

PSEUDO-DISCRIMINANT AND DICKSON INVARIANT

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1. Let E be a vector space of finite dimension over a field K . To a bilinear symmetric form $f(x, y)$ defined over $E \times E$ is attached classically the notion of *discriminant*: it is an element of K which is not entirely defined by f ; however, it is entirely determined when in addition a basis of E is chosen, and when the basis is changed, the discriminant is multiplied by a *square* in K . More precisely, let u be a linear mapping of E into E , and let $f_1(x, y) = f(u(x), u(y))$ the form "transformed" by u ; if $\Delta(f)$, $\Delta(f_1)$ are the discriminants of f and f_1 with respect to the *same* basis of E , and $D(u)$ the determinant of u with respect to that basis, then one has the classical relation

$$(1) \quad \Delta(f_1) = (D(u))^2 \Delta(f).$$

When K has characteristic $\neq 2$, the preceding results may be expressed in terms of the "quadratic form" $f(x, x)$ associated to $f(x, y)$. However, when K has characteristic 2, the one-to-one association between bilinear symmetric forms and quadratic forms no longer subsists. More precisely, to a given *alternate* symmetric form $f(x, y)$ (that is, $f(x, x) = 0$ for all $x \in E$) is associated a whole family of quadratic forms $Q(x)$, satisfying the fundamental identity

$$(2) \quad Q(x+y) = Q(x) + Q(y) + f(x, y)$$

and to all these Q is associated the same discriminant of f (with respect to a given basis).

Now C. Arf [1] has introduced an element $\Delta(Q)$ attached to Q and to a given *symplectic basis* of E (with respect to the form f) which we shall call the *pseudo-discriminant* of Q . He proved moreover that under a change of symplectic basis, $\Delta(Q)$ is transformed in the following way: let $\mathcal{P}$ be the homomorphism $\xi \rightarrow \xi + \xi^2$ of the additive group K into itself; then the pseudo-discriminants of Q with respect to two different symplectic bases have a *difference* which has the form $\mathcal{P}(\lambda)$. Arf's proof is rather lengthy and proceeds by induction on n . We propose to show how the pseudo-discriminant is related to the *Clifford algebra* of Q in a way which parallels the well-known relation between the discriminant of f and the Clifford algebra of f over a field of characteristic $\neq 2$. At the same time, this will clear up the origin of a curiously isolated result obtained by L. E. Dickson for the orthogonal

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group $O_n(K, Q)$ over a finite field of characteristic 2: the transformations u of that group are defined by the condition $Q(u(x))=Q(x)$, and Dickson showed [4, p. 206] that a certain bilinear polynomial $D(u)$ in the elements of the matrix of u (with respect to a symplectic basis), turns out to be always equal to 0 or 1 for elements of $O_n(K, Q)$ (the first case occurring if and only if u is a product of an *even* number of transvections of $O_n(K, Q)$; see [6, p. 301]). Now the connection with the Clifford algebra which we mentioned above leads one in a natural way to form the polynomial $D(u)$ for an arbitrary *symplectic* transformation u ; if $Q_1(x)=Q(u(x))$ is then the "transformed" of Q by u , and $\Delta(Q)$, $\Delta(Q_1)$ and $D(u)$ are computed with respect to the *same* symplectic basis, we will prove the following identity, which can be considered as the counter-part of (1)

$$(3) \quad \Delta(Q_1) = \Delta(Q) + \mathcal{P}(D(u)).$$

Dickson's result follows obviously from this relation.

2. We shall always suppose that the alternate form f is nondegenerate, which implies that $n=2m$ is *even*, and that the forms Q associated with f are *nondefective* [5, p. 39-40]. For the definition of the *Clifford algebra* $C(Q)$ of a quadratic form Q associated to f , we refer the reader to [3] or [6]. If $(e_i)_{1 \leq i \leq n}$ is a symplectic basis of E , such that

$$f(e_i, e_{m+j}) = \delta_{ij}, \quad f(e_i, e_j) = 0, \quad f(e_{m+i}, e_{m+j}) = 0 \quad 1 \leq i, j \leq m,$$

then the unit element and the e_i ($1 \leq i \leq n$) constitute a system of generators for $C(Q)$, with the relations

$$(4) \quad \begin{cases} e_i^2 = Q(e_i), & e_{m+i}^2 = Q(e_{m+i}), & e_i e_j = e_j e_i \\ e_{m+i} e_{m+j} = e_{m+j} e_{m+i}, & e_i e_{m+j} + e_{m+j} e_i = \delta_{ij} \end{cases} \quad 1 \leq i, j \leq m.$$

From this it follows that $C(Q)$ is an algebra of rank 2^{2m} over K . Moreover, the elements of *even degree* of $C(Q)$ (generated by the products of an even number of the e_i 's) constitute a subalgebra $C^+(Q)$ of rank 2^{2m-1} over K , and it can be shown that the center Z of that algebra has rank 2 over K [3, p. 44]. Now, it is readily verified from (4) that the element

$$(5) \quad z = e_1 e_{m+1} + e_2 e_{m+2} + \cdots + e_m e_{2m}$$

commutes with all products $e_i e_j$, and therefore constitutes with the unit element a basis for Z over K . From (4) it follows that $z^2 + z = \Delta(Q)$, where

$$(6) \quad \Delta(Q) = Q(e_1)Q(e_{m+1}) + Q(e_2)Q(e_{m+2}) + \cdots + Q(e_m)Q(e_{2m})$$

is precisely the *pseudo-discriminant* of Q relative to the basis (e_i) considered by Arf. Now the fact that $\Delta(Q)$ has the form $\mathcal{P}(\lambda)$ expresses the fact that the equation $z^2+z=\Delta(Q)$ has a solution in K , in other words, that Z is not a field. When Z is a field, it is a separable quadratic field over K , and if it is generated by the roots of any equation $t^2+t=\mu$, then μ and $\Delta(Q)$ differ by an element of the form $\mathcal{P}(\lambda)$ [2, p. 177, exerc. 8]. This proves immediately that when the pseudo-discriminant is computed with respect to two different symplectic bases, the values obtained have a difference of the form $\mathcal{P}(\lambda)$.

3. We are now going to make the above result more precise by proving (3). If u is a symplectic transformation, the elements $u(e_i)$ ($1 \leq i \leq 2m$) constitute again a symplectic basis for E , hence also a system of generators for the Clifford algebra $C(Q)$, satisfying relations similar to (4) (with $Q(u(e_i))$ replacing $Q(e_i)$). The element

$$(7) \quad z' = u(e_1)u(e_{m+1}) + \cdots + u(e_m)u(e_{2m})$$

constitutes therefore, with the unit element, a basis for Z over K , in other words, z' has the form $p+qz$, where p, q are in K . Now it is easy to compute z' as a function of the coefficients of the matrix of u with respect to (e_i) : let

$$\begin{aligned} u(e_i) &= \sum_{j=1}^m a_{ij}e_j + \sum_{j=1}^m b_{ij}e_{m+j} \\ u(e_{m+i}) &= \sum_{j=1}^m c_{ij}e_j + \sum_{j=1}^m d_{ij}e_{m+j}. \end{aligned}$$

Let on the other hand $Q(e_i) = \alpha_i$, $Q(e_{m+i}) = \beta_i$. Then z' is a linear combination of elements $e_h e_k$, and it follows from (4) and (5) that we need only consider among those elements the squares e_i^2 and the products $e_i e_{m+i}$, $e_{m+i} e_i$ since we know in advance that z' can contain no other elements from the basis of $C^+(Q)$. We thus obtain

$$(8) \quad p = \sum_{i=1}^m \sum_{j=1}^m (\alpha_j a_{ij} c_{ij} + \beta_j b_{ij} d_{ij} + b_{ij} c_{ij})$$

$$(9) \quad q = \sum_{i=1}^m (a_{ij} d_{ij} + b_{ij} c_{ij}).$$

But it follows, from the fact that the transposed matrix of u is again the matrix of a symplectic transformation, that $q=1$. The expression on the right of (8) is the *Dickson invariant* $D(u)$; as the relation $z' = p + z$ yields $z'^2 + z' = z^2 + z + p^2 + p$, the identity (3) follows immediately from (6).

4. We cannot expect, of course, that the mapping $u \rightarrow D(u)$ should be a homomorphism of the symplectic group $Sp_{2m}(K)$ into the additive group of K , if only because we know that $Sp_{2m}(K)$ is a simple group. However, there are some relations between the Dickson invariants of

two symplectic transformations u , v and the Dickson invariant of their product. In fact, it follows immediately from the expression of z' obtained in § 3, that we have

$$(10) \quad D(vu) = D(u) + D_a(v)$$

where $D(u)$ and $D(vu)$ are the Dickson invariants of u and vu with respect to the basis (e_i) , and $D_a(v)$ the Dickson invariant of v with respect to the basis $(u(e_i))$. This general formula takes a simpler shape when u is an *orthogonal transformation*, because then $Q(u(e_i)) = Q(e_i)$ for $1 \leq i \leq 2m$; on the other hand, the matrix of v with respect to the basis $(u(e_i))$ is the same as the matrix of $u^{-1}vu$ with respect to (e_i) , and we thus obtain

$$(11) \quad D(vu) + D(u^{-1}vu) = D(u).$$

But in this identity we can replace v by uvu^{-1} ; therefore we also have

$$(12) \quad D(uv) = D(u) + D(v)$$

when u is an *orthogonal transformation*, v an arbitrary *symplectic transformation* ($D(u)$ being equal to 0 or 1, as recalled above).

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Added in proof (November 1955): Since this paper was submitted for publication, the following papers, containing substantially the result of § 2, have appeared:

M. Kneser, *Bestimmung des Zentrums der Cliffordschen Algebren einer quadratischen Form über einem Körper der Charakteristik 2*, J. Reine Angew. Math., **193** (1954), 123-125.

E. Witt, *Über eine Invariante quadratische Formen mod. 2*, J. Reine Angew. Math., **193** (1954), 119-120.

E. Witt and W. Klingenberg, *Über die Arfsche Invariante quadratischer Formen mod. 2*, J. Reine Angew. Math., **193** (1954), 121-122.