Math Proc. Camb. Phil. Soc. (1985), **97**, 7
Printed in Great Britain

Homogeneous elements of free algebras have free idealizers

BY WARREN DICKS
Bedford College, London

(*Received* 29 *March* 1984)

Dedicated to P. M. Cohn on his 60th birthday

Abstract

Let k be a field, X a set, $F = k\langle X\rangle$ the free associative k-algebra, and b an element of F that is homogeneous with respect to the grading of F induced by some map $X \to \mathbb{N}^+$. We show that the idealizer of b in F, $S = \{f \in F | fb \in bF\}$, is a free algebra.

Part of the interest of this result lies in its usefulness in the study of one-relator graded algebras; it will be applied in [**5**] to show that much of the homological information about F/FbF is already contained in the finite-dimensional commutative subalgebra S/bF.

The first half of the proof of the freeness of S, Section 1, is concerned with 'trivializable' equations $\Sigma\, a_i b_i c_i = 0$, cf. Definition 1·2. In Proposition 1·14 and Theorem 1·18 we give two different methods for reducing a homogeneous equation $\Sigma\, a_i b_i c_i = 0$ to a family of other such equations; in Theorem 1·11 and Proposition 1·16 we show that two very simple types of equations, called 'chains of atoms' and 'staircases' respectively, are trivializable. What this implies is that an equation is trivializable provided that it can be transformed, by repeated applications of the two reducing methods, to a family of equations, each of which is a chain of atoms or a staircase. The second half of the proof, Section 2, is then a purely formal argument, divorced from the free algebra, showing that any homogeneous equation $\Sigma\, a_i b_i c_i = 0$ can be reduced in this way, and is therefore trivializable. The desired result then follows immediately, cf. Theorems 1·19 and 1·22.

This proof takes many of its ideas from the extremely original paper of Gerasimov [**6**]. The author is grateful to J. Backelin for a helpful outline of [**6**], and to V. N. Gerasimov for a kind letter correcting a misunderstanding of one of the arguments in [**6**].

1. *Trivializing relations*

1·1. *Notation.* Throughout this section we fix an integer $n \geqslant 2$, and a graded ring $F = \bigoplus_{\alpha \in \mathbb{N}} F_\alpha$ such that F_0 is a field in the centre of F. For convenience, let us call such an F a *graded algebra.*

Let $F^\times$ denote the set of non-zero elements of F. For $\alpha \in \mathbb{N}$, let $F_\alpha^\times$ be the set of non-zero elements of F_α; if $a \in F_\alpha^\times$ then we say a is *homogeneous* of *degree* α, and write $|a| = \alpha$. We understand that 0 is homogeneous, but that it does not have any degree. For $b \in F$ let $\bar{b}$ denote the homogeneous component of b of maximum degree; here $\bar{0} = 0$. The meanings of $\bar{F}$ and $\bar{F}^\times$ are clear.

For $\alpha, \beta \in \mathbb{Z}$, let $[\alpha, \beta] = \{i \in \mathbb{Z} | \alpha \leqslant i \leqslant \beta\}$.

1·2. *Definitions.* A relation

$$\sum_{i=1}^{m} a_i b_i c_i = 0, \quad a_i, b_i, c_i \in \bar{F}, \quad m \in [1, n] \tag{1}$$

is said to be *homogeneous* if $|a_i| + |b_i| + |c_i|$ is the same for all the i for which it is defined, that is, the $i \in [1, m]$ such that a_i, b_i, c_i are non-zero. A homogeneous relation (1) is said to be *trivializable* in F if there is a partition of $[1, m]$ into two sets H, K, and a set of homogeneous relations $a_k = \sum_H a_h d_{hk}$ $(k \in K)$ and $c_h = -\sum_K d'_{hk} c_k$ $(h \in H)$ such that $d_{hk} b_k = b_h d'_{hk}$ for all $(h, k) \in H \times K$. In particular, if H is empty then all the a_i are zero, and if K is empty then all the c_i are zero.

1·3. *Definitions.* A *format B in F* is a triple (α, b, γ) where $\alpha, \gamma \in \mathbb{N}$ and $b \in \bar{F}^{\times}$. The *degree* of B is $|B| = \alpha + |b| + \gamma$. In this situation we call B a *b-format.*

1·4. *Definitions.* A *tableau T* in F is a finite set of formats all of which have the same degree; this value is called the *degree* of T, denoted $|T|$. If for some $b \in \bar{F}^{\times}$ all the elements of T are b-formats then T is said to be a *b-tableau.*

A relation (1) is a *T-relation* if $(|a_i|, b_i, |c_i|) \in T$ for each $i \in [1, m]$ for which a_i, c_i are non-zero. If every T-relation (1) is trivializable in F then T *satisfies n-term weak algorithm*; if this holds for all n then T *satisfies weak algorithm.*

If every 1-tableau in F satisfies (n-term) weak algorithm then F itself is said to *satisfy (n-term) weak algorithm.* (Since F_0 is already a field, this agrees with the usage in [**1**].)

Our aim is to show that for any $b \in \bar{F}^{\times}$, every b-tableau in F satisfies n-term weak algorithm if F itself does. It will then follow easily that if F is a free algebra then so is the idealizer of b in F.

1·5. PROPOSITION. *If T is a tableau in the graded algebra F then the following are equivalent.*

(*a*) *T satisfies n-term weak algorithm.*

(*b*) *For every T-relation* (1) *and each $j \in [1, m]$ either*

there is a homogeneous relation $\sum_{i=1}^{m} a_i e_i = 0$ such that $e_j = 1$ and for each $i \in [1, m]$, $e_i b_j \in b_i F$ (2)

or *there is a homogeneous relation $\sum_{i=1}^{m} e_i c_i = 0$ such that $e_j = 1$ and for each $i \in [1, m]$, $b_j e_i \in F b_i$.* (3)

(*c*) *For every T-relation* (1) *with all a_i, c_i non-zero, there exists $j \in [1, m]$ such that* (2) *or* (3) *holds.*

(*d*) *For every T-relation* (1) *with some c_i non-zero, there exists $j \in [1, m]$ such that* (2) *holds.*

(*e*) *For every T-relation* (1) *with some a_i non-zero, there exists $j \in [1, m]$ such that* (3) *holds.*

Proof. $(a) \Rightarrow (b), (c), (d), (e)$ are clear.

$(d) \Rightarrow (a)$. Let (1) be a T-relation. We proceed by induction on m. For $m = 0$ there is nothing to prove, so we may assume $m \geqslant 1$ and the implication holds for all smaller values of m. If all the c_i are zero we take $H = [1, m]$, $K = \varnothing$ and the conditions of 1·2 are satisfied vacuously. Thus we may assume some c_i is non-zero; by (*d*), and renumbering if necessary, $a_1 = -a_2e_2 - \dots - a_me_m$ and $e_ib_1 = b_ie'_i$ for all $i \in [2,m]$. Thus $\sum_{i=2}^m a_ib_ic'_i = 0$, where $c'_i = c_i - e'_ic_1$ for all $i \in [2,m]$. By the induction hypothesis, $[2,m]$ can be partitioned into two sets H, K as in 1·2. It is then not difficult to check that $[1,m]$ can be partitioned into two sets H, $\{1\} \cup K$ and there exist relations as in 1·2. Thus (a) holds.

The proofs that (b), (c) and (e) each imply (a) are similar. ∎

For convenience we now recall four results from [**1**].

1·6. Lemma. *If F satisfies n-term weak algorithm, and $a_1, \dots, a_m \in \bar{F}^\times$ are right F-dependent with $m \leqslant n$ and $|a_1| \leqslant \dots \leqslant |a_m|$ then some a_j is a right F-linear combination of $a_1, \dots, a_{j-1}$.*

Proof. We have a relation $\sum_{i=1}^m a_ic_i = 0$ with some c_i non-zero. By taking the leading homogeneous part we may assume this is a homogeneous relation. It is clearly a relation in a 1-tableau, so by n-term weak algorithm in the form (d) of Proposition 1·5 there is a $j \in [1,m]$ such that (2) holds. Consider the largest k such that $e_k \neq 0$; here $k \geqslant j$ since $e_j = 1$. Thus $|e_k| = |a_j| - |a_k| \leqslant 0$, so $e_k \in F_0^\times$ and $a_k = \sum_{i=1}^{k-1} a_ie_i(-e_k^{-1})$. ∎

1·7. Corollary. *If F satisfies 2-term weak algorithm, and $a, b, c, d \in \bar{F}^\times$ are such that $ab = cd$ and $|a| \leqslant |c|$ then there is a unique $e \in F^\times_{|c|-|a|}$ such that $ae = c$, $b = ed$. It follows that every element of $\bar{F}^\times$ can be written as a product of homogeneous atoms in a unique way, up to associates (and here all units are central).* ∎

1·8. Corollary. *If F satisfies n-term weak algorithm, and $a_1, \dots, a_m \in \bar{F}$, $m \in [1,n]$, then some subfamily $(a_i | i \in I)$ is a basis of the right ideal $\mathfrak{a} = \sum_{i=1}^m a_iF$.* ∎

The cardinality of I is called the *rank* of $\mathfrak{a}$, denoted $\operatorname{rk}\mathfrak{a}$. Here the rank of a free F-module is well defined since there is a ring homomorphism to a field, $F \to F_0$.

One can show that each n-generator right ideal of F is free (but not necessarily on a subset of the generating set), if F satisfies n-term weak algorithm, cf. [**1**].

1·9. Theorem. *The graded algebra F satisfies weak algorithm if and only if F is a free F_0-algebra with a homogeneous free generating set.*

Proof. Suppose F satisfies weak algorithm, and let $H = \bar{F} \backslash F_0$. By the well-ordering principle, we can index H with an ordinal in such a way that the elements of H are listed $x_1, x_2, \dots, x_\alpha, \dots$ with $|x_1| \leqslant |x_2| \leqslant \dots \leqslant |x_\alpha| \leqslant \dots$. By Lemma 1·6, if we delete each x_α which is a right F-linear combination of earlier elements, then we get a right F-independent homogeneous set X which is easily seen to freely generate F as F_0-algebra.

Conversely, suppose $F = F_0\langle X\rangle$ is a free F_0-algebra on a homogeneous set X. Each $f \in F$ can be written uniquely in the form $f = \sum_{x \in X} f_x x + f_0$, $f_0 \in F_0$, $f_x \in F$ almost all zero. We wish to show that we can trivialize any relation $\Sigma a_ic_i = 0$ in a 1-tableau T in F. We may assume $|c_1| \leqslant |c_2| \leqslant \dots$. If $c_1 \in F_0^\times$ then it is easy to see that (3) holds. Thus we may assume $c_1 \notin F_0$. Here $(c_1)_x \neq 0$ for some $x \in X$, so

$$\Sigma a_i(c_i)_x = (\Sigma a_ic_i)_x = 0.$$

Since X is homogeneous, this is a relation in a 1-tableau T' of smaller degree than that of T. By induction on $|T|$, there is a $j \in [1,m]$ such that $a_j \in \sum_{i=j+1}^m a_iF$, so (2) holds, and F satisfies weak algorithm in the form (c) of Proposition 1·5. ∎

1·10. *Definitions.* A format $B = (\alpha, b, \gamma)'$ has *left margin* $\lambda(B) = \alpha$ and *right margin* $\rho(B) = \alpha + |b|$. If b is an atom in $\bar{F}^\times$ we say that B is an *atomic* format.

If $B_1 = (\alpha_1, b_1, \gamma_1)$, $B_2 = (\alpha_2, b_2, \gamma_2)$ are two formats of the same degree we say B_1 is *left* of B_2, denoted $B_1 < < B_2$, if $\lambda(B_1) < \lambda(B_2)$ and $\rho(B_1) < \rho(B_2)$. If $\lambda(B_1) \leqslant \lambda(B_2)$ and $\rho(B_1) \leqslant \rho(B_2)$ we write $B_1 \leqslant \leqslant B_2$. By $B_1 \leqslant < B_2$ we mean $B_1 < < B_2$ or $\lambda(B_1) = \lambda(B_2)$ and $\rho(B_1) \leqslant \rho(B_2)$; similarly, by $B_1 < \leqslant B_2$ we mean $B_1 < < B_2$ or $\lambda(B_1) \leqslant \lambda(B_2)$ and $\rho(B_1) = \rho(B_2)$.

A tableau is said to be a *chain* if its elements can be listed $B_1, \ldots, B_t$ so that either $B_1 \leqslant < B_2 < < \ldots < < B_{t-1} < < B_t$ or $B_1 < < B_2 < < \ldots < < B_{t-1} < \leqslant B_t$.

We can now prove one of the results fundamental to the development of the theory.

1·11. THEOREM. *If the graded algebra F satisfies n-term weak algorithm then any tableau T in F which is a chain of atomic formats also satisfies n-term weak algorithm.*

Proof. By symmetry we may assume the elements of T are $C_1, \ldots, C_t$ and

$$C_1 \leqslant < C_2 < < \ldots < < C_t.$$

We shall show by induction on $|T|$ that (c) of Proposition 1·5 is satisfied. Let (1) be a T-relation with all a_i, c_i non-zero. By renumbering if necessary, we may assume $|a_1 b_1| \leqslant \ldots \leqslant |a_m b_m|$, and hence $|a_1| \leqslant \ldots \leqslant |a_m|$, by the ordering of the elements of T.

If $|T| = 0$ the result is clear. Thus we may assume that $|T| > 0$ and that the result holds for all tableaux of smaller degree.

Consider first the case where $|a_1| > 0$. View (1) as a relation $\Sigma a_i(b_i c_i) = 0$ in a 1-tableau; by n-term weak algorithm in F, in the form of Proposition 1·5(e), there is a homogeneous relation $\Sigma a_i'(b_i c_i) = 0$ with $a_j' = 1$ for some j. Let

$$T' = \{(|a_i'|, b_i, |c_i'|) : i \in [1, m],\ a_i' \neq 0\}.$$

Here $|T'| = |b_j| + |c_j| < |a_j| + |b_j| + |c_j|$ since $0 < |a_1| \leqslant |a_j|$, so $|T'| < |T|$. But T' is still a chain of atomic formats, so satisfies n-term weak algorithm by the induction hypothesis. By Proposition 1·5(e), for some $k \in [1, m]$ there is a homogeneous relation $\Sigma e_i c_i = 0$ with $e_k = 1$ and $b_k e_i \in F b_i$ for all i. Thus (3) holds.

This leaves the case $|a_1| = 0$.

We next dispose of the case where $|a_3| = 0$. For each $i \in [1, m]$ let $B_i = (|a_i|, b_i, |c_i|)$, an element of T. If $|a_3| = 0$ then B_1, B_2, B_3 all have left margin 0, but at most two elements of T have the same left margin so two of B_1, B_2, B_3 are equal, say $B_1 = B_2$. Here $b_1 = b_2$ and $a_1, a_2 \in F_0^{\times}$. Taking $e = a_2^{-1} a_1$ we have $a_1 = a_2 e$, $e b_1 = b_1 e = b_2 e \in b_2 F$, and (2) holds. Similarly if $B_1 = B_3$ or $B_2 = B_3$ then (2) holds.

Thus we may assume $|a_1| < |a_3|$. Hence for any i, $j \in [1, m]$, if $j \geqslant 3$ and either $|a_i| = |a_j|$ or $|c_i| = |c_j|$ then $B_i = B_j$ and $b_i = b_j$.

Define a descending chain of right ideals $\mathfrak{a}_2 \supseteq \mathfrak{a}_3 \supseteq \ldots \supseteq \mathfrak{a}_m$ by

$$\mathfrak{a}_j = a_1 b_1 F + \ldots + a_j b_j F + a_{j+1} F + \ldots + a_m F \quad \text{for} \quad j \in [2, m].$$

Fix $j \in [3, m]$. Then $\mathfrak{a}_{j-1} = \mathfrak{a}_j + a_j F$. Set

$$a_i' = \begin{cases} a_i b_i & \text{if} \quad i \in [1, j] \\ a_i & \text{if} \quad i \in [j+1, m] \end{cases} \quad \text{and} \quad c_i' = \begin{cases} c_i & \text{if} \quad i \in [1, j] \\ b_i c_i & \text{if} \quad i \in [j+1, m]. \end{cases}$$

Thus
$$\sum_{i=1}^{m} a_i' c_i' = 0 \tag{4}$$

and $\mathfrak{a}_j = \sum_{i=1}^{m} a_i' F$. By Corollary 1·8, we can choose a subset $H = H_j$ of $[1, m]$ such that $(a_i' \mid h \in H)$ is a right F-basis of $\mathfrak{a}_j$. Let $K = K_j$ be the complement of H in $[1, m]$. Then for each $k \in K$ there is a homogeneous relation

$$a_k' = \sum_H a_h' d_{hk}. \tag{5}$$

It follows that for all $h \in H$,

$$c_h' + \sum_K d_{hk} c_k' = 0. \tag{6}$$

If it happens that $j \in H$ then we write e_k for d_{jk} and (6) takes the form

$$c_j + \sum_{\substack{k \in K \\ k<j}} e_k c_k + \sum_{\substack{k \in K \\ k>j}} e_k b_k c_k = 0. \tag{7}$$

For $k < j$, if $e_k \neq 0$ then $|e_k| = |c_j| - |c_k| \leqslant 0$; here $e_k \in F_0^\times$, $|c_j| = |c_k|$ so $b_j = b_k$ and $b_j e_k = e_k b_j = e_k b_k \in F b_k$. Thus (7) shows that (3) holds.

This leaves the case $j \in K_j$ for every $j \in [3, m]$. Here we write e_h for d_{hj}, and (5) takes the form

$$a_j b_j = \sum_{\substack{h \in H \\ h<j}} a_h b_h e_h + \sum_{\substack{h \in H \\ h>j}} a_h e_h. \tag{8}$$

If some e_h lies in $F_0^\times$ then we can alter the set H by exchanging j for h, and then we are in the previous case of $j \in H$. Thus we may assume no e_h lies in $F_0^\times$. Now notice that $\{a_h' \mid h \in H\} \cup \{a_j\}$ is a generating set for $\mathfrak{a}_{j-1}$ satisfying a relation (8). By Corollary 1·8, $\operatorname{rk} \mathfrak{a}_{j-1} < \operatorname{card}(H) + 1 = \operatorname{rk} \mathfrak{a}_j + 1$; thus $\operatorname{rk} \mathfrak{a}_{j-1} \leqslant \operatorname{rk} \mathfrak{a}_j$, and we have

$$1 \leqslant \operatorname{rk} \mathfrak{a}_2 \leqslant \operatorname{rk} \mathfrak{a}_3 \leqslant \dots \leqslant \operatorname{rk} \mathfrak{a}_m. \tag{9}$$

But $\mathfrak{a}_m$ is generated by m elements which are right dependent, by (1), so $\operatorname{rk} \mathfrak{a}_m < m$. Thus one of the steps in (9) must be equality.

Consider the case where $\operatorname{rk} \mathfrak{a}_2 = 1$. This means that one of the elements

$$a_1 b_1,\ a_2 b_2,\ a_3,\ a_4,\ \dots,\ a_m \tag{10}$$

is a left factor of all the others. But $a_1 b_1$ is an atom because $a_1 \in F_0^\times$ and B_1 is an atomic format; moreover, $0 < |a_3| \leqslant |a_4| \leqslant \dots$, so none of the elements in (10) is a unit. It follows that $a_1 b_1$ is a left factor of $a_2 b_2$, say $a_2 b_2 = a_1 b_1 c$. Since $n \geqslant 2$, Corollary 1·7 shows there is a unique e such that $a_2 = a_1 e$, $e b_2 = b_1 c \in b_1 F$, and (2) holds.

The remaining case is where $\operatorname{rk} \mathfrak{a}_{j-1} = \operatorname{rk} \mathfrak{a}_j$ for some $j \in [3, m]$, and we consider (8) for this j. Here one of the elements of $\{a_h' \mid h \in H\} \cup \{a_j\}$ can be expressed as a right F-linear combination of the others. Consider first the case where there is a (homogeneous) relation

$$a_j = \sum_{\substack{h \in H \\ h<j}} a_h b_h f_h + \sum_{\substack{h \in H \\ h>j}} a_h f_h. \tag{11}$$

For $h > j$, if $f_h \neq 0$ then $|f_h| = |a_j| - |a_h| \leqslant 0$, which means $f_h \in F_0^\times$ and $|a_j| = |a_h|$, so $b_j = b_h$ and $f_h b_j = b_j f_h = b_h f_h \in b_h F$. This shows that (2) is satisfied. We are now left with the case where, for some $h_0 \in H$, a_{h_0}' can be written as a right F-linear combination of a_j and the a_h', $h \in H \setminus \{h_0\}$. Here $\{a_h' \mid h \in H,\ h \neq h_0\} \cup \{a_j\}$ is a generating set for $\mathfrak{a}_{j-1}$, and hence is a basis, since $\operatorname{rk} \mathfrak{a}_{j-1} = \operatorname{rk} \mathfrak{a}_j = \operatorname{card}(H)$. We thus have a homogeneous expression

$$a_{h_0}' = a_j f_j + \sum_{h \in H \setminus \{h_0\}} a_h' f_h. \tag{12}$$

Right multiplying (12) by e_{h_0} gives an expression for $a'_{h_0} e_{h_0}$ in terms of our basis of $\mathfrak{a}_{j-1}$; but (8) also gives an expression for $a'_{h_0} e_{h_0}$ in terms of our basis. Comparing the coefficients of a_j we see $f_j e_{h_0} = b_j$. Since b_j is an atom and $e_{h_0} \notin F_0^\times$ we see $f_j \in F_0^\times$. Right multiplying (12) by f_j^{-1} gives an expression of the form (11) and we are in the previous case. This completes the proof. ∎

1·12. *Example.* One cannot weaken the hypotheses of Theorem 1·11 to allow $B_1 \leqslant < B_2 < < \ldots < < B_{t-1} < \leqslant B_t$. Let F_0 be any field and F be the free F_0-algebra on seven generators, t, u, v, w, x, y, z with grading such that $|vx| = |wz|$, $|uw| = |ty|$. Take $B_1 = (0, t, |yz|)$, $B_2 = (0, uw+ty, |z|)$, $B_3 = (|u|, vx+wz, 0)$, $B_4 = (|uv|, x, 0)$. Then $B_1 \leqslant < B_2 < < B_3 < \leqslant B_4$ and this is a tableau, T, of atomic formats which does not satisfy weak algorithm because the T-relation

$$(1)(t)(yz) + (-1)(uw+ty)(z) + (u)(vx+wz)(1) + (uv)(x)(-1) = 0$$

cannot be trivialized in F.

1·13. *Definitions.* Two formats $(\alpha_1, b_1, \gamma_1)$, $(\alpha_2, b_2, \gamma_2)$ of the same degree are said to *overlap* if $[\alpha_1+1, \alpha_1+|b_1|] \cap [\alpha_2+1, \alpha_2+|b_2|] \neq \varnothing$.

The *components* of a tableau T are the classes of the equivalence relation on T that is generated by the (reflexive symmetric) overlap relation. These components are again tableaux in their own right.

1·14. PROPOSITION. *Let T be a tableau in the graded algebra F. If F and each component of T satisfy n-term weak algorithm then T satisfies n-term weak algorithm.*

Proof. Let (1) be a T-relation with all the a_i, c_i non-zero, and for each $i \in [1, m]$ let $B_i = (|a_i|, b_i, |c_i|)$, an element of T. Renumbering if necessary, we may assume $|a_1| \leqslant |a_2| \leqslant \ldots \leqslant |a_m|$, and that for some $p \in [1, m]$, $B_1, \ldots, B_p$ all lie in one component T' of T, while $B_{p+1}, \ldots, B_m$ do not lie in T'. Hence $|a_i b_i| \leqslant |a_{p+1}|$ for all $i \in [1, p]$. If $a_1 b_1, \ldots, a_p b_p$ are right F-independent, then, by Lemma 1·6, they can be extended to a right F-basis of $\sum_{i=1}^p a_i b_i F + \sum_{i=p+1}^m a_i F$; it follows that $c_p \in \sum_{i=p+1}^m F b_i c_i$ and (3) holds. If $a_1 b_1, \ldots, a_p b_p$ are right F-dependent then, by Lemma 1·6, $\sum_{i=1}^p a_i b_i e_i = 0$ with some $e_j = 1$, and then $\sum_{i=1}^p a_i b_i e_i c_j = 0$ is a T'-relation. By n-term weak algorithm in T', in the form of (*d*) of Proposition 1·5, some $a_k = \sum_{i=1}^p a_i d_i$, $d_i b_k \in b_i F$, $d_k = 0$ and thus (2) holds. This proves (*c*) of Proposition 1·5, so T satisfies n-term weak algorithm. ∎

1·15. *Definitions.* A format $(\alpha_1, b_1, \gamma_1)$ is said to *left merge* with another format $(\alpha_2, b_2, \gamma_2)$ of the same degree if they overlap and there exist $d \in F_{\alpha_2 - \alpha_1}$, $d' \in F_{\gamma_1 - \gamma_2}$ such that $db_2 = b_1 d' \neq 0$. If F satisfies 2-term weak algorithm then, by Corollary 1·7, this is equivalent to the existence of factorizations $b_1 = dx$, $b_2 = xd'$.

Two formats of the same degree are said to *merge* if they overlap and one left merges with the other.

A tableau T is said to be a *staircase* if each pair of overlapping elements of T merges. In this event $\leqslant \leqslant$ is a total order on T.

1·16. PROPOSITION. *Let T be a staircase tableau in the graded algebra F. If F satisfies n-term weak algorithm then so does T.*

Proof. Let (1) be a T-relation with all the a_i, c_i non-zero, and for each $i \in [1, m]$ let $B_i = (|a_i|, b_i, |c_i|)$, an element of T. Renumbering if necessary we may assume $B_1 \leqslant \leqslant \ldots \leqslant \leqslant B_m$. For each $i \in [1, m-1]$ we choose a factorization $b = x_i y_i$ as follows: if B_i overlaps B_m (for example, if $B_i = B_m$) then by the staircase property there exist factorizations $b_i = x_i y_i$, $b_m = y_i z_i$ with $|a_m| = |a_i x_i|$; if B_i and B_m do not overlap then $|a_i b_i| \leqslant |a_m|$ and we set $x_i = b_i$, $y_i = 1$. We then have

$$a_m b_m c_m = -\sum_{i=1}^{m-1} a_i x_i y_i c_i \in \sum_{i=1}^{m-1} a_i x_i F$$

with $|a_m| \geqslant |a_i x_i|$ for all $i \in [1, m-1]$. By Corollary 1·8 and Lemma 1·6, there is a homogeneous relation $a_m = \sum_{i=1}^{m-1} a_i x_i d_i$. For each $i \in [1, m-1]$, $(x_i d_i) b_m \in b_i F$ since either $x_i = b_i$, or $d_i \in F_0$ and

$$x_i d_i b_m = x_i d_i y_i z_i = x_i y_i d_i z_i = b_i d_i z_i \in b_i F.$$

Thus Proposition 1·5 (*c*) holds. ∎

1·7. *Definitions.* A format $B_1 = (\alpha_1, b_1, \gamma_1)$ is said to be a *multiple* of another format $B_2 = (\alpha_2, b_2, \gamma_2)$ of the same degree, if there is a factorization $b_1 = a b_2 c$ with $a \in F_{\alpha_2 - \alpha_1}$, $c \in F_{\gamma_2 - \gamma_1}$. We also say B_2 *divides* B_1. Since it is possible that $\alpha_2 = \alpha_1$, the properties of 'merging' and 'dividing' are not mutually exclusive. Indeed, if F satisfies 2-term weak algorithm, and b_2 is an atom, then the only way for B_2 to merge with B_1 is for it to divide B_1.

1·18. THEOREM. *Let T be a tableau in the graded algebra F, and B a format in F with $|B| = |T|$. Write*

$$T_B^+ = \{B\} \cup \{B' \in T \mid B' \text{ is not a multiple of } B\},$$

$$T_B^- = \{B' \in T \mid B' \text{ is a multiple of } B, \text{ or } B' \text{ merges with } B, \text{ or } B' \text{ does not overlap } B\}.$$

If T_B^+ and T_B^- satisfy n-term weak algorithm then so does T.

Proof. Let (1) be a T-relation with all the a_i, c_i non-zero. Let $I = [1, m]$ and for each $i \in I$, let $B_i = (|a_i|, b_i, |c_i|)$, an element of T. Set

$$I^+ = \{i \in I \mid B_i \text{ overlaps } B, \text{ but } B \text{ does not merge with, nor divide, } B_i\},$$

$$I^- = \{i \in I \mid B_i \text{ is a multiple of } B\},$$

$$I^0 = I \setminus (I^+ \cup I^-).$$

Thus $\{B_i \mid i \in I^+ \cup I^0) \subseteq T_B^+$, $\{B_i \mid i \in I^- \cup I^0\} \subseteq T_B^-$.

Say $B = (\alpha, b, \gamma)$, so for each $i \in I^-$ there is a factorization $b_i = b_i' b b_i''$. Then

$$\sum_{I^+ \cup I^0} a_i b_i c_i + \sum_{I^-} (a_i b_i') b (b_i'' c_i) = 0$$

is a T_B^+-relation, so is trivializable. Thus I is partitioned into two sets H, K and there are homogeneous relations

$$\sum_{H^+ \cup H^0} a_h d_{hk} + \sum_{H^-} a_h b_h' d_{hk} = \begin{cases} a_k & \text{for all } k \in K^+ \cup K^0 \\ a_k b_k' & \text{for all } k \in K^- \end{cases}, \tag{13}$$

$$-\sum_{K^+ \cup K^0} d_{hk}' c_k - \sum_{K^-} d_{hk}' b_k'' c_k = \begin{cases} c_h & \text{for all } h \in H^+ \cup H^0 \\ b_h'' c_h & \text{for all } h \in H^- \end{cases}, \tag{14}$$

such that
$$\begin{Bmatrix} d_{hk} b_k & \text{for } k \in K^+ \cup K^0 \\ d_{hk} b & \text{for } k \in K^- \end{Bmatrix} = \begin{Bmatrix} b_h d_{hk}' & \text{for } h \in H^+ \cup H^0 \\ b d_{hk}' & \text{for } h \in H^- \end{Bmatrix} \tag{15}$$

where $H^+ = H \cap I^+$, etc. From (15) we see that

$$d_{hk} b = b_h d_{hk}' = 0 \quad \text{for all} \quad (h, k) \in H^+ \times K^- \tag{16}$$

since B, B_h overlap but do not merge. Similarly,

$$d_{hk}b_k = bd'_{hk} = 0 \quad \text{for all} \quad (h,k)\in H^- \times K^+. \tag{17}$$

We claim
$$\sum_{I^-} a_i b_i c_i + \sum_{H^0} a_h b_h (c_h + \sum_{K^+} d'_{hk} c_k) + \sum_{K^0} (a_k - \sum_{H^+} a_h d_{hk})\, b_k c_k = 0. \tag{18}$$

In fact,
$$\sum_{H^0}\sum_{K^+} a_h b_h d'_{hk} c_k = \sum_{K^+}\sum_{H^0} a_h d_{hk} b_k c_k \quad \text{by (15)}$$
$$= \sum_{K^+} (a_k - \sum_{H^+} a_h d_{hk})\, b_k c_k \quad \text{by (13) and (17)}.$$

Similarly,
$$-\sum_{K^0}\sum_{H^+} a_h d_{hk} b_k c_k = -\sum_{H^+} a_h b_h (\sum_{K^0} d'_{hk} c_k) \quad \text{by (15)}$$
$$= \sum_{H^+} a_h b_h (c_h + \sum_{K^+} d'_{hk} c_k) \quad \text{by (14) and (16)}.$$

Substituting these into the left-hand side of (18) transforms the sum into $\Sigma\, a_i b_i c_i$ where i ranges over $I^- \cup H^0 \cup K^+ \cup K^0 \cup H^+$ which is just a partition of $I = [1, m]$; thus (18) reduces to (1).

If I^- is empty then (13)–(15) show (1) is trivializable; we may therefore assume I^- is non-empty and, by renumbering if necessary, $1 \in I^-$. Now (18) is a $T_{\overline{B}}$-relation, so by hypothesis, (18) can be trivialized. Taking $j = 1$ in (*b*) of Proposition 1·5, we see that either there is a homogeneous relation

$$\sum_{I^-} a_i e_i + \sum_{H^0} a_h e_h + \sum_{K^0} (a_k - \sum_{H^+} a_h d_{hk})\, e_k = 0 \tag{19}$$

with $e_1 = 1$ and $e_i b_1 \in b_i F$ for all $i \in I^- \cup H^0 \cup K^0$, or there is a dual expression. We wish to verify (*c*) of Proposition 1·5, so by symmetry it suffices to consider (19). Here

$$\sum_{I^-} a_i e_i + \sum_{H^0} a_h e_h + \sum_{K^0} a_k e_k - \sum_{H^+} a_h (\sum_{K^0} d_{hk} e_k) = 0$$

with $e_1 = 1$, $e_i b_1 \in b_i F$ for all $i \in I^- \cup H^0 \cup K^0$, and so for $h \in H^+$

$$(\sum_{K^0} d_{hk} e_k)\, b_1 \in \sum_{K^0} d_{hk} b_k F = \sum_{K^0} b_h d'_{hk} F \subseteq b_h F.$$

Thus (*c*) of Proposition 1·5 is satisfied, as desired. ∎

We are now ready to begin proving the main result.

1·19. THEOREM. *Let b be a non-zero homogeneous element of the graded algebra F, and let T be a b-tableau in F. If F satisfies n-term weak algorithm then so does T.*

Beginning of the proof. If $|b| = 0$ the result is clear, so we may assume $|b| \geqslant 1$. By Corollary 1·7, b can be written as a product of homogeneous atoms in a unique way, up to associates. Let $X = \{a_1, \ldots, a_z\}$ be a set of representatives of the associativity classes of atoms occurring in such a factorization of b. In $F^\times$, X freely generates a free semigroup (without 1), X^*, and b is an associate of an element of X^*. There is no loss of generality in assuming $b \in X^*$.

Henceforth we consider only those formats (α', b', γ') of degree $|T|$, where $b' \in X \cup \{b\}$; given the integer $|T|$ one can recover such a format from the pair $(\alpha', b') \in \mathbb{Z} \times X^*$. Using the four results, Theorems 1·11, 1·18, Propositions 1·14, 1·16, we shall show that T satisfies n-term weak algorithm if F does. To highlight the combinatorial nature of this part of the proof, we abstract these four results to conditions on finite subsets of

$\mathbb{Z} \times X^*$, and show how the desired conclusion follows; this programme is carried out in the next section, and the present theorem will follow from Theorem 2·8. ∎

1·20. *Notation.* For any element b of a ring R we define the *idealizer* of b in R as $\mathbb{I}(b) = \{r \in R \mid rb \in bR\}$.

1·21. Corollary. *Let b be a homogeneous element of the graded algebra F. If F satisfies n-term weak algorithm then so dos the graded algebra $\mathbb{I}(b)$.*

Proof. It is clear that $\mathbb{I}(b)$ is a graded subalgebra of F. If $b = 0$ then $\mathbb{I}(b) = F$, so we may assume $b \in \bar{F}^{\times}$. Suppose $m \in [1, n]$ and

$$\sum_{i=1}^{m} a_i c_i = 0 \tag{20}$$

is a relation in a 1-tableau in $\mathbb{I}(b)$ with some c_i non-zero. In particular, for each $i \in [1, m]$ there is a $c_i' \in F$ such that $c_i b = bc_i'$ and some c_i' is non-zero. Right multiplying (20) by b we get $\sum_{i=1}^{m} a_i bc_i' = 0$, and this is a T-relation for some b-tableau T in F. By Theorem 1·19, T satisfies n-term weak algorithm, so by (d) of Proposition 1·5, there is a homogeneous relation $\sum_{i=1}^{m} a_i d_i = 0$ with some $d_j = 1$, and $d_i b \in bF$ for all $i \in [1, m]$. Thus the d_i lie in $\mathbb{I}(b)$, and we have shown that every 1-tableau in $\mathbb{I}(b)$ satisfies (d) of Proposition 1·5. Hence $\mathbb{I}(b)$ satisfies n-term weak algorithm. ∎

Combining this with Theorem 1·19, we get the result stated in the title.

1·22. Theorem. *Let k be a field, X a graded set (that is, given with a map $X \to \mathbb{N}^+$), and $F = k\langle X\rangle$ the free algebra with the naturally induced grading. If b is a homogeneous element of F then $\mathbb{I}(b)$ is a free subalgebra of F with a homogeneous free generating set.* ∎

We can deduce from this that certain non-homogeneous elements also have free idealizers.

1·23. Theorem. *Let k be a field, X a graded set, $F = k\langle X\rangle$ the free algebra with the naturally induced grading, and b an element of $F^{\times}$. Suppose b has the property that, for each expression for $\bar{b}$ in the form $(ac)^m a$ with $m \geqslant 1$, there exists $s \in \mathbb{I}(b)$ such that $\bar{s} = ac$. Then $\overline{\mathbb{I}(b)} = \overline{\mathbb{I}(\bar{b})}$, and $\mathbb{I}(b)$ is a free subalgebra of F with a free generating set Y, such that $\bar{Y}$ is a free generating set of $\mathbb{I}(\bar{b})$.*

Proof. It is clear that $\overline{\mathbb{I}(b)} \subseteq \overline{\mathbb{I}(\bar{b})}$, with no hypotheses on b. Now suppose that $r \in \mathbb{I}(\bar{b})$, say $r\bar{b} = \bar{b}t$. There is an $m \in \mathbb{N}$ such that $\bar{b} = r^m a$ for some a that does not have r as a left factor. Here $r^{m+1} a = r\bar{b} = \bar{b}t = r^m at$, so $ra = at$. Since a does not have r as a left factor, Corollary 1·7 shows that $r = ac$ for some c. If $m = 0$ then

$$r = ac = \bar{b}c = \overline{(bc)} \in \overline{\mathbb{I}(b)}.$$

If $m \geqslant 1$ then $\bar{b} = r^m a = (ac)^m a$, and by the hypothesis, there is an $s \in \mathbb{I}(b)$ such that $\bar{s} = ac = r$. This proves $\overline{\mathbb{I}(b)} = \overline{\mathbb{I}(\bar{b})}$.

By Theorem 1·22, $\mathbb{I}(\bar{b})$ has a homogeneous free generating set Z, and we can choose a subset Y of $\mathbb{I}(b)$ such that $\bar{Y} = Z$. It is then straightforward to show that Y freely generates $\mathbb{I}(b)$. ∎

1·24. *Examples.* Let $X = \{x, y\}$, $F = k\langle X\rangle$ where k is a field, and let X_1^* be the free monoid on X.

(i) If $b = xyx$ then $\mathbb{I}(b) = k + kxy + bF$. By Theorem 1·22, this has a homogeneous free generating set; in practice, finding one amounts to choosing a homogeneous k-basis of the augmentation ideal modulo its square. Here the augmentation ideal, $kxy+bF$, has square $kxyxy+byxF+bFxy+bFbF$. It follows that a free generating set of $\mathbb{I}(b)$ is given by

$$\{xy\} \cup b(X_1^* \backslash(\{y\} \cup yx\,X_1^* \cup X_1^* xy \cup X_1^* xyx\,X_1^*)). \tag{21}$$

(ii) If $b = xyx-x$ then $\mathbb{I}(b) = k+kxy+bF$. By Theorem 1·23, this has a free generating set, given by (21).

(iii) If $b = xyx-y$ then $\mathbb{I}(b) = k+bF$, and this does not satisfy the hypotheses of Theorem 1·23 for any grading of X. However, this idealizer is free since it is a graded algebra with respect to the y-degree, and it is not difficult to show that it satisfies weak algorithm, cf. [3]. Using the equations

$$bwxyxz = bwyz+bw.bz, \quad bwxy^2z = bwy^2xz+bwxy.bz-bw.bxyz \ (w,z \in X_1^*)$$

one finds that a free generating set for $\mathbb{I}(b)$ is given by

$$bX_1^* \backslash b(X_1^* xyx\,X_1^* \cup X_1^* xy^2 X_1^*).$$

(iv) If $b = xyx-1$ then $\mathbb{I}(b) = k+bF$, and this is not free; but it is left and right hereditary, cf. [4].

1·25. *Conjecture.* For every element b of a free algebra F, the idealizer of b in F is left and right hereditary.

We conclude this section with the following result.

1·26. THEOREM (Gerasimov). *If F and b are as in the hypotheses of Theorem 1·23 then $\overline{FbF} = \overline{F\bar{b}F}$. Here the word problem for the one-relator algebra $F/(b)$ is solvable and, if, for some $x \in X$, the image of $X\backslash\{x\}$ freely generates a free subalgebra of $F/(\bar{b})$ then the same holds for $F/(b)$.*

Proof. Clearly $\overline{F\bar{b}F} \subseteq \overline{FbF}$. Now suppose $d \in FbF$, say $d = \sum_{i=1}^m a_i b c_i$. We wish to show that $\bar{d} \in F\bar{b}F$, and we may assume that the a_i, c_i are non-zero and chosen so as to minimize the integer

$$\alpha = \max\{|\bar{a}_i|+|\bar{c}_i| : i \in [1,m]\}.$$

Let $I = \{i \in [1,m] : |\bar{a}_i|+|\bar{c}_i| = \alpha\}$. Then $\sum_{i\in I}\bar{a}_i\bar{b}\bar{c}_i$ is either $\bar{d}$ or 0. The former gives the desired conclusion; the latter will yield a contradiction, as follows. By Theorem 1·19, there is a partition H, K of I, and we have homogeneous relations as in 1·2, involving elements of $\overline{\mathbb{I}(\bar{b})} = \overline{\mathbb{I}(b)}$. Thus for each $(h,k) \in H\times K$ there is an element $s_{hk} \in \mathbb{I}(b)$ such that $s_{hk}b = bs'_{hk}$, and $\bar{a}_k = \sum_H \bar{a}_h \bar{s}_{hk}\,(k \in K)$, $\bar{c}_h = -\sum_K \bar{s}'_{hk}\bar{c}_k$ $(h \in H)$. For each $k \in K$ we replace a_k with $a_k - \sum_H a_h s_{hk}$; for each $h \in H$ we replace c_h with $c_h + \sum_K s'_{hk} c_k$. This gives a new expression for d, and contradicts the minimality of α. Hence $\overline{F\bar{b}F} = \overline{FbF}$.

The remaining results are easily verified by considering homogeneous components of highest degrees, cf. [6] or [2]. ∎

1·27. *Notes.* Corollary 1·21 answers, in the affirmative, Problem 2 of [3]. Easy examples of tensor rings show that for $n = 3$ the centrality of F_0 cannot be omitted.

Certain cases of Theorem 1·22 were known previously. In [3] the easy case where b is a monomial in X was noted, while [7] handled the case where b is a *strong prime* in F, that is, $F/(b)$ has no non-zero zerodivisors. (Notice that homogeneous strong primes are necessarily atoms, and the atomic case of Theorem 1·22 follows directly from Theorem 1·11. In fact, M. C. Hedges has shown that the proof of Theorem 1·11 can be adapted to prove that, in free algebras, homogeneous atoms are strong primes.)

If k has characteristic zero then the final part of Theorem 1·26 is a very special case of the Freiheitssatz of Makar-Limanov [**8**].

Although none of the results so far, other than Theorem 1·26, appears explicitly in [**6**], the spirit of Gerasimov's paper pervades the entire section. The statements of Propositions 1·14, 1·16 and Theorem 1·18 were suggested by Lemmas 4·3, 5·2·2 and 2·3, respectively, of [**6**]; even the idea behind Theorem 1·11 was inspired, somehow, by the proof of Lemma 5·2 of [**6**].

2. *Positioned words*

2·1. *Notation.* Throughout this section we fix a finite graded set X; that is, there is given a map $X \to \mathbb{N}^+, a \mapsto |a|$. We write X^*, X_1^*, for the free semigroup on X, and the free monoid on X, respectively. The resulting monoid homomorphism $X_1^* \to \mathbb{N}$ is denoted $b \mapsto |b|$, and $|b|$ is called the *degree* of b.

2·2. *Definitions.* An element B of $\mathbb{Z} \times X^*$ is called a *positioned word*; for brevity, and to emphasize the link with the previous section, we shall call positioned words *formats* in X. If $B = (\alpha, b)$, say, then we call B a *b-format*; if $b \in X$, then B is an *atomic* format. We say that B has *left margin* $\lambda(B) = \alpha$, *right margin* $\rho(B) = \alpha + |b|$, *support*

$$\sigma(B) = [\alpha+1, \alpha+|b|],$$

and *weight* $w(B) = |b|$. For $\delta \in \mathbb{Z}$, we set $\delta + B = (\delta+\alpha, b)$.

2·3. *Definitions.* Let $B_1 = (\alpha_1, b_1)$, $B_2 = (\alpha_2, b_2)$ be two formats (in X). If

$$\lambda(B_1) < \lambda(B_2) \quad \text{and} \quad \rho(B_1) < \rho(B_2)$$

we say that B_1 is *left* of B_2, and B_2 is *right* of B_1; we write $B_1 < < B_2$. If $\lambda(B_1) \leqslant \lambda(B_2)$ and $\rho(B_1) \leqslant \rho(B_2)$ we write $B_1 \leqslant \leqslant B_2$. By $B_1 \leqslant < B_2$ we mean that either $B_1 < < B_2$, or $\lambda(B_1) = \lambda(B_2)$ and $\rho(B_1) \leqslant \rho(B_2)$; similarly, by $B_1 < \leqslant B_2$ we mean that either $B_1 < < B_2$, or $\rho(B_1) = \rho(B_2)$ and $\lambda(B_1) \leqslant \lambda(B_2)$.

If $\rho(B_1) \leqslant \lambda(B_2)$ we write $B_1 < < < B_2$. If $\rho(B_1) = \lambda(B_2)$ then we say that B_1 and B_2 are *adjacent*.

If $\sigma(B_1) \cap \sigma(B_2) \neq \varnothing$ then B_1 and B_2 are said to *overlap*. If $\lambda(B_2)$, $\lambda(B_2)+1 \in \sigma(B_1)$ then B_1 *overlaps the left margin* of B_2; similarly, if $\rho(B_2)$, $\rho(B_2)+1 \in \sigma(B_1)$ then B_1 *overlaps the right margin* of B_2.

If there exist $a, c \in X_1^*$, $b \in X^*$, such that $|a| = \alpha_2 - \alpha_1$ and $b_1 = ab$, $b_2 = bc$ then B_1 *left merges* with B_2. We say B_1 and B_2 *merge* if one of them left merges with the other.

If there exists $a, c \in X_1^*$ such that $|a| = \alpha_2 - \alpha_1$ and $b_1 = ab_2c$ then B_2 is said to *divide* B_1, B_2 is a *b_2-factor* of B_1, and B_1 is a *multiple* of B_2.

Let $a \in X$. If there is an a-factor A of B_1 that overlaps B_2 but does not divide B_2, then B_1 is said to *a-discord* with B_2; otherwise, B_1 *a-concords* with B_2. If B_1 and B_2 *a-concord* with each other then B_1 and B_2 are *a-concordant*; otherwise they are *a-discordant*. In the case where B_1 is itself an atomic format overlapping B_2 but not dividing B_2 then we say B_1 *discords* with B_2; the element a will be clear from the context.

2·4. *Definitions.* A *tableau* T (in X) is a finite subset of $\mathbb{Z}\times X^*$. We say T has *left margin* $\lambda(T) = \min\{\lambda(B)|B\in T\}$, *right margin* $\rho(T) = \max\{\rho(B)|B\in T\}$, *support* $\sigma(T) = \bigcup_{B\in T}\sigma(B)$, and *weight* $w(T) = \sum_{B\in T}w(B)$.

For $b\in X^*$, the set of b-formats in T is denoted T_b. If $b_1,\dots,b_s\in X^*$ and

$$T = T_{b_1}\cup\dots\cup T_{b_s}$$

then T is said to be a $b_1,\dots,b_s$-*tableau.*

We say that T is a *chain* if its elements can be listed $B_1,\dots,B_t$ so that

$$B_1 \leqslant < B_2 << \dots << B_{t-1} << B_t \quad\text{or}\quad B_1 << B_2 << \dots << B_{t-1} <\leqslant B_t;$$

it is a *strong* chain if $B_1 << B_2 << \dots << B_{t-1} << B_t$.

We say that T is a *staircase* if every pair of overlapping elements of T merges.

The *components* of T are the classes of the equivalence relation on T that is generated by the overlap relation.

For a format A we say T *overlaps* A if $\sigma(A)\cap\sigma(T)\neq\varnothing$, we write $T <<< A$ if $\rho(T)\leqslant\lambda(A)$ and so on.

For an atomic format A we define

$$T^+_A = \{A\}\cup\{B\in T|A \text{ does not divide } B\},$$
$$T^-_A = \{B\in T|\text{either } A \text{ divides } B, \text{ or } A \text{ does not overlap } B\}.$$

For $a\in X$, we say that T is *a-breakable* if there is an a-format A dividing some element of $T\setminus\{A\}$ and discording with some (other) element of T; such an A is said to *break* T. If T is not a-breakable then it is *a-unbreakable.*

We now introduce the notion of a *satisfactory* tableau. This is defined as the most restrictive property satisfying the following four conditions.

2·4·1. All staircase tableaux are satisfactory.

2·4·2. Every tableau that is a chain of atomic formats is satisfactory.

2·4·3. A tableau is satisfactory whenever all its components are.

2·4·4. If an atomic format A breaks a tableau T and T^+_A, T^-_A are satisfactory then T is satisfactory.

Thus, by definition, the set of satisfactory tableaux is the smallest set having these four properties. We shall show that for $b\in X^*$, every b-tableau is satisfactory, which will complete the proof of Theorem 1·19.

2·5. *Definition.* Let $a\in X$, $b\in X^*$. Set

$$\Delta = \Delta(a,b) = \{\delta\in\mathbb{Z}|(\delta, b)\ a\text{-concords with } (0,b)\}.$$

Thus for $\delta\in\mathbb{Z}$, if the absolute value of δ is at least $|b|$ then $\delta\in\Delta$, since (δ,b) does not overlap $(0,b)$; if a does not occur in b then $\Delta=\mathbb{Z}$. Set

$$\Delta^0_+ = \{\delta\in\mathbb{N}^+|\delta\in\Delta\} = \{\delta\in\mathbb{N}^+|(\delta,b)\ a\text{-concords with } (0,b)\},$$
$$\Delta^0_- = \{\delta\in\mathbb{N}|-\delta\in\Delta\} = \{\delta\in\mathbb{N}^+|(0,b)\ a\text{-concords with } (\delta,b)\},$$
$$\Delta^0 = \Delta^0_+\cap\Delta^0_-,$$
$$\Delta_+ = \Delta^0_+\setminus\Delta^0,$$
$$\Delta_- = \Delta^0_-\setminus\Delta^0.$$

Write $\delta^0_+ = \min \Delta^+_0$, and similarly for δ^0_-, δ^0, δ_+, δ_- with the convention that the minimum of the empty set is $+\infty$.

In situations where the particular choice of a, b is to be emphasized, we write $\delta_+(a,b)$ for δ_+, etc.

Notice that Δ^0_+, Δ^0_- are closed under addition, and hence so is Δ^0.

2·6. THEOREM. *Let $a \in X$, $b \in X^*$ with $\delta_-(a,b) \geqslant \delta_+(a,b)$. Suppose that T is an a, b-tableau satisfying the following conditions*:

(a 1) *No element of T_a discords with two a-concordant elements of T_b.*

(a 2) *If T_a overlaps the right margin of some $B \in T_b$ then B is the rightmost b-format in some component of T.*

(a 3) *T is a-breakable.*

(a 4) *Each a-format A which breaks T either overlaps the right margin of an element of T_b or discords with two a-concordant elements of T_b.*

Then there is an a-format A which breaks T such that (a 1) *and* (a 2) *hold with T replaced with T^+_A.*

2·6·1. *Remark.* Notice that if (a 4) fails for some A then the conclusion holds for this A; thus (a 4) is superfluous – but makes the proof smoother.

2·6·2. *Remark.* A dual result holds for $\delta_-(a,b) \leqslant \delta_+(a,b)$ with the three occurrences of 'right' replaced with 'left'. This can be proved by adapting the argument, or by introducing the notion of the *dual* B^o of a format B, where $(\alpha, b)^o = (-\alpha - |b|, b^o)$ and $b \mapsto b^o$ is the anti-automorphism of X^* fixing X. The details are left to the reader.

The proof of Theorem 2·6 is broken up into a sequence of lemmas.

2·6·3. LEMMA (Gerasimov). *Suppose $B_0 << B_1 << B_2$ are elements of T_b such that B_1, B_2 are a-concordant, and thus $\delta_2 = \lambda(B_2) - \lambda(B_1)$ lies in Δ^0. If A is an a-format dividing B_0 and discording with B_1 and B_2, then the a-format $A' = -\delta_2 + A$ discords with B_1 and divides B_0.*

Proof. Let $\delta_1 = \lambda(B_1) - \lambda(B_0)$. Since A discords with B_2, $A' = -\delta_2 + A$ discords with $B_1 = -\delta_2 + B_2$. Also, A divides $B_0 = -\delta_1 + B_1$, so the a-format $A'' = \delta_1 + A$ divides B_1 so concords with B_2. Furthermore,

$$\lambda(B_2) < \rho(A) < \rho(A'') \leqslant \rho(B_1) < \rho(B_2)$$

so A'' overlaps B_2; thus A'' divides B_2. Hence $-\delta_1 - \delta_2 + A''$ divides $-\delta_1 - \delta_2 + B_2$; that is, A' divides B_0. ∎

2·6·4. *Notation.* Throughout the remainder of the proof of Theorem 2·6, let $B_1 << B_2 << \ldots$ be the elements of T_b, and let t be the largest integer such that $B_1, \ldots, B_t$ are pairwise a-concordant. For each $B_i \in T_b$, let $\delta_i = \lambda(B_i) - \lambda(B_t)$.

2·6·5. LEMMA. *If A is any a-format breaking T then A divides some $B_i \in T_b$, and each such i exceeds t. In particular, δ_{t+1} exists.*

Proof. By the definition of breaking, it is clear that A divides some $B_i \in T_b$. Suppose $i \leqslant t$ and assume (without loss of generality) that A is the leftmost a-format that divides B_i and breaks T. If $B_{i'} << A$ for some i' then $\rho(B_{i'}) < \rho(A) \leqslant \rho(B_i)$ so $i' < i \leqslant t$; but A divides B_i so, by definition of t, A concords with $B_{i'}$. Hence A does not overlap the right margin of any $B_{i'} \in T_b$. By (a 4), A discords with two a-concordant elements $B_{i_1} << B_{i_2}$ of T_b. In particular B_i a-discords with B_{i_1}, which means $i_1 \notin [1,t]$, and thus $i \leqslant t < i_1$ and $i < i_1 < i_2$. By Lemma 2·6·3, there is an a-format A' discording with B_{i_1}, dividing B_i, and left of A. Thus A' breaks T and this contradicts the choice of A. Hence $i > t$.

Since A exists by (a 3), T_b has an element B_i with $i > t$, and thus B_{t+1} and δ_{t+1} exist. ❙

2·6·6. LEMMA. *For each $B_i \in T_b$, if $i > t$ then $\delta_i \in \Delta^0_-$; moreover $\delta_{t+1} \in \Delta_-$.*

Proof. By Lemma 2·6·5 there is no a-format which breaks T and divides B_t, so $B_t = -\delta_i + B_i$ a-concords with B_i, so $-\delta_i \in \Delta$, which means $\delta_i \in \Delta^0_-$.

In particular, $\delta_{t+1} \in \Delta^0_-$; if $\delta_{t+1} \in \Delta^0$ then $B_1, \ldots, B_t, B_{t+1}$ are pairwise a-concordant (since Δ^0 is closed under addition) and this contradicts the definition of t. Thus $\delta_{t+1} \in \Delta_-$. ❙

2·6·7. COROLLARY. *$\Delta_- \neq \varnothing$ and $\infty > \delta_{t+1} \geqslant \delta_- > \delta_+$.*

Proof. By Lemma 2·6·6, $\delta_{t+1} \in \Delta_-$ so $\delta_{t+1} \geqslant \delta_-$, and by hypothesis $\delta_- \geqslant \delta_+$. Since Δ_+, Δ_- are disjoint by construction, $\delta_- \neq \delta_+$. ❙

2·6·8. *Notation.* Let $\alpha = |a|$, $\beta = |b|$, $B = (0,b)$ and let $A_1 <<< A_2 <<< \ldots$ be the a-factors of B. Since $\delta_+ \in \Delta^0_+$, all the $\delta_+ + A_j$ concord with B. As there are only finitely many of these A_j there is some integer s such that $\delta_+ + A_1, \ldots, \delta_+ + A_{s-1}$ divide B, while $B <<< \delta_+ + A_s$. Since A_s divides B, there is a factorization $b = b'ab''$ with $b', b'' \in X_1^*$, $\lambda(A_s) = |b'|$. Now $\beta = \rho(B) \leqslant \lambda(\delta_+ + A_s) = \delta_+ + |b'|$ which means that the integer $\epsilon = |b'| + \delta_+ - \beta$ is non-negative.

All of this notation will be fixed throughout the remainder of the proof of Theorem 2·6.

2·6·9. LEMMA. *$\beta - \delta_+ + \epsilon + \alpha \leqslant \delta^0_-$.*

Proof. Suppose not, so $\delta^0_- < \beta - \delta_+ + \epsilon + \alpha = |b'| + \alpha = \lambda(A_s) + \alpha = \rho(A_s)$. Consider any A_j. If $j \geqslant s$, then $\lambda(B) = 0 < -\delta^0_- + \rho(A_s) \leqslant \rho(-\delta^0_- + A_j) < \rho(A_j) \leqslant \rho(B)$; thus $-\delta^0_- + A_j$ overlaps B, and since it concords with B by definition of Δ_-, it must divide B. Hence for $j \geqslant s$, $\delta_+ - \delta^0_- + A_j$ concords with B. On the other hand, for $j < s$, $\delta_+ + A_j$ divides B, by definition of s, so $-\delta^0_- + \delta_+ + A_j$ concords with B. In summary, $\delta_+ - \delta^0_- + B$ a-concords with B, which means that the integer $\delta = \delta_+ - \delta^0_-$ lies in Δ.

If $\delta > 0$ then $\delta \in \Delta^0_+$ but is smaller than δ_+ so $\delta \notin \Delta_+$, which forces $\delta \in \Delta^0 \subseteq \Delta^0_-$. Thus $\delta^0_- + \delta \in \Delta^0_-$, which means that $\delta_+ \in \Delta^0_-$, a contradiction.

If $\delta = 0$ then $\delta_+ = \delta^0_- \in \Delta^0_-$, a contradiction.

If $\delta < 0$ then $\delta^0_- - \delta_+ = -\delta \in \Delta^0_-$, but $\delta^0_- - \delta_+ < \delta^0_-$, the final contradiction. ❙

2·6·10. COROLLARY (Gerasimov). *$\alpha + \beta \leqslant \delta^0_- + \delta_+$ and $\alpha + \beta \leqslant \delta^0_+ + \delta_-$.*

Proof. The first inequality is clear from Lemma 2·6·9, since $\epsilon \geqslant 0$. The second inequality follows by symmetry, since the proof of Lemma 2·6·9 used only the fact that $\delta_+ < \infty$. ❙

2·6·11. LEMMA. *If A is any a-format breaking T, and $B_i \in T_b$ with $i < t$, then $B_i <<< A$, so A concords with $B_1, \ldots, B_{t-1}$.*

Proof. By definition of t, $-\delta_i \in \Delta^0 \subseteq \Delta^0_+$ so $-\delta_i \geqslant \delta^+_0$. Thus

$$-\delta_i + \delta_{t+1} \geqslant \delta^0_+ + \delta_- \geqslant \alpha + \beta$$

by Corollaries 2·6·7 and 2·6·10. Hence $\lambda(B_{t+1})-\lambda(B_i) \geqslant \alpha+\beta$, so

$$\lambda(B_{t+1}) \geqslant \lambda(B_i)+\beta = \rho(B_i) \quad \text{and} \quad B_i < < < B_{t+1}.$$

By Lemma 2·6·5, A divides some B_j with $j \geqslant t+1$; thus

$$\rho(B_i) \leqslant \lambda(B_{t+1}) \leqslant \lambda(B_j) \leqslant \lambda(A) \quad \text{and} \quad B_i < < < A. \,|$$

2·6·12. Lemma. *If A is any a-format breaking T and $\rho(A) \leqslant \rho(B_{t+1})$, then either A overlaps the right margin of B_t or A discords with two a-concordant elements of T_b.*

Proof. By (a 4) we may assume A overlaps the right margin of some B_i. By Lemma 2·6·11, $i > t-1$, and by hypothesis $\rho(A) \leqslant \rho(B_{t+1})$, which forces $i < t+1$ and thus $i = t$. |

2·6·13. *Notation.* Throughout the remainder of the proof of Theorem 2·6 let $A_* = \lambda(B_{t+1})+A_s$, an a-factor of B_{t+1}. Notice

$$\lambda(A_*) = \lambda(B_{t+1})+\lambda(A_s) = \lambda(B_{t+1})+\beta-\delta_++\epsilon$$

and
$$\rho(A_*) = \lambda(A_*)+\alpha = \lambda(B_{t+1})+\beta-\delta_++\epsilon+\alpha.$$

2·6·14. Lemma. *If A is any a-format breaking T and A concords with B_t then $A_* < < A$.*

Proof. Suppose not, so $\rho(A) \leqslant \rho(A_*) = \lambda(B_{t+1})+\beta-\delta_++\epsilon+\alpha$. By Lemma 2·6·12, A discords with two a-concordant elements $B_{i_1} < < B_{i_2}$ of T_b. By Lemma 2·6·11, $i_1 > t-1$ and thus $i_1 \geqslant t+1$, since $i_1 \neq t$. The positive integer $\delta = \lambda(B_{i_2})-\lambda(B_{i_1})$ lies in $\Delta^0 \subseteq \Delta^0_-$ so $\delta \geqslant \delta^0_-$. Thus

$$\lambda(B_{i_2})-\lambda(B_{t+1}) \geqslant \lambda(B_{i_2})-\lambda(B_{i_1}) = \delta \geqslant \delta^0_- \geqslant \beta-\delta_++\epsilon+\alpha \geqslant \rho(A)-\lambda(B_{t+1}),$$

where the second last inequality holds by Lemma 2·6·9. Hence $\lambda(B_{i_2}) \geqslant \rho(A)$ which contradicts A discording with B_{i_2}. |

2·6·15. Corollary. *$B_t < < < A_*$, and A_* concords with every element of T.*

Proof.

$$\begin{aligned}\lambda(A_*)-\rho(B_t) &= (\lambda(B_{t+1})+\beta-\delta_++\epsilon)-(\lambda(B_t)+\beta)\\ &= \delta_{t+1}-\delta_++\epsilon \geqslant \delta_--\delta_+ > 0,\end{aligned}$$

by Corollary 2·6·7. Thus $B_t < < < A_*$.

Suppose A_* discords with some element of T. Since A_* divides B_{t+1} we see A_* breaks T. By Lemma 2·6·14, A_* discords with B_t, contradicting $B_t < < < A_*$. |

2·6·16. *Notation.* Let $\Delta' = \Delta(a, b')$ where b' is as defined in 2·6·8, and $\Delta(\ ,\ )$ as in 2·5. Let Δ'^0_-, δ'^0_-, etc., be defined similarly. Write $\beta' = |b'| = \beta-\delta_++\epsilon$. For each $B_i \in T_b$ let $B'_i = (\lambda(B'_i), b')$, a factor of B_i. This notation will be fixed throughout the remainder of the proof of Theorem 2·6.

2·6·17. Lemma (Gerasimov). *$\delta_--\delta_+ \in \Delta'_-$, so $\delta_--\delta_+ \geqslant \delta'_-$.*

Proof. By definition of b', the a-factors of $B' = (0, b')$ are $A_1, \ldots, A_{s-1}$; for each such a-factor A of B', δ_++A is an a-factor of B, by definition of s. Thus $-\delta_-+\delta_++A$ concords with B, and, in particular, with B'. This proves $-\delta_-+\delta_+ \in \Delta'$ and thus $\delta_--\delta_+ \in \Delta'^0_-$.

If $\delta_--\delta_+ \in \Delta'^0$ we shall deduce $\delta_- \in \Delta^0_+$, a contradiction. Consider any a-factor A_j of B. If $\lambda(A_j) \geqslant \beta-\delta_-$ then $B < < < \delta_-+A_j$ and δ_-+A_j concords with B. If $\lambda(A_j) <$

$\beta-\delta_-$ then $\lambda(A_j) < \beta-\delta_+$ which means $j < s$ and A_j divides B'. Assuming $\delta_- - \delta_+ \in \Delta'^0$ this means that $\delta_- - \delta_+ + A_j$ concords with B'. But

$$\lambda(\delta_- - \delta_+ + A_j) = \delta_- - \delta_+ + \lambda(A_j) < \delta_- - \delta_+ + \beta - \delta_- = \beta - \delta_+,$$

which means that $\delta_- - \delta_+ + A_j$ must actually divide B', and hence B. Thus

$$\delta_+ + (\delta_- - \delta_+ + A_j)$$

concords with B. In summary, for all the A_j, $\delta_- + A_j$ concords with B, that is, $\delta_- \in \Delta_+^0$, the desired contradiction. ∎

2·6·18. LEMMA. *For each $B_i \in T_b$, if $i > t$ and $\delta_i < \alpha+\beta$ then $\delta_i \in \Delta_-$.*

Proof. By Lemma 2·6·6, $\delta_{t+1} \in \Delta_-$, and this leaves the case $i \geqslant t+2$. By Corollary 2·6·10 with b' in place of b,

$$\delta'^0_+ + \delta'_- \geqslant \alpha + \beta' = \alpha + \beta - \delta_+ + \epsilon.$$

Thus

$$\begin{aligned} \delta'^0_+ + \delta_{t+1} &\geqslant \delta'^0_+ + \delta_- \quad \text{by Corollary 2·6·7} \\ &\geqslant \delta'^0_+ + \delta'_- + \delta_+ \quad \text{by Lemma 2·6·17} \\ &\geqslant \alpha + \beta + \epsilon \quad \text{by the above} \\ &> \delta_i \quad \text{by hypothesis.} \end{aligned}$$

Hence $\delta'^0_+ > \delta_i - \delta_{t+1} = \lambda(B_i) - \lambda(B_{t+1}) > 0$. This means that $\lambda(B_i) - \lambda(B_{t+1}) \notin \Delta'^0_+$, so $\lambda(B_i) - \lambda(B_{t+1}) + B'$ a-discords with B', so B'_i a-discords with B'_{t+1}. Thus there is an a-format A that divides B'_i (and hence B_i) and discords with B'_{t+1} (and hence with B_{t+1}), so $\lambda(A) < \rho(B'_{t+1})$. Hence $\rho(A) = \lambda(A) + \alpha < \rho(B'_{t+1}) + \alpha = \rho(A_*)$ so $A << A_*$. By Lemma 2·6·14, A discords with B_t. This shows that B_i a-discords with B_t so $\delta_i \notin \Delta^0$. By Lemma 2·6·6, $\delta_i \in \Delta^0_-$ and thus $\delta_i \in \Delta_-$, as desired. ∎

2·6·19. LEMMA. *For any a-format A, if A breaks T then $B_t << A$.*

Proof. Suppose to the contrary that there is an a-format A breaking T with $\rho(A) \leqslant \rho(B_t)$. By Lemma 2·6·12, A discords with two a-concordant elements $B_{i_1} << B_{i_2}$ of T_b. By Lemma 2·6·11, $t \leqslant i_1 < i_2$. Since A discords with B_{i_2},

$$\lambda(B_{i_2}) < \rho(A) \leqslant \rho(B_t) = \lambda(B_t) + \beta,$$

which means that $\delta_{i_2} < \beta$. Thus $0 \leqslant \delta_{i_1} < \delta_{i_2} < \beta$. By Lemma 2·6·18, $\delta_{i_2} \in \Delta_-$. Now

$$\delta_{i_2} - \delta_{i_1} = \lambda(B_{i_2}) - \lambda(B_{i_1}) \in \Delta^0 \subseteq \Delta^0_+ \quad \text{so} \quad \delta_{i_2} - \delta_{i_1} \geqslant \delta^0_+.$$

If $i_1 = t$ then $\delta_{i_1} = 0$ and $\delta_{i_2} \in \Delta^0$, contradicting $\delta_{i_2} \in \Delta_-$. Thus $i_1 > t$, and by the same argument as for δ_{i_2} we have $\delta_{i_1} \in \Delta_-$, and thus $\delta_{i_1} \geqslant \delta_-$. Hence

$$\delta_{i_2} = (\delta_{i_2} - \delta_{i_1}) + \delta_{i_1} \geqslant \delta^0_+ + \delta_- \geqslant \alpha + \beta$$

by Corollary 2·6·10. This contradicts $\delta_{i_2} < \beta$. ∎

2·6·20. LEMMA. *There is a unique a-format A which discords with B_t and breaks T. Moreover A concords with all $B_i \neq B_t$ in T_b, and divides B_{t+1}.*

Proof. By Lemma 2·6·6, $\delta_{t+1} \in \Delta_-$ which means that $\delta_{t+1} \notin \Delta_+$, so there is an a-factor A of B_{t+1} discording with B_t. Thus A discords with B_t, breaks T and divides B_{t+1}.

Suppose that A' is any a-format breaking T and discording with B_t. By Lemma 2·6·5, A' divides some B_i with $i \geqslant t+1$. By Lemma 2·6·19, A and A' overlap the right margin of B_t, and if $A \neq A'$ then A, A' must discord and $i \neq t+1$. Notice $\lambda(A) < \rho(B_t)$ and thus

$$\lambda(\delta_+ + A) = \delta_+ + \lambda(A) < \delta_+ + \rho(B_t) < \delta_- + \rho(B_t) \leqslant \delta_{t+1} + \rho(B_t) = \rho(B_{t+1}).$$

Hence $\delta_+ + A$ overlaps B_{t+1}; but we know $\delta_+ + A$ concords with B_{t+1}, so actually divides B_{t+1}. Similarly, $\lambda(\delta_+ + A') \leqslant \delta_{t+1} + \rho(B_t) < \delta_i + \rho(B_t) = \rho(B_i)$ and hence $\delta_+ + A'$ divides B_i. Now $\delta_+ + A$ discords with $\delta_+ + A'$ so discords with B_i; thus $\delta_+ + A$ breaks T. Among all $\delta \in \Delta_- \cup \{0\}$, choose the largest one such that the a-format $A_\delta = -\delta + \delta_+ + A$ breaks T. Then A_δ concords with B_{t+1} by definition of Δ_-. By Lemma 2·6·19, $B_t << A_\delta$. Thus $\rho(A_\delta) \geqslant \rho(B_t) \geqslant \lambda(B_{t+1})$, but $\lambda(A_\delta) \leqslant \lambda(A_0) \leqslant \rho(B_{t+1})$, so A_δ must divide B_{t+1}. Now $\delta \neq \delta_+$ so $A_\delta \neq A$; also, A_δ, A both divide B_{t+1} so they do not overlap, and by Lemma 2·6·19, $A <<< A_\delta$ and $B_t <<< A_\delta$. By Lemma 2·6·12, A_δ discords with two a-concordant elements $B_{i_1} << B_{i_2}$ of T_b. Moreover, $i_1 > t$, $i_1 \neq t+1$, so $t+1 < i_i < i_2$, and by Lemma 2·6·3, there exists $\delta' = \lambda(B_{i_2}) - \lambda(B_{i_1}) \in \Delta^0$ such that $-\delta' + A_\delta = A_{\delta+\delta'}$ divides B_{t+1} and discords with B_{i_1}. Thus $A_{\delta+\delta'}$ breaks T; this contradicts the maximality of δ and proves the uniqueness of A.

Now suppose A discords with some $B_i \neq B_t$ in T_b, so $i \neq t, t+1$. By Lemma 2·6·11, $i > t-1$ and thus $i \geqslant t+2$. Since A discords with B_t $\lambda(A) < \rho(B_t)$, and since A discords with B_i, $\lambda(B_i) < \rho(A)$. Thus

$$\delta_i = \lambda(B_i) - \lambda(B_t) = \lambda(B_i) - \rho(B_t) + \beta < \rho(A) - \lambda(A) + \beta = \alpha + \beta.$$

By Lemma 2·6·18 $\delta_i \in \Delta_-$, which means that some a-factor A' of B_i discords with B_t. By the uniqueness $A' = A$ so A divides B_i, a contradiction. ∎

2·6·21. LEMMA. *For each $B_i \in T_b$ if $t+1 < i$ then $A_* <<< B_i$.*

Proof. Suppose to the contrary that $\lambda(B_i) < \rho(A_*) = \lambda(B_{t+1}) + \beta - \delta_+ + \epsilon + \alpha$. Then

$$\lambda(B_i) - \lambda(B_{t+1}) < \beta - \delta_+ + \epsilon + \alpha \leqslant \delta_-^0,$$

by Lemma 2·6·9, so $\lambda(B_i) - \lambda(B_{t+1}) \notin \Delta_-^0$, which means that

$$B_{t+1} = -(\lambda(B_i) - \lambda(B_{t+1})) + B_i$$

a-discords with B_i. Let A' be the leftmost a-factor of B_{t+1} discording with some $B_{i'}$, $i' > t+1$. By Lemma 2·6·20, A' concords with B_t and then by Lemma 2·6·12, A' discords with two a-concordant elements of T_b. By Lemma 2·6·3 this gives a contradiction to the choice of A'. ∎

Proof of Theorem 2·6. Take A as in Lemma 2·6·20 and consider $T^+ = T_A^+$. By Corollary 2·6·15 $A << A_*$, so by Lemma 2·6·21 the only multiple of A in T_b is B_{t+1}. Thus $T^+ = \{A\} \cup (T \setminus \{B_{t+1}\})$. By Lemma 2·6·19, A overlaps the right margin of B_t. By Lemmas 2·6·15 and 2·6·21, B_t is the rightmost b-format in a component of T^+, since no element of T^+ has both $\lambda(A_*)$, $\lambda(A_*)+1$ in its support. By Lemma 2·6·20, it is now easy to see that (a 1), (a 2) hold with T replaced with T^+. ∎

2·7. LEMMA. *Let $b \in X^* \setminus X$, and let $a_1, \ldots, a_z$ be the elements of X occurring in b, listed so that $|a_1| \leqslant \ldots \leqslant |a_z|$. Let T be a $b, a_1, \ldots, a_z$-tableau and write T_j for T_{a_j}, $j \in [1, z]$. Then T is satisfactory if there is a b-tableau U and an $i \in [1, z]$ such that the following hold:*

(0) *For all $j \in [1, i-1]$, $T_j = \varnothing$.*

(a) *No component of* $T_{i+1} \cup \ldots \cup T_z$ *overlaps two elements of* U.

(a 1) *No element of* T_i *discords with two* a_i*-concordant elements of* T_b.

(a 2) *If* $\delta_-(a_i, b) \geqslant \delta_+(a_i, b)$ *(repectively,* $\delta_-(a_i, b) < \delta_+(a_i, b)$*) then any element of* T_b *whose right (respectively, left) margin is overlapped by* T_i *is the rightmost (respectively, leftmost) b-format in a component of* $T_b \cup T_i$.

(b) *No element of* $T_{i+1} \cup \ldots \cup T_z$ *divides an element of* U.

(b 1) *No element of* T_i *divides an element of* T_b.

(c) *For all* $j \in [i+1, z]$, $U \cup T_{i+1} \cup \ldots \cup T_j$ *is* a_j*-unbreakable.*

(c 1) $T_b \subseteq U$, *and each element of* T_i *breaks* U.

(d) $T_{i+1} \cup \ldots \cup T_z$ *is a strong chain of atomic formats.*

Proof. We proceed by induction on $w(T)$. If $w(T) = 0$ then T is empty, so satisfactory. Thus we may assume $w(T) > 0$ and that the result is true for all tableaux of smaller width. Since the hypotheses of the lemma are satisfied by subsets of T (with the same U and i), we see that all proper subsets of T are satisfactory.

By symmetry we may assume $\delta_-(a_i, b) \geqslant \delta_+(a_i, b)$, as in Remark 2·6·2.

Consider first the case where $T_b \cup T_i$ is a_i-breakable. By Theorem 2·6, there is an a_i-format A which breaks $T_b \cup T_i$ and such that (a 1) and (a 2) are satisfied with T^+ in place of T, where $T^+ = T_A^+$. Here T_A^- is a proper subset of T, so is satisfactory by the induction hypothesis; by 2·4·4 it suffices to show that T^+ is satisfactory. As it is clear that $w(T^+) < w(T)$, it suffices to show that the conditions of the lemma hold for T^+, with the same U and i. Now (a), (b), (c), (d) hold for T^+ since $T_{i+1}^+ \cup \ldots \cup T_z^+$ is a subset of $T_{i+1} \cup \ldots \cup T_z$. Clearly (0) holds for T^+. By (b 1) and (c 1) for T, each element of $T_b \cup T_i$ divides an element of U; since A breaks $T_b \cup T_i$ we see that A divides some element of $T_b \cup T_i$ (and hence some element of U) and discords with some other element of $T_b \cup T_i$ (and hence discords with some element of U). Thus A breaks U. Since $T_i^+ \subseteq T_i \cup \{A\}$ and $T_b^+ = \{B \in T_b | A \text{ does not divide } B\}$, it follows that (b 1) and (c 1) hold for T^+. Thus T is satisfactory in this case.

It remains to consider the case where $T_b \cup T_i$ is a_i-unbreakable. Here the elements of T_b are pairwise a_i-concordant. We shall show that no component of T_i overlaps two elements of T_b. Thus, suppose to the contrary that a component T' of T_i overlaps two elements $B_1 < < B_2$ of T_b. Let A be the rightmost element of T' which overlaps B_1. By (a 2), A does not overlap the right margin of B_1, since B_1 and B_2 lie in the same component of $T_b \cup T_i$. Thus $\rho(A) \leqslant \rho(B_1)$ and $\rho(B_1) + 1 \notin \sigma(T')$. Hence

$$\rho(A) = \rho(T') > \lambda(B_2),$$

and A overlaps the a_i-concordant elements B_1 and B_2, contradicting (a 1) or (b 1). Thus no component of T_i overlaps two elements of T_b. We claim that the statements (a′), (b′), (c′), (d′), obtained from (a), (b), (c), (d) by replacing $i+1$ with i, and U with T_b, are all true. To see (a′), notice that by (a) and (c 1), no component of $T_{i+1} \cup \ldots \cup T_z$ overlaps two elements of $T_i \cup T_b$, and we have just seen that no component of T_i overlaps two elements of T_b, so no component of $T_i \cup T_{i+1} \cup \ldots \cup T_z$ overlaps two elements of T_b; that is, (a′) holds. By (b), (c 1), and (b 1), we see that (b′) holds. It is clear from (c) and (c 1) that $T_b \cup T_i \cup T_{i+1} \cup \ldots \cup T_j$ is a_j-unbreakable if $j \in [i+1, z]$; for $j = i$, we know $T_b \cup T_i$ is a_i-unbreakable, so (c′) holds. To see (d′), suppose that some $A \in T_i$ overlaps some $A' \in T_{i+1} \cup \ldots \cup T_z$, so A' is an a_j-format for some $j > i$, and thus $w(A) \leqslant w(A')$. By (c 1), A divides an element B of U and discords with another element B' of U. Thus A' overlaps A, which divides B, so A' overlaps B; but by (a), A' does not overlap B', while A does, so $\sigma(A) \nsubseteq \sigma(A')$. Hence, either $\lambda(A) < \lambda(A')$ (in which case $\rho(A) = \lambda(A)+w(A) < \lambda(A')+w(A') = \rho(A')$) or $\rho(A') < \rho(A)$ (in which case $\lambda(A') = \rho(A')-w(A') < \rho(A)-w(A) = \lambda(A)$); that is, either $A << A'$ or $A' << A$. It follows that $<<$ is a total order on $T_i \cup T_{i+1} \cup \ldots \cup T_z$, so (d') holds.

If $i > 1$, then the hypotheses of the lemma still hold if i is replaced by $i-1$ and U is replaced by T_b, since the analogues of (0), (a 1), (a 2), (b 1), (c 1) are trivially true, with $T_{i-1} = \varnothing$. Without loss of generality, we may assume that i was chosen to be as small as possible; thus $i = 1$. Write $T' = T_1 \cup \ldots \cup T_z$. Here we have the following statements holding.

(a') No component of T' overlaps two elements of T_b.

(b') No element of T' divides an element of T_b.

(c') For each $j \in [1,z]$, $T_b \cup T_1 \cup \ldots \cup T_j$ is a_j-unbreakable.

(d') T' is a strong chain of atomic formats.

By (c'), T_b is a_j-concordant for all $j \in [1,z]$, so T_b is a staircase. If T_b and T' do not overlap, then by (d'), each component of T is either a staircase or a (strong) chain of atomic formats, so T is satisfactory by 2·4·1, 2·4·2 and 2·4·3. Thus we may assume that T' overlaps some element B of T_b. From (b'), and the $j = z$ case of (c'), we see that the a_z-factors of B (which exist since a_z occurs in b) do not overlap T'. Thus there exist atomic factors of B overlapping T', and other atomic factors of B not overlapping T'. Hence there must exist adjacent atomic factors A, A_0 of B, such that A overlaps an element A' of T', while A_0 does not overlap T'; here, either

$$\rho(A) = \lambda(A_0) \quad \text{or} \quad \rho(A_0) = \lambda(A),$$

and by symmetry, we may assume $\rho(A) = \lambda(A_0)$. Since T_b is a staircase, A does not discord with any element of T_b. Thus, if A overlaps any element B' of T_b then B' is divisible by A, which overlaps A'; by (a'), $B' = B$. Thus B is the unique element of T_b overlapping A, so $T_A^+ = \{A\} \cup (T \backslash \{B\})$. Let V be the union of all components of T' that overlap A; by (a'), $\{A\} \cup V$ is a component of T_A^+. We know that A_0 does not overlap V or A, and $A <\!<\!< A_0$ so $V <\!<\!< A_0$ and $\rho(V) \leqslant \lambda(A_0) = \rho(A)$. If A is an a_j-format, say, then $T_b \cup V$ is not a_j-concordant, so by (c'), if A'' is any element of V then A'' is an $a_{j''}$-format for some $j'' > j$. Thus $w(A) \leqslant w(A'')$. Now $\rho(A'') \leqslant \rho(V) \leqslant \rho(A)$, so that either $\rho(A'') < \rho(A)$ (in which case

$$\lambda(A'') = \rho(A'')-w(A'') < \rho(A)-w(A) = \lambda(A))$$

or $\rho(A'') = \rho(A)$ (in which case

$$\lambda(A'') = \rho(A'')-w(A'') \leqslant \rho(A)-w(A) = \lambda(A)).$$

Thus $A'' <\leqslant A$, and $V \cup \{A\}$ is a chain of atomic formats. All the other components of T_A^+ are proper subsets of T, and these are satisfactory by the induction hypothesis. Thus T_A^+ is satisfactory by 2·4·3. By the induction hypothesis, T_A^- is satisfactory, so T is satisfactory by 2·4·4. ∎

2·8. Theorem. *If $b \in X^*$ then every b-tableau is satisfactory.*

Proof. Let T be a b-tableau. If $b \in X$, then T is a (strong) chain of atomic formats, so is satisfactory by 2·4·2. If $b \in X^* \backslash X$, then T is satisfactory by Lemma 2·7, with

$U = T_b$ and $i = z$; the conditions are vacuously true since the sets T_j and $[i+1, z]$ are all empty. ∎

This completes the proof of Theorem 1·19.

2·9. *Notes*. Theorem 2·6 is closely modelled on lemma B″ of [**6**], the new steps being 2·6·9, 11–15, 20, 21; the proof of Lemma 2·7 is essentially contained in the proofs of lemma 5·2 and lemma A of [**6**].

REFERENCES

[**1**] P. M. COHN. *Free Rings and their Relations*. London Math. Soc. Monographs, no. 2 (Academic Press, 1971).

[**2**] W. DICKS. On one-relator associative algebras. *J. London Math. Soc.* (2) **5** (1972), 249–252.

[**3**] W. DICKS. Idealizers in free algebras. Ph.D. Thesis, London, 1974.

[**4**] W. DICKS. A free algebra can be free as a module over a non-free subalgebra. *Bull. London Math. Soc.* **15** (1982), 373–377.

[**5**] W. DICKS. On the cohomology of one-relator associative algebras. *J. Algebra* (in the Press).

[**6**] V. N. GERASIMOV. Distributive lattices of subspaces and the equality problem for algebras with a single relation. *Algebra i Logika* **15** (1976), 384–435 [Russian]. (English translation: *Algebra and Logic* **15** (1976), 238–274.)

[**7**] J. LEWIN and T. LEWIN. On ideals of free associative algebras generated by a single element. *J. Algebra* **8** (1968), 248–255.

[**8**] L. MAKAR-LIMANOV. On algebraically closed skew fields. *J. Algebra* (in the Press).