ANNALS OF MATHEMATICS
Vol. 75, No. 1, January, 1962
Printed in Japan

DECOMPOSITION THEOREMS FOR $S(n)$-COMPLEXES

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(Received December 7, 1960)

I. The purpose of this note is to give unified proofs for several known (or partially known) decomposition theorems for chain-complexes on which the symmetric group $S(n)$ operates. The method is elementary (although abelian categories occur), with transfer homomorphism playing an essential role.

$S(n)$ denotes the group of permutations of the integers $1, 2, \cdots, n$. For $k \leqq n$, we identify $S(k)$ with the subgroup of $S(n)$ whose elements leave $n-k+1, n-k+2, \cdots, n$ fixed. The first theorem concerns the homology groups of $S(n)$:

THEOREM 1. (Nakaoka [4]). *If G is an abelian group on which $S(n)$ operates trivially then the homology homomorphism*

$$\iota_{n,k*}\colon H_*(S(k), G) \to H_*(S(n), G) ,$$

induced by the inclusion $\iota_{n,k}\colon S(k) \subset S(n)$, is a monomorphism and its image is a direct summand. In other words, $\iota_{n,k*}$ has a left inverse.

Dually the cohomology homomorphism

$$\iota^*_{n,k}\colon H^*(S(n), G) \to H^*(S(k), G)$$

has a right inverse.

Nakaoka's proof is geometric (and rather complicated); it uses properties of $\smile$ - and $\frown$ - products in symmetric products of spheres.

THEOREM 2 (Steenrod [6, 22]). *Let Y be a semi-simplicial complex, and SP^nY its n-fold symmetric product ($=$ orbit complex of Y^n under the action of $S(n)$). Then the chain map*

$$i_{n,k}\colon C(SP^kY) \to C(SP^nY) , \qquad k \leqq n$$

induced by the inclusion $SP^kY \subset SP^nY$ (using a base point) has a left inverse (CY denotes the chain complex of Y).

Our argument here is essentially a conceptual version of Nakaoka's proof for Proposition 2.6 in [3]. There is, of course, no simpler proof than Steenrod's own in [6, 22].

Dually to Theorem 2 we have

THEOREM 3. *Let $\Gamma_nY \subset C(Y^n)$ be the chains of the n^{th} cartesian power which are invariant under permutations of factors in Y^n. Then the*

* Work supported by the Office of Naval Research under contract Nonr. 266 (57).

following composite chain map has a right inverse

$$i^{k,n}\colon \Gamma_n Y \to C(Y^n)^{S(k)} \to \Gamma_k Y\,, \qquad k \leqq n\,;$$

the middle term consists of chains in Y^n *which are invariant under permutation of the first k factors; the first map is the inclusion, the second is induced by the projection* $Y^n \to Y^k$ *(first k factors).*

The following theorem is still of the same nature though slightly more difficult to prove.

THEOREM 4. *Let* Y *be a semi-simplicial complex,* $u \in H^{2m}(Y, \mathbf{Z})$ *an even-dimensional integral cohomology class such that* $\langle u, z\rangle = 1$ *for some homology class* $z \in H_{2m}(Y, \mathbf{Z})$. *Then the following composite homomorphism has a right inverse*

$$H_j(SP^n Y, G) \xrightarrow{t_*} H_j((SP^{n-1} Y) \times Y, G) \xrightarrow{/u} H_{j-2m}(SP^{n-1} Y, G)\,,$$

where t is the transfer homomorphism (*see* (4) *and the proof of Theorem* 2), *and* $/u$ *is the slant product* (*see* [5]) *with* u.

Dually, the composite

$$H^j(SP^{n-1} Y, G) \xrightarrow{\times u} H^{j+2m}((SP^{n-1} Y) \times Y, G) \xrightarrow{t_*} H^{j+2m}(SP^n Y, G)$$

has a left inverse ($\times u$ = *cross-product with* u).

In the case where $Y = S^{2m} = 2m$-sphere, and u is a generator of $H^{2m}(S^{2m}, \mathbf{Z})$ this result is contained in [4] (in a different formulation; see [4], 5.5); in fact, Nakaoka uses it to prove Theorem 1.

The proof of Theorem 4 will only be sketched in this note.

Theorem 4 can be generalized to coefficient rings Λ other than $\mathbf{Z}$ (and G a Λ-module). In case Λ has characteristic 2 we do not have to require that u be even-dimensional (only of dimension > 1).

II. Two lemmas precede the proofs. In order to avoid repetition in dualizing, we formulate and prove them in an arbitrary *abelian category* $\mathfrak{A}$ (see A. Grothendieck, Tôhoku Math. J., 9 (1957), 119–221).

Let A be an object in $\mathfrak{A}$, and π a group. We say π *operates on* A on the left (resp. on the right) if a map $\varphi\colon \pi \to \mathrm{Hom}(A, A)$ is given which satisfies

(1) $\varphi(1) = \mathrm{id}_A$, $\varphi(xy) = \varphi(x)\cdot\varphi(y)$ resp. $\varphi(xy) = \varphi(y)\cdot\varphi(x)$, for $x, y \in \pi$.

We put

(2) $A^\pi = \bigcap_{x\in\pi} \ker(\mathrm{id}_A - \varphi(x))$, $A_\pi = A/\bigcup_{x\in\pi} \mathrm{im}(\mathrm{id}_A - \varphi(x))$, if these objects exist. Recall that $\bigcap_\gamma \ker(\psi_\gamma\colon A \to B_\gamma)$ is equivalent to $\ker(\psi\colon A \to \prod_\gamma B_\gamma)$ if the direct product $\prod_\gamma B_\gamma$ exists (ψ being the morphism with components ψ_γ). In particular, A^π always exists if π is finite. A_π is just the

dual of A^π (i.e., apply the definition of A^π to the dual category $\mathfrak{A}^*$); so it exists for finite π, too.

If $\rho \subset \pi$ is a subgroup, we have natural morphisms

$$(3) \qquad i^{\rho,\pi}: A^\pi \to A^\rho \quad \text{resp.} \quad i_{\pi,\rho}: A_\rho \to A_\pi;$$

obtained from id(A) by passing to sub-objects resp. quotients (see [1, XII, 8]).

If the index $m = [\pi : \rho]$ is finite, we also have *transfer* morphisms

$$(4) \qquad t^{\pi,\rho}: A^\rho \to A^\pi \quad \text{resp.} \quad t_{\rho,\pi}: A_\pi \to A_\rho ,$$

defined as follows: Let $x_1, x_2, \cdots, x_m \in \pi$ be representative elements of the right resp. left cosets of ρ in π (which we assume to operate on the *right*). Then $t^{\pi,\rho}$ resp. $t_{\rho,\pi}$ is obtained from $\sum_{i=1}^m \varphi(x_i): A \to A$ by passing to sub-objects resp. quotients. Clearly

$$(5) \qquad t^{\pi,\rho} i^{\rho,\pi} = m \cdot id(A^\pi), \quad i_{\pi,\rho} t_{\rho,\pi} = m \cdot id(A_\pi) .$$

Finally, if π is a finite group we have the *norm* morphism

$$(6) \qquad N(\pi): A_\pi \to A^\pi \qquad \text{(see [1, XII. 1])} ,$$

induced by $\sum_{x \in \pi} \varphi(x): A \to A$, and commutative diagrams

$$(7) \qquad \begin{array}{ccc} A_\pi & \xrightarrow{t_{\rho,\pi}} & A_\rho \\ {\scriptstyle N(\pi)}\downarrow & & \downarrow{\scriptstyle N(\rho)} \\ A^\pi & \xrightarrow[i^{\rho,\pi}]{} & A^\rho , \end{array} \qquad\qquad \begin{array}{ccc} A^\pi & \xleftarrow{t^{\pi,\rho}} & A^\rho \\ {\scriptstyle N(\pi)}\uparrow & & \uparrow{\scriptstyle N(\rho)} \\ A_\pi & \xleftarrow[i_{\pi,\rho}]{} & A_\rho , \end{array}$$

dual to each other.

LEMMA 1. *Let A be an object in $\mathfrak{A}$ on which $S(n)$ operates on the right, and let $0 < k < n$. Then, in the diagram*

$$(8) \qquad \begin{array}{ccccc} & & A_{S(k)\times S(n-k-1)\times 1} & & \\ & {\scriptstyle i'}\swarrow & & \nwarrow{\scriptstyle t'} & \\ A_{S(k)\times S(n-k)} & \xleftarrow{t} & A_{S(n)} & \xleftarrow{i} & A_{S(n-1)\times 1} \\ {\scriptstyle i''}\uparrow & & & & \downarrow{\scriptstyle t''} \\ A_{S(k-1)\times 1\times S(n-k)} & & \xleftarrow[\xi_*]{} & & A_{S(k-1)\times S(n-k)\times 1} \end{array}$$

the middle row is the sum of the top row and the bottom row, i.e., $ti = i't' + i''\xi_* t''$.

The dual lemma is obtained by reversing the arrows, lifting the indices, and replacing "right" by "left".

The notation in (8) is as follows: $S(k-1) \times 1 \times S(n-k)$ is the subgroup of $S(n)$ which leaves $(1, 2, \cdots, k-1)$, (k), $(k+1, \cdots, n)$ fixed (as a whole, not pointwise); similarly for the other groups which appear as indices. The symbol $S(0)$ should be deleted where it occurs (e.g., $S(0) \times S(r) = S(r)$). For i, i', i'' and t, t', t'' see (3) and (4); we dropped the indices. The morphism ξ_* (actually an isomorphism) is induced by $\varphi(\xi)$ where $\xi \in S(n)$ is the permutation $(\xi(1), \cdots, \xi(n)) = (1, 2, \cdots, k-1, n, k, k+1, \cdots, n-1)$.

PROOF. For every (unordered) subset $K \subset (1, 2, \cdots, n-1)$ of k elements pick a permutation $\gamma_K \in S(n)$ such that $(\gamma_K(1), \gamma_K(2), \cdots, \gamma_K(k)) = K$ and $\gamma_K(n) = n$. Then $\{\gamma_K\}$ is a system of representatives of the left cosets of $S(k) \times S(n-k-1) \times 1$ in $S(n-1)$. Similarly, representatives of left cosets of $S(k-1) \times S(n-k) \times 1$ in $S(n-1)$ are obtained by picking a $\gamma_L \in S(n)$ such that $(\gamma_L(1), \gamma_L(2), \cdots, \gamma_L(k-1)) = L, \gamma_L(n) = n$, for every (unordered) subset $L \subset (1, 2, \cdots, n-1)$ of $(k-1)$ elements. As is easily seen, the elements $\{\gamma_K\} \cup \{\gamma_L \cdot \xi\}$ then form a system of representatives of left cosets of $S(k) \times S(n-k)$ in $S(n)$ (recall that $\xi(i) = i$ for $i < k$, $\xi(k) = n$). The lemma follows, because the middle, top, and bottom rows of (8) are induced (passage to quotients) by the following endomorphisms of A respectively: $\sum_K \varphi(\gamma_K) + \sum_L \varphi(\gamma_L \cdot \xi)$, $\sum_K \varphi(\gamma_K)$, and $(\varphi(\xi) \circ (\sum_L \varphi(\gamma_L)) = \sum_L \varphi(\gamma_L \cdot \xi)$.

The following lemma is implicitly contained in [4] (see proof of 5.5).

LEMMA 2. *Let*

$$0 \xrightarrow{\sigma_0} B_0 \xrightarrow{\sigma_1} B_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_m} B_m$$

be morphisms in $\mathfrak{A}$. *Assume there exists a system of morphisms*

$$\tau_{k,n}: B_n \to B_k\,, \qquad k \leqq n \leqq m$$

such that

$$\tau_{k,n}\sigma_n \equiv \tau_{k,n-1} \mod \operatorname{im}(\sigma_k) \qquad \text{for } k < n \tag{9}$$

and

$$\tau_{k,k} = \mathrm{id}\,. \tag{10}$$

Then the morphism

$$T_n: B_n \to \textstyle\bigoplus_{k=0}^{n} B_k/\operatorname{im}(\sigma_k)$$

with components

$$B_n \xrightarrow{\tau_{k,n}} B_k \xrightarrow{\text{projection}} B_k/\operatorname{im}(\sigma_k) = \operatorname{coker}(\sigma_k)$$

is an isomorphism, and σ_n *has a left inverse,* $n = 0, 1, \cdots, m$.

The dual lemma is obtained by reversing arrows and compositions, lifting indices, and replacing coker (σ_k) *by ker* (σ^k) (e.g., (9) *becomes* $(\sigma^n\tau^{n,k} - \tau^{n-1,k})\,|\,\ker(\sigma^k) = 0)$.

PROOF by induction on n. The case $n = 0$ is trivial.

Condition (9) implies commutativity in

$$\begin{array}{ccc} & B_n & \\ \sigma_n \nearrow & & \searrow T'_n \\ B_{n-1} & \xrightarrow[T_{n-1}]{} & \bigoplus_{k=0}^{n-1} B_k/\mathrm{im}(\sigma_k) \end{array} \tag{11}$$

where T'_n has the same components (for $k < n$) as T_n. Since T_{n-1} is an isomorphism by induction, we have $(T_{n-1})^{-1}T'_n\sigma_n = \mathrm{id}$, i.e., σ_n has a left inverse, and

$$B_n \cong B_{n-1} \oplus \mathrm{coker}(\sigma_n) \cong \bigoplus_{k=0}^{n} B_k/\mathrm{im}(\sigma_k) \ .$$

Condition (10) guarantees that this isomorphism has the desired n^{th} component, q.e.d.

III. PROOF OF THEOREM 1. If W is a right $S(n)$-free acyclic complex over the integers $\mathbf{Z}$, then $H_*(S(n), G) = H(W \otimes_{S(n)} G)$ (see [1]); more generally $H_*(\pi, G) = H(W \otimes_\pi G)$ for every subgroup $\pi \subset S(n)$ since W is also π-free.

Now let $A = W \otimes G$, and let $S(n)$ operate on A by

$$(w \otimes g)x = wx \otimes x^{-1}g \ , \qquad w \in W, g \in G, x \in S(n) \ .$$

Then $W \otimes_\pi G = A_\pi$ for every subgroups $\pi \subset S(n)$. Further, the homomorphism $H_*(\rho, G) \to H_*(\pi, G)$ induced by the inclusion of a subgroup, $\rho \subset \pi$, corresponds to $H(i_{\pi,\rho})$: $H(A_\rho) \to H(A_\pi)$ (see (3)). Now consider the diagram

$$\begin{array}{ccccccc}
H_*(S(k)) & \xleftarrow{p_{k,n-k-1*}} & H_*(S(k) \times S(n-k-1) \times 1) & & & \nwarrow^{t_{n-k-1,n-1*}} & \\
\Vert & & \downarrow i'_* & & & & \\
H_*(S(k)) & \xleftarrow{p_{k,n-k*}} & H_*(S(k) \times S(n-k)) & \xleftarrow{t_{n-k,n*}} & H_*(S(n)) & \xleftarrow{i_{n*}} & H_*(S(n-1)) \\
\uparrow i_{k*} & & \uparrow i''_* & & & \swarrow_{\xi_* t''_*} & \\
H_*(S(k-1)) & \xleftarrow[p_{k-1,n-k*}]{} & H_*(S(k-1) \times 1 \times S(n-k)) & & & &
\end{array} \tag{12}$$

Its right part (bold face arrows) is obtained from diagram (8) by passing to homology. The maps in the left part are induced by natural homomorphisms between the corresponding permutation groups (injections and projections). The left part is obviously commutative; therefore Lemma 1 shows

(13) $$\tau_{k,n} i_{n*} \equiv \tau_{k,n-1} \quad \text{mod} \quad \text{im}(i_{k*})$$

where

(14) $$\tau_{k,n} = p_{k,n-k*} t_{n-k,n*} \ .$$

Lemma 2 now applies and asserts that i_{n*}, and hence $\iota_{n,k*} = i_{n*} i_{n-1*} \cdots i_{k+1*}$ has a left inverse. Moreover, it describes explicitly the resulting splitting

(15) $$H_*(S(n)) \cong \bigoplus_{k=0}^{n} H_*(S(k))/\text{im}\,\{H(S(k-1)) \to H(S(k))\}$$

in terms of the $\tau_{k,n}$, i.e., transfers and projections $S(k) \times S(n-k) \to S(k)$.

The cohomology part of Theorem 1 is strictly dual: We define $A = \text{Hom}_Z(W, G)$ and let $S(n)$ operate by

$$(xf)(w) = f(wx) \ , \qquad f \in A,\ w \in W,\ x \in S(n) \ .$$

Then $H^*(\pi, G) = H(A^\pi)$ for every subgroup $\pi \subset S(n)$; the cohomology homomorphism induced by an inclusion, $\rho \subset \pi$, corresponds to $H(i^{\rho,\pi})$: $H(A^\pi) \to H(A^\rho)$. We look at the dual diagram (12) (arrows and compositions reversed, indices lifted), and deduce from the dual Lemma 1 that

(16) $$(i^{n*} \tau^{n,k} - \tau^{n-1,k}) \,|\, \ker(i^{k*}) = 0$$

where $\tau^{n,k}$ is the composition

$$H^*(S(k)) \xrightarrow{p^*} H^*(S(k) \times S(n-k)) \xrightarrow{t^*} H^*(S(n)) \ .$$

Then the dual Lemma 2 applies and shows that i^{n*}: $H^*(S(n)) \to H^*(S(n-1))$ has a right inverse, q.e.d.

COROLLARY. *If G is an abelian group on which $S(n)$ operates trivially then the homomorphisms* $H_*(S(n-1), G) \underset{t_n*}{\overset{i_n*}{\rightleftarrows}} H_*(S(n), G)$ *(t_n = transfer, i_{n*} induced by $S(n-1) \subset S(n)$) have the property*

(18) $$i_{n*} t_{n*} = n \cdot \text{id},\ t_{n*} i_{n*} = n \cdot \text{id} \ .$$

In particular, i_{n*} and t_{n*} are isomorphisms if the order of every $g \in G$ is finite and prime to n.

A dual result holds for cohomology.

Indeed, the first relation (18) follows from (5), the second from

$$i_{n*}(t_{n*} i_{n*}) = (i_{n*} t_{n*}) i_{n*} = n \cdot i_{n*} = i_{n*}(n \cdot \text{id}) \ ,$$

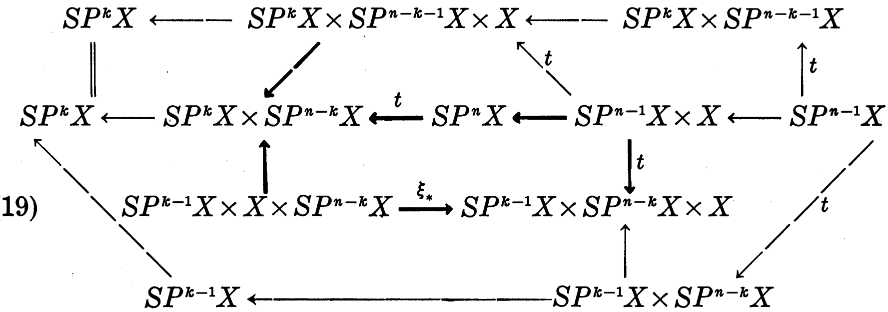
because i_{n*} is monomorphic.

PROOF OF THEOREMS 2 AND 3. Let $X = CY$ be the FD-complex (=chain complex with face- and degeneracy-operators) of Y, and put $A = C(Y^n) = X^n = n^{\text{th}}$ cartesian power of X on which $S(n)$ operates by permutation of factors. Then

$$C(SP^n Y) = SP^n X = A_{S(n)},$$
$$C(SP^k Y \times SP^{n-k} Y) = SP^k X \times SP^{n-k} X = A_{S(k)\times S(n-k)}\ , \text{ etc.}$$

(compare [2, 6.2]). Consider the diagram

(19)



where all arrows without letters denote natural inclusions or projections (using a base point in Y if necessary), and t denotes the appropriate transfer. The center part (bold face arrows) is again diagram (8), the pasted-on pieces are commutative. It follows from Lemma 1 that

(20) $$\tau_{k,n} i_n \equiv \tau_{k,n-1} \mod \operatorname{im}(i_k)\ ,$$

where $i_n\colon SP^{n-1}X \to SP^n X$ is the inclusion, and $\tau_{k,n}$ the composite

(21) $$SP^n X \xrightarrow{\ t\ } SP^k X \times SP^{n-k} X \longrightarrow SP^k X\ .$$

Lemma 2 now produces a left inverse of i_n, the corresponding splitting

(22) $$SP^n X \cong \bigoplus_{k=0}^{n} SP^k X / SP^{k-1} X$$

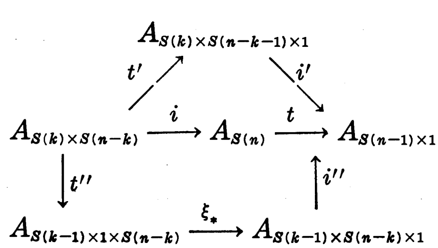
being induced by the maps (21).

The proof of Theorem 3 is strictly dual: We use the same A; then $\Gamma_n Y = A^{S(n)}$. We take the dual diagram (19) (reverse arrows, replace $SP^n X = A_{S(n)}$ by $A^{S(n)}$ etc.), get the dual of (20), and apply the dual Lemma 2.

PROOF OF THEOREM 4 (sketch). This proof requires a modified Lemma 1 as follows (notations as in Lemma 1).

LEMMA 1′. *Let A be an object in $\mathfrak{A}$ on which $S(n)$ operates on the left. Then in the diagram*

(23)

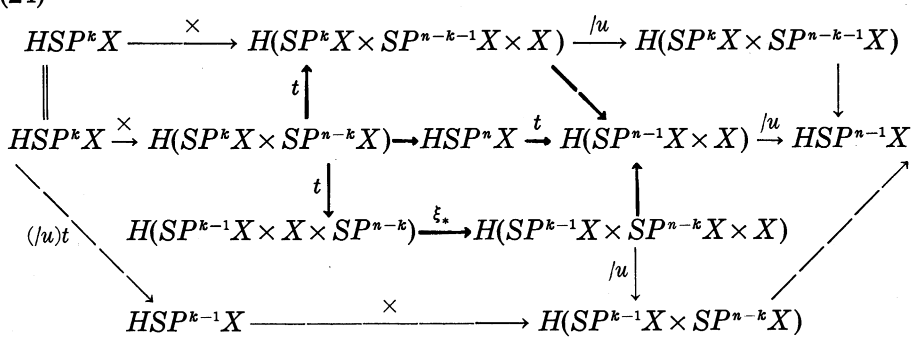


the middle row is the sum of the top row and the bottom row. i.e., $ti = i't' + i''\xi_* t''$.

One can verify Lemma 1′ similarly to Lemma 1. Another argument which shows the connection to Lemma 1 runs as follows: The norm morphisms (6) give a map of (23) into the dual diagram (8) (because the diagrams (7) are commutative). In the special case where $A = \mathbf{Z}[S(n)] =$ group ring of $S(n)$ over the integers, the norm morphisms are isomorphisms, hence Lemma 1 and Lemma 1′ are equivalent. Now, an arbitrary object $A \in \mathfrak{A}$ on which $S(n)$ operates can be interpreted as an additive functor $\Phi\colon \mathbf{Z}[S(n)] \to \mathfrak{A}$; ($\mathbf{Z}[S(n)]$ is a subcategory of the category of abelian groups; it has a single object, and its morphisms are the left multiplications of the group ring). This functor extends to objects of the form $\mathbf{Z}[S(n)]_\pi$ where $\pi \subset S(n)$. In particular, (23) is obtained from the corresponding diagram for $\mathbf{Z}[S(n)]$ by applying Φ; therefore Lemma 1′ is true in general.

The proof of Theorem 4 is similar to the one for Theorem 2. We look at the diagram ($H =$ homology with coefficients in G)

(24)



where arrows without letters are natural projections, t denotes transfers, $/u$ is the slant product with u and $\times$ is the cross product with a *divided power* $\gamma_r(z) \in H_{2mr}(SP^r X, \mathbf{Z})$ or with $\gamma_r(z) \times z \in H_{2mr+2m}(SP^r X \times X, \mathbf{Z})$. These operations γ_r have been defined by J.C. Moore (unpublished) in the generality needed here; they will be explained and used elsewhere. A characterization of them is formulated below without proof.

The central part of (24) (bold face arrows) is (23) after applying H. The pasted-on pieces are commutative: For the upper left piece this is (25); for the other pieces, one uses standard properties of t and the slant- and cross-product. Since

$$(a \times \gamma_{k-n-1}(z) \times z)/u = (a \times \gamma_{n-k-1}(z))\cdot\langle u, z\rangle = a \times \gamma_{n-k-1}(z)$$

for $a \in HSP^kX$, the first line in (24) equals $\times \gamma_{n-k-1}(z)$. Therefore Lemma 1′ shows

$$(\sigma^n \tau^{n,k} - \tau^{n-1,k}) \,|\, \ker(\sigma^k) = 0$$

where

$$\sigma^n = (/u)t\colon HSP^nX \to HSP^{n-1}X$$

and $\tau^{n,k}$ is the composite

$$HSP^kX \xrightarrow{\times} H(SP^kX \times SP^{n-k}X) \longrightarrow HSP^nX\ .$$

The dual Lemma 2 then proves the homology part of Theorem 4.

A characterization of γ_r. For every $n>0$ there exists a unique sequence of natural transformations

$$\gamma_r\colon H_{2n}(X, \mathbf{Z}) \to H_{2nr}(SP^rX, \mathbf{Z})\ , \qquad r = 1, 2, \cdots$$

in the category of all FD*-complexes X, such that*

$$\gamma_1 = \text{id}, \quad t(\gamma_{r+1}(z)) = \gamma_r(z) \times z \tag{25}$$

for $z \in H_{2n}(X, \mathbf{Z})$ and $t =$ transfer: $H(SP^{r+1}Y, \mathbf{Z}) \to H(SP^rX \times X, \mathbf{Z})$.

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