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COBORDISM EXACT SEQUENCES FOR DIFFERENTIAL AND COMBINATORIAL MANIFOLDS

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Several exact sequences have been established recently which relate cobordism groups of various types [1], [3], [8]. These have all been established in the differential case. The main object of this paper is to establish the same results in the combinatorial case. We first define a combinatorial analogue to Thom's notion of t-regularity, and prove some basic lemmas on submanifolds representing double coverings (Part I).

After this, we find that the same proofs are valid as in the differential case, so we treat the two cases together. This gives us an opportunity to collect together all the geometrical arguments used in the proofs of these theorems, which have hitherto been somewhat scattered in the literature. We establish the three exact sequences in Theorem 1; the proofs are so simple that one could axiomatise a category of manifolds in which they work. The proof would be valid for topological manifolds if only some corresponding version of Lemma 7 could be established in that case.

We then give a proof, in the same order of ideas, that one of our maps is a derivation. The behaviour of the others with respect to multiplication is rather more complicated, and we hope to return to it in a subsequent paper.

We shall assume known the definitions of combinatorial and of differential manifolds (which may have boundary). All manifolds occurring in this paper will be compact. A compact manifold without boundary is called closed. To avoid logical difficulties concerning sets, we may regard all manifolds as imbedded in a finite dimensional subspace of a given Hilbert space; however, it will be more convenient to state our constructions for abstract manifolds. We shall usually denote manifolds by the symbols: V, W, M, N.

PART I. SUBMANIFOLDS AND DOUBLE COVERINGS

Elementary lemmas

We recall that there is a (1–1) correspondence between elements of $H^1(V, \mathbf{Z}_2)$, (which may be regarded as homomorphisms of $\pi_1(M)$ to $\mathbf{Z}_2$), and double coverings of V. Now a submanifold W of V of unit codimension determines a dual cohomology class in $H^1(V, \mathbf{Z}_2)$. W thus determines a double covering of V, which may be described as follows. The covering is trivial on $V - W$, so has two sheets there, and these are cross-joined along W. We shall call this the double covering of V defined by W. In this part we prove that conversely, to each double covering of V corresponds a submanifold W, to some extent unique; this is the sole property of differential and combinatorial manifolds needed for the proofs in the second part of this paper. We first deal with some easy lemmas.

For a submanifold W of codimension 1 in a manifold V of any sort, one can define, using the fact that W locally separates V into two sides, a double covering of W describing this separation, and generalising the normal bundle in the differential case. We call this the *normal covering* of W in V.

LEMMA 1. *The restriction to W of the double covering of V defined by W is isomorphic to the normal covering of W in V.*

PROOF. Designate the sheets over $V - W$ as upper and lower. Then a point of the covering lying over W, if moved slightly to one side of W, appears on the upper sheet; if moved to the other, on the lower. We regard it as determining that side of W (defined locally) above which it appears on the upper sheet. Thus it determines one point of the normal covering. This correspondence defines an isomorphism of the two coverings.

Any manifold V admits a double covering corresponding to its two possible local orientations; this we call the *orientation covering* of V. Let W be a submanifold of V of unit codimension; over it we have the orientation covering, the normal covering in V, and a covering induced from the orientation covering of V. Now there exists a composition for double coverings which is well-known and may be described in various ways (e.g., by adding the corresponding cohomology classes).

LEMMA 2. *The composition of the orientation covering of W and the normal covering of W in V is the covering induced on W by the orientation covering of V.*

PROOF. This simply follows from the well-known fact that locally, designating an orientation of W and one side of W in V gives rise to an orientation of V, which is changed if either of these two is.

We now make a precise definition of the word 'submanifold', as it will be used for the rest of this paper. We shall call a map between two manifolds *smooth* if, in the differential case, it is infinitely differentiable, or in the combinatorial case, it is piecewise linear. Let V be an n-dimensional manifold.

DEFINITION. A closed subset W of V is a *submanifold of codimension*

r if for each point x of W there is a neighbourhood U of x in V, and a smooth homeomorphism h of U on E^n taking $U \cap W$ on E^{n-r}.

Here, E^n is euclidean space of dimension n, with coordinates (x_i); E^{n-r} is the subspace defined by setting the last r coordinates equal to zero. If V has a boundary, we permit also neighbourhoods mapped to (E^n_+, E^{n-r}_+), where the subscript $+$ denotes the subset defined by $x_1 \geqq 0$. Thus $\partial V \cap W = \partial W$, where ∂V denotes the boundary of the manifold V.

In the case $r = 1$, it follows that V cut along W is a manifold, with boundary determined by W — in fact, isomorphic to the normal covering of W. It is well-known that the boundary has a product neighbourhood; in the differential case this follows, using a riemannian structure on V, from standard properties of geodesics; in the combinatorial case, it is a theorem of Whitehead [10]. Sewing V back together, we see that W has a tubular neighbourhood in V in the following sense.

DEFINITION. A *tubular neighbourhood* of W in V is a smooth homeomorphism as a neighbourhood of W in V of the bundle over W associated to the normal covering of W in V, with fibre $[-1, 1]$, and group operating by change of sign. W is identified with the zero cross-section. For definition of associated bundles see Steenrod [6]; in this case it is simply the mapping cylinder of the projection map of the covering. Observe that the bundle has natural differential resp. piecewise linear structure, derived from the local product structure, and the word smooth makes reference to this.

The existence of a tubular neighbourhood may be similarly seen if W has a boundary — we shall not give the proof, since we have omitted the details in the other case, and we do not need this result.

Regularising maps

We are now ready to introduce the methods of regularising maps on submanifolds of unit codimension. In the differential case, let N be a submanifold of M, V a manifold, $f\colon V \to M$ a smooth map. Denote the tangent space to V at x by V_x.

DEFINITION. f is *t-regular* on N if for every x in V such that $f(x) = y$ is in N, we have $df(V_x) + N_y = M_y$.

In the combinatorial case let L be a subset of the simplicial complex K.

DEFINITION. *L is in general position for r in K* if for every closed simplex σ of K meeting L, $\sigma \cap L$ is the intersection of σ with a hyperplane of codimension r.

We now make the important observation that if N is a submanifold of unit codimension in M, then M has a triangulation with N in general position for 1. For it is clearly sufficient to show this for a tubular neighbourhood of N (and then extend the triangulation arbitrarily to the rest of M). We first triangulate N, and remark that over each simplex we have a product with the interval $[-1, 1]$, and give each of these a product triangulation (see [4 Ch. 2]), making sure that these fit together. This is easily accomplished, e.g., as follows. To give a product triangulation we must order the vertices of each simplex. It is a little simpler to consider the orderings of the simplices of the double covering of N. The orderings of the two simplices over a simplex of N are the reverse of each other (since the product is 'turned the other way up'). Subject to this, and the condition that the ordering of a face of a simplex is induced by that of the simplex, any ordering is possible. We find one, e.g., by choosing one vertex of the covering over each vertex of N, ordering these arbitrarily; ordering the other vertices in the order corresponding to the reverse of this, and placing them after the vertices first chosen. This total ordering of the vertices of the covering of N induces the desired ordering of the vertices of each simplex. We triangulate the bundle accordingly, and N, or the zero cross-section, is then in general position.

We now give the technical lemmas which allow us to prove our results on induced submanifolds.

LEMMA 3_c. *If L is in general position for r in K, g: $V \to K$ a simplicial map, then $g^{-1}(L)$ is in general position for r in V.*

LEMMA 3_d. *If N is a submanifold of M, f: $V \to M$ a map, we can approximate f by a smooth map g, t-regular on N.*

LEMMA 4_c. *W is a submanifold of unit codimension in V if and only if V has an allowed triangulation with W in general position for* 1.

LEMMA 4_d. *If N is a submanifold of M, g: $V \to M$ a smooth map, t-regular on N, then $g^{-1}(N) = W$ is a submanifold of V.*

The differential Lemmas 3_d and 4_d are results of Thom [7, Th. I.5]. The second part of Lemma 4_c follows from the remark above. The proofs of the first part (for arbitrary codimension) and of Lemma 3_c we defer for the moment. (The second part of Lemma 4_c is false for arbitrary codimension.)

We can now prove

LEMMA 5. *Any double covering p of V can be defined by a submanifold W (of unit codimension in V).*

PROOF. Since real projective space of sufficiently high dimension is a universal space for the group $\mathbf{Z}_2$ (Steenrod [6]), we may first represent the covering p by a map f: $V \rightarrow P_A(\mathbf{R})$ for sufficiently large A. We fix A, and write $M = P_A(\mathbf{R})$, $N = P_{A-1}(\mathbf{R})$. Then N is a submanifold of M of unit codimension. Now in the differential case, by Lemma 3_d we approximate f by a map g, t-regular on N, and deduce by Lemma 4_d that $g^{-1}(N) = W$ is a submanifold of V. In the combinatorial case, by Lemma 4_c we triangulate M with N in general position for 1; we may suppose V triangulated, and approximate f by a simplicial map g. Then by Lemma 3_c, $g^{-1}(N) = W$ is in general position for 1 in V, so by Lemma 4_c again, it is a submanifold.

The covering p is induced by g from the standard double covering of M. This consists of two sheets over $M - N$, cross-joined along N. Since by our choice of g, $g(W)$ meets N transversely (i.e., the two sides of W in V map into locally different sides of N in M), p consists of two sheets over $V - W$, cross-joined along W. Thus W does indeed define p.

To obtain a uniqueness clause to Lemma 5, we first prove a converse. Let us say that g: $V \rightarrow M$ induces W in the sense of Lemma 5, if $W = g^{-1}(N)$, and g is smooth and either t-regular on N, or simplicial for some triangulation of M with N in general position.

LEMMA 6. *Let the submanifold W of V define the double covering p. Then there is a map g: $V \rightarrow M$ which induces W in the sense of Lemma* 5.

PROOF. First represent the restriction $p \mid W$ of p to W by a smooth map f: $W \rightarrow N$. The standard covering of N (which is the normal covering of N in M) then induces the covering $p \mid W$ of W. By Lemma 1, this is the normal covering of W in V. Now take tubular neighbourhoods $\bar{W}$, $\bar{N}$ of W, N in V, M. Since f carries the normal covering of W in V to that of N in M, f extends in a natural way to a smooth map $\bar{f}$ of $\bar{W}$ $\bar{N}$, (as these bundles are associated with the coverings). But $M - \bar{N}$ is contractible (it is, in fact, a cell), so we may now extend $\bar{f}$ to a smooth map g of V in M, with $g(V - \bar{W}) \subset (M - \bar{N})$. Then g is the required map. In the differential case, it is clear from our construction that g is t-regular on N. In the combinatorial case, we may triangulate W, N with f simplicial. Then give $\bar{N}$ the twisted product triangulation mentioned above, and $\bar{W}$ a twisted product triangulation compatible with it (i.e., the ordering of vertices of a simplex of $\bar{W}$ is subject to: $x < y$ if $f(x) < f(y)$). Then N is in general position in $\bar{N}$, and $\bar{f}$ is a simplicial map. The extension to V now presents no difficulty.

LEMMA 7. *Let p be a double covering of V, X a submanifold of ∂V defining $p \mid \partial V$. Then there is a submanifold W of V, defining p, with $\partial W = X$.*

PROOF. By Lemma 6, there is a map $g\colon \partial V \to M$ which induces X in the sense of Lemma 5. Extend g to a product neighbourhood $\partial V \times I$ of ∂V in V by $\bar{g}(v, t) = g(v)$. Here, I denotes the unit interval $[0, 1]$. Extend $\bar{g}$ to a map f of V in M, representing p. This is possible since g represents $p \mid \partial V$. Now apply the proof of Lemma 5: we remark that in each case modifying f to an f' which induces a submanifold W of V, defining p, can be done leaving f fixed on a neighbourhood of ∂V. For the t-regularity deformation (Lemma 3_d) is done in local steps; and the simplicial approximation theorem can be refined [11] to leave f fixed on a subcomplex where it is already simplicial. It follows that $W \cap \partial V = X$, and since W is a submanifold, this is ∂W.

Lemma 7 provides the sought 'uniqueness' of the W of Lemma 5. For if W, W' both define the covering p, we consider the covering q of $V \times I$ corresponding to p. Then $W \times 0 \cup W' \times I$ defines $q \mid \partial(V \times I)$, and by Lemma 7 we can find a submanifold Y of $V \times I$ with these as boundary. Hence W, W' are L-equivalent in the sense of Thom (*loc. cit.*), and W is unique up to L-equivalence. However, the form stated for Lemma 7 is that best adapted for our applications. We shall need one further slight extension of these results.

LEMMA 8. *If the normal covering of W in V is trivial, and W defines p, then p can be induced by a map of V to a circle S^1. Conversely, if p can be induced by a map of V in S^1, we may define p by a submanifold W with trivial normal covering in V.*

PROOF. We note that S^1 is simply the projective space of dimension 1, and the result follows by applying the proofs of Lemmas 5 and 6, taking M as a circle, and N as a point in it. For the normal covering of W in V is trivial if and only if it can be induced by a map in a point.

Proof of the combinatorial lemmas

It now remains only to prove the auxiliary Lemmas 3_c and 4_c. We define an *element-pair* to be a polyhedral pair, piecewise linearly equivalent to $(\sigma_{n-r} * \partial\sigma_r, \sigma_{n-r})$, where σ_i denotes a simplex of dimension i, and $*$ denotes the join. Then in Lemma 4_c, instead of giving an open neighbourhood of each point of W as a euclidean pair, it is clearly sufficient to give a closed neighbourhood U, which is an element-pair (and this holds also for bounded manifolds). Suppose then V a triangulated combinatorial manifold of dimension n, and W in general position for r in V.

PROOF OF LEMMA 4_c. It is clear that W is a piecewise linear subset of V. Let x be any point of W; we suppose it interior to a k-simplex α_k of

V, which meets W in a cell β_{k-r}. Then a neighbourhood of x is provided by the union of the n-simplices δ_n^j of V with α_k as a face. We shall choose U somewhat smaller. Specifically, for each simplex γ_{k+1}^i with α_k as face, pick a point p^i in $(\gamma_{k+1}^i - \alpha_k) \cap W$. Then for each δ_n^j there are just $n - k$ $(k + 1)$-faces of it which contain α_k, and we denote the simplex which is the join of the corresponding p^i by ζ_{n-k-1}^i, and its join to α_k by ε_n^i. Then ε_n^j is a neighbourhood of x in δ_n^j, and we take U to be the union of all the ε_n^j.

Let D be the union of all the ζ_{n-k-1}^j. Then U is the join $\alpha_k * D$. Moreover, D is contained in W (since the p^i are), and so $U \cap W$ is just $\beta_{k-r} * D$. So it remains only to prove $(\alpha_k * D, \beta_{k-r} * D)$ an element-pair. But (α_k, β_{k-r}) is merely a simplex with a plane section, and we can easily find a piecewise linear homeomorphism on the standard element-pair (which we may suppose to be concentric and lie in the same hyperplanes) by projecting the vertices of a suitable triangulation from x and extending to a simplicial map. D is (isomorphic to) the link of a simplex α_k in the combinatorial manifold V, hence is equivalent to a simplex boundary (or to a simplex, if x is on the boundary of V). Finally, the join of an element-pair with a simplex boundary (or simplex) is clearly another element-pair, so the result follows.

For Lemma 3_c, recall that L is in general position for r in the complex K, and that g: $V \to L$ is a simplicial map.

Proof of Lemma 3_c. If g is an imbedding, the lemma is easy, for we may identify V with $g(V)$, and then $g^{-1}(L) = L \cap V$ is clearly in general position for r. But we can reduce the general case to this. For replace K by $V \times K$, and g by f, where $f(x) = (x, g(x))$, so that f is an imbedding. Triangulate $V \times K$ as follows: first order the vertices of K; then order those of V compatibly (so that if $g(x) < g(y)$ then $x < y$). Then use the product triangulation [4, Ch. 2]. This makes f a simplicial map. Clearly $V \times L$ is in general position in $V \times K$, and $g^{-1}(L) = f^{-1}(V \times L)$, so the general result follows.

Part II. The Exact Sequences

Definitions of the cobordism groups and maps

Consider the set of closed manifolds of dimension k: (manifolds are to be understood in the same sense — differential or combinatorial — throughout) on it we introduce the relation: $V \backsim_2 V'$ if there is a (compact) $(k + 1)$-manifold M with boundary $V \cup V'$. If this condition is satisfied, V and V' are call *cobordant*, and M is said to provide a cobordism between them. The relation $\sim_2$ is clearly symmetric, it is reflexive since $\partial(V \times I)$ is the union of two copies of V, and it is transitive since if $\partial M = V \cup V'$, $\partial M' = V' \cup V''$, we may glue M to M' along V' giving a manifold M'' which is easily seen to be combinatorial (resp. differential) if M, M' are, and with $\partial M'' = V \cup V''$. Thus $\sim_2$ is an equivalence relation. We shall denote the equivalence class of V by $\{V\}$.

Disjoint union gives an addition clearly compatible with this relation, and the classes form a group. For the empty manifold acts as zero, and, since $V \cup V$ bounds $V \times I$, each class is its own inverse. Hence we have a group $\mathfrak{N}_k$. Topological product is also compatible with the equivalence relation, for if $V \sim_2 V'$, say $\partial M = V \cup V'$, and if W is closed, then $\partial(M \times W) = V \times W \cup V' \times W$, so $V \times W \sim_2 V' \times W$. Thus we may introduce products $\mathfrak{N}_k \times \mathfrak{N}_l \to \mathfrak{N}_{k+l}$, and these are clearly commutative, associative, and distributive over addition. We obtain a graded commutative ring $\mathfrak{N}$, which is called the cobordism ring. In the differential case it was defined, and its structure completely elucidated, by Thom [7].

Essentially the same remarks go also for the oriented case: we let $V \sim V'$ if there is an oriented manifold M with oriented boundary $\partial M = V \cup (-V')$, where $-$ denotes reversal of orientation. The relation $\sim$ is an equivalence relation; we denote the class of V by $[V]$. Disjoint union again gives an addition turning the set of classes into a group, and here $-$ yields the inverse. Finally, the topological product turns the sequence of these groups into a graded skew-commutative ring Ω.

We also consider a third ring. $\mathfrak{W}$ is the subset of $\mathfrak{N}$ consisting of classes containing a manifold M satisfying

(A) *The orientation covering of M can be induced by a map of M in the circle S^1.*

It is clear that if M, M' satisfy (A), then so do $M \cup M'$ and $M \times M'$ (the two maps in S^1 can be multiplied since S^1 is a group), hence $\mathfrak{W}$ is a subalgebra of $\mathfrak{N}$.

These are the objects which will appear in our exact sequences; we must now define the maps. Four of these are easy to find. We denote the inclusion map by $i: \mathfrak{W} \to \mathfrak{N}$. An element of Ω determines an element of $\mathfrak{N}$ by simply ignoring orientation (this is clearly compatible with the equivalence relations); this element is moreover in $\mathfrak{W}$, since an orientable manifold satisfies (A) *a fortiori*. Hence there are maps $r: \Omega \to \mathfrak{N}$, $s: \Omega \to \mathfrak{W}$ with $is = r$. All these three are clearly ring homomorphisms. We also have an additive homomorphism $2: \Omega \to \Omega$, defined by doubling each element.

We need two more maps, whose definition is less simple, involving the results of Part I. We shall define below $\partial: \mathfrak{N} \to \Omega$, of degree -1, by letting $\partial\{M\}$ be the class of a manifold V defining the orientation covering of M, and $d: \mathfrak{N} \to \mathfrak{N}$, of degree -2, by letting $d\{M\}$ be the class of a submanifold B of V defining the restriction of that covering to V. We first need a lemma.

LEMMA 9. *Let $p: \tilde{M} \to M$ be a double covering of the closed oriented manifold M. Take $\tilde{M}$ with the orientation induced by p. Then* $[\tilde{M}] = 2[M]$.

PROOF. Let V be a submanifold of M defining p (by Lemma 5), and T (the image of) a tubular neighbourhood for V in M. Then $\tilde{T} = p^{-1}(T)$ is a tubular neighbourhood of $\tilde{V} = p^{-1}(V)$, and is a fibre bundle over V associated to $p \mid V$ (by Lemma 1), with fibre two segments, as in Fig. 1 (where the group acts by reflection in the dotted line).

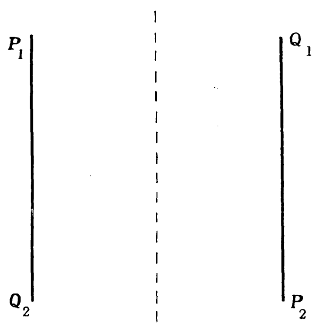

Fig. 1

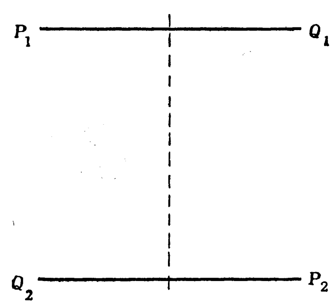

Fig. 2

The covering p is trivial over $M - V$; we choose the notation in Fig. 1 so that P_1, Q_1 are in one sheet of this; P_2, Q_2 in the other. Then if from $\tilde{M}$ we remove $\tilde{T}$ and replace by $2T$, which may be considered a bundle over V again with fibre two segments, as in Fig. 2, where again the group acts by reflection in the dotted line, we simply obtain two copies of M.

We have described a modification of $\tilde{M}$ which gives $2M$; we now say that there exists a corresponding cobounding manifold N. N is obtained by gluing to the closure of $(\tilde{M} - \tilde{T}) \times I$ the bundle over V, again associated to $p \mid V$, with fibre an octagon, described by Fig. 3, which also indicates how the gluing is to take place. It is immediate that N is a combinatorial manifold; that, in the differential case it is a differential manifold, follows by rounding off corners — indeed, we could take a curvilinear octagon with all angles right and eliminate the need for this.

Finally, we must check that N is orientable. The bundle described by

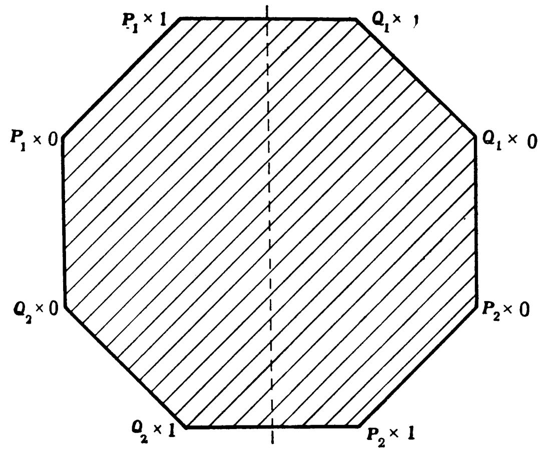

Fig. 3

Fig. 3 is orientable, since it is homeomorphic to $T \times I$. To this must be attached $(\tilde{M} - \tilde{T}) \times I$, which consists of orientable components. The fact that orientations match up follows (taking the product with I) from the fact that T glued to $M - T$ leads to the orientable M.

REMARK. This proof (with the discussion of orientation deleted) establishes also the same result in the non-oriented case, viz., *If* p: $\tilde{M} \to M$ *is a double covering of the closed manifold* M, *then* $\{\tilde{M}\} = 0$. However, this result is obvious *a priori*, since the mapping cylinder of p is a manifold with boundary $\tilde{M}$. This argument establishes also the result: *If* M *is non-orientable, and* p: $\tilde{M} \to M$ *its orientation covering, then* $[\tilde{M}] = 0$. For in this case, the mapping cylinder of p is also orientable (it is a bundle over M, with fibre a segment, whose orientation changes with that of M).

LEMMA 10. *There is a map* $\partial: \mathfrak{N} \to \Omega$, *of degree* -1, *such that* $\partial\{M\}$ *is the class of a submanifold* V *of* M *defining the orientation covering of* M.

PROOF. By Lemma 5, there is a submanifold V defining the covering. If M is cobordant to M', and N provides the cobordism, then if V, V' define the orientation coverings of M, M', by Lemma 7, there is a submanifold W of N, with boundary $V \cup V'$, which defines the orientation covering of N.

We now check that all these manifolds are orientable; from this it will follow that $\{M\}$ defines $[V]$, at least up to sign — we have not yet chosen an orientation for V. Now by Lemma 1, the orientation covering of N induces on W the normal covering of W in N. It follows by Lemma 2 (since double coverings under composition form a group $H^1(W, Z_2)$) that the orientation covering of W is trivial; i.e., that W is orientable, as required.

Finally, let T be a tubular neighbourhood of V in M. Then the closure of $M - T$ is an orientable manifold, and its boundary is that of T, a double cover of V. Hence by Lemma 9, $2[V] = 0$. Hence $\{M\}$ determines $[V]$, the sign of the latter being irrelevant.

Now for V also have a double covering, the restriction of that of M. By Lemma 5, this may be defined by a submanifold B of V.

LEMMA 11. *There is a map* $d: \mathfrak{N} \to \mathfrak{N}$, *of degree* -2, *such that* $d\{M\} = \{B\}$.

PROOF. We have just noted that B exists. If $\{M\} = \{M'\}$, and M, M' give rise to B, B', let $\partial N = M \cup M'$. By Lemma 7, we can first extend $V \cup V'$ to a submanifold W of N, defining the orientation covering and then again extend $B \cup B'$ to a submanifold C defining the restriction of that covering to W. Thus $\partial C = B \cup B'$, and so $\{B\} = \{B'\}$ as required.

It is clear that both the above constructions are compatible with disjoint unions, so ∂ and d are homomorphisms.

The exact sequences

We are now ready to state and prove our main result.

THEOREM 1. *The following sequences are exact*:

$$\Omega \xrightarrow{2} \Omega \xrightarrow{s} \mathfrak{W} \xrightarrow{\partial i} \Omega \xrightarrow{2} \Omega \tag{1}$$

$$0 \longrightarrow \mathfrak{W} \xrightarrow{i} \mathfrak{N} \xrightarrow{d} \mathfrak{N} \longrightarrow 0 \tag{2}$$

$$\Omega + \mathfrak{N} \xrightarrow{(2,0)} \Omega \xrightarrow{r} \mathfrak{N} \xrightarrow{(\partial, d)} \Omega + \mathfrak{N} \xrightarrow{(2,0)} \Omega\,. \tag{3}$$

PROOF. We note that (3) is obtained by splicing (1) and (2), and its exactness follows from theirs by trivial arguments which we leave to the reader. We split the proof into a number of stages.

Sequence (1) *is of order* 2. $s(2x) = 2s(x) = 0$ since s is a homomorphism; $2\partial i(x) = \partial i(2x) = 0$ since ∂, i are homomorphisms (and $\mathfrak{W}$ has exponent 2); and $\partial is(x) = \partial r(x) = 0$, since if M is orientable, its orientation covering is trivial, and we may choose V empty.

(2, s) *is exact*. If $[V]$ is in the kernel of s, V bounds a manifold M (in general non-orientable). By Lemma 7, we may extend the empty submanifold of ∂M to a submanifold A of M defining the orientation covering of M. Let T be a tubular neighbourhood of A, W the closure of $M - T$, and C their common boundary. Since W is orientable, with boundary

$V \cup C$, $[V] = \pm [C]$. A is orientable, and C a double covering, so by Lemma 9, $[C] = 2[A]$. Thus $[V]$ is in the image of 2.

(∂i, 2) is exact. If $[V]$ is in the kernel of 2, let M' be an orientable manifold with $\partial M' = 2V$. Let M be obtained from M' by gluing together the copies of V. Then the orientation double covering of M consists of two sheets over $M - V$, cross-joined along V, so is defined by V. The normal covering of V is trivial, so by Lemma 8, the orientation covering of M can be induced by a map in a circle; i.e., M satisfies (A). Hence $\{M\}$ is in $\mathfrak{W}$, and $\partial i\{M\} = [V]$.

(s, ∂i) is exact. Let $\{M\}$ be in the kernel of ∂i. We may suppose that M satisfies (A); then by Lemma 8, the orientation covering of M can be defined by a submanifold V with trival normal covering in M. We cut M along V to obtain an orientable manifold M' with boundary $2V$. Our hypothesis gives $[V] = 0$, so let $V = \partial N$, N orientable. Define N' by gluing a copy of N on to each boundary component V of M'. Then N' is clearly orientable. Hence the result will follow if we prove $\{M\} = \{N'\}$.

Consider $N' \times I$: in the boundary component $N' \times 1$ are two copies of N. If these are glued together, we may round off the corners along $V \times 1$ (in the differential case), and still have a manifold, W say. ∂W has two components: $N' \times 0$, and one obtained from $N' \times 1$ by deleting the two copies of N, and gluing together along the boundary. But this gives just a copy of M, as desired.

Sequence (2) *is of order* 2. If M satisfies (A), by Lemma 8, we may suppose that V (defining its orientation covering) has trivial normal covering in M, hence that B (which defines this covering) is empty.

i is $(1-1)$ by definition of $\mathfrak{W}$.

(i, d) is exact. Let $\{M\}$ be in the kernel of d, form V and B from M in the usual way. Then B bounds some manifold, say C. Let T be a tubular neighbourhood for B in V; this bundle is associated with the orientation covering of B, since V is orientable, by Lemma 2. Let U be the bundle over C with fibre $[-1, 1]$, associated with the orientation bundle of C. We may identify T with the part of U lying over B. We then glue U to $V \times I$ along T (in $V \times 1$), and round the corners, thus obtaining a manifold W. Let $C' = C \cup (V \times I)$. (See Fig. 4.)

We now repeat this process, extending the cobordism C of B, *via* the cobordism W of V, to a cobordism N of M (for this process, cf. [9]). The normal covering of V in M consists of two sheets over $V - B$, cross-joined along B. This extends to the covering of W defined by C'. We then form the associated bundle with fibre $[-1, 1]$, and glue to $M \times I$ along a tubular neighbourhood of V in M. Rounding off the corners, this gives a

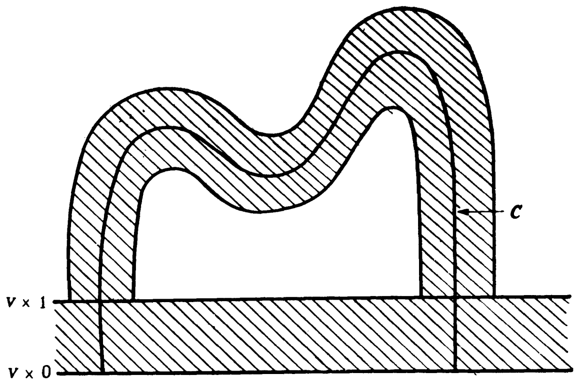

Fig. 4

manifold N. We set $W' = W \cup (V \times I)$, $C'' = C' \cup (B \times I)$. Then it is clear from our construction that W' defines the orientation covering of N, and that C'' defines the restriction of that covering to W'. ∂N consists of two components, M and M' say; correspondingly $\partial W = V \cup V'$, and $\partial C = B$. The normal covering of V' in M' is defined by the empty submanifold, so is trivial, so by Lemma 8, M' satisfies (A). But $\partial N = M \cup M'$, so $\{M\} = \{M'\}$ is in $\mathfrak{W}$.

d is onto. Let B be any manifold, M the bundle over it, associated with the orientation covering, with fibre $P_2(\mathbf{R})$ (the real projective plane), on which the group acts by $(x, y, z) \rightarrow (-x, y, z)$. Let V be the subbundle with fibre a circle given by $z = 0$, and identify B with the subbundle of this with point fibre given by $y = z = 0$.

The orientation cover of M is the associated bundle with fibre S^2 and action as described (this is orientable since the orientation of the fibre changes with that of B). This is trivial over $M - V$, and the sheets are cross-joined along V, so V is derived from M in the usual way. Similarly, the restriction of this to V is trivial on $V - B$, with two sheets cross-joined along B, so is defined by B. Hence indeed $d\{M\} = \{B\}$, and B was arbitrary, so d is onto.

This concludes the proof of Theorem 1. It seems appropriate at this point to acknowledge where the above proofs (in the differential case) first appeared. The first use of V and B was made by Rohlin [5], who also proved that (2, s) wase xact (not by the above method). Lemmas 9 and 10 and the above proof of the exactness of (2, s) are due to Dold [2], who was also the first to note the exactness of sequence (3) in [3]. Sequence (2), with essentially the above proof that d is onto is due to Atiyah [1]. Sequence (1) and the remaining proofs are due to the author [8].

Multiplicative structure

We shall now write $s\partial i = \partial'$.

THEOREM 2. *∂' is a derivation* of $\mathfrak{W}$.

PROOF. Let M, M' satisfy (A). By Lemma 8, we may define the orientation coverings of M, M' by submanifolds V, V' with trivial normal coverings, induced by maps in a circle. We seek to prove

$$\partial'\{M \times M'\} = \{V \times M'\} + \{M \times V'\} .$$

We now take a tubular neighbourhood of V in M; this may be identified with $V \times [-1, 1]$. Let S^1 be the circle obtained by identifying the end points of the interval $[-1, 1]$, and let $\varphi\colon M \to S^1$ be the map induced by projecting the tubular neighbourhood on its second factor, and mapping the rest of M to -1. Similiarly we obtain a map $\varphi'\colon M' \to S^1$. But S^1 is a group, with the group operation addition (reduced mod 2), which is a map $\pi\colon S^1 \times S^1 \to S^1$. Then

$$M \times M' \xrightarrow{\varphi \times \varphi'} S^1 \times S^1 \xrightarrow{\pi} S^1$$

gives a map $\psi = \pi \circ (\varphi \times \varphi')$ representing the orientation covering of $M \times M'$ (since this is the composition of the coverings induced from M and M').

Thus $\partial'\{M \times M\}$ is represented by $\psi^{-1}(0) = W$, say, the inverse image under $\varphi \times \varphi'$ of the dotted lines in Fig. 5. W is a manifold, since it agrees with the union of $V \times M'$ and $M \times V'$ except at points corresponding to the interior of the square , where $\varphi \times \varphi'$ is trivial, and the corresponding points form two copies of $V \times V' \times (0, 1)$. In the differential case, to ensure differentiability we take instead the dotted lines in Fig. 6; this makes no real difference to the argument.

Then W can be derived from the union of $V \times M'$ and $M \times V'$, the inverse image of the unbroken lines in Fig. 5, by deleting two copies of

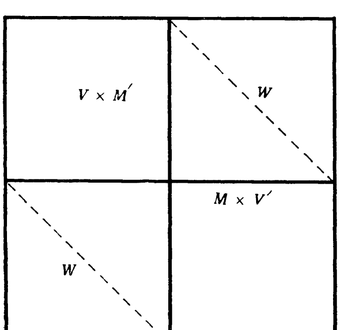

Fig. 5

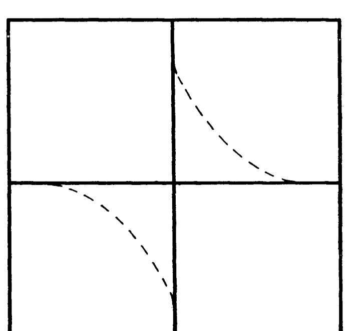

Fig. 6

$V \times V' \times (0, 1)$, and replacing them by two others. This situation is the same as we had in Lemma 9 (except that here the fiberings are trivial), and as there we can use Fig. 3 to yield a cobordism of W to the disjoint union of $V \times M'$ and $M \times V'$, as desired.

It is easy to see that $r\partial: \mathfrak{N} \rightarrow \mathfrak{N}$ is not a derivation. For, in the differential case, it annihilates orientable manifolds, and hence Dold's odd dimensional generators for $\mathfrak{N}$, and sends a typical even dimensional generator $\{P_{2n}(\mathbf{R})\}$ to $\{P_{2n-1}(\mathbf{R})\}$, which is zero. (This is well known: all the Stiefel numbers vanish. A simple geometric proof is: S^{2n-1}, hence also $P_{2n-1}(\mathbf{R})$, is a principal S^1-bundle over $P_{n-1}(\mathrm{C})$. We fill up the fibre S^1 to a disc D^2; this gives a manifold with $P_{2n-1}(\mathbf{R})$ as boundary.) Hence it annihilates a complete set of generators for $\mathfrak{N}$, so, were it a derivation, it would be zero. But this it certainly is not. A particular example is (x_i denoting any choice of an i-dimensional generator for $\mathfrak{N}$) $r\partial(x_2) = r\partial(x_4) = 0$, $r\partial(x_4x_2) = x_5$. Nor is $d: \mathfrak{N} \rightarrow \mathfrak{N}$ a derivation, for here $d(x_2) = 1$, $d(x_4) = x_2$, but $d(x_4x_2) = x_4$. Since the differential groups are contained in the combinatorial ones (see below), it follows that neither $r\partial$ nor d is a derivation in that case either.

It is possible, on lines similar to, but more complicated than those of Theorem 2, to consider $r\partial$ and d, and attempt to prove them derivations. It is possible to carry through the argument up to the point at which we use Fig. 3, where essentially we have a bundle with fibre S^1, and fill this up to a disc D^2. In these two cases, however, the fibre is a real or complex projective plane, so does not bound, and this introduces a new term into the equation. We hope to return to this point in a later paper.

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