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The HNN Construction for Rings

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1. INTRODUCTION

The two most important constructions in combinatorial group theory are the coproduct with amalgamation and the construction introduced by Higman, Neumann, and Neumann in [17] and therefore subsequently called the HNN construction; these two constructions generally occur closely intertwined. By contrast, the efforts of combinatorial ring theory have been concentrated on coproducts with amalgamation (e.g., [2]) while interest in the HNN ring construction has been confined to algebraic K-theory (e.g., [16, 26]). The purpose of this article is to present a unified foundation for the theory of the HNN ring construction based largely on analogy with what is known for coproducts.

Let us fix our conventions and notation.

All rings will be associative with a 1, and the 1 is to be respected by ring homomorphisms and module actions.

Let K, A be two rings, and $\alpha: K \to A$, $\beta: K \to A$ two ring homomorphisms. The associated *HNN construction* is the ring R presented on the generators and relations of A together with two new generators t, t^{-1} and new relations saying that t, t^{-1} are mutually inverse and that $t^{-1}\alpha(k)\, t = \beta(k)$ for all $k \in K$. With this presentation, R is an A-*ring*; that is, there is specified a ring homomorphism $\eta: A \to R$.

Throughout this article the above symbols will retain the same meaning.

One can specify R in terms of universal properties as the A-ring universal

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with a distinguished unit t such that the inner automorphism $i_t: R \to R$, $r \mapsto t^{-1}rt$, makes the diagram

$$\begin{array}{ccccc} & & A & \xrightarrow{\ \eta\ } & R \\ & \overset{\alpha}{\nearrow} & & & \\ K & & & & \big\downarrow i_t \\ & \underset{\beta}{\searrow} & & & \\ & & A & \xrightarrow{\ \eta\ } & R \end{array}$$

commute. There is another way of describing this. A right A-module M_A can be viewed as a right K-module by pullback along either α or β and the resulting K-module will be denoted M_α or M_β, respectively, and a similar convention applies on the left. Right multiplication by t then defines an isomorphism ${}_RR_\alpha \to {}_RR_\beta$ of (R, K) bimodules, and R is the A-ring universal with a distinguished such isomorphism. In the notation of [3], $R = A\langle t, t^{-1}: \bar{A}_\alpha \cong \bar{A}_\beta\rangle$, and in the modified notation introduced in [4], $R = A\langle t, t^{-1}: {}^{\otimes}A_\alpha \cong {}^{\otimes}A_\beta\rangle$.

Throughout we shall use α to make A into a K-ring and shall view β as a homomorphism from K to the K-ring A. With this convention in mind we denote the HNN construction by $A_K\langle t, t^{-1}; \beta\rangle$. This notation is intended to be reminiscent of the familiar case where α is an isomorphism, so we can identify K and A, and the HNN construction reduces to the skew Laurent-polynomial ring $A[t, t^{-1}; \beta]$.

It is useful to know that R can be written as the skew Laurent polynomial ring $S[t, t^{-1}; \sigma]$, where S is the countable coproduct $S = \cdots {}_\alpha A_\beta \coprod_K {}_\alpha A_\beta \coprod_K {}_\alpha A_\beta \cdots$ whose image in R is $\cdots t^2At^{-2} \coprod_{tKt^{-1}} tAt^{-1} \coprod_K A \cdots$ and σ is the right-shift automorphism corresponding to conjugation by t. It is straightforward to verify that $A \to S[t, t^{-1}; \sigma]$ has the universal HNN property and hence $S[t, t^{-1}; \sigma] = A_K\langle t, t^{-1}; \beta\rangle$. Since skew Laurent-polynomial rings are reasonably well understood it might be hoped that information about coproducts now easily translates to HNN constructions; this is certainly true in some cases, but usually information gets lost in passing to the skew Laurent-polynomial ring and we must resort to other methods.

So far we have recounted the basic folklore on HNN ring constructions, and we can now briefly sketch what is done in this article. In Section 2 we recall some global dimension inequalities which have appeared elsewhere [14, 21]. Under quite mild hypotheses that cannot be altogether omitted, r.gl.dim R is bounded above by

$$\begin{cases} \text{r.gl.dim}\, A & \text{if r.gl.dim}\, K < \text{r.gl.dim}\, A \\ 1 + \text{r.gl.dim}\, A & \text{if r.gl.dim}\, K \geqslant \text{r.gl.dim}\, A. \end{cases}$$

In Section 3, the procedure developed by Cohn [8] for analysing a coproduct as a direct limit is tailored to fit the description of the HNN construction used by Waldhausen [26] who attributes it to Cappell. In Section 4, still following Cohn, we find the machinery of Section 3 works best where ${}_{\alpha}A$, ${}_{\beta}A$ are faithfully flat; it is shown in Section 5 that this hypothesis suffices for, among other things, R to be right coherent whenever A is right coherent and K is right Noetherian, which generalizes [26, Proposition 4.1(2)]. An example adapted from [14] shows that faithful flatness cannot be weakened to, say, flatness. (It should be noted that Cohn's result permits a similar generalization of the corresponding theorem for coproducts [7]; [16, Remark 1.10]; [26, Proposition 4.1(1)]).

The next section, 6, presents quite general conditions under which the direct limit structure on R can be converted into a graded structure on left R-modules *induced* from A, that is, of the form $R \otimes {}_A M_0$ for some left A-module M_0. (Here again, the corresponding remarks for coproducts have yet to appear in the literature.) At this point we restrict α, β to be injective and K to be *completely reducible*, that is, a finite direct product of matrix rings over skew fields or, equivalently, $\text{r.gl.dim}\, K = 0$. Sections 7–9 are devoted to reproducing the HNN analogue of Bergman's thorough coproduct analysis [2]. In Section 7 an argument based on refining the graded structure of $R \otimes {}_A M_0$ shows that each of the submodules of $R \otimes {}_A M_0$ is (isomorphic to) an R-module induced from A. Since every free R-module is induced from A, this says in particular that every projective R-module is induced from A. A similarly based argument in Section 8 gives a useful decomposition for surjective homomorphisms between finitely generated R-modules induced from A. One consequence is that the semigroup (under $\oplus$) of (isomorphism classes of) finitely generated left R-modules induced from A can be described as the coequalizer of the two semigroup homomorphisms from "finitely generated K-modules" to "finitely generated A-modules." The subsemigroup of all finitely generated projective R-modules can similarly be described as the coequalizer of the same two homomorphisms with their common codomain restricted to the subsemigroup of all finitely generated projective A-modules.

In Section 9, K is (further) restricted to be a skew field, and this forces R to acquire many of the module theoretic properties of A. From the preceding result, A and R have isomorphic semigroups of finitely generated projectives. If one of A, R is a fir or semifir or n-fir then so is the other. If A is an n-fir then the general linear group $GL_n(R)$ is generated by the subgroup $GL_n(A)$ together with

$$\begin{pmatrix} t & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

and the *elementary* matrices, that is, matrices which differ from the identity matrix in one off-diagonal entry. For $n = 1$, this says that if A has no zerodivisors then R has no zerodivisors, and the group of units of R is generated by t and the units of A (and is actually an HNN group extension, as was pointed out to me by Alexander Lichtman). The fact that if A is a fir then $A_K\langle t, t^{-1}; \beta\rangle$ is a fir is a substantial generalization of the well-known fact that the skew Laurent-polynomial ring $K[t, t^{-1}; \beta]$ is a fir. (Recall that K is a skew field here.)

In Section 10 we apply the theory to give a relatively short proof of the Lewin–Lewin embedding theorem [21]: If G is a torsion-free one-relator group and K is a skew field then the group ring KG can be embedded in a skew field.

In Section 11 another application of the theory gives purely algebraic proofs of some of Waldhausen's results [26]; for example, if G is a torsion-free one-relator group and K is a regular Noetherian ring then the natural map $K_0(K) \to K_0(KG)$ is an isomorphism.

2. Homological Generalities

It is a law of nature that an A-ring R with a universal property has much of its homological character encoded in the "multiplication map" $R \otimes_A R \to R$, and in this respect HNN constructions are exemplary. All of the arguments and some of the results have been noted previously [4, 14, 21] but are recalled here for completeness.

As Cartan or Eilenberg observed [6, Proposition IX.3.2], for any ring homomorphism $\eta: A \to R$, the kernel of $R \otimes_A R \to R$, $x \otimes y \mapsto xy$, is the R-bimodule $\Omega_{R/A}$ presented on generators

$$dr \text{ (mapping to } 1 \otimes r - r \otimes 1), \qquad r \in R$$

and relations

$$\begin{aligned} &d(\eta a) = 0, && a \in A, \\ &d(r + s) = dr + ds, && r, s \in R, \\ &d(rs) = dr \cdot s + r \cdot ds, && r, s \in R. \end{aligned}$$

To see this, notice there is a well-defined additive map $R \otimes_A R \to \Omega_{R/A}$ sending each $r \otimes s$ to $r \cdot ds$, and the composite $\Omega_{R/A} \to R \otimes_A R \to \Omega_{R/A}$ must be the identity since it fixes the additive generators $q \cdot dr \cdot s$, $q, r, s \in R$; it follows readily that

$$0 \to \Omega_{R/A} \to R \otimes_A R \to R \to 0$$

is exact.

Applying this to the A-ring $R = A_K\langle t, t^{-1}; \beta\rangle$ with generators t, t^{-1} and relations $t\,t^{-1} = t^{-1}\,t = 1$, $t^{-1}\,\alpha(k)\,t = \beta(k)$ $(k \in K)$, we compute that $\Omega_{R/A}$ is the R-bimodule with generators dt, dt^{-1} and relations saying that the second generator $dt^{-1} = -t^{-1}\,dt\,t^{-1}$ is superfluous and that $\alpha(k)\,dt = dt\,\beta(k)$ $(k \in K)$. But the R-bimodule $R\,{}_\alpha\!\otimes_K {}_\beta R$ also is presented on one generator $1 \otimes 1$ with relations $\alpha(k)(1 \otimes 1) = (1 \otimes 1)\,\beta(k)$ $(k \in K)$. It follows that

$$R_\alpha \otimes_K {}_\beta R \cong \Omega_{R/A} \cong \operatorname{Ker}(R \otimes {}_A R \to R).$$

Hence the R-bimodule sequence

$$\begin{gathered} 0 \to R_\alpha \otimes_K {}_\beta R \to R \otimes {}_A R \to R \to 0, \\ 1 \otimes 1 \mapsto 1 \otimes t - t \otimes 1 \end{gathered}$$

is exact. In keeping with our preference for α over β, let us use the isomorphism ${}_\beta R \cong {}_\alpha R$ to write another exact R-bimodule sequence

$$\begin{gathered} 0 \to R \otimes {}_K R \to R \otimes {}_A R \to R \to 0, \\ 1 \otimes 1 \mapsto 1 \otimes 1 - t \otimes t^{-1}, \end{gathered} \tag{1}$$

where R is a K-ring via $\eta\alpha$.

As is usual with this procedure, there are immediate consequences.

THEOREM 1. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$ and M_R, N_R be right R-modules. If R_A and R_K are projective (or ${}_A R$ and ${}_K R$ are flat) then there is an exact triangle*

$$\begin{array}{ccc} \operatorname{Ext}_A(M,N) & \xrightarrow{\alpha^* - \beta^*} & \operatorname{Ext}_K(M,N) \\ & {\scriptstyle \eta^*}\nwarrow \quad \swarrow{\scriptstyle \delta} & \\ & \operatorname{Ext}_R(M,N) & \end{array} \tag{2}$$

of graded groups, natural in M and N. Here δ has degree $+1$, and α^*, β^*, η^* are the canonical homomorphisms coinduced by pullback along α, β, η, respectively. Further,

$$\text{r.gl.dim}\,R \leqslant \begin{cases} \text{r.gl.dim}\,A & \textit{if} \ \ \text{r.gl.dim}\,K < \text{r.gl.dim}\,A \\ 1 + \text{r.gl.dim}\,A & \textit{if} \ \ \text{r.gl.dim}\,K \geqslant \text{r.gl.dim}\,A. \end{cases} \tag{3}$$

Analogous results are true for homology, and the statements can be obtained by changing "N_R" to "${}_R N$," "projective" to "flat," "Ext" to "Tor," "+1" to "−1," "coinduced" to "induced," "r.gl.dim" to "w.gl.dim," and changing the direction of the arrows.

Proof. Since the last term of (1) is projective as right R-module, the sequence will remain exact under $\mathrm{Hom}_R(-, N)$ giving an exact sequence

$$\begin{gathered} 0 \to N \to \mathrm{Hom}_A(R, N) \to \mathrm{Hom}_K(R, N) \to 0, \\ f \mapsto (r \mapsto f(r) - f(rt) \cdot t^{-1}) \end{gathered} \tag{4}$$

of right R-modules. Now applying $\mathrm{Ext}_R(M, -)$ to (4) gives an exact triangle

$$\begin{array}{ccc} \mathrm{Ext}_R(M, \mathrm{Hom}_A(R, N)) & \longrightarrow & \mathrm{Ext}_R(M, \mathrm{Hom}_K(R, N)) \\ & \nwarrow \quad \swarrow \delta & \\ & \mathrm{Ext}_R(M, N) & \end{array} \tag{5}$$

of graded groups, where δ has degree $+1$.

Let us now proceed as far as possible on the supposition that R_A is projective. Here any injective A-resolution of N_A lifts under $\mathrm{Hom}_A(R, -)$ to an injective R-resolution of $\mathrm{Hom}_A(R, N)$, so the canonical map $\mathrm{Ext}_R(M, \mathrm{Hom}_A(R, N)) \to \mathrm{Ext}_A(M, N)$ is an isomorphism. Further, $\mathrm{Hom}_K(R, N) = \mathrm{Hom}_A(R, \mathrm{Hom}_K(A, N))$ so that

$$\begin{aligned} \mathrm{Ext}_R(M, \mathrm{Hom}_K(R, N)) &= \mathrm{Ext}_R(M, \mathrm{Hom}_A(R, \mathrm{Hom}_K(A, N))) \\ &\cong \mathrm{Ext}_A(M, \mathrm{Hom}_K(A, N)). \end{aligned}$$

Now (5) can be rewritten as

$$\begin{array}{ccc} \mathrm{Ext}_A(M, N) & \longrightarrow & \mathrm{Ext}_A(M, \mathrm{Hom}_K(A, N)) \\ & \nwarrow \quad \swarrow \delta & \\ & \mathrm{Ext}_R(M, N). & \end{array}$$

Writing *pd* for projective dimension and *id* for injective dimension we have

$$pd_R M \leqslant 1 + pd_A M \qquad \text{and} \qquad id_R N \leqslant \max\{id_A N,\ 1 + id_A \mathrm{Hom}_K(A, N)\}.$$

It follows from the latter that

$$\text{r.gl.dim}\, R \leqslant \max\{\text{r.gl.dim}\, A,\ 1 + \sup\{id_A \mathrm{Hom}_K(A, N)\}\},$$

where the supremum is taken over all right A-modules N. (Notice that an A-module of the form $\mathrm{Hom}_K({}_AA_K, N_K)$, where N is a right A-module, is, in the terminology of [18], *injective relative to* K, and we are considering the supremum of the injective A-dimensions of the A-modules that are already injective relative to K.)

To go any farther we must now suppose further that R_K is projective. As before, $\text{Ext}_R(M, \text{Hom}_K(R, N)) \to \text{Ext}_K(M, N)$ is an isomorphism, so (5) can be rewritten in the form (2). Thus

$$\text{r.gl.dim}\, R \leqslant \max\{\text{r.gl.dim}\, A, 1 + \text{r.gl.dim}\, K\}$$

and from the previous paragraph, $\text{r.gl.dim}\, R \leqslant 1 + \text{r.gl.dim}\, A$ so we have proved (3).

The above argument was based on applying $\text{Ext}_R(M, \text{Hom}_R(-, N))$ to (1); had we assumed ${}_AR$, ${}_KR$ flat, we would have applied $\text{Ext}_R(M \otimes_R -, N)$. For homology, the arguments are similar and begin by applying $\text{Tor}^R(M, -\otimes_R N)$ and $\text{Tor}^R(M \otimes_R -, N)$ to (1). ∎

Let us now use a technique of [4] to show that if $0 = \text{r.gl.dim}\, K \leqslant \text{r.gl.dim}\, A \leqslant 1$ then $\text{r.gl.dim}\, R \leqslant 1$. In terms of (3) this says that (3) holds whenever the right hand side is at most 1 (without any restriction on the structure of R as A or K-module). (In connection with this, it is of interest that neither side of (3) can equal 0; since R can be expressed as a skew Laurent-polynomial ring it has non-unit non-zerodivisors (such as $t+1$) or is trivial.)

We begin by analysing $\Omega_{R/K}$ in much the same way as was done for $\Omega_{R/A}$. Since R is generated as K-ring by t, t^{-1} and all $a \in A$ with certain relations, so $\Omega_{R/K}$ is generated as R-bimodule by dt and all $da \in \Omega_{A/K}$ with corresponding relations obtained by "differentiating" the given ring relations. Thus we have the following.

THEOREM 2. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$ and write $\Omega_{A/K}$ for the kernel of the multiplication map $A \otimes_K A \to A$. Then there is an exact sequence of R-bimodules*

$$\begin{array}{ccc} 0 \to R \otimes_A \Omega_{A/K} \otimes_A R \to \Omega_{R/K} \to & \Omega_{R/A} & \to 0 \\ & \wr\| & \\ 1 \otimes da \otimes 1 \mapsto da \qquad & R \otimes_K R & \end{array} \qquad (6)$$

Recall that A is *right hereditary* means $\text{r.gl.dim}\, A \leqslant 1$.

THEOREM 3. *If K is completely reducible and A is right hereditary then the HNN construction $R = A_K\langle t, t^{-1}; \beta\rangle$ is right hereditary.*

An analogous result holds for weak global dimension.

Proof. Let M_R be any right R-module. Applying $M \otimes_A -$ to the split exact sequence $0 \to \Omega_{A/K} \to A \otimes_K A \to A \to 0$ of left A-modules gives an exact sequence $0 \to M \otimes_A \Omega_{A/K} \to M \otimes_K A \to M \to 0$ of right A-modules. Now M_K is K-projective so $M \otimes_K A$ is A-projective, and thus so is the submodule $M \otimes_A \Omega_{A/K}$ because A is right hereditary.

Since R is left K-projective, (6) is a split exact sequence of left R-modules and so remains exact under $M \otimes_R -$ to give an exact sequence of right R-modules

$$0 \to M \otimes_A \Omega_{A/K} \otimes_A R \to M \otimes_R \Omega_{R/K} \to M \otimes_K R \to 0.$$

But $(M \otimes_A \Omega_{A/K})_A$ and M_K are projective so the sequence is split and $(M \otimes_R \Omega_{R/K})_R$ is also projective. By the reasoning in the first paragraph of the proof, but with R in place of A, we have an exact sequence of right R-modules $0 \to M \otimes_R \Omega_{R/K} \to M \otimes_K R \to M \to 0$ which happens to be a projective resolution of M of length 1. This proves that R is right hereditary. ∎

Later we will see that if α, β are injective and K is completely reducible then r.gl.dim $R = \max\{1, \text{r.gl.dim}\, A\}$, and a corresponding result holds for weak global dimension. The following three examples illustrate the sort of aberration that can occur if α or β is not injective. Let F be an arbitrary ring with r.gl.dim $F = n$, say; we take K, A to be F-rings and α, β to be F-ring homomorphisms. All statements about r.gl.dim hold also for w.gl.dim.

EXAMPLE 4. If $K = F[e \mid e^2 = e] \simeq F \times F, A = F$, $\alpha(e) = 1$, $\beta(e) = 0$ then R is trivial. ∎

EXAMPLE 5 [14; Sect. 5]. Let $K = F[e \mid e^2 = e]$, $A = F\langle e, x, y, z \mid e^2 = e, xey = z^2\rangle$, $\alpha(e) = e$, $\beta(e) = 0$; then $R = F\langle x, y, z, t, t^{-1} \mid z^2 = 0\rangle$ and we know that r.gl.dim $K = n$, r.gl.dim $A = n + 2$ (see [14]) and r.gl.dim $R = \infty$. ∎

EXAMPLE 6. If $K = F[x]$, $A = F[y]$, $\alpha(x) = y^2$, $\beta(x) = 0$ then $R = F\langle y, t, t^{-1} \mid y^2 = 0\rangle$. Clearly r.gl.dim K = r.gl.dim $A = n + 1$, r.gl.dim $R = \infty$. ∎

Taking $n = 0$ in Examples 5 and 6, we see that (3) can fail if the right hand side exceeds 1; and we are obliged to accept Theorems 1 and 3 as a best possible description of an indescribable situation. By contrast, imposing the natural hypotheses that α and β be injective may improve the behaviour of R, and we can no longer discern where the hypotheses on the structure of R as A- or K-module are relevant. It is ironic that here we do not know the actual importance of the module structures and yet it is here that we will best understand how they can be described in terms of the K-module structures on A; cf. Sections 4, 6.

Our final topic of this section is the Euler characteristic. Recall that if a right R-module M_R has a finite R-resolution by finitely generated projectives,

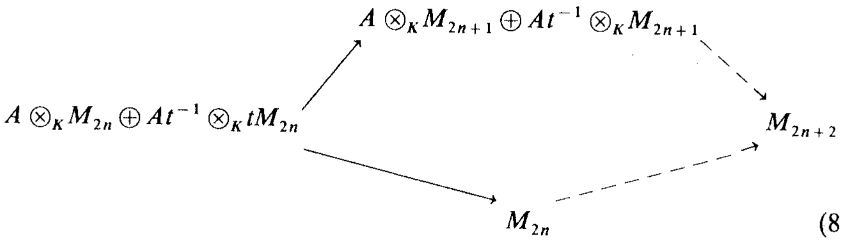
$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$

then we define $\chi_R(M)$ to be the element $\sum_{i=0}^{n} (-1)^i [P_i]$ of $K_0(R)$ (the commutes. Thus A and tAt^{-1} contain an image of K as K-subbimodule, and these are to be identified in R. We will write A/K and tAt^{-1}/K to denote the corresponding quotient bimodules, isomorphic to ${}_{\alpha}A_{\alpha}/\alpha(K)$, ${}_{\beta}A_{\beta}/\beta(K)$, respectively.

Let us construct inductively around our conditions

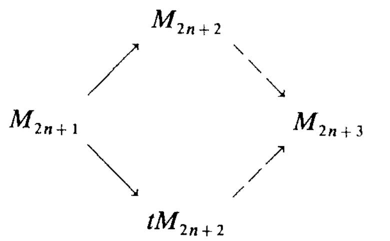
$$M_{2n} \text{ is a left } A\text{-module and } M_{2n+1} \text{ is a left } K\text{-module given with } K\text{-linear maps } M_{2n} \to M_{2n+1},\ tM_{2n} \to M_{2n+1}. \tag{7}$$

For $n < 0$ we set $M_n = 0$ and (7) is satisfied. For $n = 0$ we are given M_0, and for $n = 1$ we define $M_1 = M_0 \oplus tM_0$, and again (7) is satisfied. Suppose that $n \geqslant 0$ and that (7) holds for n. Then we define M_{2n+2} as the pushout of the diagram



(8)

of A-linear maps, where the upper arrow is defined componentwise using the given maps $M_{2n} \to M_{2n+1}$, $tM_{2n} \to M_{2n+1}$, and the lower arrow is a multiplication map defined using the A-module structure on M_{2n}. As a pushout of A-linear maps, M_{2n+2} is a left A-module. We turn now to the definition of M_{2n+3} which requires the K-linear map $M_{2n+1} \to tM_{2n+2}$ corresponding to $m \mapsto t(t^{-1}m)$ and formally defined as follows. From (8) there is a map $At^{-1} \otimes_K M_{2n+1} \to M_{2n+2}$ and hence a map $tAt^{-1} \otimes_K M_{2n+1} \to tM_{2n+2}$; composing with $M_{2n+1} = K \otimes_K M_{2n+1} \to tAt^{-1} \otimes_K M_{2n+1}$ gives the required map $M_{2n+1} \to tM_{2n+2}$. Now M_{2n+3} is defined as the pushout of the diagram

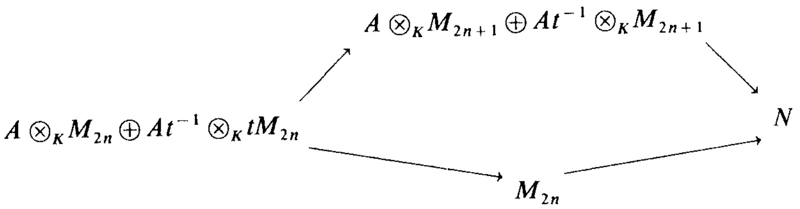


(9)

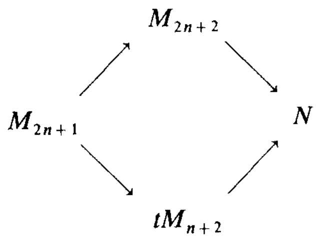
and this fulfills (7) for $n + 1$, so we have defined the procedure for inductively constructing the system (M_n).

The definition (8) of M_{2n+2} yields a map $A \otimes_K M_{2n+1} \to M_{2n+2}$ and thus a map $M_{2n+1} = K \otimes_K M_{2n+1} \to A \otimes_K M_{2n+1} \to M_{2n+2}$. This makes (M_n) into a directed system of left K-modules, and we want its direct limit M_∞ to be isomorphic to $R \otimes_A M_0$. From (8) the composite homomorphism $M_{2n} \to M_{2n+1} \to M_{2n+2}$ is A-linear, so $M_\infty = \varinjlim M_{2n}$ has a left A-module structure extending the K-module structure. To show that M_∞ is a left R-module we present an isomorphism ${}_\beta M_\infty \to {}_K M_\infty$, corresponding to left multiplication by t. The isomorphism ${}_\beta M_{2n} \simeq {}_K tM_{2n}$, $m \mapsto tm$, induces an isomorphism ${}_\beta M_\infty = \varinjlim {}_\beta M_{2n} \simeq \varinjlim {}_K tM_{2n} = {}_K M_\infty$, so M_∞ is a left R-module with this t-action. Notice that the action of A, t and t^{-1} are all as suggested by the notation and the constructions (8), (9). It remains to show that the A-linear map $M_0 \to M_\infty$ has the universal property of $M_0 \to R \otimes_A M_0$, namely, that every A-linear map from M_0 to a left R-module N lifts uniquely to an R-linear map $R \otimes_A M_0 \to N$.

Suppose we are given an A-linear map $M_0 \to {}_n N$, where N is a left R-module. There is then a unique extension to a K-linear map $M_1 = M_0 \oplus tM_0 \to N$ that respects t. Suppose further that for some $n \geqslant 0$ there is a K-linear map $M_{2n+1} \to N$ such that the composite $M_{2n} \to M_{2n+1} \to N$ is A-linear and the composite $tM_{2n} \to M_{2n+1} \to N$ respects t. Then the diagram



commutes, so there is a unique lifting to an A-linear map $M_{2n+2} \to N$. Further the K-linear map $tM_{2n+2} \to N$ gives rise to another commuting diagram



and hence there is a unique lifting to a K-linear map $M_{2n+3} \to N$ such that the composites $M_{2n+2} \to M_{2n+3} \to N$, $tM_{2n+2} \to M_{2n+3} \to N$ respect the actions of A and t, respectively.

We have thus proved by induction that for each n there is a unique K-linear map $M_n \to N$ which respects the A, t and t^{-1} actions. Passing to the direct limit gives a unique lifting to an R-linear map $M_\infty \to N$, and therefore $M_\infty \cong R \otimes_A M_0$. It remains to describe each quotient $M_n/im\, M_{n-1}$; for convenience we abbreviate this to M_n/M_{n-1} although we are not assuming that the map $M_{n-1} \to M_n$ is injective.

THEOREM 8. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$ and M_0 be a left A-module. Then the R-module $M = R \otimes_A M_0$ is the direct limit of a directed system of left K-modules M_n, where*

$$M_n = 0 \quad \text{for} \quad n < 0,$$

$$M_0 \text{ is as given},$$

$$M_1 = M_0 \oplus tM_0,$$

$$M_{2n+2}/M_{2n+1} \cong A/K \otimes_K M_{2n+1}/M_{2n} \oplus At^{-1} \otimes_K M_{2n+1}/tM_{2n} \qquad \text{for } n \neq -1, \tag{10}$$

$$M_{2n+3}/M_{2n+2} \cong tA \otimes_K M_{2n+1}/M_{2n} \oplus tAt^{-1}/K \otimes_K M_{2n+1}/tM_{2n} \qquad \text{for } n \neq -1 \tag{11}$$

and additional information is given by

$$M_{2n+3}/M_{2n+1} \cong tM_{2n+2}/M_{2n+1} \oplus M_{2n+2}/M_{2n+1} \qquad \text{for all } n, \tag{12}$$

$$M_{2n+3}/M_{2n+2} \cong tM_{2n+2}/M_{2n+1} \qquad \text{for all } n, \tag{13}$$

$$M_{2n+3}/tM_{2n+2} \cong M_{2n+2}/M_{2n+1} \qquad \text{for all } n. \tag{14}$$

Further (M_{2n}) is a directed subsystem of A-modules such that

$$M_{2n+2}/M_{2n} \cong A \otimes_K M_{2n+1}/M_{2n} \oplus At^{-1} \otimes_K M_{2n+1}/tM_{2n} \qquad \text{for } n \neq -1. \tag{15}$$

Proof. Since (9) is a pushout, (12), (13) and (14) can be checked by elementary diagram chasing, the details of which are omitted; of course these results are immediate if one knows the fact that pushouts commute with cokernels, itself a consequence of the more general principle that colimits commute with colimits; cf. [23, Chap. II, Corollary 12.2]. Similarly (8) gives (15) for $n \neq -1$. Here the image of M_{2n+1}/M_{2n} corresponds to $K \otimes M_{2n+1}/M_{2n} \oplus 0$, so the quotient $M_{2n+2}/M_{2n+1} \cong M_{2n+2}/M_{2n}/M_{2n+1}/M_{2n}$ has the form (10). It remains to prove (11). Left multiplying (15) by t gives $tM_{2n+2}/tM_{2n} \cong tA \otimes_K M_{2n+1}/M_{2n} \oplus tAt^{-1} \otimes_K M_{2n+1}/tM_{2n}$, and the image of M_{2n+1}/tM_{2n} corresponds to $0 \oplus K \otimes M_{2n+1}/tM_{2n}$. Now using (13) the quotient

$$M_{2n+3}/M_{2n+2} \cong tM_{2n+2}/M_{2n+1} \cong tM_{2n+2}/M_{2n}\big/M_{2n+1}/M_{2n}$$

which has the form (11). ∎

Notice that any structure on M_0 commuting with the A-action is, by virtue of the construction, automatically inherited by the M_n. For example, starting with the A-bimodule, $R_0 = {}_AA_A$ gives a directed system (R_n) of (K, A) bimodules, and $\varinjlim R_n = {}_RR_A$. Now the M_n of the preceding theorem could have been defined as $R_n \otimes_A M_0$; the only reason it was not so defined was to emphasize the one-sided nature of the construction by eliminating unnecessary structure.

Theorem 8 gives an inductive description of the M_n/M_{n-1} from which we see that $M_{2n+3}/M_{2n+1} \cong M_{2n+3}/M_{2n+2} \oplus M_{2n+2}/M_{2n+1}$ is built up as a direct sum of K-tensor products $C_{n+1} \otimes_K C_n \otimes \cdots \otimes_K C_0$ of "length" $n+1$, where $C_0 = M_0$ or tM_0 and for $i = 1,..., n+1$, C_i is one of the K-bimodules A/K, At^{-1}, tA, tAt^{-1}/K and the tensor products occurring are those where no cancellation would occur in R. To put this more formally, let us assign each of the four bimodules a left and right sign, and M_0, tM_0 a left sign, as follows:

$$\begin{array}{lll} +\,tA\,+ & +\,tAt^{-1}/K\,- & +\,tM_0 \\ -\,A/K\,+ & -\,At^{-1}\,- & -\,M_0. \end{array}$$

The permitted tensor products $C_{n+1} \otimes \cdots \otimes C_0$ are those which are *sign-linked*, that is for each $i = n,..., 0$ the right sign of C_{i+1} equals the left sign of C_i. If the left sign of C_{n+1} is $+$ the tensor product is a summand of M_{2n+3}/M_{2n+2}, and if the left sign of C_{n+1} is $-$ then it is a summand of M_{2n+2}/M_{2n+1}.

4. Lifting of Faithful Flatness

Our first application of Theorem 8 will verify sufficient conditions for ${}_AR$ to be flat. This will require the notion of a *faithfully flat* left A-module, that is, an ${}_AN$ such that a sequence $N_1 \to N_2 \to N_3$ of right A-modules is exact if and only if the abelian group sequence $N_1 \otimes_A N \to N_2 \otimes_A N \to N_3 \otimes_A N$ is exact. We will only be interested in the case where N is an A-ring so the following characterization applies: ${}_AR$ is faithfully flat if and only if $\eta: A \to R$ is injective and ${}_A(R/A)$ is flat; cf. [5, I.3.5, Proposition 9].

Theorem 9. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$. If ${}_KA$, ${}_\beta A$ are faithfully flat then ${}_AR$*

is faithfully flat. If further M_0 is a left A-module such that ${}_KM_0$, ${}_\beta M_0$ are flat then the following hold:

(i) *Each of the maps $M_n \to M_{n+1}$ of Theorem* 8 *is an embedding and ${}_K(M_{n+1}/M_n)$ is flat.*

(ii) *The map $M_0 \to M = R \otimes_A M_0$ is an embedding and ${}_AM/M_0$ is flat.*

(iii) $wd_R M \geqslant wd_A M_0$.

Proof. To show that ${}_AR$ is faithfully flat it suffices to show that $A \to R$ is an embedding and ${}_AR/A$ is flat; this will follow from (ii) in the case $M_0 = A$.

(i) Notice that each $M_{n+1}/im\ M_n$ is, by Theorem 8, a direct sum of K-tensor products of A/K, tA, At^{-1}, tAt^{-1}/K, M_0, tM_0 which are all left K-flat. Hence each $M_{n+1}/im\ M_n$ is flat. Suppose that $n \geqslant 0$ and that $M_{2n} \to M_{2n+1}$, $tM_{2n} \to M_{2n+1}$ are embeddings, as happens for $n = 0$. Since the quotients M_{2n+1}/M_{2n}, $M_{2n+1}/tM_{2n} = M_{2n}/im\ M_{2n-1}$ are known to be left K-flat, the given embeddings lift to embeddings under $A \otimes_K -$, $At^{-1} \otimes_K -$, respectively. From (8) it follows that $M_{2n} \to M_{2n+2}$ is an embedding. To show $M_{2n+1} \to M_{2n+2}$ injective it now suffices to compute mod M_{2n}:

$$M_{2n+1}/M_{2n} \to M_{2n+2}/M_{2n} \cong A \otimes_K M_{2n+1}/M_{2n} \oplus At^{-1} \otimes_K M_{2n+1}/tM_{2n}.$$

But this map is injective since ${}_KM_{2n+1}/M_{2n}$ is flat. We have now verified that $M_{2n+1} \to M_{2n+2}$ is an embedding, and the verification for $tM_{2n+1} \to M_{2n+2}$ is similar. It then follows from (9) that $M_{2n+2} \to M_{2n+3}$, $tM_{2n+2} \to M_{2n+3}$ are embeddings, and we have lifted our inductive hypothesis to $n + 1$, and (i) is now proved.

(ii) It is clear from (i) that each $M_0 \to M_n$ is an embedding and hence $M_0 \to M$ is. For each $n \geqslant 0$ we have, from (15), an exact A-module sequence

$$0 \to M_{2n}/M_0 \to M_{2n+2}/M_0 \to A \otimes_K M_{2n+1}/M_{2n} \oplus At^{-1} \otimes_K M_{2n+1}/tM_{2n} \to 0$$

and the last term is left A-flat since M_{2n+1}/M_{2n}, M_{2n+1}/tM_{2n} are left K-flat. Thus if we inductively assume ${}_AM_{2n}/M_0$ is flat, which certainly holds for $n = 0$, then ${}_AM_{2n+2}/M_0$ is flat. Hence, by induction, every ${}_AM_{2n}/M_0$ is flat, and since flatness is preserved by direct limits, ${}_AM/M_0 = \varinjlim {}_AM_{2n}/M_0$ is flat.

(iii) Since ${}_AR$ is flat by (ii), every flat R-resolution of ${}_RM$ is a flat A-resolution, so $wd_R M \geqslant wd_A M$. Now by (ii), $wd_A M = wd_A M_0$. ∎

In this theorem we have an example of a statement about $R = A_K\langle t, t^{-1}; \beta\rangle$ that can be deduced from known facts about coproducts and skew Laurent-polynomial rings, namely, that ${}_AR$ is faithfully flat if ${}_KA$, ${}_\beta A$ are; cf. [14, Sect. 5]. However, part (iii) of Theorem 9 is *not* readily obtained this way.

An application of these results gives the following.

COROLLARY 10. *If K is von Neumann regular and α, β are injective then* w.gl.dim $R = \max\{1, \text{w.gl.dim}\, A\}$.

Proof. Since ${}_K A$, ${}_\beta A$ are faithfully flat by our hypotheses, ${}_A R$ is faithfully flat by Theorem 9, and hence so is ${}_K R$. Thus by Theorem 1, w.gl.dim $R \leqslant \max\{1, \text{w.gl.dim}\, A\}$. By Theorem 9(iii), and the fact that ${}_K M_0$, ${}_\beta M_0$ are automatically flat for any ${}_A M_0$, w.gl.dim $R \geqslant$ w.gl.dim A. Finally, since R is a skew Laurent-polynomial ring containing the non-trivial ring K, R is not a von Neumann regular ring and is non-trivial, so w.gl.dim $R \geqslant 1$. ∎

A phenomenon exemplified by Theorem 9 is that module properties of ${}_K A/K$, ${}_\beta A/\beta(K)$ lift to ${}_A R/A$. By contrast, the failure of properties to lift from ${}_K A$, ${}_\beta A$ to ${}_A R$ is quite common.

EXAMPLE 11 [3, Sect. 10]. Let F be a field, $K = F[x]$, $A = F\langle x, x^{-1}, y\rangle$, $\alpha(x) = x$, $\beta(x) = y$. Now A has no zerodivisors and so is torsion-free, and hence flat, as K-module via α or β. But ${}_A R$ is not flat since y becomes invertible in R and this introduces right R-dependence relations on right A-independent elements, e.g.,

$$\begin{array}{ccc} [(x+1)A + yA] \otimes_A R & \rightarrow & A \otimes_A R \\ \wr\| & & \wr\| \\ (x+1)R \oplus yR & & R \end{array} \quad \text{is not injective.}$$ ∎

5. MAYER–VIETORIS PRESENTATIONS OF MODULES

Fix a right R-module M_R.

A *Mayer–Vietoris presentation* of M is defined as any exact sequence

$$0 \rightarrow M(K) \otimes_K R \xrightarrow{f} M(A) \otimes_A R \rightarrow M \rightarrow 0 \tag{16}$$

of right R-modules, such that $M(K)$, $M(A)$ are right K-, A-modules, respectively, and f is constructed from two K-linear maps

$$f_K : M(K) \rightarrow M(A)_K \quad f_\beta : M(K) \rightarrow M(A)_\beta$$

by $f(m \otimes r) = f_K(m) \otimes r - f_\beta(m) \otimes t^{-1} r$.

That M has a Mayer–Vietoris presentation can be seen by applying $M \otimes_R -$ to (1), which gives

$$0 \rightarrow M \otimes_K R \xrightarrow{f} M \otimes_A R \rightarrow M \rightarrow 0$$

and here f_K is the identity map, and f_β is right multiplication by t. The disadvantage of this Mayer–Vietoris presentation is that the A-module M bears almost no relation to the R-module M; for example, M could have a finite R-presentation and the underlying A-module would have no knowledge of this. Fortunately, it is often possible to find a Mayer–Vietoris presentation that takes account of such data. The basic example is $M = {}^XR$, the direct sum of copies of R indexed by a set X. Here we have

$$0 \to {}^XK \otimes_K R \xrightarrow{f} {}^X(A \oplus tA) \otimes_A R \to {}^XR \to 0, \tag{17}$$

where the notation indicates how the middle term is to be mapped to ${}^XR$, and $f_K(m) = \alpha(m)$, $f_\beta(m) = t\beta(m)$, or more suggestively $f(m \otimes r) = m \otimes r - mt \otimes t^{-1}r$. (Of course we could have taken $M(K) = 0$, $M(A) = {}^XA$, but (17) will be needed later, and is a better illustration of the general situation.)

Basically then, we want to start from an R-presentation of M and build up a sequence (16), where $M(K), M(A)$ have K-, A-presentations, respectively, that are connected in some way to the original R-presentation; this should then give useful information about conditions that are related to presentations, for example, coherence. Our approach owes much to [7, 26].

For simplicity let us concentrate on finitely presented modules, say we have a presentation

$${}^YR \to {}^XR \to M \to 0,$$

where X, Y are finite sets. Viewing the elements of the free modules as columns allows us to view the presentation as a matrix. Further, new presentations of M can be obtained by choosing matrices stably associated to the given one; recall that two matrices U, V are said to be *stably associated* if there exist invertible matrices P, Q and identity matrices I, I' such that

$$\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} = P \begin{pmatrix} V & 0 \\ 0 & I' \end{pmatrix} Q.$$

By Higman's linearization-by-enlargement trick (cf. [13, p. 152]) every matrix over R is stably associated to a matrix with entries from $\eta A \cup \{t, t^{-1}\}$. Multiplying by t and linearizing again we arrive at a matrix with entries from $\eta A \cup \{t\}$. Let us assume that our original presentation is of this form. Here the A-submodule N of ${}^XR$ generated by the image of Y lies in ${}^X(\eta A + t\eta A)$ and we have a diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \to & P & \to & N \otimes_A R & \to & NR & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & {}^X(\eta A + t\eta A) \otimes_A R & \to & {}^X R & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Q & \to & M(A) \otimes_A R & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $M(A) = {}^X(\eta A + t\eta A)/N$, P is the kernel of $N \otimes_A R \to NR$, and Q is the kernel of $M(A) \otimes_A R \to M$.

To get any more information we have to make some assumptions on R. Our immediate requirements are injectivity of $A \oplus tA \to \eta A + t\eta A$, and flatness of ${}_A R$. Looking ahead, let us make the stronger assumption that ${}_K A$, ${}_\beta A$ are faithfully flat (cf. Theorem 9). Now recalling (17) we have a diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P & \to & N \otimes_A R & \to & NR & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & {}^X K \otimes_K R & \to & {}^X(A \oplus tA) \otimes_A R & \to & {}^X R & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Q & \to & M(A) \otimes_A R & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

of exact rows and columns, where the left column is induced. We want to express Q in the form $M(K) \otimes_K R$ and this will be achieved if we express the image of P in ${}^X K \otimes_K R$ in the form $Z \otimes_K R$. In the following technical lemma we find that P can be so expressed *provided that* $N \cap {}^X K = Nt^{-1} \cap {}^X K$. Notice that with a little (necessary) loss of generality we can ensure that $N \cap {}^X K = Nt^{-1} \cap {}^X K$ by replacing N with $N + (NR \cap {}^X K)(A + tA)$. This does not change NR and now $N \cap {}^X K = NR \cap {}^X K = Nt^{-1} \cap {}^X K$.

LEMMA 12. *Let* $R = A_K\langle t, t^{-1}; \beta\rangle$, *where* ${}_K A$, ${}_\beta A$ *are faithfully flat. Let* X *be a set and* N *a right* A-*submodule of* ${}^X(A + tA)$ *in* ${}^X R$. *If* $N \cap {}^X K =$

$Nt^{-1} \cap {}^X K$ and this set is denoted Z then there is an exact R-module sequence

$$0 \to Z \otimes_K R \to N \otimes_A R \to NR \to 0,$$

where $z \otimes r \mapsto z \otimes r - zt \otimes t^{-1} r$ for all $z \in Z, r \in R$.

Proof. By Theorem 9, ${}_K R$, ${}_A R$ are flat and $A \cap tA = 0$. Hence $Z \otimes_K R \subseteq {}^X K \otimes_K R = {}^X R$ and $N \otimes_A R \subseteq {}^X(A \oplus tA) \otimes_A R = {}^X R \oplus {}^X R$ so the map $Z \otimes_K R \to N \otimes_A R$ is the restriction of $v \mapsto (v, -v)$ which is clearly injective. Thus it remains to prove exactness at $N \otimes_A R$, and since the kernel P of $N \otimes_A R \to NR$ clearly contains $\operatorname{im}(Z \otimes_K R)$, we need only show the reverse inclusion. By Theorem 9(i) in the case $M_0 = A$ (and writing R_n for M_n) $R = \bigcup R_{2n+2}$, and for $n \geqslant 0$ we have a commutative diagram

$$\begin{array}{lcl}
N \otimes_A R_{2n+2} & \longrightarrow & NR_{2n+2} \subseteq {}^X R_{2n+3} \\
\downarrow & & \downarrow \\
N \otimes_A R_{2n+2}/R_{2n} & & {}^X(R_{2n+3}/R_{2n+2}) \\
\wr| & & \\
N \otimes_A (A \otimes_K tR_{2n}/R_{2n-1} & & \\
\quad \oplus At^{-1} \otimes_K R_{2n}/R_{2n-1}) & & \wr| \\
\wr| & & \\
N \otimes_K tR_{2n}/R_{2n-1} & & {}^X(A + tA/K) \otimes_K tR_{2n}/R_{2n-1} \\
\oplus & & \oplus \\
Nt^{-1} \otimes_K R_{2n}/R_{2n-1} & \longrightarrow & {}^X(At^{-1} + tAt^{-1}/K) \otimes_K R_{2n}/R_{2n-1}.
\end{array}$$

Now by flatness of ${}_K tR_{2n}/R_{2n-1}$, ${}_K R_{2n}/R_{2n-1}$ the kernel of the bottom map is $(N \cap {}^X K) \otimes_K tR_{2n}/R_{2n-1} \oplus (Nt^{-1} \cap {}^X K) \otimes_K R_{2n}/R_{2n-1}$, which can be written $Z \otimes_K tR_{2n}/R_{2n-1} \oplus Z \otimes_K R_{2n}/R_{2n-1}$. The inverse image of this in $N \otimes_A R_{2n+2}$ can be written in a suggestive notation as $Z \otimes tR_{2n} + Zt \otimes t^{-1} R_{2n} + N \otimes_A R_{2n} \subseteq N \otimes_A R_{2n+2}$. Thus we have described the kernel of the lower route; since the kernel of the upper route contains $P \cap (N \otimes_A R_{2n+2})$ we have

$$\begin{aligned}
P \cap (N \otimes_A R_{2n+2}) & \\
&\subseteq P \cap [Z \otimes tR_{2n} + Zt \otimes t^{-1} R_{2n} + N \otimes_A R_{2n}] \\
&= P \cap [Z(1 \otimes 1 - t \otimes t^{-1})\, tR_{2n} + Z(1 \otimes 1 - t \otimes t^{-1})\, R_{2n} + N \otimes_A R_{2n}] \\
&\subseteq P \cap [im(Z \otimes_K R) + N \otimes_A R_{2n}] \\
&= im(Z \otimes_K R) \; + \; P \cap (N \otimes_A R_{2n}).
\end{aligned}$$

Now for $n = 0$, $P \cap (N \otimes_A R_0) = P \cap N = 0 \subseteq im(Z \otimes_K R)$, and it follows easily by induction that $P \cap (N \otimes_A R_{2n}) \subseteq im(Z \otimes_K R)$ for all n. Taking the union over all n gives $P \subseteq im(Z \otimes_K R)$ as desired. ∎

We now have the following generalization of [26, Proposition 4.1(2)].

THEOREM 13. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where ${}_KA$, ${}_\beta A$ are faithfully flat. If K is right Noetherian then every finitely presented right R-module M has a Mayer–Vietoris presentation*

$$0 \to M(K) \otimes_K R \to M(A) \otimes_A R \to M \to 0,$$

where $M(K)_K$ and $M(A)_A$ are finitely presented.

Proof. We have seen that if M_R has a presentation ${}^{Y'}R \to {}^{X'}R \to M \to 0$, where X', Y' are finite, then M_R has a linearized presentation ${}^{Y}R \to {}^{X}R \to M \to 0$, where X, Y are finite. We constructed a Mayer–Vietoris presentation by taking N to be the A-submodule of ${}^XR$ generated by the image of Y and defining

$$M(K) = {}^XK/Z, \qquad \text{where } Z = NR \cap {}^XK,$$
$$M(A) = {}^X(A + tA)/(N + Z(A + tA)),$$

which are clearly finitely generated. Also K is right Noetherian so Z_K is finitely generated as it is a submodule of ${}^XK$. Since N_A is finitely generated it is clear that $M(K)_K$ and $M(A)_A$ are finitely presented. ∎

Recall that K is said to be *right coherent* if every finitely presented right K-module has a resolution by finitely generated free K-modules. We say that K is *right regular* if every finitely presented right K-module has finite projective dimension over K.

COROLLARY 14. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where ${}_KA$, ${}_\beta A$ are faithfully flat. If K is right Noetherian and A is right coherent then R is right coherent. If in addition K and A are right regular then R is right regular.*

Proof. If K is right Noetherian and A is right coherent it is clear from Theorem 13 and the flatness of ${}_KR$, ${}_AR$ that R is right coherent, and similarly if we further assume K, A to be right regular then so is R. ∎

The preceding arguments made frequent use of the faithful flatness of ${}_KA$, ${}_\beta A$ and we now wish to show that these hypotheses are not entirely superfluous.

EXAMPLE 15 [4, Example 4.2]. Let F be a field, let $K = F[x_0]$, and let

$$A = F\langle w, w^{-1}, x_n, y_n, z \mid x_n = zx_{n+1}, y_n = y_{n+1}z, n = 0, 1, 2, \ldots\rangle$$

be a K-ring in the manner suggested by the notation. Let β be the F-algebra homomorphism that sends x_0 to w. Then the HNN extension is

$$R = F\langle x_n, y_n, z, (x_0)^{-1}, t, t^{-1} \mid x_n = zx_{n+1}, y_n = y_{n+1}z, n = 0, 1, 2, \ldots\rangle.$$

Now K is Noetherian, and we know from [4] that A is a semifir, and hence coherent, but that R is *not* right coherent since the principal right ideal y_0R is not finitely related.

In this example, ${}_KA$ is faithfully flat and ${}_\beta A$ is flat but not faithfully flat.

Notice that the R-module $M = R/y_0R$ has a Mayer–Vietoris presentation with $M(K) = 0$, $M(A) = A/y_0A$, but that it fails to provide useful information. ∎

(Added December 1982: The recent paper by H. Åberg (Coherence of amalgamations, *J. Alg.* **78** (1982), 372–385) gives a slick proof of a result slightly more general than the first part of Corollary 14.

THEOREM (Åberg). *Let $R = A_K\langle t, t^{-1}; \beta\rangle$ and suppose ${}_KR$, ${}_AR$ are flat. If K is right Noetherian and A is right coherent, then R is right coherent.*

Proof. Let M be a right R-module, I a set, and R^I the direct power viewed as left R-module. It suffices to show $\operatorname{Tor}_1^R(M, R^I) = 0$, for then R^I is left R-flat, which is one of the characterizations of right coherence. From (1) we get two exact sequences $0 \to R \otimes_K (R^I) \to R \otimes_A (R^I) \to R^I \to 0$ and $0 \to M \otimes_K R \to M \otimes_A R \to M \to 0$ by applying $- \otimes_R (R^I)$ and $M \otimes_R -$, respectively. Now applying $M \otimes_R -$ and $(-)^I$ to these two exact sequences, respectively, we get a commuting diagram with exact rows

$$\begin{array}{l} \operatorname{Tor}_1^R(M, R \otimes_K (R^I)) \\ \qquad \to \operatorname{Tor}_1^R(M, R^I) \to M \otimes_K (R^I) \xrightarrow{f} M \otimes_A (R^I) \to M \otimes_R (R^I) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow g \qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad 0 \to (M \otimes_K R)^I \xrightarrow{h} (M \otimes_A R)^I \to M^I \to 0. \end{array}$$

Since K is right coherent and ${}_KR$ is flat, ${}_K(R^I)$ is flat so $R \otimes_K (R^I)$ is left R-flat and the leftmost term in the top row vanishes. It suffices then to show that f is injective, and since h is injective it suffices to show that g is injective. If M_K is finitely presented, then g is an isomorphism; as K is right Noetherian, M is a directed union of finitely presented K-submodules so g is a directed union of injective maps since R^I, R are left K-flat. Hence g is injective as desired. ∎)

6. Induced Modules with an Induced Grading

Theorem 8 described an induced module $M = R \otimes_A M_0$ as a direct limit; we now wish to examine the case where this direct limit can be viewed as a direct sum of the quotient modules.

Throughout this section we use the notation set up in Theorem 8.

Suppose we are given for each of the maps $M_{n-1} \to M_n$ a *retraction* $M_n \to M_{n-1}$; that is, the composite $M_{n-1} \to M_n \to M_{n-1}$ is the identity; and suppose further that each of the retractions $M_n \to M_{n-1}$ is K-linear and that each "even" composite $M_{2n} \to M_{2n-1} \to M_{2n-2}$ is A-linear. Such a system of data will be called an *induced grading* on M. It is appropriate to call this a grading because the K-linear retractions make M isomorphic to the graded K-module $\oplus M_n/M_{n-1}$, and what is more important, the A-linear retractions $M_{2n} \to M_{2n-2}$ make M isomorphic to the graded A-module $\oplus M_{2n}/M_{2n-2}$ and we know

$$\bigoplus M_{2n}/M_{n-2} \cong M_0 \oplus A \otimes_K \Big(\bigoplus_n M_{2n+1}/M_{2n} \Big) \oplus At^{-1} \otimes_K \Big(\bigoplus_n M_{2n+1}/tM_{2n} \Big) \tag{18}$$

by Theorem 8. It is obvious that such a breakdown will provide rather detailed information on M; what is not obvious is the set of circumstances under which M has an induced grading. We begin with a rather general criterion which will be ideal for our purposes.

Theorem 16. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$ and ${}_AM_0$ be a left A-module. If each of the maps*

$$M_{2n+1}/M_{2n} \to A \otimes_K M_{2n+1}/M_{2n}, \tag{19}$$

$$M_{2n+1}/tM_{2n} \to {}_\beta A_\beta \otimes_K M_{2n+1}/tM_{2n} \tag{20}$$

has a K-linear retraction then $R \otimes_A M_0$ has an induced grading.

Proof. We begin the induction by supposing that $n \geqslant 0$ and that each of the maps $M_{2n} \to M_{2n+1}$, $tM_{2n} \to M_{2n+1}$ has a K-linear retraction, which certainly holds for $n = 0$, by definition of M_1. These then induce an A-linear retraction of $M_{2n} \to M_{2n+2}$, as can be seen by looking at the pushout definition (8) of M_{2n+2}. Now consider the diagram

$$\begin{array}{ccccccc}
M_{2n} & \to & M_{2n+1} & \to & M_{2n+1}/M_{2n} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
M_{2n} & \to & M_{2n+2} & \to & A \otimes_K M_{2n+1}/M_{2n} \oplus At^{-1} \otimes_K M_{2n+1}/tM_{2n} & \to & 0.
\end{array}$$

We have just seen that the bottom sequence is A-split and we assumed the top sequence is K-split; further, by the hypotheses of the theorem, the right hand vertical map has a K-linear retraction. Thus, starting from M_{2n+2} there are two paths leading to M_{2n+1}, one around each square, and their sum is a K-linear retraction $M_{2n+2} \to M_{2n+1}$. The latter when composed with the K-linear retraction $M_{2n+1} \to M_{2n}$ gives our previous (A-linear) retraction $M_{2n+2} \to M_{2n}$. Since this is precisely what we are trying to obtain for all n, it suffices to lift our original inductive hypotheses to $n+1$. Now we have a K-linear retraction of $M_{2n+1} \to M_{2n+2}$ and hence, from the pushout definition (9) of M_{2n+3}, we have a K-linear retraction of $tM_{2n+2} \to M_{2n+3}$. A similar argument using the diagram

$$\begin{array}{ccccc}
tM_{2n} \to & M_{2n+1} & \longrightarrow & M_{2n+1}/tM_{2n} & \to 0 \\
\| & \downarrow & & \downarrow & \\
tM_{2n} \to & tM_{2n+2} & \to & tA \otimes_K M_{2n+1}/M_{2n} \oplus tAt^{-1} \otimes_K M_{2n+1}/tM_{2n} & \to 0
\end{array}$$

gives a K-linear retraction of $M_{2n+1} \to tM_{2n+2}$ and thus by (9) we have a K-linear retraction of $M_{2n+2} \to M_{2n+3}$ which completes the inductive cycle. ∎

The hypotheses of the theorem are most readily verified in the situation where the ring homomorphisms α, $\beta: K \to A$ have *K-bimodule* retractions; that is, α, β are injective and ${}_{\alpha}A_{\alpha} = \alpha(K) \oplus {}_{\alpha}A'_{\alpha}$, ${}_{\beta}A_{\beta} = \beta(K) \oplus {}_{\beta}A''_{\beta}$ for suitable subbimodules A', A'' of A. (This holds, for example, if K, A are group rings and α, β arise from injective group homomorphisms.) Then for any left K-module ${}_KN$, $A \otimes_K N = N \oplus A' \otimes_K N$, ${}_{\beta}A_{\beta} \otimes_K N = N \oplus A'' \otimes_K N$ and it is clear that (19),(20) have K-linear retractions for any left A-module M_0. The resulting description of $R \otimes_A M_0$ as a direct sum of sign linked tensor products (cf. (18) and Section 3) can be made natural by identifying A/K, tAt^{-1}/K with the images of A', $tA''t^{-1}$ in R. Thus the presence of K-bimodule retractions makes manipulations on R conceptually quite simple; it is a very powerful hypothesis and we shall see some of its consequences in Section 11. What we want to investigate now is the sort of induced grading we can expect if K is completely reducible. We begin with the analogue of Theorem 9.

THEOREM 17. *Let* $R = A_K\langle t, t^{-1}; \beta\rangle$. *If* α, β *are monomorphisms and* ${}_K(A/\alpha(K))$, ${}_{\beta}(A/\beta(K))$ *are projective then* η *is a monomorphism and* ${}_A(R/A)$, ${}_AR$ *are projective. If, further,* M_0 *is a left A-module such that* ${}_KM_0$, ${}_{\beta}M_0$ *are projective then the following hold*:

(i) *There is an induced grading on* $M = R \otimes_A M_0$, *so, as filtered module,* ${}_AM$ *is isomorphic to* (18).

(ii) *The map* $M_0 \to M$ *is an embedding and* ${}_AM/M_0$ *is projective.*

(iii) $pd_R M \geqslant pd_A M_0$.

Proof. The first part will follow from (ii) in the case $M_0 = A$.

(i) From Section 3 we know that, for each n, ${}_K(M_n/M_{n-1})$ is a direct sum of K-tensor products $C_m \otimes C_{m-1} \otimes \cdots \otimes C_0$, where $m = [n/2]$. Here our hypotheses imply each ${}_KC_i$ is projective so it follows that ${}_K(M_n/M_{n-1})$ is projective. For any projective left K-module P the short exact sequence

$$0 \to P \to A \otimes_K P \to A/K \otimes_K P \to 0 \tag{21}$$

of projective left K-modules splits. In particular (19), and similarly (20), have left K-linear retractions, so by Theorem 16, M has an induced grading, and so is isomorphic to (18) as filtered A-module.

(ii) This is clear since M is isomorphic to (18) as A-module.

(iii) This follows as in the proof of Theorem 9(iii). ∎

COROLLARY 18. *If K is completely reducible and α, β are injective then* $\text{l.gl.dim}\, R = \max\{1, \text{l.gl.dim}\, A\}$ *and* $\text{r.gl.dim}\, R = \max\{1, \text{r.gl.dim}\, A\}$.

Proof. Since K is completely reducible the modules ${}_\alpha A/\alpha(K)$, ${}_\beta A/\beta(K)$ are projective, so ${}_AR$ is projective by Theorem 17. Hence by the left–right dual of Theorem 1, $\text{l.gl.dim}\, R \leqslant \max\{1, \text{l.gl.dim}\, A\}$. Further, for any left A-module M_0, the left K-modules ${}_\alpha M_0$, ${}_\beta M_0$ are projective, so by Theorem 17(iii), $pd_R(R \otimes_A M_0) \geqslant pd_A M_0$, and $\text{l.gl.dim}\, R \geqslant \text{l.gl.dim}\, A$. Finally, since R is a nontrivial skew Laurent-polynomial ring, $\text{l.gl.dim}\, R \geqslant 1$, which proves the first equality. The second equality follows by symmetry. ∎

At this stage we know that under suitable hypotheses, such as α, β being monomorphisms and K being completely reducible, we can decompose any induced module M as a nice direct sum of certain strings of K-tensor products $C_m \otimes C_{m-1} \otimes \cdots \otimes C_0$. Notice that if each ${}_KC_i$ is free with a specified basis we get a very useful basis of ${}_KM$. This applies in particular if K is a skew field; if K is an arbitrary completely reducible ring we need a slight generalization of freeness defined as follows.

Let K be any ring and E a *complete* set of orthogonal idempotents for K; that is, $E = \{e_1, \dots, e_n\}$, $e_ie_j = \delta_{ij}e_i$, $\sum e_i = 1$, no $e_i = 0$. A left K-module M is said to be *free-relative-to-E* if it has a subset X such that $M = \bigoplus_{x \in X} Kx$, and for each $x \in X$, the left annihilator of x is of the form $K(1-e)$ for a (unique) $e \in E$ called the *left index* of x; in this event X is said to be a *K-basis-relative-to-E* of M. Where E consists of the identity element, these concepts coincide with the usual notions of freeness and bases.

If M is a K-bimodule we have a left K-module decomposition $M = \bigoplus_{e \in E} Me$; if each ${}_KMe$ is free-relative-to-E, say X_e is a basis relative-to-E, then $X = \bigcup_{e \in E} X_e$ is a basis-relative-to-E of M such that for each $x \in X$,

$xe = x$ for a (unique) $e \in E$ called the *right index* of x. In this event we call X a left *bi-basis-relative-to-E* of ${}_KM_K$.

Now suppose X is a left bi-basis-relative-to-E of ${}_KM_K$ and Y a basis-relative-to-E of ${}_KN$ for some N. For each $y \in Y$ let $e_y \in E$ be the left index of y. Then $M \otimes_K N = \oplus_{y \in Y} M \otimes Ky \simeq \oplus_{y \in Y} Me_y$. Thus $M \otimes N$ has a basis-relative-to-E given by the family of all $x \otimes y$, $x \in X$, $y \in Y$, such that the right index of x equals the left index of y.

THEOREM 19. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, E some complete set of orthogonal idempotents of K, and M_0 a left A-module. Assume α, β are monomorphisms. Suppose ${}_\alpha A_\alpha$, ${}_\beta A_\beta$, ${}_\beta A_\alpha$, ${}_\alpha A_\beta$ have left bi-bases-relative-to-E $X \cup \alpha(E)$, $Y \cup \beta(E)$, W, Z, respectively, and ${}_\alpha M_0$, ${}_\beta M_0$ have bases-relative-to-E C, D, respectively. Then a left K-basis-relative-to-E of $M = R \otimes_A M_0$ is given by the family U of all linked expressions $u = c_n c_{n-1} \cdots c_0$; that is, the c_i are chosen from the sets*

$$\begin{array}{ccc} -X+ & -Zt^{-1}- & -C \\ +tW+ & +tYt^{-1}- & +tD \end{array}$$

with c_0 chosen from the third column and for $i = 1, \ldots, n$, c_i is chosen from the first two columns so that the right index and right sign of c_i coincide with the left index and left sign of c_{i-1}, respectively.

Such a family U is called a *Schreier-basis-relative-to-E* of M; later we shall make use of the fact that a Schreier basis is closed under taking terminal segments, hence the terminology.

Proof of Theorem 19. By hypothesis, ${}_\alpha(A/\alpha K)$, ${}_\beta(A/\beta K)$, ${}_\alpha M_0$, ${}_\beta M_0$ have K-bases-relative-to-E, $X(\text{mod } \alpha K)$, $Y(\text{mod } \beta K)$, C, D, respectively, and so in particular are projective. Thus by Theorem 17, M can be expressed as a direct sum of sign-linked tensor products; by the remarks preceding the theorem these tensor products have K-bases-relative-to-E given by the elements of U.

Alternatively, one can prove Theorem 19 directly by constructing a left K-module N having U as a K-basis-relative-to-E, and then defining on N an A-action and a t-action, and verifying $N \simeq R \otimes_A M_0$. The latter verification is accomplished by checking the universal property—for any left R-module N' and any A-linear map $M_0 \to N'$ there is a unique lifting to an R-linear map $N \to N'$. ∎

7. Submodules of Induced Modules

Throughout this section we assume that K is completely reducible, and that α, β are monomorphisms. We further fix a left A-module M_0.

This section, and the next two, are very closely patterned on [2].

Our objective now is to show that any R-submodule L of the induced left R-module $M = R \otimes_A M_0$ has an A-submodule L_0 such that the canonical map $R \otimes_A L_0 \to L$ is an isomorphism. In other words, a submodule of an induced module is induced. In spirit the proof will be a reduction argument similar to the Euclidean algorithm for a polynomial ring over a field; the essential difference is the quantity of apparatus required.

Let E be a fixed complete set of orthogonal idempotents in K such that each $e \in E$ is *primitive*; that is, Ke is a simple left K-module. Any left K-module then has a basis-relative-to-E and we shall call this a left K-basis, without reference to E. Similarly any K-bimodule has a left bi-basis-relative-to-E, and we shall call this a left bi-basis.

Let us fix left bi-bases $X \cup \alpha(E)$, $Y \cup \beta(E)$, W, Z of ${}_\alpha A_\alpha$, ${}_\beta A_\beta$, ${}_\beta A_\alpha$, ${}_\alpha A_\beta$, respectively, and left K-bases C, D of ${}_\alpha M_0$, ${}_\beta M_0$, respectively. By Theorem 19, M has a Schreier K-basis, U, consisting of all linked expressions from

$$\begin{matrix} -X+ & -Zt^{-1}- & -C \\ +tW+ & +tYt^{-1}- & +tD. \end{matrix} \tag{22}$$

Consider any linked expression $u = c_n c_{n-1} \cdots c_0$ in U. We define the *length* of u to be n, the *left sign* of u to be the left sign of c_n, and the *left index* of u to be the left index of c_n, denoted e_u (an element of E). There is then a *coefficient-of-u* map $\varphi_u : M \to Ke_u$ arising as the composite $M = \bigoplus_{v \in U} Kv \to Ku \simeq Ke_u$.

For each n let U_{2n} be the set of $u \in U$ of length n and left sign $-$, and U_{2n+1} the set of $u \in U$ of length n and left sign $+$. Let M_n/M_{n-1} denote the K-submodule of M spanned by U_n, and $M_n = \sum_{i \leqslant n} M_i/M_{i-1}$. Thus $M_n = 0$ for $n < 0$, $M_1 = M_0 + tM_0$ and for all $n \geqslant 0$ $M_{2n+2} = (A + At^{-1}) M_{2n+1}$ and $M_{2n+3} = M_{2n+2} + tM_{2n+2}$. This just elaborates on the grading constructed in the previous section.

Well-order each of the sets W, X, Y, Z, C, D arbitrarily, and order U_n lexicographically reading from *right* to *left*. Then $U = \bigcup U_n$ is well-ordered, first by the subscripts n and then by the well-ordering within each U_n. For each $x \in M$ and basis element $u \in U$, if $\varphi_u(x) \neq 0$, u is said to be a *K-support* of x. For nonzero $x \in M$, we define the *leading* K-support of x to be the K-support of x that is greatest under the ordering of U.

Let x be a non-zero element of M and consider the least n such that $x \in M_{2n+3}$. Recalling that M_{2n+3}/M_{2n+1} is isomorphic to the direct sum of all sign-linked tensor products $C_{n+1} \otimes \cdots \otimes C_0$, we are motivated to say that x has *length* $n+1$. Let us examine the image of x in

$$\begin{array}{lcccl} M_{2n+3}/M_{2n+1} & & & & \\ & M_{2n+3}/M_{2n+2} & & tA \otimes_K M_{2n+1}/M_{2n} & \oplus\ tAt^{-1}/K \otimes_K M_{2n+1}/tM_{2n} \\ \cong & \oplus & \cong & & \oplus \\ & M_{2n+2}/M_{2n+1} & & A/K \otimes_K M_{2n+1}/M_{2n} \oplus & At^{-1} \otimes_K M_{2n+1}/tM_{2n}. \end{array} \tag{23}$$

If the component of x in M_{2n+3}/M_{2n+2} is zero we say x is *A-pure* and this happens precisely if $x \in M_{2n+2}$. If x is *not* A-pure we define the *t-leading K-support* of x to be the greatest K-support of x lying in U_{2n+3} (that is, in fact, the leading K-support). (Notice that the t-leading K-support of an A-pure element is not defined.) If the component of x in M_{2n+2}/M_{2n+1} is zero we say x is *t-pure* and this happens precisely if $x \in tM_{2n+2}$. If x is *not* t-pure we define the *A-leading* K-support of x to be the greatest K-support of x lying in U_{2n+2}. (Again, the A-leading K-support of a t-pure element is not defined.) Let us call an element *pure* if it is A-pure or t-pure, and call the remaining elements *K-pure.* Thus a K-pure element is one with an A-leading and a t-leading K-support.

If $u \in U_{2n+1}$ for some n then the K-linear map $\varphi_u : M_{2n+1}/M_{2n} \to K$ lifts to an A-linear map $\Phi_u : A \otimes_K M_{2n+1}/M_{2n} \to A$. Similarly, if $u \in U_{2n}$ then φ_u lifts to an A-linear map $\Phi_u : At^{-1} \otimes_K M_{2n}/M_{2n-1} \to At^{-1}$. Now the induced grading allows an identification of ${}_AM$ with

$$M_0 \oplus A \otimes_K \left(\bigoplus_n M_{2n+1}/M_{2n} \right) \oplus At^{-1} \otimes_K \left(\bigoplus_n M_{2n}/M_{2n-1} \right)$$

and so each $u \in U$ induces a left A-linear map $\Phi_u : M \to A \oplus At^{-1}$. An *A-support* of an element x of M is a basis element u such that $\Phi_u(x) \neq 0$, and the greatest such is called the *leading* A-support. Notice that the set of elements of M which have no A-supports is precisely M_0.

Let L be an R-submodule of M. Let $L(K)$ be the K-submodule of L consisting of all elements of L which have no K-support that is the leading K-support of a pure element of L. Let $L(A)$ be the A-submodule of L consisting of all elements of L which have no A-support that is the A-leading or t-leading K-support of an element of L. Let $L_0 = AL(K) + L(A)$, an A-submodule of L.

It is clear that $RL_0 \subseteq L$; to see that equality holds let us prove by induction that $L \cap M_{2n+1} \subseteq RL_0$ for all n. Since this is true for $n = -1$ we may suppose it holds for some $n \geqslant -1$. Let $x \in L \cap M_{2n+3}$. Since we have to show that $x \in RL_0$ there is no loss of generality in assuming $x \notin M_{2n+1}$ and $x \notin L_0$. If $x \in M_{2n+2}$ then consider the fact that $x \notin L(A)$: some A-support u

of x is the A-leading or t-leading K-support of an element y of L, and we may assume $\varphi_u(y)=e_u$ and $e_u y=y$. Since $x\in M_{2n+2}$, it follows from the definition of A-support and (15) that $y\in M_{2n+1}$, so $y\in RL_0$ by the induction hypothesis. Let $x'=x-\Phi_u(x)\,y$. Since there is no "cancellation" in $\Phi_u(x)\,u$ the leading A-support of $\Phi_u(x)\,y$ is u; thus the (finite) set of A-supports of x', arranged in descending order, is lexicographically less than the set of A-supports of x. But the set of descending sequences in a well-ordered set is itself well-ordered, so we may inductively assume that $x'\in RL_0$; hence $x\in RL_0$. This proves that $L\cap M_{2n+2}\subseteq RL_0$. Now consider the case where x is t-pure: here $t^{-1}x\in M_{2n+2}$ so by the preceding step $t^{-1}x\in RL_0$ and hence $x\in RL_0$. This proves that all pure elements of $L\cap M_{2n+3}$ lie in RL_0. Finally consider the case where x is K-pure. Since $x\notin L(K)$ some K-support u of x is the leading K-support of a pure element y of L, and we may assume $\varphi_u(y)=e_u$ and $e_u y=y$. Since y is a pure element of $L\cap M_{2n+3}$ it belongs to RL_0 by the preceding step. Let $x'=x-\varphi_u(x)\,y$. The set of K-supports of x' is lexicographically less than the set of K-supports of x so we may inductively assume that $x'\in RL_0$; hence $x\in RL_0$. This proves that $L\cap M_{2n+3}\subseteq RL_0$, and by induction this holds for all n. Hence $RL_0=L$.

It remains to show that the map $R\otimes_A L_0\to M$ is injective. It will be important that the only data needed on $L(A)$, $L(K)$ is the following.

Every element of $L(K)$ is K-pure. (24)

Every element of $L(A)$ is A-pure. (25)

Every element of $tL(A)$ is t-pure. (26)

No element of $L(K)$ has a K-support u which is the leading K-support of a pure element ry, where $r\in R$, and either $y\in L(K)$ with $\varphi_u(y)=0$ or $y\in L(A)$. (27)

No element of $L(A)$ has an A-support u which is the A-leading or t-leading K-support of an element ry, where $r\in R$, and either $y\in L(K)$ or $y\in L(A)$ with $\Phi_u(y)=0$. (28)

It is immediate from the definitions of $L(K)$, $L(A)$ that (24), (25), (27), (28) hold. To see (26) let $tx\in tL(A)$, say, $x\in M_{2n+2}-M_{2n}$. Then $tx\in tM_{2n+2}$ and if tx is not t-pure it must lie in $M_{2n+1}=M_{2n}+tM_{2n}$. But then $x\in t^{-1}M_{2n}+M_{2n}$ and the leading A-support u of x lies in U_{2n}. Now u is easily seen to be the A-leading K-support of tx. This contradicts the fact that $x\in L(A)$, so tx must be t-pure.

We are now ready to begin proving that $f\colon R\otimes_A L_0\to M$ is injective. To avoid the ambiguity of the R-action that arises from viewing L_0 as a subset of both $R\otimes_A L_0$ and M, we use a copy N_0 of L_0 for the remainder of the proof. The result that we want is (i) of the following.

LEMMA 20. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is completely reducible and α, β are injective. Let N_0, M_0 be left A-modules and $f: R \otimes_A N_0 \to R \otimes_A M_0$ be any R-linear map. Suppose that N_0 can be written in the form $N_0 = N(A) \oplus A \otimes_K N(K)$ in such a way that the images $L(A) = f(N(A)), L(K) = f(N(K))$ satisfy (24)–(28) with respect to a Schreier basis U of $M = R \otimes_A M_0$.*

(i) *If the restrictions of f, $N(A) \to M$, $N(K) \to M$, are injective then f is injective.*

(ii) *If f is surjective then $L(A) = M_0$, $L(K) = 0$ and f is induced from an A-linear map $N_0 \to M_0$.*

Proof. (i) Given a K-submodule P of $N = R \otimes_A N_0$, for each $u \in U$ that occurs as the leading K-support of an element of $f(P)$, we can choose an $x_u \in P$ such that the leading K-support of $f(x_u)$ is u and further $\varphi_u(f(x_u)) = e_u$, $e_u x_u = x_u$. We call e_u the *left index* of x. Let us speak of a subset $\{x_u\}$ so chosen as *representing* the leading K-supports for P. It will be a K-basis of P if the restriction of f, $P \to M$, is injective. Similarly we define subsets *representing* the A-leading K-supports for P, and subsets *representing* the t-leading K-supports of P. Still assuming that $P \to M$ is injective, these will be K-bases if no element of $f(P)$ is t-pure, A-pure, respectively.

Assume the hypotheses for (i) hold and let ${}^+B$, ${}^-B$ represent all t-leading (= leading by (24)) and A-leading K-supports for $N(K)$, respectively. Further, let ${}^-C$ represent all A-leading (= leading) K-supports for $N(A)$ and ${}^+C$ represent all t-leading (= leading) K-supports for $tN(A)$. By the preceding paragraph each of these is a K-basis since (24)–(26) hold.

Let V' denote the complement of ${}^+B$ in the family of all linked expressions constructed from

$$\begin{matrix} -X+ & -Zt^{-1}- & -{}^-C & -{}^-B \\ +tW+ & +tYt^{-1}- & +{}^+C & +{}^+B; \end{matrix} \tag{29}$$

that is, V' looks like the family of all linked expressions constructed from

$$\begin{matrix} -X+ & -Zt^{-1}- & -({}^-C \cup X({}^+B) \cup {}^-B) \\ +tW+ & +tYt^{-1}- & +({}^+C \cup tW({}^+B)) \end{matrix}$$

with the obvious interpretation of $tW({}^+B)$, $X({}^+B)$. Here the third column gives K-bases of $N(A) + AN(K) = N_0$ and $tN(A) + tAN(K) = tN_0$, respectively. It follows that V' is a K-basis of $N = R \otimes_A N_0$. Further ${}^+B$ and ${}^-B$ are K-bases for the same K-submodule, $N(K)$, so the set

$$V = (V' - {}^-B) \cup {}^+B$$

is again a K-basis for N, and is the complement of ${}^{-}B$ in the family of all linked expressions constructed from (29).

To prove that f is injective we shall show that the $f(v)$ $(v \in V)$ have distinct leading K-supports.

Let $d_m \cdots d_0$ be a linked expression in V with $m \geqslant 1$. We claim that the leading K-support of $f(d_m \cdots d_0) = d_m \cdots d_1 f d_0$ is $d_m \cdots d_1 u$, where

$$u \text{ is } \begin{cases} \text{the } A\text{-leading } K\text{-support of } fd_0 \text{ if the right sign of } d_1 \text{ is } - \\ \text{the } t\text{-leading } K\text{-support of } fd_0 \text{ if the right sign of } d_1 \text{ is } +. \end{cases}$$

Since $d_m \ldots d_1 d_0 \neq 0$ it follows that $d_m \ldots d_1 e_u \neq 0$ (since $e_u d_0 = d_0$) so $d_m \ldots d_1 u \neq 0$ is therefore a K-support of $d_m \ldots d_1 f d_0$. Any other K-support can, for some K-support v of fd_0, be written $d'_m \ldots d'_i v$, and this is either of shorter length, or of the same length and smaller in the lexicographic ordering since v must then be lexicographically less than u. This proves that $f(d_m \ldots d_0)$ has leading K-support as claimed. It also proves that every element of $f(V - {}^{+}B)$ is pure.

For each n and each $u \in U_{2n}$ let $b_u \in {}^{-}B$, $c_u \in {}^{-}C$ denote the element, if any, whose image has A-leading K-support u. To emphasize the t-purity of elements of U_{2n+1}, let us denote them by tu, where u need not be an element of U. For each $tu \in U_{2n+1}$ let $b_{tu} \in {}^{+}B$, $c_{tu} \in {}^{+}C$ denote the element, if any, whose image has t-leading K-support tu. Then by the preceding paragraph, the leading K-support of $f(v)$, $v = d_m \ldots d_0 \in V$, is $d_m \ldots d_1 tu$ if $d_0 = b_{tu}$ or c_{tu} and is $d_m \ldots d_1 u$ if $d_0 = b_u$ (so $m \geqslant 1$) or c_u.

With the aim of getting a contradiction suppose that two distinct elements $v = d_m \ldots d_0$, $v' = d'_{m'} \ldots d'_0$ have images with the same leading K-supports $d_m \ldots d_1 (t) u = d'_{m'}, \ldots d'_1 [t]\, u'$, where, say, $m \geqslant m'$, and $(t)u$, $[t]\, u' \in U$ and there are four possibilities as to the presence or absence of t's. From the construction of U, $d_n \ldots d_1 (t)\, u = [t]\, u'$, where $n = m - m'$.

If $n = 0$ then $(t)\, u = [t]\, u'$ so $u = u'$ and $v = d_m \ldots d_1 b_{(t)u}$, $v' = d_m \ldots d_1 c_{(t)u}$ or the other way around. Then either b_u, c_u are both defined, or b_{tu}, c_{tu} are both defined. In the former case $f(b_u)$ is an element of $L(K)$ with a K-support u that is the leading K-support of a pure element $f(c_u) \in L(A)$, which contradicts (27). Similarly, in the latter case, $f(b_{tu})$ is an element of $L(K)$ with a K-support tu that is the leading K-support of a pure element $t \cdot t^{-1} f(c_{tu})$, where $t \in R$ and $t^{-1} f(c_{tu}) \in L(A)$, which also contradicts (27). Hence $n > 0$.

There are now essentially two cases: $d'_0 = b_{[t]u'}$, $d'_0 = c_{[t]u'}$.

If $d'_0 = b_{[t]u'}$ then the element $f(d'_0)$ of $L(K)$ has a K-support $[t]\, u' = d_n \ldots d_1 (t)\, u$ that is the leading K-support of a pure element $f(d_n \ldots d_1 d_0)$, where either

(i) $d_0 = b_{(t)u}$ so $d_n \ldots d_1 \in R$ and $fd_0 \in L(K)$ with $\varphi_{d_n \ldots d_1 (t) u}(fd_0) = 0$ since $n \geqslant 1$, or

(ii) $d_0 = c_{(t)u}$ so $d_n \dots d_1(t) \in R$ and $(t^{-1})\, fd_0 \in L(A)$,

and (27) is contradicted in any event.

If $d_0' = c_{[t]u'}$ then the element $[t^{-1}]fd_0' \in L(A)$ has an A-support $d_{n-1} \dots d_1(t)\, u$ that is the A-leading or t-leading K-support of an element $f(d_{n-1} \dots d_0)$, where one of the following holds,

(i) $d_0 = b_{(t)u}$ so $d_{n-1} \dots d_1 \in R$ and $fd_0 \in L(K)$, or

(ii) $d_0 = c_u$ so $d_{n-1} \dots d_1 \in R$ and $fd_0 \in L(A)$ with $\Phi_{d_{n-1} \dots d_1 u}(fd_0) = 0$ since u is the A-leading K-support of the A-pure fd_0 and is therefore longer than any A-support of fd_0, or

(iii) $d_0 = c_{tu}$ so $d_{n-1} \dots d_1 t \in R$ and $t^{-1}fd_0 \in L(A)$ with $\Phi_{d_{n-1} \dots d_1 tu}(t^{-1}fd_0) = 0$ since tu is the leading K-support of the t-pure $fd_0 \in tM_{2i+2} - M_{2i+1}$, say, and is therefore longer than any A-support of the A-pure $t^{-1}fd_0 \in M_{2i+2}$,

which contradicts (28).

Hence the images of distinct elements of V have distinct leading K-supports, which means that V has a faithful image which is a K-basis of the image of f. Hence f is injective.

(ii) Now suppose that f is surjective. Without loss of generality we may retain the hypotheses of (i) by dividing out of $N(A)$, $N(K)$ their respective intersections with the kernel of f. Let V be as in the proof of (i); by the surjectivity of f any element of M_0 can be written as a K-linear combination of elements $f(v)$, $v \in V$, and by considering the leading term of such an expression we see that all $f(v)$ must lie in M_0, and so in particular are A-pure. From the form of V, the only possibility is for the v to lie in ${}^{-}C$. Hence $f(N(A)) \supseteq M_0$. Let $N'(A)$ be the inverse image in $N(A)$ of M_0, so the composite

$$R \otimes_A N'(A) \to R \otimes_A N_0 \to R \otimes_A M_0$$

is the identity. Since the right hand map is an isomorphism by (i), so is the left hand map. But this map is induced from the inclusion $N'(A) \to N_0$, and R_A is faithfully flat, so $N'(A) = N_0$. Thus $L(K) = 0$, $L(A) = M_0$. ∎

THEOREM 21. *If $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is completely reducible and α, β are injective then any R-submodule of an induced R-module is isomorphic to an induced R-module.* ∎

COROLLARY 22. *Every projective left R-module P is of the form $R \otimes_A P_0$, where P_0 is a projective A-module.*

Proof. As a submodule of a free, hence induced, R-module, P is of the form $R \otimes_A P_0$ for some left A-module P_0. But by Theorem 17(iii), $hd_A P_0 \leqslant hd_R P = 0$ so ${}_A P_0$ is projective. ∎

8. Homomorphisms between Induced Modules

In this section we again take K to be completely reducible (given with the set E) and α, β to be monomorphisms.

Suppose that N_0, M_0 are left A-modules and we wish to determine the R-linear maps $f: R \otimes_A N_0 \to R \otimes_A M_0$ (or equivalently the A-linear maps $N_0 \to R \otimes_A M_0$). The immediate example is that of a homomorphism $R \otimes_A f_0$: $R \otimes_A N_0 \to R \otimes_A M_0$ induced from an A-linear map $f_0: N_0 \to M_0$. In very trivial circumstances these are the only homomorphisms, but in general there are others, and a description depends on a certain type of R-linear automorphism of $N = R \otimes_A N_0$ defined as follows. Let $\varphi: N \to R$ be an R-functional and n any element of the kernel of φ; then writing $\hat{n}$ for the R-linear map $R \to N$ which sends 1 to n, we see $\varphi\hat{n}: R \to R$ is zero and hence $1_N - \hat{n}\varphi: N \to N$ is an automorphism with inverse $1_N + \hat{n}\varphi$. Such an automorphism is called a *transvection*. We will say that a transvection $1_N - \hat{n}\varphi$ of $N = R \otimes_A N_0$ is *N_0-based* (or less precisely *A-based*) if φ is an induced functional $\varphi = R \otimes_A \varphi_0: R \otimes_A N_0 \to R \otimes_A A$, and $n = rn_0$, where $r \in R$ and n_0 is in the kernel of $\varphi_0: N_0 \to A$. In this case $N_0 \to R \otimes_A N_0$ by $x \mapsto x - \varphi_0(x)\, rn_0$.

Since an N_0-based transvection depends on the choice of N_0 in N, we are interested in methods for finding new N_0's. One such method is applicable if N_0 is written in the form $N(A) \oplus A \otimes_K Kx$, $x \in N_0$, for then we can replace it with $N(A) \oplus At^{-1} \otimes_K Kx$; and vice-versa. This will be called a *t-factor change of basis*; it is not an automorphism of N but a rewriting of the presentation of N as an induced module. Combined with the notion of A-based transvections it provides a group of automorphisms of N that is sufficiently large for our purposes.

In the next section we will reap the interesting consequences of the following technical result.

Theorem 23. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is completely reducible and α, β are injective; let N_0, M_0 be left A-modules and $f: R \otimes_A N_0 \to R \otimes_A M_0$ be an R-linear map. If f is surjective and N_0 is finitely generated then f can be written as a composite of finitely many t-factor changes of basis and A-based transvections of $N = R \otimes_A N_0$, followed by a homomorphism induced from a surjective A-linear map f_0 to M_0.*

In particular M_0 is finitely generated.

Notice that f_0 will be an isomorphism if and only if f is an isomorphism.

Proof. Obviously N_0 can be written as $N(A) \oplus A \otimes_K N(K)$ for a finitely generated A-submodule $N(A)$, and a finitely generated K-submodule $N(K)$, of N_0; for example, take $N(A) = N_0$, $N(K) = 0$. Given such a representation we write $L(A) = f(N(A))$, $L(K) = f(N(K))$. We wish to alter $N(A)$, $N(K)$ and f so as to transform $L(A)$ into M_0, $L(K)$ into 0.

Let U be a Schreier basis for $M = R \otimes_A M_0$. As $N(K)$ is finitely generated, only finitely many elements of U occur as K-supports of elements of $L(K)$; similarly, only finitely many elements of U occur as A-supports of elements of $L(A)$. To measure the effectiveness of our procedures we associate with $L(A)$, $L(K)$ a totally ordered set as follows: For each n let $L(A)_n$ denote the (finite) set of elements of U_n which occur as an A-support of an element of $L(A)$; and similarly let $L(K)_n$ be the set of K-supports for $L(K)$ in U_n. Now totally order the disjoint union of these totally ordered sets by putting

$$\cdots L(K)_{2n+2} > L(A)_{2n+1} > L(A)_{2n} > L(K)_{2n+1} > L(K)_{2n} > L(A)_{2n-1} \cdots. \quad (30)$$

The resulting totally ordered (finite) set will be called the *index* of the pair $L(A)$, $L(K)$. The set of indices is well-ordered by comparing two by the largest elements in the greatest of the sets (30) in which they differ.

If $L(A)$, $L(K)$ fail to satisfy any one of conditions (24)–(28) then a remedial operation can be performed that will reduce the index.

If (24) fails then some $x \in N(K)$ is such that fx is pure. If the leading K-support is u we assume $e_u x = x$ and $\varphi_u(fx) = e_u$. The map $\varphi_u f$: $N(K) \to M \to K$ has image Ke_u and so splits and $N(K) = N'(K) \oplus Kx$, where $N'(K)$ is the K-submodule of $N(K)$ consisting of all elements whose image in M does not have u as a K-support. Now if fx is t-pure, with $u \in L(K)_{2n+1}$, say, then apply a t-factor change of basis and replace $N(K)$ with $N'(K)$ and $N(A)$ with $N(A) \oplus At^{-1} \otimes_K Kx$, so the A-supports of $f(t^{-1}x)$ add elements to $L(A)_{2n-1}, L(A)_{2n-2}, \ldots$, but the element u has been removed from $L(K)_{2n+1}$ which reduces the index. Otherwise fx is A-pure with $u \in L(K)_{2n+2}$, say, and we replace $N(K)$ with $N'(K)$, $N(A)$ with $N(A) \oplus A \otimes_A Kx$, which removes u from $L(K)_{2n+2}$ and adds the A-supports of fx to $L(A)_{2n+1}, L(A)_{2n}, \ldots$, which reduces the index.

If (25) fails then some $x \in N(A)$ is such that fx is not A-pure. Thus $fx \in (M_{2n} + tM_{2n}) - M_{2n}$ for some n, and hence the leading A-support, tu, say, of fx is the t-leading K-support of fx and $\Phi_{tu}(fx) = \varphi_{tu}(fx)$. We may assume $e_{tu} x = x$ and $\varphi_{tu}(fx) = e_{tu}$. Now the map $\Phi_{tu} f$: $N(A) \to M \to A$ has image Ae_{tu} and so splits and $N(A) = N'(A) \oplus A \otimes_K Kx$, where $N'(A)$ is the A-submodule of $N(A)$ consisting of all elements whose image in M does not have tu as an A-support. Replace $N(A)$ with $N'(A)$ and $N(K)$ with $N(K) \oplus Kx$. This removes tu from $L(A)_{2n+1}$ and adds the K-supports of fx to $L(K)_{2n+1}, L(K)_{2n}, \ldots$, which reduces the index.

If (26) fails then some $x \in N(A)$ is such that tfx is not t-pure. Thus $fx \in (M_{2n} + t^{-1}M_{2n}) - M_{2n+1}$ for some n, and hence the leading A-support, u, say, of fx, is the A-leading K-support of tfx, and $\Phi_u(fx) = t^{-1}\varphi_u(tfx) \in At^{-1}$. We may assume $e_u tx = tx$ and $\varphi_u(tfx) = e_u$. Now the map $\Phi_u f: N(A) \to M \to At^{-1}$ has image $At^{-1}e_u$ and so splits and $N(A) = N'(A) \oplus At^{-1} \otimes_K Ktx$, where $N'(A)$ is the A-submodule of $N(A)$ consisting of all elements whose image in M does not have u as an A-support. Apply a t-factor change of basis to replace $N(A)$ with $N'(A)$ and $N(K)$ with $N(K) \oplus Ktx$. Now $f(tx) \in tM_{2n} + M_{2n} = M_{2n+1}$ so we have removed u from $L(A)_{2n}$, and added the K-supports of $f(tx)$ to $L(K)_{2n+1}$, $L(K)_{2n}$,..., which reduces the index.

If (27) fails then some element of $L(K)$ has a K-support u that is the leading K-support of a pure element rfy, where $r \in R$ and either $y \in N(K)$ with $\varphi_u(fy) = 0$ or $y \in N(A)$. We may assume $e_u ry = ry$ and $\varphi_u(f(ry)) = e_u$. Extend the composite $\varphi_u f: N(K) \to M \to K$ to an A-linear functional $\psi_u: N(A) \oplus A \otimes_K N(K) \to A$ vanishing on $N(A)$, so that $\psi_u(y) = 0$ in either case. There is then an N_0-based transvection sending each $x \in N_0$ to $x - \psi_u(x)\, ry$. Replacing f by its composite with this automorphism of N does not affect $L(A)$ and it removes u from some $L(K)_n$ and adds the lower K-supports of $f(ry)$ to $L(K)_n$, $L(K)_{n-1}$,..., which reduces the index.

If (28) fails then some element of $L(A)$ has an A-support u (or tu) that is the A-leading (or t-leading) K-support of an element rfy, where $r \in R$ and either $y \in N(K)$ or $y \in N(A)$ with $\Phi_{(t)u}(fy) = 0$. We may assume $e_{(t)u} ry = ry$ and $\varphi_{(t)u}(f(ry)) = e_{(t)u}$. Extend the composite $t\Phi_u f: N(A) \to M \to At^{-1} \to A$ (or $\Phi_{tu} f: N(A) \to M \to A$) to an A-functional $\Psi_{(t)u}$: $N_0 \to A$ vanishing on $N(K)$, so that $\Psi_{(t)u}(y) = 0$ in either case, There is then an N_0-based transvection sending each $x \in N_0$ to $x - \Psi_u(x)\, t^{-1}ry$ (or $x - \Psi_{tu}(x)\, ry$). Replacing f by its composite with this transvection does not affect $L(K)$ and it removes $(t)\, u$ from some $L(A)_n$ and adds the lower A-supports of $f(ry)$ to $L(A)_n$, $L(A)_{n-1}$,..., which reduces the index.

Thus we can continue reducing the index of the pair $L(A)$, $L(K)$ by composing f with A-based transvections, by performing t-factor changes of basis, and by performing summand transfers between $N(A)$ and $N(K)$ (which do not change N_0) as long as conditions (24)–(28) are not satisfied. By the well-ordering of indices, a finite number of these operations suffice to make (24)–(28) satisfied. Then by Lemma 20(ii), $L(A) = M_0$, $L(K) = 0$ and f is induced from a surjective A-homomorphism to M_0.

What this proves is that there is an automorphism g of N that is the composite of finitely many t-factor changes of basis and A-based transvections such that $fg: N \to N \to M$ is induced from a surjective A-linear map. Since g^{-1} is made up of the same types of automorphisms as was g, the result follows. ∎

COROLLARY 24. *If an induced R-module* $R \otimes_A M_0$ *is finitely generated then the A-module* M_0 *is finitely generated.*

Proof. If $f: R^n \to R \otimes_A M_0$ is surjective, the preceding theorem applies with $N_0 = A^n$, so ${}_A M_0$ is finitely generated. ∎

9. Projective Modules, Free Ideals and the General Linear Group

Let us consider Theorem 23 in the context of categories. We write K-mod for the category of finitely generated left K-modules and let $S_\oplus(K\text{-mod})$ denote the additive semigroup of isomorphism classes of objects of K-mod under the operation induced by the direct sum, $\oplus$.

The homomorphisms $K \rightrightarrows^{\alpha}_{\beta} A \to^{\eta} R$ induce semigroup homomorphisms

$$S_\oplus(K\text{-mod}) \underset{\tilde{\beta}}{\overset{\tilde{\alpha}}{\rightrightarrows}} S_\oplus(A\text{-mod}) \xrightarrow{\tilde{\eta}} S_\oplus(R\text{-mod})$$

under tensor product, and the two composite homomorphisms are equal. Let us write $S \oplus (R \otimes_A \text{mod})$ for the image of $\tilde{\eta}$. Then the following can be deduced.

THEOREM 25. *Let* $R = A_K\langle t, t^{-1}; \beta\rangle$, *where K is completely reducible and* α, β *are injective. Then* $\tilde{\eta}': S_\oplus(A\text{-mod}) \to S_\oplus(R \otimes_A \text{mod})$ *is the coequalizer of the semigroup homomorphisms* $\tilde{\alpha}$, $\tilde{\beta}$. *In particular, if K is a skew field then* $\tilde{\eta}'$ *is an isomorphism.*

Proof. From the preceding remarks $\tilde{\eta}$ factors through the coequalizer and it remains to determine the conditions under which $\eta([M_0]) = \eta([N_0])$ for $[M_0]$, $[N_0] \in S_\oplus(A\text{-mod})$. Thus we are considering an R-isomorphism

$$R \otimes_A M_0 \cong R \otimes_A N_0$$

of two induced modules, which by Theorem 23 can be decomposed as

$$R \otimes_A M_0 = R \otimes_A M_0^{(1)} \cong R \otimes_A M_0^{(1)} = R \otimes_A M_0^{(2)} \cong \cdots \cong R \otimes_A M_0^{(n)},$$

where $M_0^{(n)} = N_0$, where the automorphisms are A-based transvections or induced automorphisms, and each $M_0^{(i+1)}$ is obtained from $M_0^{(i)}$ by a t-factor change of basis. We wish to show that $[M_0]$, $[M_0^{(1)}]$,..., $[M_0^{(n)}]$ are identified in the coequalizer of $\tilde{\alpha}$, $\tilde{\beta}$, so we are reduced to considering the case where N_0 is obtained from M_0 by a t-factor change of basis. Thus we may write

$$M_0 \cong L_0 \oplus A \otimes_K Kx, \qquad N_0 \cong L_0 \oplus At^{-1} \otimes_K Kx \cong L_0 \oplus A_\beta \otimes_K Kx$$

and hence $[M_0]=[L_0]+\tilde{\alpha}[Kx]$, $[N_0]=[L_0]+\tilde{\beta}[Kx]$ in $S_{\oplus}(A\text{-mod})$. From this it is clear that $[M_0]$, $[N_0]$ have equal images in the coequalizer of $\tilde{\alpha}$, $\tilde{\beta}$, so the coequalizer factors through $\tilde{\eta}'$ and the result follows. ∎

Our interest in Theorem 25 lies in its applicability to projective modules. Let us write A-pmod for the full subcategory of A-mod whose objects are the finitely generated projective A-modules. Since K-pmod $=K$-mod for K completely reducible, we have the following.

THEOREM 26. *Let $R=A_K\langle t,t^{-1};\beta\rangle$, where K is completely reducible and α,β are injective. Then*

$$S_{\oplus}(K\text{-mod}) \rightrightarrows S_{\oplus}(A\text{-pmod}) \to S_{\oplus}(R\text{-pmod})$$

is a coequalizer diagram in the category of semigroups.

Proof. Since shrinking the common codomain of α, β has the effect of shrinking the coequalizer, it suffices to show that $\eta(S\oplus(A\text{-pmod}))=S\oplus(R\text{-pmod})$. But this equality is immediate from Corollaries 22 and 24. ∎

COROLLARY 27. *If K is a skew field then $S_{\oplus}(A\text{-pmod})\to S_{\oplus}(R\text{-pmod})$ is an isomorphism.* ∎

The completely reducible rings K for which Corollary 27 holds are precisely the finite direct products of skew fields of distinct characteristics. For if K is such a ring then for any A, $\tilde{\alpha}$, $\tilde{\beta}$ are obviously equal. Conversely, if K is not such a ring, let F be the smallest completely reducible subring of K, necessarily a finite direct product of prime fields of distinct characteristics. Let A be the coproduct $K \amalg_F K$, and let α, $\beta: K\to A$ be the canonical maps into the first and second factors, respectively, so $A=\alpha(K)\amalg_F\beta(K)$ and $R=A_K\langle t,t^{-1};\beta\rangle\cong K\amalg_F F[t,t^{-1}]$. In the diagram

$$\begin{array}{ccccc}
 & & S_{\oplus}(K\text{-mod}) & & \\
 & \nearrow & & \searrow & \\
S_{\oplus}(F\text{-mod}) & & & & S_{\oplus}(A\text{-pmod}) \twoheadrightarrow S_{\oplus}(R\text{-pmod}) \\
 & \searrow & & \nearrow & \\
 & & S_{\oplus}(K\text{-mod}) & &
\end{array}$$

of semigroup homomorphisms, $S_{\oplus}(F\text{-mod})$ is a proper subsemigroup of the semigroup $S_{\oplus}(K\text{-mod})$, and by [2, Corollary 2.11], $S_{\oplus}(A\text{-pmod})$ is the abelian-semigroup coproduct of two copies of $S_{\oplus}(K\text{-mod})$ amalgamating

$S_{\oplus}(F\text{-mod})$. From the form the relations take it is not difficult to show that $S_{\oplus}(K\text{-mod}) \to S_{\oplus}(A\text{-pmod})$ is not surjective. Now by Theorem 26, $S_{\oplus}(R\text{-pmod})$ is obtained from $S_{\oplus}(A\text{-pmod})$ by identifying the two images of $S_{\oplus}(K\text{-mod})$. Thus $S_{\oplus}(R\text{-pmod})$ collapses down to $S_{\oplus}(K\text{-mod})$ and $S_{\oplus}(A\text{-pmod}) \to S_{\oplus}(R\text{-pmod})$ is not injective.

Henceforth we assume that K is a skew field. It is now unnecessary to specify that α, β be injective, for if they are not then $A = 0$ and all our results hold trivially.

We say a ring A is *projective-free* if every finitely generated projective A-module is free of unique rank, or equivalently $S_{\oplus}(\mathbb{Z}\text{-pmod}) \to S_{\oplus}(A\text{-pmod})$ is bijective.

THEOREM 28. *If $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is a skew field then each of the following classes of rings contains R if and only if it contains A:*

(i) *projective-free rings;*
(ii) *left firs;*
(iii) *n-firs, where n is a natural number;*
(iv) *semifirs.*

Proof. Part (i) is immediate from Theorem 26; (ii) now follows by Corollary 18 and the fact that left firs can be characterized as projective-free left hereditary rings; cf. [11, Theorem 0.2.9]. To see (iii), let A be an n-fir and M an n-generator R-submodule of a free (hence induced) left R-module F. By Theorem 21, M is induced, say, $M \cong R \otimes_A M_0$. By Theorem 23 any surjection $R^n \to M$ can be written as $R^n \cong R \otimes_A L_0 \to R \otimes_A M_0$, where $L_0 \to M_0$ is a surjective A-linear map. By Theorem 25, $R \otimes_A A^n \cong R \otimes_A L_0$ implies $L_0 \cong A^n$ so M_0 is an n-generator submodule of the A-module ${}_AF$. But ${}_AF$ is free by Theorem 17, so ${}_AM_0$ is free and hence ${}_RM$ is free. The uniqueness of rank is clear from Corollary 27. Conversely, let R be an n-fir and M_0 an n-generator A-submodule of a free A-module F_0. By the left–right dual of Theorem 9, R_A is flat and $R \otimes_A M_0 \to R \otimes_A F_0$ is injective. So $R \otimes_A M_0$ is an n-generator R-submodule of a free R-module and so is free of unique rank. Now by Theorem 25, ${}_AM_0$ is free of unique rank which proves (iii). Now (iv) follows by considering all natural numbers n. ∎

For our next applications we consider the general linear group $GL_n(R)$ of $n \times n$ invertible matrices over R. If x and z are a column and a row vector of length n over A such that $zx = 0$ then for any $y \in R$, $I - xyz$ is an element of $GL_n(R)$, and we shall call such a matrix an *A-based transvection matrix.*

THEOREM 29. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is a skew field. Suppose* *that n is an integer such that for all left A-modules P, if* $A \oplus P \cong A^n$ *then* $P \cong A^{n-1}$. *Then* $GL_n(R)$ *is generated by* $GL_n(A)$,

$$\begin{pmatrix} t & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

and the A-based transvection matrices.

For $n = 1$ this says that if one-sided inverses in A are two-sided inverses then the group of units of R is generated by the units of A, t, and $\{1 - xyz \mid zx = 0,\ z,\ x \in A,\ y \in R\}$.

Proof. Any element of $GL_n(R)$ acts as an R-automorphism of R^n by right multiplication. By Theorem 23 any automorphism of $R^n = R \otimes_A A^n$ can be written as a composite of isomorphisms of the following types:

induced isomorphism $R \otimes_A M_0 \simeq R \otimes_A N_0$,

A-based transvection $R \otimes_A M_0 \simeq R \otimes_A M_0$,

t-factor change of basis $R \otimes_A M_0 = R \otimes_A N_0$.

In the first and last factors of this decomposition $R \otimes_A M_0$ and $R \otimes_A N_0$, respectively, are given in the form $R^n = R \otimes_A A^n$ which means that the specified R-bases are induced from A-bases of M_0, N_0, respectively. It is clear that we are free to specify n-element A-bases for all the other M_0, N_0 that occur, and to let the $R \otimes_A M_0$, $R \otimes_A N_0$ have the induced R-bases. This then gives a matrix factorization of our element of $GL_n(R)$. Clearly, the matrices corresponding to A-based transvections in our factorization are A-based transvection matrices, and the matrix corresponding to the induced isomorphism is an element of $GL_n(A)$. Finally, for a t-factor change of basis, we know from our hypotheses on n, A that some automorphism of M_0 carries the given A-basis of M_0 to an A-basis $v_1, \ldots, v_n$ such that $t^{\pm 1}v_1, \ldots, v_n$ is an A-basis of N_0, so the matrices representing t-factor changes of basis belong to

$$GL_n(A) \begin{pmatrix} t^{\pm 1} & 0 \\ 0 & I_{n-1} \end{pmatrix} GL_n(A).$$

This completes the proof. ∎

A description of A-based transvection matrices is possible in the case where A is an n-fir. Here, for a row z and column x of length n over A, $zx = 0$ implies the existence of a $U \in GL_n(A)$ and a partitioning such that $zU = (0\ \ *)$, $U^{-1}x = \binom{*}{0}$. Then $U^{-1}(I_n - xyz)\,U = I_n - \binom{*}{0}\,y(0\ \ *) = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix}$, which is a product of elementary matrices (that is, matrices that differ from the identity matrix in one off-diagonal entry).

COROLLARY 30. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is a skew field and A is an n-fir. Then $GL_n(R)$ is generated by $GL_n(A)$,*

$$\begin{pmatrix} t & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

and the elementary matrices. ∎

Alexander Lichtman has pointed out to me the interesting fact that in the case $n = 1$ we get an HNN group extension; let us write $Un(R)$ for the group of units of R.

COROLLARY 31. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is a skew field and A has no zerodivisors. Then $Un(R)$ is the HNN group extension determined by the two maps from $Un(K)$ to $Un(A)$.*

Proof. For each of the subgroups $Un(\alpha K)$, $Un(\beta K)$ of $Un(A)$ choose a set of left coset representatives $\{1\} \cup X$, $\{1\} \cup Y$, respectively. Then the left action of $Un(K)$ on $Un(A)$, $Un(tAt^{-1})$, $Un(tA)$, $Un(At^{-1})$ has as a transversal $\{1\} \cup X$, $\{1\} \cup tYt^{-1}$, $\{t\} \cup tY$, $\{t^{-1}\} \cup Xt^{-1}$, respectively. We consider the following signed sets:

$$\begin{matrix} - & X & + & & - & \{t^{-1}\} \cup Xt^{-1} & - \\ + & \{t\} \cup tY & + & & + & tYt^{-1} & - \end{matrix}$$

Notice that a sign-linked expression $c_n c_{n-1} \cdots c_0$ cannot vanish in the corresponding tensor product $C_n \otimes C_{n-1} \otimes \cdots \otimes C_0$ (cf. the last paragraph of Section 3). In particular, each nonempty sign-linked expression is different from 1 in R. It follows that that HNN group extension embeds in $Un(R)$, and by Corollary 30 is all of $Un(R)$. ∎

COROLLARY 32. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is a skew field, and suppose that X is a column of length n with entries from an induced left R-module. Then there is a $U \in GL_n(R)$ such that $R \otimes_A A^n(UX) \to R^n(X)$ is an isomorphism.*

Proof. Essentially this was obtained in the first part of the proof of Theorem 28(iii): there is an A-submodule M_0 of R^nX such that we may identify R^nX with $R \otimes_A M_0$. The surjection $R^n \to R^nX$ can be factored as $R^n \simeq R \otimes_A A^n \to R \otimes_A M_0 = R^nX$, where the first factor, call it U^{-1}, can be viewed as an element of $GL_n(R)$, and $M_0 = A^nUX$. ∎

COROLLARY 33. *Let $R = A_K\langle t, t^{-1}; \beta\rangle$, where K is a skew field and suppose that X, Z are matrices over R such that $XZ = 0$. Then there exist an* *invertible square matrix U over R, and (not necessarily square) matrices B, C over A such that*

$$X = X' B U^{-1}, \qquad U C Z' = Z, \qquad BC = 0.$$

In particular, if $xz = 0$ *in R there exist a unit u of R and elements* b_i, c_j *of A such that*

$$x = (x_1' b_1 + \cdots + x_p' b_p) u^{-1}, \quad u(c_1 z_1' + \cdots + c_q z_q') = z, \quad b_i c_j = 0.$$

Proof. Say X is $r \times n$, and Z is $n \times c$. We view X as a row vector $(x_1, \ldots, x_n)$ with entries from ${}^rR$, and Z as a column vector

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

with entries from R^c. Corollary 32 implies that for some $U \in GL_n(R)$, $R \otimes_A A^n UZ \to R^n Z$ is an isomorphism. Let us replace Z with UZ and X with XU^{-1}. Then under the isomorphism ${}^rR \otimes_A A^n Z \to {}^rR^n Z$ the expression $\sum_{m=1}^n x_m \otimes z_m$ is mapped to $XZ = 0$ so is already 0 in ${}^rR \otimes_A A^n Z = {}^rR \otimes_A (A^n/\mathrm{Ker}(Z\colon A^n \to R^c))$ which means that X is in the image of ${}^rR \otimes_A \mathrm{Ker}(Z\colon A^n \to R^c)$ in ${}^rR \otimes_A A^n = {}^rR^n$; that is, there is a matrix B over A such that $X = X' B$, $BZ = 0$. Say B is $p \times n$. By flatness of ${}_AR^c$, $\mathrm{Ker}(B\colon {}^nR^c \to {}^pR^c) = \mathrm{Ker}(B\colon {}^nA \to {}^pA) \otimes_A R^c$ so Z is in the image of the latter in ${}^nR^c$; that is, there is a matrix C over A such that $BC = 0$, $CZ' = Z$. ∎

The above proof incorporates the following correction to [2] supplied by Bergman: In the proof of [2, Corollary 2.16(ii)] the set V appearing at [2, p. 11, lines 6–8] should be replaced with a *finite* subset of itself such that VR still contains y.

10. The Lewin–Lewin Embedding Theorem

Fix a skew field K and a torsion-free one-relator group G.

Using combinatorial group theory and combinatorial ring theory Lewin–Lewin [21] constructed a skew field having the group ring KG as a subring. The difficulty inherent in obtaining this result was further aggravated by the limited information then available on the HNN ring construction. Now that more is known we can clarify the ring-theoretic part of their proof by translating the coproduct-and-skew-Laurent-polynomial arguments into HNN arguments. Our account is self-contained apart from the group-theoretic result [21, Proposition 2] and some fundamental facts about semifirs which we summarise in the next three results.

THEOREM 34 (Cohn). *Let $A \to B$, $A \to C$ be injective ring homomorphisms.*

(i) *If ${}_AB$, ${}_AC$ are free on bases $X \cup \{1\}$, $Y \cup \{1\}$, respectively, then the coproduct $B \amalg_A C$ amalgamating A is free as left B-module on the family of all sequences of alternating strings $y_1 x_1 y_2 x_2 \ldots$ in X, Y not beginning with an element of X (and including the empty sequence).*

(ii) *If A is a skew field and B, C are semifirs then $B \amalg_A C$ is a semifir.*

Proof. (i) See [8] or [2].

(ii) See [10] or [2]. ∎

THEOREM 35. *Let $\alpha: A \to B$, $\beta: A \to B$ be injective ring homomorphisms.*

(i) *If ${}_\alpha B$, ${}_\beta B$ are free on bases $X \cup \{1\}$, $Y \cup \{1\}$, respectively, then the HNN extension $B_A\langle t, t^{-1}; \beta\rangle$ is free as left B-module on the family of all linked expressions constructed from*

$$\begin{array}{cc} - \quad X \quad + & - Xt^{-1} \cup \{t^{-1}\} - \\ + \; tY \cup \{t\} + & + \quad tYt^{-1} \quad - \end{array}$$

not beginning with an element of X or Xt^{-1} (and including the empty expression).

(ii) *If A is a skew field and B is a semifir then $B_A\langle t, t^{-1}; \beta\rangle$ is a semifir.*

Proof. (i) This follows from Theorem 19 and is not difficult to prove directly.

(ii) This is the "if" half of Theorem 28(iv). ∎

Notice that, by induction, Theorem 34(ii) and Theorem 35(ii) each imply that for a free group F the group ring KF is a semifir. (It is in fact a fir.)

Recall that for any ring R and set Σ of matrices over R there is a ring homomorphism $R \to R\langle \Sigma^{-1}\rangle$ that is universal with the property that each element of Σ is carried to an invertible matrix. We call $R\langle \Sigma^{-1}\rangle$ the *matrix localization* of R at Σ. An $n \times n$ matrix A over R is said to be *full* over R if it cannot be factored $A = BC$, where B is $n \times n-1$ and C is $n-1 \times n$.

THEOREM 36 (Cohn). *If R is a semifir and Φ the set of full matrices over R then $R\langle \Phi^{-1}\rangle$ is a skew field, denoted $U(R)$.*

Proof. See [11, p. 283] or [22]. ∎

We record two simple consequences, essentially due to Cohn.

COROLLARY 37. *If R is a semifir and Σ a set of matrices over R such that $R\langle\Sigma^{-1}\rangle$ is again a semifir then $U(R\langle\Sigma^{-1}\rangle) = U(R)$.*

Proof. Let Φ denote the set of all full matrices over R and write $\bar{\Phi}$ for the image of Φ over $R\langle\Sigma^{-1}\rangle$. Since $R\langle\Sigma^{-1}\rangle$ is a semifir, every invertible matrix over $R\langle\Sigma^{-1}\rangle$ is full so $\Sigma \subseteq \Phi$ and by universal properties $R\langle\Sigma^{-1}\rangle\langle\bar{\Phi}^{-1}\rangle = R\langle\Phi^{-1}\rangle = U(R)$. Thus every element of $\bar{\Phi}$ is full over $R\langle\Sigma^{-1}\rangle$ and $U(R\langle\Sigma^{-1}\rangle)$ is a matrix localization of $U(R)$. But these are both skew fields so $U(R) = U(R\langle\Sigma^{-1}\rangle)$. ∎

COROLLARY 38. *If A, B are K-rings that are semifirs then $U[U(A) \amalg_K B] = U(A \amalg_K B)$.*

Proof. $U(A) \amalg_K B$ is a matrix localization of $A \amalg_K B$ and the result follows by Corollary 37. ∎

The basis of the Lewin–Lewin proof is a delicate induction based on lifting up information through HNN extensions. To simplify the exposition we introduce the following somewhat technical definition.

Let $A \to B$ be a ring homomorphism, where A is a semifir and ${}_AB$ is free on a basis containing 1 (so $A \to B$ is injective). A ring homomorphism $B \to C$ will be said to *lock* $A \to B$ if the composite $A \to B \to C$ factors through the natural map $A \to U(A)$, and the multiplication map $U(A) \otimes_A B \to C$, is injective. The latter condition is equivalent to the left A-basis of B being left $U(A)$-independent in C, and in particular $B \to C$ is injective. Where B is viewed as a subring of C, we say C *locks* $A \to B$; if, further, A is viewed as a subring of B then C is said to *lock* A *in* B.

We begin with the transitivity property.

LEMMA 39 (Lewin–Lewin). *Suppose $A \subseteq B \subseteq C \subseteq D$ are rings with A, B semifirs and ${}_AB$, ${}_BC$ free on bases containing 1. If D locks A in B and B in C then it locks A in C.*

Proof. $U(A) \otimes_A C = U(A) \otimes_A B \otimes_B C \subseteq U(B) \otimes_B C \subseteq D$, where the first inclusion holds since $U(B)$ locks A in B, and ${}_BC$ is flat. ∎

LEMMA 40 (Lewin–Lewin). *Suppose $A \subseteq B \subseteq C$, and D are all K-rings with A, C, D semifirs and ${}_AB$ free on a basis containing 1. If C locks A in B then $U(C \amalg_K D)$ locks $A \amalg_K D$ in $B \amalg_K D$.*

Proof. By Corollary 38, $U(C \amalg_K D) = U([U(A) \amalg_K D] \amalg_{U(A)} C) = U(U[U(A) \amalg_K D] \amalg_{U(A)} C) = U(U[A \amalg_K D] \amalg_{U(A)} C)$, which contains $U[A \amalg_K D]$. Let $Y \cup \{1\}$ be a left K-basis of $U(A)$ containing a left K-basis of A, and let $Z \cup \{1\}$ be a left K-basis of D. By Theorem 34(i) the sequences of alternating strings in Y, Z not beginning with an element of Y form a left $U(A)$-basis of $U(A) \amalg_K D$ containing a left A-basis of $A \amalg_K D$. Thus there exists a left $U(A)$-basis $W \cup \{1\}$ of $U[A \amalg_K D]$ containing a left A-basis of $A \amalg D$. Also since C locks A in B there exists a left $U(A)$-basis $X \cup \{1\}$ of C containing a left A-basis of B. By Theorem 34(i) again, the sequences of alternating strings in W, X not beginning with an element of W form a left $U[A \amalg_K D]$-basis of $U[A \amalg_K D] \amalg_{U(A)} C \subseteq U(C \amalg_K D)$ containing a left $A \amalg_K D$-basis of $[A \amalg_K D] \amalg_A B = B \amalg_K D$. Thus $U(C \amalg_K D)$ locks $A \amalg_K D$ in $B \amalg_K D$. ∎

COROLLARY 41 (Lewin–Lewin). *Suppose B, D are K-rings which are semifirs. Then $U(B \amalg_K D)$ locks D in $B \amalg_K D$.*

Proof. This is the case $A = K$, $C = B$ of Lemma 40. ∎

LEMMA 42. *Let $A \rightrightarrows^{\alpha}_{\beta} B \subseteq C$ with A a semifir and ${}_{\alpha}B$, ${}_{\beta}B$ free on bases containing 1. Suppose C locks α and β. Then there are induced maps $\alpha: U(A) \to C$, $\beta: U(A) \to C$ and a natural identification $C_{U(A)}\langle t, t^{-1}; \beta\rangle = C_A\langle t, t^{-1}; \beta\rangle$.*

Suppose further $D \subseteq B$ with D a semifir and ${}_{D}B$ free on a basis containing 1. If C locks D in B then $C_A\langle t, t^{-1}; \beta\rangle$ locks D in $B_A\langle t, t^{-1}; \beta\rangle$.

Proof. In $R = C_A\langle t, t^{-1}; \beta\rangle$ the skew subfields of C generated by αA and βA are conjugate under t, and are isomorphic to $U(A)$ so we have a natural map $C_{U(A)}\langle t, t^{-1}; \beta\rangle \to R$ with an obvious inverse, and we treat this as an identification.

Let $X \cup \{1\}$, $Y \cup \{1\}$ be left $U(A)$-bases of ${}_{\alpha}C$, ${}_{\beta}C$ containing left A-bases of ${}_{\alpha}B$, ${}_{\beta}B$, respectively. By Theorem 35(i), a left C-basis of R is given by the set of all linked expressions constructed from

$$\begin{array}{ccc} - \quad X \quad + & & - \; Xt^{-1} \cup \{t^{-1}\} \; - \\ + \; tY \cup \{t\} \; + & & + \quad tYt^{-1} \quad - \end{array}$$

not beginning with an element of X or Xt^{-1}. This contains the correspondingly constructed left B-basis of $B_A\langle t, t^{-1}; \beta\rangle$ so $U(D) \otimes_D B_A\langle t, t^{-1}; \beta\rangle = U(D) \otimes_D B \otimes_B B_A\langle t, t^{-1}; \beta\rangle \subseteq C \otimes_B B_A\langle t, t^{-1}; \beta\rangle \subseteq R$, where the first inclusion follows from the fact that C locks D in B and $B_A\langle t, t^{-1}; \beta\rangle$ is free, and hence flat, as left B-module. Hence R locks D in $B_A\langle t, t^{-1}; \beta\rangle$. ∎

THE LEWIN–LEWIN EMBEDDING THEOREM. *If K is a skew field and G a torsion-free one-relator group then the group ring KG can be embedded in a skew field. More precisely, if X is a set, w a cyclically reduced word in the free group on X which is not a proper power, and $G = \langle X \mid w\rangle$ the group presented on X with single defining relator w, then there exists a skew field that locks KG_x in KG for every $x \in \operatorname{supp}(w)$. Here G_x denotes the subgroup* *of* G *generated by the image of* $X - \{x\}$, *and* $\mathrm{supp}(w)$ *the subset of* X *involved in* w.

Proof. Define the *complexity* of w (with respect to X) to be $c(w) =$ (length of w) $- |\mathrm{supp}(w)|$. Where the presentation of G is clearly indicated, we shall refer to the *complexity* of G, denoted $c(G)$.

We argue by induction on $c(G)$. If $c(G) = 0$ then G is obviously a free group such that $G_x = G$ for all $x \in \mathrm{supp}(w)$, so KG is a semifir and the skew field $U(KG)$ satisfies the conclusion of the theorem.

Now assume $c(G) > 0$ and that the conclusion of the theorem holds for all one-relator groups of smaller complexity. By Lemma 39 and Corollary 41 there is no harm in adjoining a new indeterminate to X so we may assume without loss of generality that $X \neq \mathrm{supp}(w)$. We wish to express G as an HNN extension of a one-relator group of smaller complexity. Suppose we have a $t \in X$ and a map $X \to \mathbb{Z}$, $x \mapsto n_x$, with $n_t = 1$, such that the resulting homomorphism to $\mathbb{Z}$ from the free group on X sends w to 0. The kernel of this homomorphism is freely generated by the elements $x_n = t^{-n}xt^{n-n_x}$ ($x \in X$, $x \neq t$, $n \in \mathbb{Z}$) so w can be expressed (uniquely) as a (cyclically reduced) word w' in the x_n (and w' is not a proper power). For a suitable choice of $t \in X$ and map $X \to \mathbb{Z}$ one can arrange for $c(w')$ to be smaller than $c(w)$. There are essentially two cases. If w has exponent sum zero on some $t \in \mathrm{supp}(w)$ we take

$$\begin{aligned} n_x &= 1 \quad \text{if} \quad x = t \\ &= 0 \quad \text{if} \quad x \neq t. \end{aligned}$$

Here $\mathrm{length}(w') \leqslant \mathrm{length}(w) - 2$, $|\mathrm{supp}(w')| \geqslant |\mathrm{supp}(w)| - 1$ so $c(w') < c(w)$ in this case. If no $x \in \mathrm{supp}(w)$ has exponent sum zero in w then by [21, Proposition 2] for any $t \in X - \mathrm{supp}(w)$ there exists a map $X \to \mathbb{Z}$ as above such that the resulting w' has $\mathrm{length}(w') = \mathrm{length}(w)$, $|\mathrm{supp}(w')| \geqslant |\mathrm{supp}(w)| + 1$, so $c(w') < c(w)$ in this case. Thus in any event we may assume $c(w') < c(w)$.

For each $x \neq t$ in $\mathrm{supp}(w)$ let $m(x)$, $M(x)$ denote, respectively, the least and greatest n such that $x_n \in \mathrm{supp}(w')$. Let

$$Y = \{x_n \mid x \in \mathrm{supp}(w),\ x \neq t,\ m(x) \leqslant n \leqslant M(x)\},$$
$$Z = \{x_n \mid x \notin \mathrm{supp}(w),\ x \neq t,\ n \in \mathbb{Z}\}.$$

Let $H = \langle Y \cup Z \mid w' \rangle$ be the group presented on $Y \cup Z$ with single defining relator w'. Then $c(H) < c(G)$ so by the induction hypothesis there exists a skew field $V(KH)$ that locks KH_{x_n} in KH for each $x_n \in \mathrm{supp}(w')$. Let

$$Y_m = \{x_n \in Y \mid m(x) \leqslant n < M(x)\},$$
$$Y_M = \{x_n \in Y \mid m(x) < n \leqslant M(x)\}$$

and $H_m = \langle Y_m \cup Z | \rangle$, $H_M = \langle Y_M \cup Z | \rangle$ the free groups on $Y_m \cup Z$, $Y_M \cup Z$, respectively. There is an isomorphism $\beta: H_M \to H_m$ that shifts the subscripts down by one. Since $c(w) > 0$ we can choose an $x \neq t$ in $\operatorname{supp}(w)$ and then $x_{m(x)} \in \operatorname{supp}(w')$ and H_M is a free factor of $H_{x_{m(x)}}$. Since $V(KH)$ locks $KH_{x_{m(x)}}$ in KH it locks KH_M in KH by Corollary 41 and Lemma 39. Similarly $V(KH)$ locks KH_m in KH. By Lemma 42 we have a natural identification of HNN extensions

$$R = V(KH)_{KH_M}\langle t, t^{-1}; \beta\rangle = V(KH)_{U(KH_M)}\langle t, t^{-1}; \beta\rangle$$

and R is a semifir by Theorem 35(ii). Let the skew field $U(R)$ be denoted $V(KG)$. By generators and relations there is a natural identification $KG = KH_{KH_M}\langle t, t^{-1}; \beta\rangle$ so there is a homomorphism $KG \to R \to V(KG)$. To complete the induction step it remains to show that $V(KG)$ locks KG_x in KG for all $x \in \operatorname{supp}(w)$. For this we shall need another description of $V(KG)$.

Let us fix an $x \in \operatorname{supp}(w)$ with $x \neq t$. Let F be the free group on

$$Y_x = \{ y_n | y \in X - \{x, t\}, n \in \mathbb{Z} \}.$$

Then $H_M \cap F$ is a free factor of F, say, $F = F_1 \amalg (H_M \cap F)$. Define $H^+ = H \amalg F_1 = \langle Y \cup Z \cup Y_x | w' \rangle$, $H_M^+ = H_M \amalg F_1 = \langle Y_M \cup Z \cup Y_x | \rangle$. By Theorem 34(ii) (or alternatively Theorem 35(ii) and induction) $V(KH) \amalg_K KF_1$ is a semifir and we can define $V(KH^+) = U(V(KH) \amalg_K KF_1)$. For any $y_n \in \operatorname{supp}(w')$, $V(KH^+)$ locks $KH_{y_n}^+ = KH_{y_n} \amalg_K KF_1$ in $KH^+ = KH \amalg_K KF_1$ by Lemma 40. In particular, $V(KH^+)$ locks $KH_{x_{m(x)}}^+$ in KH^+, and H_M^+ is a free factor of $H_{x_{m(x)}}^+$ so by Corollary 41 and transitivity $V(KH^+)$ locks KH_M^+ in KH^+. As before we construct an HNN extension which is a semifir,

$$S = V(KH^+)_{KH_M^+}\langle t, t^{-1}; \beta\rangle = V(KH^+)_{U(KH_M^+)}\langle t, t^{-1}; \beta\rangle.$$

But S is then a matrix localization of $(V(KH) \amalg_K KF_1)_{(KH_M \amalg_K KF_1)}\langle t, t^{-1}; \beta\rangle$ which by generators and relations can be identified with $V(KH)_{KH_M}\langle t, t^{-1}; \beta\rangle$, that is, R. So by Corollary 37, $U(S) = U(R) = V(KG)$. For any $y_n \in \operatorname{supp}(w')$, S locks $KH_{y_n}^+$ in $KH^+_{KH_M^+}\langle t, t^{-1}; \beta\rangle = KG$ by Lemma 42, so $V(KG) = U(S) \supseteq S$ locks $KH_{y_n}^+$ in KG. In particular, $V(KG)$ locks $KH_{x_{m(x)}}^+$ in KG and F is a free factor of $H_{x_{m(x)}}^+$ so $V(KG)$ locks KF in KG by Corollary 41 and transitivity. Now notice that the semifir KG_x can be expressed as an Ore extension $KF[t, t^{-1}; \beta]$. Thus the principal ideal domain $U(KF)[t, t^{-1}; \beta]$ is a matrix localization of KG_x; its skew field of fractions, $U(KF)(t, t^{-1}; \beta)$, must be $U(KG_x)$ by Corollary 37. In the diagram

$$\begin{array}{ccc} U(KF) \otimes_{KF} KG & \searrow & \\ \downarrow & & V(KG) \\ U(KF)[t, t^{-1}; \beta] \otimes_{KG_x} KG & \nearrow & \end{array}$$

the upper arrow is injective since $V(KG)$ locks KF in KG, and the vertical arrow is easily seen to be surjective, so the lower arrow is injective. Thus a left KG_x-basis of KG remains left independent over $U(KF)[t, t^{-1}; \beta]$ in $V(KG)$, so is automatically left independent over its Ore localization $U(KG_x)$. This proves that $V(KG)$ locks KG_x in KG for any $x \neq t$ in supp(w). It remains to show that if $t \in$ supp(w) then $V(KG)$ locks KG_t in KG. Thus suppose $t \in$ supp(w). Since w is cyclically reduced there exists some $x_n \in$ supp(w') with $n \neq M(x)$. Here $H^+_{x_n}$ contains $t^{-M(x)}G_t t^{M(x)}$ as a free factor. But $V(KG)$ locks $KH^+_{x_n}$ in KG so by Corollary 41 and transitivity, $V(KG)$ locks $t^{-M(x)}KG_t t^{M(x)}$ in KG, so clearly locks KG_t in KG. This completes the proof by induction. ∎

Remark. The Embedding Theorem also holds for twisted group rings. The only adaptation needed in the proof is that when the maps β are being defined their action on K must be specified, and this is determined by conjugation by t in KG. At the beginning of the proof when the new indeterminate is added it should be specified that it commute with K.

11. K-Theory and the Mayer–Vietoris Exact Sequence

In this section we look at one of the major results concerning HNN extensions, namely, Waldhausen's exact sequence. Here we operate with the fixed hypotheses that there are given injective ring homomorphisms α, $\beta: K \to A$ and K-bimodule splittings ${}_\alpha A_\alpha = \alpha(K) \oplus X$, ${}_\beta A_\beta = \beta(K) \oplus Y$. As usual $R = A_K\langle t, t^{-1}; \beta\rangle$.

Theorem (Waldhausen [26, p. 221]). *If K is right regular, right coherent and X, Y are left K-free then there is an exact sequence of abelian groups*

$$\cdots \to K_n(K) \xrightarrow{K_n(\alpha) - K_n(\beta)} K_n(A) \xrightarrow{K_n(\eta)} K_n(R) \xrightarrow{\delta_n} K_{n-1}(K) \to \cdots. \tag{31}$$

Here the K_n, $n \geqslant 0$, are Quillen's functors [24], and the K_n, $n < 0$, are Bass's functors [1].

It is beyond the scope of a purely algebraic survey to give a proof of this, and we shall limit ourselves to outlining the least topological parts of the argument. We concentrate on verifying directly the following important consequence of the exact sequence.

Corollary (Waldhausen). *If k is a right regular right Noetherian ring and G is a torsion-free one-relator group then $K_0(k) \to K_0(kG)$ is an isomorphism.*

From Dunwoody [15] and Lewin [20], for example, we know that, even if k is a field, finitely generated projective kG-modules need not be induced from k; what the Corollary tells us is they are at least "stably" induced from k.

Let us set up the notation. Consider the $K \times K$-subbimodule of $M_2(R)$

$$M = \begin{pmatrix} tA & tYt^{-1} \\ X & At^{-1} \end{pmatrix}.$$

The tensor ring $T = K \times K\langle M\rangle$ maps in a natural way to $M_2(R)$; and since ${}^2R$ is an $(M_2(R), R)$-bimodule, it is a (T, R)-bimodule. Further, as $(K \times K, R)$-bimodule ${}^2R \simeq (K \times K) \otimes_K R$, so for any right $K \times K$-module N there is a natural isomorphism

$$(N \otimes_{K \times K} T) \otimes_T ({}^2R) \simeq N \otimes_K R.$$

We can now give Waldhausen's analysis of isomorphisms of induced modules.

THEOREM 43 (Waldhausen). *Let $M(K)$, $M(A)$ be right K, A-modules, respectively. For any R-linear isomorphism $\kappa\colon M(A) \otimes_A R \to M(K) \otimes_K R$ such that $\kappa(M(A)) \subseteq M(K) \otimes_K (A + tA)$ there exist right K-modules P, Q and a commuting diagram*

$$\begin{array}{ccc}
M(A) \otimes_A R & \xrightarrow[\sim]{\kappa} & M(K) \otimes_K R \\
\wr\downarrow{\scriptstyle \kappa_A \otimes_A R} & & \wr\downarrow{\scriptstyle \kappa_K \otimes_K R} \\
(P \otimes_K A \oplus Q \otimes_K tA) \otimes_A R & & (P \oplus Q) \otimes_K R \\
\wr & & \wr \\
(P \oplus Q) \otimes_K R & & \\
\wr & & \\
((P \oplus Q) \otimes_{K \times K} T)) \otimes_T ({}^2R) & \xrightarrow[\sim]{\kappa_T \otimes_T ({}^2R)} & ((P \oplus Q) \otimes_{K \times K} T)) \otimes_T ({}^2R)
\end{array}$$

where $\kappa_A, \kappa_T, \kappa_K$ are isomorphisms of the indicated modules, and all other isomorphisms are natural.

Proof. From Theorem 8 with $M_0 = A$, we see that T actually embeds in $M_2(R)$. Let us identify T with its image in $M_2(R)$ and write

$$T = \begin{pmatrix} K \oplus +R+ & +R- \\ -R+ & K \oplus -R- \end{pmatrix}.$$

Here $+R+$ denotes the K-subbimodule of R spanned by all (nonempty) linked expressions starting with $+$ and ending with $+$ constructed from

$$\begin{array}{ll} +\,tA\,+ & +\,tYt^{-1}\,- \\ -\,X\,+ & -\,At^{-1}\,-, \end{array}$$

and similarly for the other components. Notice that $K\oplus +R+$ and $K\oplus -R-$ are actually graded subrings of R. We shall write $+R = +R+ \oplus +R-$ and $-R = -R+ \oplus -R-$ viewed as K-subbimodules of R, and similarly for $R+$, $R-$.

Since $T = (K\times K)\oplus M\otimes_{K\times K} T$ we have a decomposition of $K\times K$ bimodules

$$\begin{pmatrix} K\oplus +R+ & +R- \\ -R+ & K\oplus -R- \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \oplus \begin{pmatrix} tA & tYt^{-1} \\ X & At^{-1} \end{pmatrix} \otimes_{K\times K} \begin{pmatrix} K\oplus +R+ & +R- \\ -R+ & K\oplus -R- \end{pmatrix}. \tag{32}$$

Thus we have K-bimodule decompositions

$$\begin{aligned} R &= K\oplus +R \oplus -R \\ &= K\oplus +R\oplus (X(K\oplus +R)\oplus At^{-1}(K\oplus -R)) \\ &= A(K\oplus +R)\oplus At^{-1}(K\oplus -R). \end{aligned}$$

The latter is an (A, K)-bimodule decomposition, and the multiplication can be viewed as $\otimes_K$.

Thus we have a decomposition of κ,

$$\begin{array}{ccc} M(A)\otimes_A R & \xrightarrow[\sim]{\kappa} & M(K)\otimes_K R \\ \| & & \| \\ & & M(K)(R+) \\ & \nearrow & \\ M(A)(K\oplus +R+)\oplus M(A)\,t^{-1}(-R+) & & \oplus \\ & \searrow & \\ \oplus & & M(K) \\ & \nearrow & \\ M(A)(+R-)\oplus M(A)\,t^{-1}(K\oplus -R-) & & \oplus \\ & \searrow & \\ & & M(K)(R-). \end{array}$$

This means that $M(K)$ has two K-submodules P, Q such that $M(K)=P\oplus Q$ and κ is the sum of two K-linear isomorphisms

$$\kappa_1 : M(A)(K\oplus +R+)\oplus M(A)\,t^{-1}(-R+) \xrightarrow[\sim]{} M(K)(R+)\oplus P,$$

$$\kappa_2 : M(A)(+R-)\oplus M(A)\,t^{-1}(K\oplus -R-) \xrightarrow[\sim]{} M(K)(R-)\oplus Q.$$

From the left-right analogue of (32) we have decompositions of R as (A,A) and (K,A)-bimodule, respectively:

$$\begin{aligned} R &= A\oplus A(K\oplus +R+)\,tA\oplus A(+R-)\,A\oplus At^{-1}(-R+)\,tA \\ &\quad \oplus At^{-1}(K\oplus -R-)\,A, \\ R &= A\oplus (R-)\,A\oplus tA\oplus (R+)\,tA. \end{aligned}$$

Thus we can express κ in the form

$$\begin{array}{ccc} M(A)\otimes_A R & \xrightarrow[\sim]{\kappa} & M(K)\otimes_K R \\ \| & & \| \\ [(M(A)(K\oplus +R+)\oplus M(A)\,t^{-1}(-R+)]\,tA & \xrightarrow{\kappa_1 tA} & (M(K)(R+)\oplus P)\,tA \\ \oplus & \nearrow & \oplus \\ M(A) & \xrightarrow{\kappa_A} & QtA\oplus PA \\ \oplus & \searrow & \oplus \\ [(M(A)(+R-)\oplus M(A)\,t^{-1}(K\oplus -R-)]\,A & \xrightarrow{\kappa_2 A} & (M(K)(R-)\oplus Q)\,A. \end{array}$$

Since $\kappa_1 tA$, $\kappa_2 A$ are isomorphisms, κ_A must also be an isomorphism of A-modules and we have a commuting diagram

$$\begin{array}{ccc} M(A)\otimes_A R & \xrightarrow[\sim]{\kappa} & M(K)\otimes_K R \\ \wr\downarrow \kappa_A\otimes_A R & & \\ (P\otimes_K A\oplus Q\otimes_K tA)\otimes_A R & & \|\, \kappa_K\otimes_K R \\ \wr\downarrow & & \\ (P\oplus Q)\otimes_K R & \xrightarrow[\sim]{\kappa'} & (P\oplus Q)\otimes_K R. \end{array}$$

It remains to analyse κ'. The restriction of κ' to $PA \oplus QtA$ is given by $\kappa \circ \kappa_A^{-1}$, and so can be expressed as the sum of the identity map on $PA \oplus QtA$ and some A-linear map $PA \oplus QtA \to PtA \oplus QA$, since we have the hypothesis $\kappa(M(A)) \subseteq M(K)(A \oplus tA) = PtA \oplus QtA \oplus PA \oplus QA$.

Thus the restriction of κ' to P is the sum of the identity map on P and some K-linear map $\kappa'_P: P \to PtA \oplus QA$. Further $\kappa'(P) = \kappa \circ \kappa_A^{-1}(P) \subseteq \kappa(M(A)) = \kappa_1(M(A))$ which meets $PtA \oplus QA = PtA \oplus Q \oplus QX$ in $PtA \oplus QX$ so we can write $\kappa'_P: P \to PtA \oplus QX$.

Similarly the restriction of κ' to Q is the sum of the identity map on Q and some K-linear map $\kappa'_Q: Q \to (PtA \oplus QA)t^{-1}$. But $\kappa'(Q) = \kappa(\kappa_A^{-1}(Qt)t^{-1}) \subseteq \kappa(M(A)t^{-1}) = \kappa_2(M(A)t^{-1})$ which meets $PtAt^{-1} \oplus QAt^{-1}$ in $PtYt^{-1} \oplus QAt^{-1}$ so we can write $\kappa'_Q: Q \to PtYt^{-1} \oplus QAt^{-1}$.

These maps determine a $K \times K$-linear map

$$(\kappa'_P, \kappa'_Q): P \oplus Q \to (P \oplus Q) \otimes_{K \times K} \begin{pmatrix} tA & tYt^{-1} \\ X & At^{-1} \end{pmatrix} \subseteq (P \oplus Q) \otimes_{K \times K} T,$$

which in turn determines a T-linear endomorphism of $(P \oplus Q) \otimes_{K \times K} T$. The sum of this endomorphism with the identity endomorphism we denote κ_T. It is easy to see that $\kappa_T \otimes_T ({}^2R) = \kappa'$ so to complete the proof we need only show that κ_T is an automorphism.

There is a decomposition of graded $(K \times K, K)$-bimodules

$${}^2R = \begin{pmatrix} K \oplus +R \\ K \oplus -R \end{pmatrix} \oplus \begin{pmatrix} -R \\ +R \end{pmatrix}.$$

Notice that the first summand is actually the left T-submodule of ${}^2R$ freely generated by $\binom{1}{1}$. Thus we have a split exact sequence of graded right K-modules

$$0 \to (P \oplus Q) \otimes_{K \times K} T \to (P \oplus Q) \otimes_{K \times K} ({}^2R) \to (P \oplus Q) \otimes_{K \times K} \begin{pmatrix} -R \\ +R \end{pmatrix} \to 0.$$

The automorphism κ' of the middle term induces the endomorphism κ_T on the first term. It is not difficult to check that the endomorphism induced on the third term differs from the identity map by a degree reducing map, so it is an automorphism. Thus, as κ' is an automorphism, we see that κ_T is also an automorphism, which completes the proof. ∎

COROLLARY 44. *Let P, Q be right K-modules. For any R-linear automorphism κ of $(P \oplus Q) \otimes_K R$ such that $\kappa(P \oplus Qt) \subseteq (P \oplus Q) \otimes (A \oplus tA)$ there exist right K-modules P', Q' and a commuting diagram*

$$\begin{array}{ccc}
(P\oplus Q)\otimes_K R & \xrightarrow[\sim]{\kappa} & (P\oplus Q)\otimes_K R \\
\wr\downarrow & & \wr\downarrow \kappa_K\otimes_K R \\
(P\otimes_K A\oplus Q\otimes_K tA)\otimes_A R & & (P'\oplus Q')\otimes_K R \\
\wr\downarrow \kappa_A\otimes_A R & & \\
(P'\otimes_K A\oplus Q'\otimes_K tA)\otimes_A R & & \wr\downarrow \\
\wr\downarrow & & \\
((P'\oplus Q')\otimes_{K\times K}T)\otimes_T R & \xrightarrow[\sim]{\kappa_T\otimes_T R} & ((P'\oplus Q')\otimes_{K\times K}T)\otimes_T R
\end{array}$$

where κ_A, κ_T, κ_K are isomorphisms of the indicated modules, and the other isomorphisms are natural.

Proof. Take $M(K) = P\oplus Q$, $M(A) = P\otimes_K A\oplus Q\otimes_K tA$ in Theorem 43. ∎

Notice that in the case where P, Q are finitely generated free K-modules, κ determines an element of $K_1(R)$ while Q, Q' determine elements of $K_0(K)$ whose difference lies in the kernel of $K_0(\alpha)-K_0(\beta)$. Since every element of $K_1(R)$ is represented by such a κ (by linearization by enlargement) we can see what Waldhausen's map $\delta_1: K_1(R)\to K_0(K)$ must look like, but proving the map exists is quite another matter. We shall look at a very special case.

Let k be the kernel of the abelian group map $\alpha-\beta: K\to A$, so k is a subring of K. Since α, β agree on k there is a ring homomorphism $k[t,t^{-1}]\to R$. Hence we have maps $K_1(A)\to K_1(R)$, $K_1(k[t,t^{-1}])\to K_1(R)$, $K_1(T)\to K_1(M_2(R))=K_1(R)$ and using Corollary 44 one can prove the following.

THEOREM 45. *If $K_0(k)\to K_0(K)$ is onto then $K_1(A)\oplus K_1(k[t,t^{-1}])\oplus K_1(T)\to K_1(R)$ is onto.* ∎

It turns out that by imposing conditions one can eliminate the $K_1(T)$ term. The basic abstract result is the following.

THEOREM 46. *Let S be a right regular right coherent ring and M an S-bimodule that is left S-flat. Then the natural map $K_1(S)\to K_1(S\langle M\rangle)$ is an isomorphism.*

Proof (sketch). We consider the category whose objects are pairs (P,f), where P is a finitely presented right S-module and f is an S-linear map $f: P\to P\otimes_S M$ such that the induced $S\langle M\rangle$-linear endomorphism of $P\otimes_S S\langle M\rangle$ is nilpotent; the morphisms in this category are to be the obvious ones. Since M is left flat, and S is right coherent, it follows that this category is abelian.

Let (P,f) be an object in this category, so the composite of the sequence

$$P \xrightarrow{f} P \otimes M \xrightarrow{f \otimes M} P \otimes M^{\otimes 2} \to \cdots \xrightarrow{f \otimes M^{\otimes n-1}} P \otimes M^{\otimes n} \qquad (33)$$

is zero for some n, say, $n = d$. Let P_n denote the kernel of the composite (33) so we have a sequence

$$0 = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_d = P \qquad \text{with} \quad f(P_n) \subseteq P_{n-1} \otimes M \quad \text{for} \quad n \geqslant 1 \qquad (34)$$

since the kernel of the composite of the last $n-1$ maps in (33) is $P_{n-1} \otimes M$ because M is left flat. As P is finitely generated we can alter the sequence (34) in such a way that each of the P_n is finitely generated. But S is right coherent and P is finitely presented so each of the P_n is finitely presented and we have a chain in the category

$$0 = (P_0, f) \subseteq (P_1, f) \subseteq \cdots \subseteq (P_d, f) = (P,f)$$

such that each of the quotients is of the form $(P_n/P_{n-1}, 0)$. Since S is right regular right coherent each $(P_n/P_{n-1}, 0)$ has a resolution $0 \to (P_{n,m}, 0) \to \cdots \to (P_{n,0}, 0) \to (P_n/P_{n-1}, 0) \to 0$, where each of the P_{nj} is finitely generated projective.

Let us call an object (P,f) of the category *elementary* if P is projective and there is a chain (34) with each P_n/P_{n-1} projective.

By standard arguments the foregoing shows that every object (P,f) of the category has a resolution by elementary objects. In particular, P can be a finitely generated free module.

By linearization by enlargement arguments, any element of $K_1(S\langle M\rangle)$ can be represented by an invertible matrix with entries in $S \oplus M$, and the foregoing shows further that the element can then be represented by an invertible matrix with entries in S; that is, $K_1(S) \to K_1(S\langle M\rangle)$ is onto, so it is an isomorphism since there is a retraction from $S\langle M\rangle$ onto S. ∎

COROLLARY 47. *If S is a right regular right coherent ring then $K_1(S) \to K_1(S[x])$ is an isomorphism.* ∎

COROLLARY 48. *If K is right regular right coherent and X, Y are left K-flat and $K_0(k) \to K_0(K)$ is onto then $K_1(A) \oplus K_1(k[t, t^{-1}]) \to K_1(R)$ is onto.*

Proof. By Theorem 46, $K_1(K \times K) \to K_1(T)$ is onto, so the result follows by Theorem 45. ∎

We now recall the fact that allows one to convert results about K_1 to results about K_0.

THEOREM 49 (Bass–Heller–Swan). *For any ring S there are maps $K_0(S) \to K_0(S[x, x^{-1}]) \to K_0(S)$ which compose to the identity, and are natural in S.*

Proof. See [25, Sect. 16] or [1, Sect. XII.7] (cf. [26, Corollary 18.2]). ∎

COROLLARY 50 (Grothendieck). *For any ring S, if $S[x]$ is right regular right coherent then so is $S[x, x^{-1}]$ and the maps $K_0(S) \to K_0(S[x]) \to K_0(S[x, x^{-1}])$ are isomorphisms.*

Proof. Since each finitely presented $S[x, x^{-1}]$-module is induced from some finitely presented $S[x]$-module, and any $S[x]$-resolution of the latter lifts to an $S[x, x^{-1}]$ resolution of the former, we see $S[x, x^{-1}]$ is right regular right coherent. By starting with a finitely generated projective $S[x, x^{-1}]$-module we see further that $K_0(S[x]) \to K_0(S[x, x^{-1}])$ is onto.

Finally, by Corollary 47 with $S[x, x^{-1}]$ in place of S we see from Theorem 49 that $K_0(S) \to K_0(S[x])$ is an isomorphism, and the result is now clear. ∎

COROLLARY 51 (Waldhausen). *If $K[x]$ is right regular right coherent and X, Y are left K-flat and $K_0(k) \to K_0(K)$ is onto then $K_0(A) \to K_0(R)$ is onto.*

Proof. By Corollary 50, $K[x, x^{-1}]$ is right regular right coherent and $K_0(k[x, x^{-1}]) \to K_0(K[x, x^{-1}])$ is onto and $X[x, x^{-1}]$, $Y[x, x^{-1}]$ are left $K[x, x^{-1}]$-flat so, by Corollary 48 and Theorem 49, $K_0(A) \oplus K_0(k[t, t^{-1}]) \to K_0(R)$ is onto. But $K_0(k[t, t^{-1}])$ has the same image as $K_0(K)$, so $K_0(A) \to K_0(R)$ is onto. ∎

Waldhausen [26] calls a group G *regular coherent* if for every right regular right Noetherian ring S, the group ring SG is right regular right coherent; for example, free groups are regular coherent by Corollary 14 and induction. In the same vein let us say G is *K_0-trivial* if for every right regular right Noetherian ring S the map $K_0(S) \to K_0(SG)$ is an isomorphism.

THEOREM 52 (Waldhausen). *Let $\alpha, \beta: L \to H$ be two group monomorphisms and G the resulting HNN group extension. If L is regular coherent K_0-trivial and H is K_0-trivial then G is K_0-trivial.*

Proof. Let S be any right regular right Noetherian ring. Then $SL[x] = S[x]L$ is right regular right coherent and $K_0(S) \to K_0(SL)$ is onto, so by Corollary 51, $K_0(SH) \to K_0(SG)$ is onto. In particular, if H is K_0-trivial then so is G. ∎

THEOREM 53 (Waldhausen). *Every torsion-free one-relator group is K_0-trivial.*

Proof. By induction it follows easily from Theorem 52 that free groups are K_0-trivial, since we have already observed they are regular coherent.

Now we proceed by induction on the complexity of the relator, as defined in the previous section. If the complexity is zero then the group is free and we have taken care of this case. Thus we may assume we have a torsion-free one relator group G of complexity greater than zero. By [21, Proposition 2] the coproduct $G \amalg \mathbb{Z}$ can be expressed as the HNN extension resulting from two group monomorphisms $\alpha, \beta: L \to H$, where L is a free group and H is a torsion-free one-relator group of smaller complexity than G. So by the induction hypothesis $G \amalg \mathbb{Z}$ is K_0-trivial, and hence so is the retract G. ∎

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