vertices v, so $G = \mathscr{G}(v_n)$ is finite and we are in case (ii)(a). Thus we may assume that the sequence is infinite. For each vertex v, the geodesic from v to v_0 must fail to traverse some e_n, so the geodesic from v to v_n does not traverse e_n, so by (2), $\mathscr{G}(v_n) \supseteq \mathscr{G}(v)$. Thus $G = \bigcup_{n \geqslant 0} \mathscr{G}(v_n)$ is countably infinite and locally finite, and we are again in case (ii)(a).

(ii) $\Rightarrow$ (i). It is clear from the preceding part of the proof how each of the types (ii)(a), (ii)(b), (ii)(c) occur as the fundamental group of a graph of finite groups, and it remains to show they do not have free subgroups of rank 2. This is clear for (ii)(a). Since factoring out a finite normal subgroup does not affect having a free subgroup of rank 2, we see that (ii)(b) also cannot have a free subgroup of rank 2. In (ii)(c), V is a finite normal subgroup of A and B, and hence of G, and moreover $G/V \simeq \mathbb{Z}_2 \coprod \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the cyclic group of order 2. Since $\mathbb{Z}_2 \coprod \mathbb{Z}_2$ has a cyclic subgroup of index 2, it follows that (ii)(c) cannot have a free subgroup of rank 2. This completes the proof.

Conditions (ii)(b) and (c) are quite well understood. It is straightforward to show that (ii)(c) is equivalent to G being finite-by-$\mathbb{Z}_2 \coprod \mathbb{Z}_2$, from which it can be deduced that (ii)(b) and (c) together are equivalent to G being finite-by-($\mathbb{Z}$-by-$\mathbb{Z}_2$), which in turn is equivalent to G being (finite-by-$\mathbb{Z}$)-by-$\mathbb{Z}_2$. It can be shown further that these latter conditions are equivalent to G being $\mathbb{Z}$-by-finite, *cf.* [**10**; pp. 178–9, p. 615] or [**3**; pp. 29–31].

In [**6**], Goursaud and Valette characterized right hereditary group rings among the group rings RG where G is *nilpotent*. Clearly a nilpotent group has no free subgroups of rank 2, and does not have $\mathbb{Z}_2 \coprod \mathbb{Z}_2$ as a homomorphic image, so to be the fundamental group of a connected graph of finite groups, a nilpotent group must be either countable and locally finite, or finite-by-$\mathbb{Z}$; these are precisely the groups arrived at by Goursaud and Valette.

2. *Background*

In this section we assemble various known results that will be used in the proof of Theorem 1.

LEMMA 3. *If G is a finite R^{-1}-group and M a right RG-module then M is isomorphic to an RG-summand of $M \otimes_R RG$.*

Proof. The map $M \to M \otimes_R RG, m \mapsto |G|^{-1}\sum_{g\in G} mg^{-1} \otimes g$ is right RG-linear and is a left inverse of the multiplication map $M \otimes_R RG \to M$. This completes the proof.

We now need the notion of a *connected graph of rings*, defined as any functor $\mathscr{S}: X \to \mathscr{R}ings$, where X is a connected graph. For any spanning tree T of X we define the *fundamental ring* S of $\mathscr{S}$ (*with respect to* T) to be the ring that is universal with the following properties: for each vertex v of X there is given a ring homomorphism $\mathscr{S}(v) \to S$; for each edge e of X there is given a unit $q(e)$ of S such that $q(e)^{-1}.s^{\iota e}.q(e) = s^{\tau e}$ for all $s \in \mathscr{S}(e)$, and $q(e) = 1$ if e is an edge of T. It can be shown, exactly as for groups, that the isomorphism class of S is independent of the choice of T.

The following is taken from [**4**; Section 5] but is given here with more details.

THEOREM 4. *If $\mathscr{S}: X \to \mathscr{R}ings$ is a connected graph of rings, and S is the fundamental ring of $\mathscr{S}$, then for any right S-module M the following sequence of right S-modules is exact:*

$$\begin{array}{ccccc} 0 \to \bigoplus_{E(X)} M \otimes_{\mathscr{S}(e)} S & \longrightarrow & \bigoplus_{V(X)} M \otimes_{\mathscr{S}(v)} S & \longrightarrow & M \to 0 \\ m \otimes_e s & \mapsto & m \otimes_{\iota e} q(e)s - mq(e) \otimes_{\tau e} s, & & m \otimes_v s \mapsto ms \end{array} \qquad (4)$$

Here M is made a right $\mathscr{S}(e)$-module via the map $\mathscr{S}(e) \to \mathscr{S}(\iota e) \to S$, and S is made a left $\mathscr{S}(e)$-module via the map $\mathscr{S}(e) \to \mathscr{S}(\tau e) \to S$.

Proof. In the case where M is the S-bimodule S, (4) is a sequence of S-bimodules

$$0 \to \bigoplus_{E(X)} S \otimes_{\mathscr{S}(e)} S \to \bigoplus_{V(X)} S \otimes_{\mathscr{S}(v)} S \to S \to 0, \qquad (5)$$

and it is the exactness of this sequence that we shall verify first.

We can present S as the S-bimodule with

(i) generators $\bar{v}, v \in V(X)$;

(ii) relations saying that for all $v \in V(X), \bar{v}$ commutes with all elements of $\mathscr{S}(v)$;

(iii) relations saying that for all $e \in E(X)$, $\overline{\iota e}.q(e) = q(e).\overline{\tau e}$.

Notice that since $q(e) = 1$ for all $e \in E(T)$, (iii) implies that all the $\bar{v}$ are equal.

Now (i) and (ii) by themselves present the S-bimodule

$$A = \bigoplus_{V(X)} S\bar{v}S = \bigoplus_{V(X)} S \otimes_{\mathscr{S}(v)} S$$

which maps onto S by sending $\bar{v} = 1 \otimes_v 1$ to 1. We claim that the kernel of this map equals the S-bimodule, B, presented with

(i′) generators $\bar{e}$, $e \in E(X)$;

(ii′) relations saying that for all $e \in E(X)$, $s^{\iota e} . \bar{e} = \bar{e} . s^{\tau e}$ for all $s \in \mathscr{S}(e)$.

It is readily verified that the (S, S)-linear map $B \to A$ sending e to $\overline{\iota e} . q(e) - q(e) . \overline{\tau e}$ is well-defined, and it clearly maps onto the kernel of the map $A \to S$.

Now

$$B = \bigoplus_{E(X)} S\bar{e}S = \bigoplus_{E(X)} S \otimes_{\mathscr{S}(e)} S$$

where S is made a right $\mathscr{S}(e)$-module via $\mathscr{S}(e) \to \mathscr{S}(\iota e) \to S$, and made a left $\mathscr{S}(e)$-module via $\mathscr{S}(e) \to \mathscr{S}(\tau e) \to S$.

We have a presentation $B \to A \to S \to 0$, and to obtain (5) we shall show that $B \to A$ has a right inverse, and hence is injective.

For any vertices u, v of X there is a unique geodesic in T from u to v, say $u = v_0, e_1^{\varepsilon_1}, v_1, \ldots, e_n^{\varepsilon_n}, v_n = v$ where for $i = 1, \ldots, n$, e_i is an edge in T with vertices v_{i-1} and v_i, and ε_i indicates the orientation of e_i in the geodesic, and is defined as

$$\varepsilon_i = \begin{cases} +1 & \text{if } \iota e_i = v_{i-1} \text{ and } \tau e_i = v_i \\ -1 & \text{if } \iota e_i = v_i \text{ and } \tau e_i = v_{i-1}. \end{cases}$$

We write $T(u, v)$ for the element $\varepsilon_1 \bar{e}_1 + \ldots + \varepsilon_n \bar{e}_n$ of B.

Recall that a *derivation* $d: S \to B$ is an additive map satisfying $d(xy) = (dx)y + x(dy)$, for all $x, y \in S$. These are precisely the maps for which

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} : \mathscr{S} \to \begin{pmatrix} S & B \\ 0 & S \end{pmatrix}$$

is a ring homomorphism. By the universal property of S, giving such a ring homomorphism is equivalent to specifying ring homomorphisms

$$\begin{pmatrix} 1 & d^v \\ 0 & 1 \end{pmatrix} : S(v) \to \begin{pmatrix} S & B \\ 0 & S \end{pmatrix}, \qquad v \in V(X),$$

together with units

$$\begin{pmatrix} q(e) & d(q(e)) \\ 0 & q(e) \end{pmatrix} \quad \text{of} \quad \begin{pmatrix} S & B \\ 0 & S \end{pmatrix}, \qquad e \in E(X),$$

satisfying the two conditions

$$\begin{pmatrix} s^{\iota e} & d^{\iota e}(s^{\iota e}) \\ 0 & s^{\iota e} \end{pmatrix} \begin{pmatrix} q(e) & d(q(e)) \\ 0 & q(e) \end{pmatrix} = \begin{pmatrix} q(e) & d(q(e)) \\ 0 & q(e) \end{pmatrix} \begin{pmatrix} s^{\tau e} & d^{\tau e}(s^{\tau e}) \\ 0 & s^{\tau e} \end{pmatrix}$$

for all $e \in E(X)$ and all $s \in \mathscr{S}(e)$; and

$$\begin{pmatrix} q(e) & d(q(e)) \\ 0 & q(e) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for all $e \in E(T)$. It follows that giving a derivation $d: S \to B$ is equivalent to specifying derivations $d^v: \mathscr{S}(v) \to B$, $v \in V(X)$, and elements $d(q(e))$ of B, $e \in E(X)$ satisfying the two conditions

$$d^{\iota e}(s^{\iota e}) \,.\, q(e) + s^{\iota e} \,.\, d(q(e)) = d(q(e)) \,.\, s^{\tau e} + q(e) \,.\, d^{\tau e}(s^{\tau e}) \tag{6}$$

for all $e \in E(X)$ and all $s \in \mathscr{S}(e)$; and

$$d(q(e)) = 0 \tag{7}$$

for all $e \in E(T)$.

For $b \in B$, $s \in S$, $[b, s]$ denotes $bs - sb \in B$. For $b \in B$, the map $ad(b): S \to B$, $s \mapsto [b, s]$ is a derivation, called the *inner derivation induced by b*. We shall usually write $[b, -]$ for $ad(b)$.

Let u be a vertex of X. Define a derivation $d_u: S \to B$ as follows:

for each $v \in V(X)$ let $d_u^v: \mathscr{S}(v) \to B$ be $[T(u, v), -]$;

for each $e \in E(X)$ let $d_u(q(e)) = T(u, \iota e) \,.\, q(e) + \bar{e} + q(e) \,.\, T(\tau e, u)$.

It is straightforward to verify that the conditions (6), (7) are satisfied, so we obtain a derivation $d_u: S \to B$.

Notice that for any two vertices u_1, u_2 of X, the derivation $d_{u_1} - d_{u_2}: S \to B$ agrees with $[T(u_1, u_2), -]$ on each $\mathscr{S}(v), v \in V(X)$, and on each $q(e), e \in E(X)$; hence $d_{u_1} - d_{u_2} = [T(u_1, u_2), -]$.

Fix a vertex v_0 of X. For each vertex u of X, d_u vanishes on $\mathscr{S}(u)$ so d_u is left $\mathscr{S}(u)$-linear. Now the left $\mathscr{S}(u)$-linear map $S \to B$, $s \mapsto d_u(s) + sT(u, v_0)$, induces a left S-linear map from $S\bar{u}S = S \otimes_{\mathscr{S}(u)} S$ to B that sends $s_1 \bar{u} s_2$ to $s_1\, d_u(s_2) + s_1 s_2 T(u, v_0)$. A left S-linear map to B is thus defined for each vertex u of X, and we can take the direct sum over all u to get a left S-linear map from $\oplus\, S\bar{u}S = A$ to B. We claim that this is a left inverse to the map $B \to A$. To see this, observe that as left S-module, B is generated by the $\bar{e}s$, $e \in E(X)$, $s \in S$, so it suffices to check the behaviour of the composite $B \to A \to B$ on the elements $\bar{e}s$. The map $B \to A$ carries $\bar{e}s$ to $\overline{\iota e} \,.\, q(e) \,.\, s - q(e) \,.\, \overline{\tau e} \,.\, s$, and the map $A \to B$ carries this element back to

$$\begin{aligned}
&(d_{\iota e}(q(e)s) + q(e) \,.\, s \,.\, T(\iota e, v_0)) - (q(e) \,.\, d_{\tau e}(s) + q(e) \,.\, s \,.\, T(\tau e, v_0)) \\
&\quad = d_{\iota e}(q(e)) \,.\, s + q(e) \,.\, d_{\iota e}(s) + q(e) \,.\, s \,.\, T(\iota e, v_0) - q(e) \,.\, d_{\tau e}(s) - q(e) \,.\, s \,.\, T(\tau e, v_0) \\
&\quad = d_{\iota e}(q(e)) \,.\, s + q(e) \,.\, (d_{\iota e} - d_{\tau e})(s) + q(e) \,.\, s \,.\, (T(\iota e, v_0) - T(\tau e, v_0)) \\
&\quad = d_{\iota e}(q(e)) \,.\, s + q(e) \,.\, [T(\iota e, \tau e), s] + q(e) \,.\, s \,.\, T(\iota e, \tau e) \\
&\quad = d_{\iota e}(q(e)) \,.\, s + q(e) \,.\, T(\iota e, \tau e) \,.\, s \\
&\quad = \bar{e}s, \quad \text{by definition of } d_{\iota e}(q(e)).
\end{aligned}$$

Hence $B \to A \to B$ is the identity map, so $B \to A$ is injective, and we have proved that (5) is exact. Now applying $M \otimes_S -$ to (5) gives (4), and this is still exact since (5) is split exact as a sequence of left S-modules. This completes the proof.

The obvious application of Theorem 4 is to the group ring of a fundamental graph of groups, and here we have the following consequence.

COROLLARY 5. *If G is the fundamental group of a connected graph of groups $\mathcal{G}\colon X \to \mathcal{G}\mathit{roups}$, then for any right RG-module M,*

$$0 \to \bigoplus_{E(X)} M \otimes_{R\mathcal{G}(e)} RG \to \bigoplus_{V(X)} M \otimes_{R\mathcal{G}(v)} RG \to M \to 0 \tag{8}$$

is an exact sequence of right RG-modules.

Proof. Let $\mathcal{S}\colon X \to \mathcal{R}\mathit{ings}$ be the composite of the functors $\mathcal{G}\colon X \to \mathcal{G}\mathit{roups}$, $R\colon \mathcal{G}\mathit{roups} \to \mathcal{R}\mathit{ings}$, where R sends each group H to the corresponding group ring RH. The corollary is now immediate from Theorem 4.

It was the following special case of Corollary 5 that originally suggested Theorem 4.

COROLLARY 6 (Chiswell). *If G is the fundamental group of a connected graph of groups $\mathcal{G}\colon X \to \mathcal{G}\mathit{roups}$, then*

$$0 \to \bigoplus_{E(X)} R \otimes_{R\mathcal{G}(e)} RG \to \bigoplus_{V(X)} R \otimes_{R\mathcal{G}(v)} RG \to R \to 0 \tag{9}$$

is an exact sequence of right RG-modules, where R is made an RG-module with trivial G-action.

If, in Corollary 6, for each $x \in \mathrm{ob}(X) = E(X) \cup V(X)$, $\mathcal{G}(x)$ is a finite R^{-1}-group, then R is an $R\mathcal{G}(x)$-summand of $R\mathcal{G}(x)$, by Lemma 3, so is projective as right $R\mathcal{G}(x)$-module, and hence (9) is a projective RG-resolution of R. In other words, $\mathrm{cd}_R G \leqslant 1$ if G is the fundamental group of a connected graph of finite R^{-1}-groups. There is an alternative way of expressing the former condition. Let us write $\varepsilon\colon RG \to R$ for the augmentation map, and $\omega = \omega(RG)$ for the augmentation ideal, $\mathrm{Ker}\,\varepsilon$, so $0 \to \omega \to RG \to R \to 0$ is an exact sequence of right RG-modules. Thus $\mathrm{cd}_R G \leqslant 1$ is equivalent to ω being projective as right RG-module. This verifies a fact proved by D. E. Cohen, that if G is the fundamental group of a connected graph of finite R^{-1}-groups then ω is projective as right RG-module. It is the *converse* of this fact that we shall need, and the following reduces the problem to the case where R is commutative.

LEMMA 7 (Swan [**12**; Proposition 3.3]). *Let $K \to R$ be a ring homomorphism.*

(i) $\mathrm{cd}_R G \leqslant \mathrm{cd}_K G$.

(ii) *If $K \to R$ is an embedding and if the right K-module R/K is projective (e.g. if K is a field) then $\mathrm{cd}_R G = \mathrm{cd}_K G$.*

(iii) *For any field k such that $k \otimes_{\mathbb{Z}} R$ is nonzero, $\mathrm{cd}_k G \leqslant \mathrm{cd}_R G$; further, there exists at least one such (prime) field k.*

Proof. (i) Let $\mathscr{P} \to K \to 0$ be a projective KG-resolution of K. Since K is projective as right K-module, this sequence is right K-split, so remains exact under

$$- \otimes_K R = - \otimes_{KG} KG \otimes_K R = - \otimes_{KG} RG.$$

Thus we get a projective RG-resolution of R, $\mathscr{P} \otimes_{KG} RG \to R \to 0$, and it follows that $\mathrm{cd}_R G \leqslant \mathrm{cd}_K G$.

(ii) Suppose that R/K is right K-projective. Then $R \simeq K \oplus (R/K)$ as right K-module, so in particular R is right K-projective. Since $RG = R \otimes_K KG$, RG is right KG-projective, so any projective RG-resolution of R is already a projective KG-resolution of R. Hence

$$\mathrm{proj.dim}_{KG} R \leqslant \mathrm{proj.dim}_{RG} R = \mathrm{cd}_R G.$$

Since K is a K-summand of R it is therefore a KG-summand of R, so

$$\mathrm{proj.dim}_{KG} R \geqslant \mathrm{proj.dim}_{KG} K = \mathrm{cd}_K G.$$

Thus $\mathrm{cd}_K G \leqslant \mathrm{cd}_R G$ and together with (i) this proves (ii).

(iii) There is a ring homomorphism $R \to k \otimes_{\mathbb{Z}} R$ so by (i), $\mathrm{cd}_{k \otimes R}\ G \leqslant \mathrm{cd}_R G$. There is a ring homomorphism $k \to k \otimes_{\mathbb{Z}} R$ so by (ii), $\mathrm{cd}_{k \otimes R}\ G = \mathrm{cd}_k G$. Finally, if $(\mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} R = 0$ for all prime integers p then R is clearly a $\mathbb{Q}$-algebra. This completes the proof.

We now come to the following beautiful result.

THEOREM 8 (Dunwoody). *If the augmentation ideal of RG is projective then G is the fundamental group of some connected graph of finite R^{-1}-groups.*

Proof. This is proved in [5]; although it is asserted in [5] only in the case where R is commutative, the proof given there does not make essential use of this hypothesis, *cf.* [13]. In fact the general case can be derived from the commutative case by the following argument of Swan. Let the augmentation ideal of RG be projective. Then $\mathrm{cd}_R G \leqslant 1$, so by Swan's result, Lemma 7 (iii), there exists a prime field k such that $\mathrm{cd}_k G \leqslant \mathrm{cd}_R G \leqslant 1$. Hence by the commutative ring case, G is the fundamental group of a connected graph of finite k^{-1}-groups, and it remains to prove these can be taken to be R^{-1}-groups. Without loss of generality the vertex groups and edge groups are all subgroups of G, so it suffices to prove that G itself is an R^{-1}-group. It is well known that this holds, and the argument is as follows. Suppose that an element g of G has order n, and consider the subgroup $\langle g \rangle$ of G generated by G. Since RG is free as right $R\langle g \rangle$-module (on any set of coset representatives for $\langle g \rangle$ in G), any projective RG-resolution of R is already a projective $R\langle g \rangle$-resolution of R, so $\mathrm{cd}_R \langle g \rangle \leqslant \mathrm{cd}_R G \leqslant 1$. Hence the augmentation ideal $(1-g)R\langle g \rangle$ of $R\langle g \rangle$ is projective, so $(1-g)R\langle g \rangle = eR\langle g \rangle$ for some idempotent e of $R\langle g \rangle$. The element $f = 1 - e$ then satisfies $f(1-g) = 0$ so $f = \sum_{1}^{n} g^i r$ for some $r \in R$. Further, $1 = 1 - \varepsilon(e) = \varepsilon(f) = nr$, so n is invertible in R. This proves that G is an R^{-1}-group and completes the proof.

3. *Necessity of* (H1), (H2) *or* (H3)

Throughout this section we have the fixed hypothesis that RG is right hereditary.

We begin by showing that R is right hereditary. For any right ideal I of R, the induced right ideal IRG of RG is right RG-projective by hypothesis, and hence is right R-projective. But as R-module, $IRG = I \oplus I\omega$, and hence I is right R-projective. This proves that R is right hereditary.

Further, ω is right RG-projective by hypothesis, so by Dunwoody's theorem, Theorem 8, G is the fundamental group of a connected graph of finite R^{-1}-groups, and we know that G is itself an R^{-1}-group. There are now three cases:

(i) G finite;

(ii) G not periodic;

(iii) G periodic and infinite.

These will be shown to entail (H3), (H1), (H2) respectively. The verification makes use of the following.

LEMMA 9. *If RG is right hereditary then for each right ideal I of R there exists a pair of RG-linear maps $\phi_1 : \omega \to I\omega$, $\phi_2 : IRG \to I\omega$, satisfying*

$$(i - \phi_2(i)) \,.\, (g - 1) = \phi_1(g - 1) \,.\, i \tag{10}$$

for all $i \in I$, $g \in G$.

Proof. Consider the following sequence of right RG-modules

$$\begin{array}{ccccccc} 0 \to \omega \cap IRG \to & \omega \oplus IRG & \to \omega + IRG \to 0. \\ x \mapsto (x, x), & (y, z) \mapsto y - z & \end{array}$$

It is obviously exact, and the last term is projective by hypothesis, so the sequence splits. Since $\omega \cap IRG = I\omega$, the splitting gives an RG-linear map $\omega \oplus IRG \to I\omega$, say $(y, z) \mapsto \phi_1(y) + \phi_2(z)$, such that for all $i \in I$, $g \in G$, $\phi_1(i \,.\, (g - 1)) + \phi_2(i \,.\, (g - 1)) = i \,.\, (g - 1)$. Since i and $g - 1$ commute, this can be written $\phi_1(g - 1) \,.\, i + \phi_2(i) \,.\, (g - 1) = i \,.\, (g - 1)$, and the lemma is proved.

Case (i). We have already seen that R is right hereditary so if G is finite then (H3) holds.

Case (ii). Suppose that G has an element g of infinite order. Let I be any right ideal of R, and let ϕ_1, ϕ_2 be as in Lemma 9. Choose a transversal T for the subgroup $\langle g \rangle$ in G; that is, every element of G can be expressed uniquely in the form tg^n, $t \in T$, $n \in \mathbb{Z}$. Define $\psi : RG \to R$ to be the right R-linear map that sends tg^n, $t \in T, n \in \mathbb{Z}$, to n. Then for any $x \in RG$, $\psi(x \,.\, (g - 1)) = \varepsilon(x)$. So applying ψ to (10) gives

$$\varepsilon(i - \phi_2(i)) = \psi(\phi_1(g - 1) \,.\, i)$$

for all $i \in I$. Now $\phi_2(i) \in \omega$, so this says $i = i_0 \,.\, i$ for all $i \in I$, where

$$i_0 = \psi(\phi_1(g - 1)) \in \psi(I\omega) \subseteq \psi(IRG) \subseteq I.$$

This proves that I is a right R-linear summand of R, for every right ideal I of R. Hence R is completely reducible and (H1) holds.

Remark. This result is well-known, and given a certain amount of homological machinery is actually rather trivial. However, we are trying to keep this account as elementary as possible. For another proof that does not require sophisticated homological methods *cf.* [7; Lemmes 1.2, 2.3].

Case (iii). Suppose that G is periodic and infinite. As G is the fundamental group of a connected graph of finite groups, it follows from Theorem 2 that G is countable and locally finite.

It remains to describe R in this case. This was accomplished by Goursaud and Valette who showed in [7; Proposition 2.10] that R is right $\aleph_0$-Noetherian and von Neumann regular.

For completeness, let us recall their elegant proof that R is right $\aleph_0$-Noetherian. Consider any right ideal I of R. By a theorem of Kaplansky [8], the projective right RG-module, $IRG + \omega$, is a direct sum of countably generated right ideals. But we know that ω is countably generated as right RG-module, so is contained in the sum of countably many of these right ideals. The sum of the remaining right ideals then has zero intersection with ω, so in particular left annihilates ω. But G is infinite so the left annihilator of ω is 0, and this proves that $IRG + \omega$ is countably generated as right RG-module. By applying ε we see that I is countably generated as right R-module, as desired.

To see that R is von Neumann regular consider any $i \in R$. For $I = iR$ let ϕ_1, ϕ_2 be as in Lemma 9. Then for every $g \in G$,

$$(i - \phi_2(i)) . (g - 1) = \phi_1(g - 1) . i \in I\omega i \subseteq iRGi.$$

Since G is infinite there exists a $g \in G$ such that the supports of $i - \phi_2(i)$ and $(i - \phi_2(i)) . g$ are disjoint, so $i - \phi_2(i) \in iRGi$. Applying ε gives $i \in iRi$, as desired. (For another proof that R is von Neumann regular *cf.* [7].)

This completes the proof of the necessity of (H1), (H2) or (H3).

4. *Sufficiency of* (H1), (H2) *or* (H3)

Sufficiency of (H1). Suppose that R is completely reducible and that G is the fundamental group of a connected graph of finite R^{-1}-groups, $\mathcal{G} : X \to \mathcal{G}roups$. Let M be a right RG-module and let $x \in \text{ob}(X) = E(X) \cup V(X)$. Since R is completely reducible, M is projective as right R-module, so $M \otimes_R R\mathcal{G}(x)$ is projective as right $R\mathcal{G}(x)$-module. But by Lemma 3, M is an $R\mathcal{G}(x)$-summand of this module, so M itself is $R\mathcal{G}(x)$-projective. Hence (8) gives a projective RG-resolution of M of length 1. This proves that RG is right hereditary in this case.

The remaining cases are essentially well known but are recalled for completeness.

Sufficiency of (H2). Suppose that R is right $\aleph_0$-Noetherian, von Neumann regular and that G is a countable, locally finite, R^{-1}-group. Thus G is the union of its finite R^{-1}-subgroups H. For any such H, consider any right RH-module M. Since R is von Neumann regular, M is flat as right R-module so $M \otimes_R RH$ is flat as right RH-module.

But by Lemma 3, M is an RH-summand of this module so M itself is RH-flat. This proves that RH is von Neumann regular for all H, and hence $RG = \bigcup RH$ is von Neumann regular. To see that RG is right $\aleph_0$-Noetherian, consider any right ideal I of RG. Since RG is countably generated as right R-module and R is $\aleph_0$-Noetherian, it follows that I is countably generated as R-module and hence as RG-module. Thus RG is right $\aleph_0$-Noetherian, von Neumann regular, and such a ring is right hereditary by a simple argument, *cf.* [**9**].

Sufficiency of (H3). Suppose that R is right hereditary and that G is a finite R^{-1}-group. Let I be a right ideal of RG. Then I is an R-submodule of the free R-module RG, so is projective as R-module, since R is right hereditary. By Lemma 3, I is isomorphic to an RG-summand of the projective RG-module $I \otimes_R RG$, so I is right RG-projective. Hence RG is right hereditary in this case.

This completes the proof of Theorem 1.

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