THE SURGERY OBSTRUCTION OF A DISJOINT UNION

ANDREW RANICKI

The surgery obstruction $\sigma_*(f, b) \in L_n(\pi_1(X))$ of an n-dimensional degree 1 normal map $(f, b) : M \to X$ (in the sense of Browder [1] and Wall [7]) was formulated in [6] as the quadratic Poincaré cobordism class of a pair (C, ψ) consisting of an n-dimensional $\mathbb{Z}[\pi_1(X)]$-module chain complex C and a chain level quadratic structure ψ inducing Poincaré duality $H^{n-*}(C) \simeq H_*(C)$. If the manifold $M = \bigcup_{i=1}^{N} M_i$ is the disjoint union of manifolds M_i it is natural to seek an expression for the surgery obstruction of (f, b) in terms of quadratic structures defined by the restrictions $(f_i, b_i) = (f, b)| : M_i \to X$, which are normal maps of degree d_i with

$$f_{i*}[M_i] = d_i[X] \in H_n(X), \quad d_i \in \mathbb{Z}, \quad \sum_{i=1}^{N} d_i = 1 .$$

The algebraic theory of surgery of [6] is here used to provide such an expression, describing the pair (C, ψ) in terms of similar pairs (C_i, ψ_i) which are associated to (f_i, b_i). For the sake of simplicity we shall be working with the oriented case—the unoriented case is exactly the same, but with more complicated terminology.

I should like to thank Julius Shaneson and William Browder for conversations which stimulated my interest in this question.

As in [6] we shall actually be working with normal maps of geometric Poincaré complexes. Some care must be exercised about the precise definition of such normal maps (cf. Brumfiel and Milgram [4], for one possible definition).

A *degree d normal map* of n-dimensional geometric Poincaré complexes

$$(f, b) : M \to X$$

is a map $f : M \to X$ such that $f_*[M] = d[X] \in H_n(X)$ $(d \in \mathbb{Z})$, together with a map of $(k-1)$-spherical Spivak normal fibrations $b : \nu_M \to \nu_X$, and with preferred spherical generators $\rho_M \in \pi_{n+k}(T(\nu_M))$, $\rho_X \in \pi_{n+k}(T(\nu_X))$. The latter are to be such that $h(\rho_M) \cap U_{\nu_M} = [M] \in H_n(M)$ (h = Hurewicz map: $\pi_{n+k}(T(\nu_M)) \to \dot{H}_{n+k}(T(\nu_M))$, U_{ν_M} = Thom class $\in H^k(T(\nu_M))$, H = reduced (co)homology, $T(\nu_M)$ = Thom space), and similarly for ρ_X. In the case $d = 1$ we no longer require $T(b)_*(\rho_M) = \rho_X \in \pi^S_{n+k}(T(\nu_X))$, as we did in the definition of a degree 1 normal map in [6].

A spherical generator $\rho_X \in \pi_{n+k}(T(\nu_X))$ for the Thom space $T(\nu_X)$ of a $(k-1)$-spherical Spivak normal fibration $\nu_X : X \to BSG(k)$ of an n-dimensional geometric Poincaré complex X determines an S-duality map

$$\alpha_X : S^{n+k} \xrightarrow{\rho_X} T(\nu_X) \xrightarrow{\Delta} X_+ \wedge T(\nu_X) ,$$

Received 2 January. 1979; revised 10 February, 1979.

[J London Math. Soc. (2), 20 (1979), 559–566]

with $X_+ = X \cup \{pt.\}$ and Δ induced by the diagonal map. If $\tilde{X}$ is a covering of X with group of covering translations π then according to [6] there is defined also a π-equivariant S-duality ("$S\pi$-duality") map

$$\alpha_{\tilde{X}} : S^{n+k} \xrightarrow{\rho_X} T(\nu_X) \xrightarrow{\Delta} \tilde{X}_+ \wedge_\pi T(\nu_{\tilde{X}}),$$

with $\nu_{\tilde{X}} : \tilde{X} \to X \xrightarrow{\nu_X} BSG(k)$ and $\tilde{X}_+ \wedge_\pi T(\nu_{\tilde{X}})$ the quotient of $\tilde{X}_+ \wedge T(\nu_{\tilde{X}})$ by the diagonal π-action.

The *chain Umkehr* of an n-dimensional degree d normal map $(f, b) : M \to X$ is the composite $\mathbb{Z}[\pi_1(X)]$-module chain map (defined up to chain homotopy)

$$f^! : C(\tilde{X}) \xrightarrow{([X]\cap -)^{-1}} C(\tilde{X})^{n-*} \xrightarrow{\tilde{f}^*} C(\tilde{M})^{n-*} \xrightarrow{[M]\cap -} C(\tilde{M}),$$

with $\tilde{X}$ the universal cover of X, $\tilde{M}$ the cover of M induced from $\tilde{X}$ by f, and

$$C(\tilde{X})^{n-*} = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(C(\tilde{X})_{n-*}, \mathbb{Z}[\pi_1(X)]).$$

The chain Umkehr is such that there is defined a chain homotopy commutative diagram

$$\begin{array}{ccc} C(\tilde{X}) & \xrightarrow{\quad f^! \quad} & C(\tilde{M}) \\ & \searrow^{d} & \downarrow{\scriptstyle \tilde{f}} \\ & & C(\tilde{X}) \end{array}$$

The *homotopy Umkehr* of (f, b) is the stable $\pi_1(X)$-equivariant homotopy class of stable $\pi_1(X)$-equivariant maps $F : \Sigma^\infty \tilde{X}_+ \to \Sigma^\infty \tilde{M}_+$ $S\pi_1(X)$-dual to the induced map of Thom spaces $T(\tilde{b}) : T(\nu_{\tilde{M}}) \to T(\nu_{\tilde{X}})$, using the $S\pi_1(X)$-duality maps $\alpha_{\tilde{M}}, \alpha_{\tilde{X}}$ determined by ρ_M, ρ_X. The homotopy Umkehr F induces the chain Umkehr $f^!$ on the chain level. The *homotopy degree* of (f, b) is the stable cohomotopy class $\delta \in [X_+, QS^0] = \pi_S^0(X_+)$ S-dual to $T(b)_*(\rho_M) \in \pi_{n+k}^S(T(\nu_X))$ under the S-duality isomorphism $\alpha_X : \pi_{n+k}^S(T(\nu_X)) \xrightarrow{\simeq} \pi_S^0(X_+)$ determined by ρ_X. The homotopy degree $\delta : X_+ \to QS^0$ sends X to the component $Q_d S^0$ of $d \in H_0(QS^0) = \mathbb{Z}$ in $QS^0 = \varinjlim_m \Omega^m S^m$. The homotopy Umkehr and the homotopy degree are related by a stable $\pi_1(X)$-equivariant homotopy commutative diagram

$$\begin{array}{ccc} \Sigma^\infty \tilde{X}_+ & \xrightarrow{\quad F \quad} & \Sigma^\infty \tilde{M}_+ \\ & \searrow^{\mathrm{adjoint}(\tilde{\delta})} & \downarrow{\scriptstyle \Sigma^\infty \tilde{f}} \\ & & \Sigma^\infty \tilde{X}_+ \end{array}$$

inducing the previous diagram on the chain level, with $\tilde{\delta} : \tilde{X}_+ \to X_+ \xrightarrow{\delta} QS^0$.

Given a group π, spaces with π-action X, Y and a stable π-equivariant map

$F:\Sigma^\infty X_+ \to \Sigma^\infty Y_+$ define the composite stable π-equivalent map

$$X_+ \xrightarrow{\text{adjoint}(F)} \Omega^\infty\Sigma^\infty Y_+ \xrightarrow{\text{stable homotopy projection}} (E\mathbb{Z}_2)_+ \wedge_{\mathbb{Z}_2}(Y_+ \wedge Y_+)\,.$$

As in [6] call the induced abelian group morphisms

$$\psi_F: H_n(X/\pi) \to Q_n(C(Y)) = H_n\Big(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C(Y)\otimes_{\mathbb{Z}[\pi]}C(Y))\Big)$$

the *quadratic construction.* Here, the generator $T\in\mathbb{Z}_2$ acts on $Y_+\wedge Y_+ = (Y\times Y)_+$ by the transposition $(a,b)\mapsto(b,a)$, on $C(Y)\otimes_{\mathbb{Z}[\pi]}C(Y)$ by the signed transposition $a\otimes b\mapsto(-)^{|a||b|}b\otimes a$, and $W = C(E\mathbb{Z}_2)$ is the free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W:\ldots\to\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1+T}\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1-T}\mathbb{Z}[\mathbb{Z}_2]\to 0\,.$$

The quadratic construction ψ_F depends only on the stable π-equivariant homotopy class of F.

The *quadratic signature* of an n-dimensional degree 1 normal map $(f,b):M\to X$ is the quadratic Poincaré cobordism class

$$\sigma_*(f,b) = \Big(C(f^!),\, e_\%\psi_F[X]\in Q_n(C(f^!))\Big)\in L_n(\pi_1(X))$$

with $C(f^!)$ the algebraic mapping cone of the chain Umkehr $f^!:C(\tilde{X})\to C(\tilde{M})$, $e:C(\tilde{M})\to C(f^!)$ the projection, and $\psi_F[X]\in Q_n(C(\tilde{M}))$ the evaluation on the fundamental class $[X]\in H_n(X)$ of the quadratic construction ψ_F on the homotopy Umkehr $F:\Sigma^\infty\tilde{X}_+\to\Sigma^\infty\tilde{M}_+$. We are not repeating here the many other definitions and constructions of [6] which might make this meaningful.

A degree 1 normal map $(f,b):M\to X$ in the sense of [1] and [7] determines a degree 1 normal map $(f,Jb):M\to X$ in the present sense, with homotopy degree $1\in\pi_S^0(X_+)$, as follows. Since M is now a manifold we can take for $(\nu_M:M\to BSG(k),\ \rho_M:S^{n+k}\to T(\rho_M))$ the Spivak normal structure given by an embedding of M in S^{n+k} $(k\gg n)$ as a manifold, not just as a Poincaré complex, and we can define $\rho_X = T(b)_*(\rho_M)\in\pi_{n+k}(T(\nu_X))$. According to [6] the surgery obstruction of (f,b) is the quadratic signature $\sigma_*(f,Jb)\in L_n(\pi_1(X))$.

Our description of the quadratic signature $\sigma_*(f,b)\in L_n(\pi_1(X))$ of a disjoint union degree 1 normal map $(f,b) = \bigcup_i (f_i,b_i):\bigcup_i M_i\to X$ is based on the following quadratic property of the quadratic construction ψ, which is an easy consequence of its construction.

LEMMA. *Given a group π, spaces with π-action X, Y_i and stable π-equivariant maps $F_i:\Sigma^\infty X_+\to\Sigma^\infty(Y_i)_+$ $(1\leqslant i\leqslant N)$ track addition defines a stable π-equivariant map*

$$F = \bigvee_i F_i:\Sigma^\infty X_+\to\Sigma^\infty\left(\bigcup_{i=1}^N Y_i\right)_+ = \bigvee_{i=1}^N\Sigma^\infty(Y_i)_+\,.$$

The quadratic construction on F is given by

$$\psi_F = \begin{pmatrix} \bigoplus_i \psi_{F_i} \\ \bigoplus_{i<j} -(f_i \otimes f_j)\Delta \end{pmatrix} : H_n(X/\pi) \to Q_n\left(C\left(\bigcup_{i=1}^N Y_i\right)\right)$$

$$= \bigoplus_{i=1}^N Q_n(C(Y_i)) \oplus \bigoplus_{i<j} H_n(C(Y_i) \otimes_{\mathbb{Z}[\pi]} C(Y_j)),$$

with $f_i : C(X) \to C(Y_i)$ *the* $\mathbb{Z}[\pi]$*-module chain map induced by* F_i*, and*

$$\Delta : H_n(X/\pi) \to H_n(C(X) \otimes_{\mathbb{Z}[\pi]} C(X))$$

the map induced by a π*-equivariant diagonal chain approximation* $\Delta : C(X) \to C(X) \otimes_{\mathbb{Z}} C(X)$.

The disjoint union of n-dimensional degree d_i normal maps $(f_i, b_i) : M_i \to X$ $(1 \leqslant i \leqslant N)$ with the same Spivak normal structure $(\nu_X : X \to BSG(k), \rho_X : S^{n+k} \to T(\nu_X))$ for X is an n-dimensional degree $d = \sum_{i=1}^N d_i$ normal map

$$(f, b) = \bigcup_{i=1}^N (f_i, b_i) : \bigcup_{i=1}^N M_i \to X$$

with chain Umkehr

$$f^! = \bigoplus_i f_i^! : C(\tilde{X}) \to C\left(\bigcup_{i=1}^N \tilde{M}_i\right) = \bigoplus_{i=1}^N C(\tilde{M}_i),$$

homotopy Umkehr

$$F = \bigvee_i F_i : \Sigma^\infty \tilde{X}_+ \to \Sigma^\infty \left(\bigcup_{i=1}^N \tilde{M}_i\right)_+ = \bigvee_{i=1}^N \Sigma^\infty(\tilde{M}_i)_+ ,$$

and homotopy degree $\delta = \sum_{i=1}^N \delta_i \in \pi_S^0(X_+)$. Applying the lemma we have:

PROPOSITION 1. *The quadratic signature of an n-dimensional degree* 1 *normal map* $(f, b) = \bigcup_{i=1}^N (f_i, b_i) : \bigcup_{i=1}^N M_i \to X$ *which is the disjoint union of degree* d_i *normal maps* $(f_i, b_i) : M_i \to X$ $\left(\sum_{i=1}^N d_i = 1\right)$ *is the quadratic Poincaré cobordism class*

$$\sigma_*(f, b) = \left(C(f^!), e_\%\psi_F[X] \in Q_n(C(f^!))\right) \in L_n(\pi_1(X)),$$

with $e : \bigoplus_{i=1}^{N} C(\tilde{M}_i) \to C(f^!)$ *the projection and*

$$\psi_F[X] = \left(\bigoplus_{i=1}^{N} \psi_{F_i}[X],\ \bigoplus_{i<j} -(f_i^! \otimes f_j^!)\Delta[X] \right) \in Q_n\left(C\left(\bigcup_{i=1}^{N} \tilde{M}_i \right) \right)$$

$$= \bigoplus_{i=1}^{N} Q_n(C(\tilde{M}_i)) \oplus \bigoplus_{i<j} H_n(C(\tilde{M}_i) \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{M}_j)).$$

One case is of particular interest: given a degree 0 normal map $(f, b) : M \to X$ there is defined a degree 1 normal map

$$(g, c) = (f \cup 1, b \cup 1) : M \cup X \to X$$

for which we can identify

$$g^! = \begin{pmatrix} f^! \\ 1 \end{pmatrix} : C(\tilde{X}) \to C(\tilde{M} \cup \tilde{X}) = C(\tilde{M}) \oplus C(\tilde{X})$$

$$e = (1 \quad -f^!) : C(\tilde{M} \cup \tilde{X}) = C(\tilde{M}) \oplus C(\tilde{X}) \to C(g^!) = C(\tilde{M})$$

$$G = F \vee 1 : \Sigma^\infty \tilde{X}_+ \to \Sigma^\infty (\tilde{M} \cup \tilde{X})_+ = \Sigma^\infty \tilde{M}_+ \vee \Sigma^\infty \tilde{X}_+ .$$

Substituting this in the expression of Proposition 1 we obtain:

PROPOSITION 2. *The quadratic signature of the degree* 1 *normal map* $(g, c) = (f \cup 1, b \cup 1)$ *is the quadratic Poincaré cobordism class*

$$\sigma_*(g, c) = \left(C(\tilde{M}),\quad \psi_F[X] + (f^! \otimes f^!)\,\Delta[X] \in Q_n(C(\tilde{M})) \right) \in L_n(\pi_1(X)),$$

where $(f^! \otimes f^!)\Delta[X] \in Q_n(C(\tilde{M}))$ *is the image of*

$$(f^! \otimes f^!)\Delta[X] \in H_n(C(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{M}))$$

under the abelian group morphism

$$H_n(C(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{M})) \to Q_n(C(\tilde{M}));$$

$$\phi \mapsto \psi, \quad \psi_s = \begin{cases} \phi \in (C(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{M}))_n & s = 0 \\ 0 \in (C(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{M}))_{n-s} & s \geqslant 1 . \end{cases}$$

In [6] there were also considered the symmetric L-groups $L^n(\pi)$ $(n \geqslant 0)$ of a group π, and the symmetric signature invariant $\sigma^*(X) \in L^n(\pi_1(X))$ of an n-dimensional geometric Poincaré complex X, both of which were originally introduced by Mishchenko. There are defined symmetrization maps

$$1 + T : L_n(\pi) \to L^n(\pi) \qquad (n \geqslant 0),$$

which are isomorphisms modulo 8-torsion. The quadratic signature $\sigma_*(f, b) \in L_n(\pi_1(X))$ of a degree 1 normal map of n-dimensional geometric Poincaré complexes $(f, b): M \to X$ has symmetrization

$$(1+T)\sigma_*(f, b) = \sigma^*(M) - \sigma^*(X) \in L^n(\pi_1(X)),$$

where $\sigma^*(M) \in L^n(\pi_1(X))$ is the image of $\sigma^*(M) \in L^n(\pi_1(M))$ under the morphism induced by $f_* : \pi_1(M) \to \pi_1(X)$. If X and Y are n-dimensional geometric Poincaré complexes and there are given group morphisms $\pi_1(X) \to \pi$, $\pi_1(Y) \to \pi$ to the same group π then

$$\sigma^*(X \cup Y) = \sigma^*(X) + \sigma^*(Y) \in L^n(\pi).$$

(The symmetric signature is defined for disconnected geometric Poincaré complexes using fundamental groupoids, exactly as in Wall [7].) Given a degree 1 normal map of n-dimensional geometric Poincaré complexes which is a disjoint union

$$(f, b) = \bigcup_{i=1}^{N} (f_i, b_i) : \bigcup_{i=1}^{N} M_i \to X$$

we thus have

$$(1+T)\sigma_*(f, b) = \sum_{i=1}^{N} \sigma^*(M_i) - \sigma^*(X) \in L^n(\pi_1(X)).$$

The semicharacteristic classes of Lee [5] are the images of the symmetric signature in appropriate Grothendieck groups of orthogonal representations, so that the semicharacteristic part of the surgery obstruction is additive on disjoint unions. I am grateful to C. T. C. Wall for drawing my attention to the relevance of [5].

For readers unfamiliar with the algebraic theory of surgery of [6] we shall express the simply-connected even-dimensional case of Proposition 2 in the language of Browder [1], using functional Steenrod squares. Indeed, this case has essentially already been worked out in §4 of Browder [2].

PROPOSITION 3. *Let $(f, b): M \to X$ be a degree* 0 *normal map of* $2i$*-dimensional geometric Poincaré complexes. The image of the quadratic signature* $\sigma_*(f \cup 1, b \cup 1) \in L_{2i}(\pi_1(X))$ *of the degree* 1 *normal map* $(f \cup 1, b \cup 1): M \cup X \to X$ *in*

$$L_{2i}(1) = \begin{cases} \mathbb{Z} & (i \equiv 0 (\mathrm{mod}\ 2)) \\ \mathbb{Z}_2 & (i \equiv 1 (\mathrm{mod}\ 2)) \end{cases}$$

is just $\begin{cases} \tfrac{1}{8}\ (\text{the signature}) \\ \text{the Arf invariant} \end{cases}$ *of the non-singular* $(-)^i$ *quadratic form* (G, λ, μ) *over the ring* A *defined by*

$$G = \begin{cases} H^i(M; \mathbb{Z})/\mathit{torsion} \\ H^i(M; \mathbb{Z}_2) \end{cases}, \quad A = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \end{cases}$$

$$\lambda : G\times G \to A;\quad (x,y)\mapsto \langle f^{!*}(x\cup y)+(f^{!*}x\cup f^{!*}y),[X]\rangle = \langle x\cup y,[M]\rangle+\langle f^{!*}x\cup f^{!*}y,[X]\rangle$$

$$\mu : G\to A/\{a-(-)^i a\,|\,a\in A\}\,(=A);\quad z\mapsto \begin{cases} \tfrac{1}{2}\lambda(z,z) \\ \langle Sq_h^{i+1}(\Sigma^k\iota),\Sigma^k[X]\rangle+\langle z\cup z,[M]\rangle \end{cases}$$

$$\big(h=(\Sigma^k z)F-\Sigma^k(f^{!*}z)\in[\Sigma^k X_+,\Sigma^k K(\mathbb{Z}_2,i)],\quad z\in H^i(M;\mathbb{Z}_2)=[M_+,K(\mathbb{Z}_2,i)],$$

ι = generator $\in H^i(K(\mathbb{Z}_2,i);\mathbb{Z}_2)=\mathbb{Z}_2$, $F:\Sigma^k X_+\to\Sigma^k M_+$ (k large) is the S-dual of the induced map of Thom spaces $T(b):T(\nu_M)\to T(\nu_X)\big)$.

(Of course, in Proposition 3—and below—we are really only using A-coefficient Poincaré duality, and not the universal $\mathbb{Z}[\pi_1(X)]$-coefficient Poincaré duality.

A finite d-sheeted covering $p:\bar{X}\to X$ of an n-dimensional geometric Poincaré complex X determines a degree d normal map $(p,b):\bar{X}\to X$ with homotopy degree the composite $X\xrightarrow{p} B\Sigma_d\to QS^0$ of the classifying map $p:X\to B\Sigma_d$ and the canonical map $B\Sigma_d\to Q_dS^0\hookrightarrow QS^0$, as follows. Let W be a closed regular neighbourhood of X in S^{n+k} for some embedding $X\hookrightarrow S^{n+k}$ ($k>n+1$), defining a Spivak normal structure $\big(\nu_X:X\to BSG(k),\rho_X:S^{n+k}\to T(\nu_X)\big)$ for X by

$$S^{k-1}\to\partial W\xrightarrow{\nu_X} W\simeq X,\quad \rho_X:S^{n+k}\xrightarrow{\text{collapse}} S^{n+k}/\overline{S^{n+k}-W}=W/\partial W=T(\nu_X).$$

The induced cover $\bar{W}$ of W has a trivialized tangent bundle, namely the pullback of the tangent bundle of W. This trivialization determines a regular homotopy class of immersions of $\bar{W}$ in S^{n+k}. Now $k>n+1$ and $\bar{W}$ has an n-dimensional spine, to wit $\bar{X}$, so that this class actually contains an embedding of $\bar{W}$ in S^{n+k}. Thus $\bar{W}$ is a closed regular neighbourhood of $\bar{X}$ for an embedding $\bar{X}\hookrightarrow S^{n+k}$, defining a Spivak normal structure $\big(\nu_{\bar{X}}:\bar{X}\to BSG(k),\rho_{\bar{X}}:S^{n+k}\to T(\nu_{\bar{X}})\big)$. (For this line of argument I am indebted to Larry Taylor.) The quadratic signature

$$\sigma_*(p\cup\bigcup 1,b\cup\bigcup 1)\in L_n(\pi_1(X))$$

of the degree 1 normal map

$$(p\cup\bigcup 1,b\cup\bigcup 1):\bar{X}\cup\bigcup_{i=2}^{d}-X\to X$$

is expressed by Proposition 1 in terms of the homotopy Umkehr $P:\Sigma^\infty\tilde{X}_+\to\Sigma^\infty\tilde{\bar{X}}_+$ $\left(=\bigvee_d\Sigma^\infty\tilde{X}_+\right)$, where $-X$ denotes X with the opposite orientation. Note that the chain Umkehr $p^!:C(\tilde{X})\to C(\tilde{\bar{X}})$ is just the usual chain level transfer of the cover $\tilde{p}:\tilde{\bar{X}}\to\tilde{X}$ of the universal cover $\tilde{X}$ of X induced from $p:\bar{X}\to X$ by $\tilde{X}\to X$. In particular, for a double cover ($d=2$) Proposition 2 gives a quadratic Poincaré cobordism class

$$\sigma_*(p\cup\bigcup 1,b\cup\bigcup 1)=\big(C(\tilde{\bar{X}}),\psi_P[X]+(p^!\otimes p^!)\Delta[X]\big)\in L_n(\pi_1(X)).$$

A double cover $p : \tilde{X} \to X$ determines yet another quadratic Poincaré cobordism class

$$\sigma_*(X, p) = \big(C(\tilde{\tilde{X}}), \psi_P[X]\big) \in L_n(\pi_1(X)).$$

If $n = 2i$ the image of $\sigma_*(X, p)$ in $L_{2i}(1)$ is just

$$\begin{cases} \frac{1}{8}(\text{the signature}) & (i \equiv 0 (\text{mod } 2)) \\ \text{the Arf invariant} & (i \equiv 1 (\text{mod } 2)) \end{cases}$$

of the non-singular $(-)^i$-quadratic form (G, λ, μ) over A defined by

$$G = \begin{cases} H^i(\tilde{X}; \mathbb{Z})/\text{torsion} \\ H^i(\tilde{X}; \mathbb{Z}_2) \end{cases}, \quad A = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \end{cases}$$

$$\lambda : G \times G \to A; \quad (x, y) \mapsto \langle p^{!*}(x \cup y) - (p^{!*}x \cup p^{!*}y), [X]\rangle = -\langle x \cup Ty, [\tilde{X}]\rangle$$

$$\mu : G \to A; \quad z \mapsto \begin{cases} \frac{1}{2}\lambda(z, z) \\ \langle Sq_h^{i+1}(\Sigma^k \iota), \Sigma^k[X]\rangle \end{cases}$$

$\big(h = (\Sigma^k z)P - \Sigma^k(p^{!*}z) \in [\Sigma^k X_+, \Sigma^k K(\mathbb{Z}_2, i)]\,, P : \Sigma^k X_+ \to \Sigma^k \tilde{X}_+, \iota \in H^k(K(\mathbb{Z}_2; i), \mathbb{Z}_2)$, T = covering translation : $\tilde{X} \to \tilde{X}\big)$,

which was used by Browder and Livesay [3] to define a desuspension invariant for fixed point free involutions on spheres, and which more recently has been studied by Brumfiel and Milgram [4]. In general, $\sigma_*(p \cup 1, b \cup 1) \neq \sigma_*(X, p)$, as has already been shown in the Arf invariant case in Proposition 5.3.1 of [4]. The Poincaré transversality obstruction for a double cover $p : \tilde{X} \to X$ of a $(4k+2)$-dimensional geometric Poincaré complex X obtained by Hambleton and Milgram [8] is the Arf invariant given by the image of $\sigma_*(X, p)$ in $L_{4k+2}(1) = \mathbb{Z}_2$.

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Department of Mathematics,
Princeton University,
Princeton, NJ 08544,
U.S.A.