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PARTITIONS OF UNITY IN THE THEORY OF FIBRATIONS

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The following properties of continuous maps play an important role in most problems involving fibrations:

(A) The *section extension property* (SEP). A map $p: E \to B$ has the SEP if every partial section over $A \subset B$ which can be extended to a halo around A has a global extension (2.2, 3.1).

(B) The *covering homotopy property* (CHP), which says that homotopies in B can be lifted to E with given initial position (4.2, 4.1).

(C) The *weak covering homotopy property* (WCHP), which holds if homotopies can be lifted with an initial position which is vertically homotopic to a given position (5.13, 5.1).

(D) The property of being a *fibre homotopy equivalence*. Given $p: E \to B$, $p': E' \to B$ ("spaces over B") a map $f: E \to E'$ is called "map over B" if $p'f = p$. A homotopy equivalence in the category of maps over B is called fibre homotopy equivalence (1.3).

Roughly speaking, we show that each of these properties P is a local property with respect to B. More precisely, P holds provided it holds over every set of a numerable covering $\{V_\lambda\}_{\lambda \in \Lambda}$ of B. *Numerable* means there exists a locally finite partition of unity $\pi_\gamma: B \to [0, 1]$, $\gamma \in \Gamma$, such that the covering $\{\pi_\gamma^{-1}(0, 1]\}$ refines $\{V_\lambda\}$.

Case (A), then, generalizes a result of Godement [7, Ch. II, 3.3, 3.4], and implies (see 2.8) that a locally trivial map $p: E \to B$ into a paracompact B admits a section provided

(i) $p^{-1}(b)$ is contractible for every $b \in B$; or

(ii) $p^{-1}(b)$ is $(n-1)$-connected, and B is locally a CW-complex of dimension $\leqq n$ (compare Steenrod [15, 12.2 and App. 3 of its 2nd ed.] for more comments see 2.9). Case (B) recovers the results of Hurewicz [10] and Huebsch [9]. Special cases of (C) were found by Fadell [5] and Fuchs [6]. The proof for the simplest case, (A), is carried out in § 2; the other cases are reduced to (A) in the succeeding §§ 3–5.

Sections 6–9 contain applications to fibre homotopy equivalence and bundle classification (in particular 6.1, 6.3, 6.4, 7.5, 8.1, 9.1). For a summary of the results, we refer the reader to the introductions of these sections.

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1. Spaces over B

Let B be a topological space. We consider categories whose objects are continuous maps into B. This section gives only the basic definitions pertaining to these categories and some elementary results (without proofs).

1.1. A continuous map $p: E \to B$ is called a *space over* B (espace découpé in Godement [7, II, 1.1]). If $p: E \to B$, $p': E' \to B$ are spaces over B then a continuous map $f: E \to E'$ is called a *map over* B if $p'f = p$. Under ordinary composition the maps over B form a category C_B. The category C_B has a privileged object, namely id_B, the identity map of B. Every space over B admits a unique map over B into id_B, namely p.

1.2. For every topological space Y define a space over B by $E = B \times Y$, $p(b, y) = b$. A space over B is called *trivial* if it is equivalent (in C_B) to a space of this form.

1.3. A homotopy $\Theta: E \times I \to E'$ is called a homotopy over B or *vertical homotopy* if

$$\Theta_t: E \to E', \qquad \Theta_t(e) = \Theta(e, t)$$

is a map over B for every $t \in I = [0, 1]$. Two maps $f_0, f_1: E \to E'$ are *vertically homotopic*, $f_0 \simeq_B f_1$, if there exists a vertical homotopy Θ with $\Theta_0 = f_0$, $\Theta_1 = f_1$. We write $\Theta: f_0 \simeq_B f_1$ (read: Θ is a vertical homotopy from f_0 into f_1). $\simeq_B$ is an equivalence relation between maps over B which is compatible with composition. By identifying equivalent maps, we get a new category $\bar{C}_B$ whose objects are those of C_B and whose morphisms are vertical homotopy classes of maps over B.

We say $p: E \to B$ is *dominated* by $p': E' \to B$ (or p' dominates p) if there exist maps $f: E \to E'$, $g: E' \to E$ over B such that $gf \simeq_B \mathrm{id}_E$ (i.e., p is a "retract" of p' in the category $\bar{C}_B$).

A map $f: E \to E'$ over B whose class in $\bar{C}_B$ is an equivalence (i.e., has left and right inverses) is called a *fibre homotopy equivalence*. Furthermore, $p: E \to B$ is called *fibre-homotopically trivial* if p is fibre homotopy equivalent to a trivial space $B \times Y \to B$.

1.4. If $p: E \to B$ is a space over B and $h: X \to E$ a continuous map then $ph: X \to B$ is a space over B, and h becomes a map over B. It makes sense, therefore, to say that two maps $h_0, h_1: X \to E$ with $ph_0 = ph_1$ are vertically homotopic, $h_0 \simeq_B h_1$, etc.

1.5. The following properties of $p: E \to B$ are clearly equivalent:

(a) p is a fibre homotopy equivalence (viewed as map over B into id_B),

(b) p is dominated by id_B,

(c) there exists a map $s\colon B \to E$ such that $ps = \mathrm{id}_B$ (a *section*) and a vertical homotopy $\Theta\colon sp \simeq_B \mathrm{id}_E$.

If one of (a), (b), (c) holds, p is called *shrinkable*.

For example, if $E = B^I$ denotes the space of all paths $w\colon I \to B$ with the compact-open topology, and $p(w) = w(0)$, then p is shrinkable: Put $s(b)(t) = b$, $\Theta(w, \tau)(t) = w(t\tau)$, $t, \tau \in I$, $b \in B$, $w \in B^I$. A fibre-homotopically trivial space over B is shrinkable if and only if Y is contractible.

1.6. If $p\colon E \to B$ is a space over B, and $\alpha\colon A \to B$ a continuous map, we define the *induced space* $p_\alpha\colon E_\alpha \to B_\alpha = A$ by

$$E_\alpha = \{(e, a) \in E \times A \mid p(e) = \alpha(a)\}\ , \qquad p_\alpha(e, a) = a\ .$$

The space E_α is determined up to homeomorphism by the following ("pull-back"-) property: *There exists a natural* (with respect to X) *bijection between continuous maps* $\beta\colon X \to E_\alpha$ and pairs $\beta_1\colon X \to A$, $\beta_2\colon X \to E$ such that $\alpha\beta_1 = p\beta_2$. In our particular construction this is given by $\beta_1 = p_\alpha\beta$, $\beta_2 = \hat{\alpha}\beta$, where $\hat{\alpha}(e, a) = e$.

If $\alpha\colon A \to B$ is the inclusion of a subset $A \subset B$, we write $p_A\colon E_A \to A$ for the induced space; it can be identified with $p \mid p^{-1}(A)\colon p^{-1}(A) \to A$.

If $f\colon E \to E'$ is a map over B the *induced map* $f_\alpha\colon E_\alpha \to E'_\alpha$ (resp. f_A if $A \subset B$) is defined by $f_\alpha(e, a) = (f(e), a)$. Induced spaces and maps then form a covariant functor $\mathcal{C}_\alpha\colon \mathcal{C}_B \to \mathcal{C}_A$. It preserves vertical homotopies, and therefore induces a functor $\bar{\mathcal{C}}_\alpha\colon \mathcal{C}_B \to \mathcal{C}_A$.

1.7. If P is a property of continuous maps then we say that $p\colon E \to B$ resp. $f\colon E \to E'$ has the *property P over $A \subset B$* if p_A (resp. f_A) has the property P. In this sense we use, for example, "p is trivial over A", "f is a fibre homotopy equivalence over A", etc. We say f has the *property P locally* if every $b \in B$ has a neighborhood V such that f has the property P over V.

2. The section extension property (SEP)

We show that a space over B which satisfies a certain local extension condition for sections, has the corresponding global extension property (2.7). This contains the known fact that a locally trivial map with contractible fibre (or n-connected fibre and locally triangulable base of dimension $\leqq n + 1$) admits a section (2.8).

2.1. DEFINITIONS. A *halo* around $A \subset B$ is a subset V of B such that there exists a continuous function $\tau\colon B \to [0, 1]$ with $A \subset \tau^{-1}(1)$, $CV \subset \tau^{-1}(0)$. For example, every $V \subset B$ is a halo around the empty set $\varnothing$ (take $\tau = 0$);

if B is normal and A closed, then every neighborhood of A is a halo. If $\tau\colon B \to [0, 1]$ is continuous then $V = \tau^{-1}(0, 1]$ is a halo around $\tau^{-1}[\varepsilon, 1]$ for every $\varepsilon > 0$ (take the function $\mathrm{Min}(1, (1/\varepsilon)\tau)$).

A (not necessarily open) covering $\{V_\lambda\}_{\lambda \in \Lambda}$, of B is called *numerable** if it admits a refinement by a locally finite partition of unity, i.e., if there exists a locally finite partition of unity $\{\pi_\gamma\colon B \to [0, 1]\}_{\gamma \in \Gamma}$ (a *numeration* of $\{V_\lambda\}$) such that every set $\pi_\gamma^{-1}(0, 1]$ is contained in some V_λ. For example, paracompact (resp. normal) spaces are characterized by the fact that every open covering (resp. every locally finite open covering) is numerable (Bourbaki, § 4, nos. 3–4). If $\{V_\lambda\}$ is numerable and $\alpha\colon X \to B$ continuous, then $\{\alpha^{-1}(V_\lambda)\}$ is numerable (by $\{\pi_\gamma \alpha\}$).

2.2. DEFINITION. A space $p\colon E \to B$ over B has the SEP if the following holds. For every $A \subset B$ and every section s over A which admits an extension to a halo V around A, there exists an extension S over B, i.e., a section $S\colon B \to E$ with $S \mid A = s$.

In particular, p then always has a section: take $A = \varnothing = V$.

2.3. PROPOSITION. *If $p\colon E \to B$ is dominated by $p'\colon E' \to B$ (1.3), and p' has the* SEP, *then p has the* SEP. *In particular, every shrinkable space has the* SEP (since it is dominated by id_B; see 1.5).

PROOF. (i) Since p' dominates p, we have maps $f\colon E \to E'$, $g\colon E' \to E$ over B and a vertical homotopy $\Theta\colon gf \simeq_B \mathrm{id}_E$ ($\Theta_0 = gf$, $\Theta_1 = \mathrm{id}$).

(ii) Take a section s over A which admits an extension to a halo V; let $\tau\colon B \to [0, 1]$ be a haloing function, and denote the extension of s by the same letter, so $s\colon V \to E$. We have to find a section $S\colon B \to E$ with $S \mid A = s \mid A$.

Because p' has the SEP, there exists a section $S'\colon B \to E'$ with $S' \mid \tau^{-1}[\frac{1}{2}, 1] = fs \mid \tau^{-1}[\frac{1}{2}, 1]$. Then define S by

$$S(b) = \begin{cases} gS'(b) & \text{for } \tau(b) \leqq \frac{1}{2} \\ \Theta(s(b), 2\tau(b) - 1) & \text{for } \tau(b) \geqq \frac{1}{2}\,. \end{cases}$$

2.4. PROPOSITION. *Let $p\colon E \to B$ be a space over B, and $r\colon B' \to B$ a retraction* (i.e. there exists $i\colon B \to B'$ with $ri = \mathrm{id}$, so $B \subset B'$). *If the induced space $p_r\colon E_r \to B'$ has the* SEP *then p has the* SEP.

PROOF. Let $A \subset V$, s, τ as in 2.3 (ii). Define $A' = r^{-1}(A)$, $V' = r^{-1}(V)$, $\tau' = \tau r$. The map $\sigma = sr$, together with the inclusion map $V' \to B'$, defines a section $s'\colon V' \to E_r$ (see pull-back characterization in 1.6) with $s' \mid V = s$. By assumption, there exists a section $S'\colon B' \to E_r$ with $S' \mid A' = s'$.

* or *normal* as is customary in the literature. These coverings were used by Michael and by Hurewicz for problems which are related to ours.

Then $S = S' \mid B$ is the required extension of s.

2.5. EXAMPLE. *If $p: E \to B$ is dominated by a trivial space $B \times Y$ with $\pi_i Y = 0$ for $i < n$* ($\pi_0 = 0$ means Y is arcwise connected), *and if B is a retract of a* CW-*complex of dimension $\leqq n$ then p has the* SEP, $n \leqq \infty$.

PROOF. By 2.3 we can assume $E = B \times Y$, and by 2.4 that B is itself a CW-complex of dimension $\leqq n$. Let $A \subset V, s, \tau$ be as in 2.3 (ii). Composing s with the projection $B \times Y \to Y$, we get a map $\sigma: V \to Y$ with $s(b) = (b, \sigma(b))$, and we have to find $\Sigma: B \to Y$ with $\Sigma \mid A = \sigma \mid A$. Let B^i denote the i-skeleton of B, and $T^i = B^i \cup \tau^{-1}[(i+1)/(i+2), 1]$, $i = 0, 1, \cdots$; clearly $A \subset \tau^{-1}(1) \subset T^i$ for all i. By induction on i we construct $\Sigma^i: T^i \to Y$ such that

$$\Sigma^i \mid B^{i-1} = \Sigma^{i-1} \mid B^{i-1} \qquad \text{and} \qquad \Sigma^i \mid A = \sigma \mid A .$$

The first equation then shows that $\Sigma = \lim (\Sigma^i)$ for $i \to \infty$ is well-defined and continuous, the second gives $\Sigma \mid A = \sigma \mid A$.

To start the induction, define Σ^0 to be σ on $\tau^{-1}[\frac{1}{2}, 1]$ and let Σ^0 have arbitrary values on the remaining vertices of B. Assume then Σ^{i-1} has already been found, $i > 0$. Pick an i-cell e^i_λ, and let $\Phi = \Phi^i_\lambda: \Delta^i \to B$ be its characteristic map (Δ^i = standard i-simplex). For large N the N-fold barycentric subdivision of Δ^i contains a subcomplex K with

$$(\tau\Phi)^{-1}\left[\frac{i+1}{i+2}, 1\right] \subset K \subset (\tau\Phi)^{-1}\left[\frac{i}{i+1}, 1\right]$$

(make N so large that every simplex which meets $(\tau\Phi)^{-1}[(i+1)/(i+2), 1]$ lies in $(\tau\Phi)^{-1}(i/(i+1), 1]$). Now the map $\Sigma^{i-1}\Phi$ is defined on K because $\Phi(K) \subset \tau^{-1}[i/(i+1), 1]$, and it is defined on the boundary $\dot{\Delta}^i$ of Δ^i because $\Phi(\dot{\Delta}^i) \subset B^{i-1}$; hence $\Sigma^{i-1}\Phi$ is defined on the subcomplex $K \cup \dot{\Delta}^i$ of Δ^i. Since $\pi_\mu(Y) = 0$ for $\mu < i$ this map can be extended by the usual skeleton-after-skeleton construction to $\Sigma^i_\lambda: \Delta^i \to Y$. Do this for all i-cells e^i_λ and define

$$\Sigma^i(x) = \begin{cases} \Sigma^{i-1}(x) & \text{for } x \in B^{i-1} \cup \tau^{-1}\left[\dfrac{i+1}{i+2}, 1\right] \supset B^{i-1} \cup A , \\ \Sigma^i_\lambda(\Phi^{-1}(x)) & \text{for } x \in e^i_\lambda . \end{cases}$$

This is easily checked to be well-defined. It is continuous on closed i-cells, hence on B^i. It is also continuous on the closed set $\tau^{-1}[(i+1)/(i+2), 1]$, hence on the union $T^i = B^i \cup \tau^{-1}[(i+1)/(i+2), 1]$, q.e.d.

2.6. PROPOSITION. *If $p: E \to B$ has the* SEP, *and if $W \subset B$ is an open set such that $W = \rho^{-1}(0, 1]$ for some continuous function $\rho: B \to [0, 1]$, then $p_W: E_W \to W$* (see 1.6) *has the* SEP.

PROOF. Given a function τ: $W \to [0, 1]$ and a section s of p over $\tau^{-1}(0, 1]$, we have to find a section S over W which agrees with s on $\tau^{-1}(1)$. The difference from the usual situation (2.3, (ii)) is, of course, that τ is not defined over all of B.

We first choose continuous functions μ_n, λ_n: $[0, 1] \to [0, 1]$ such that

$$\mu_n(x) = \begin{cases} = \dfrac{1}{n} & \text{for } x \leqq 1 - \dfrac{1}{n} \\ = \dfrac{1}{n+2} & \text{for } x \geqq 1 - \dfrac{1}{n+1} \\ \leqq \dfrac{1}{n} & \text{for all } x \in [0, 1]\,, \end{cases}$$

$\lambda_n(x) = 1/(n + 1)$ for $x \leqq 1 - 1/n$, and $\varepsilon < \lambda_n(x) < \mu_n(x)$ for all x and some $\varepsilon > 0$; $n = 1, 2, \cdots$. This can clearly be done.

We then inductively construct sections S_n: $B \to E$, $n = 2, 3, \cdots$ such that

(α) $S_{n+1}(b) = S_n(b)$ for $\rho(b) > 1/n$,

(β) $S_n(b) = s(b)$ for $\{\tau(b) > 1 - 1/n$ and $\rho(b) > 1/(n + 1)\}$.

The condition (α) then shows that $S = \lim (S_n)$ (for $n \to \infty$) is a well-defined section over $W = \{b \in B \mid \rho(b) > 0\}$, and ($\beta$) implies that $S(b) = s(b)$ for $\tau(b) = 1$.

Note that the function $\tau' = \tau \cdot \rho$ is defined on all of B ($\tau' = 0$ on CW), and $s(b)$ is defined for $\tau'(b) > 0$. Because p has the SEP, we can therefore find a section S_2: $B \to E$ which agrees with s on $\tau'^{-1}[\frac{1}{6}, 1]$, in particular on $\{b \in B \mid \tau(b) > \frac{1}{2}$ and $\rho(b) > \frac{1}{3}\}$. This starts the induction.

To get from n to $n + 1$, define a section s_{n+1} over $V_{n+1} = \{b \in W \mid \rho(b) > \lambda_n \tau(b)\}$ by

$$s_{n+1}(b) = \begin{cases} S_n(b) & \text{for } \rho(b) > \dfrac{1}{n+1} \\ s(b) & \text{for } \tau(b) > 1 - \dfrac{1}{n} \end{cases}$$

(one of the two inequalities always holds in V_{n+1}, by definition of λ_n, and if both hold, then $S_n = s$ by condition (β)).

V_{n+1} is a halo around $A_{n+1} = \{b \in W \mid \rho(b) > \mu_n \tau(b)\}$ (a haloing function is 1 for $b \in A_{n+1}$, 0 for $b \in CV_{n+1}$, and $[\rho(b) - \lambda_n \tau(b)]/[\mu_n \tau(b) - \lambda_n \tau(b)]$ otherwise). Therefore (by the SEP for p) there exists a section S_{n+1} over B with $S_{n+1} \mid A_{n+1} = s_{n+1} \mid A_{n+1}$. We check ($\alpha$), ($\beta$):

If $\rho(b) > 1/n$, then $\rho(b) > \mu_n \tau(b)$, so $b \in A_{n+1}$ and $S_{n+1}(b) = s_{n+1}(b) = S_n(b)$.

If $\tau(b) > 1 - 1/(n + 1)$ and $\rho(b) > 1/(n + 2)$, then $\rho(b) > \mu_n \tau(b)$, so

$b \in A_{n+1}$ and $S_{n+1}(b) = s_{n+1}(b) = s(b)$, q.e.d.

2.7. SECTION EXTENSION THEOREM. *Let* $p\colon E \to B$ *be a space over* B. *If there exists a numerable covering* $\{V_\lambda\}_{\lambda\in\Lambda}$ *of* B *such that* p *has the* SEP *over each* V_λ (see 1.7), *then* p *has the* SEP.

2.8. COROLLARY. *Let* $p\colon E \to B$ *be a space over* B, $A \subset B$, *and* s *a section over* A *which admits an extension to a halo* V *around* A. *Assume there exists a numerable covering* $\{V_\lambda\}_{\lambda\in\Lambda}$ *of* $B - A$ *such that*

(α) p_{V_λ} *is shrinkable for each* λ (i.e., fibre homotopy equivalent to a trivial space $V_\lambda \times Y$ with contractible Y; see 1.5 and 3.1 for equivalent conditions), *or*

(β) p_{V_λ} *is dominated by a trivial space* $V_\lambda \times Y_\lambda$ *with* $\pi_i(Y_\lambda) = 0$ *for* $i < n_\lambda$, *and* V_λ *is a retract of a* CW*-complex of dimension* $\leqq n_\lambda$, $n_\lambda \leqq \infty$. *Then there exists a section* $S\colon B \to E$ *with* $S \mid A = s$.

If S, S' *are two sections of* p *with* $S \mid V = S' \mid V$, *then* $S \simeq_B S'$ rel A, *provided that, in case* (β), *we have* $\pi_i(Y_\lambda) = 0$ *for* $i < n_\lambda + 1$.

2.9. REMARKS. An example of a $p\colon E \to B$ which is not locally trivial but to which 2.8 applies is as follows: $E = \{(x, y) \in \mathbf{R}^2 \mid |y| \leqq |x|\}$, $B = \mathbf{R}$, $p(x, y) = x$. Clearly p is shrinkable to the central section $y = 0$.

In Steenrod [15, 12.2 (see also App. 1 of its 2nd ed.], and Fadell [5, 3.9, 5.3]) which corresponds to the locally trivial case of 2.8α, the fibre Y is assumed to be *solid* which implies contractibility (provided $Y \times I$ is normal). An example of a contractible space which is not solid is the cone over a converging sequence of circles as indicated in the figure.

On the other hand Steenrod [15, 12.2] does not assume an extension s' of s to a neighborhood of A. Under suitable local conditions on B or Y, s' will always exist; we shall not pursue this question because in most applications the construction of s' is obvious.

PROOF OF 2.8. (α) (resp. (β)) implies by 2.3 (resp. 2.5) that p has the

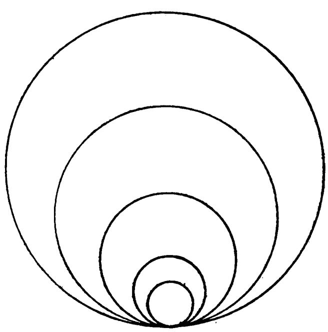

Fig. 1

SEP over each V_λ; therefore, by Theorem 2.7, also over $B' = B - A = \bigcup_\lambda V_\lambda$. Hence we can find a section $S'\colon B' \to E$ with $S' \mid A' = s \mid A'$ where $A' = B' \cap \tau^{-1}[\frac{1}{2}, 1]$ and $\tau\colon B \to [0, 1]$ is a function for the halo V around A (s can be assumed to be defined over V). Then define $S\colon B \to E$ by $S(b) = s(b)$ for $\tau(b) > \frac{1}{2}$, $S(b) = S'(b)$ for $\tau(b) < 1$.

The second statement, $S \simeq_B S'$ rel A, reduces to the first by taking the induced space $p_\alpha = p \times I\colon E \times I \to B \times I$ (where $\alpha\colon B \times I \to B$ is the projection), and starting with the section $\tilde{s}$ over

$$\tilde{V} = B \times [0, \tfrac{1}{2}) \cup B \times (\tfrac{1}{2}, 1] \cup V \times [0, 1]$$

which is given by $S \times [0, \frac{1}{2})$, $S' \times (\frac{1}{2}, 1]$, $S \times [0, 1]$. Then $\tilde{s}$ is defined over $\tilde{\tau}^{-1}(0, 1]$ where

$$\tilde{\tau}\colon B \times [0, 1] \to [0, 1],\ \tilde{\tau}(b, t) = \operatorname{Min}\big(\tau(b) + |2t - 1|, 1\big)\ .$$

Therefore a section $\tilde{S}\colon B \times I \to E \times I$ exists with $\tilde{S} \mid \tilde{A} = \tilde{s} \mid \tilde{A}$, and $\tilde{A} = B \times \{0\} \cup B \times \{1\} \cup A \times [0, 1]$. The composite

$$B \times I \xrightarrow{\tilde{S}} E \times I \to E$$

is then the required vertical homotopy rel A, q.e.d.

PROOF OF 2.7. By 2.6 we can assume that $\{V_\lambda\}$ itself (not only a refinement) is given by a locally finite partition of unity $\{\pi_\lambda\colon B \to [0, 1]\}$, i.e., $V_\lambda = \pi_\lambda^{-1}(0, 1]$. Take $A \subset V$, $s\colon V \to E$, $\tau\colon B \to [0, 1]$ as in 2.3 (ii); we have to find a section $S\colon B \to E$ with $S \mid A = s \mid A$.

Let $\pi'_\lambda = (1 - \tau)\pi_\lambda$, $\pi'_0 = \tau$, $\Lambda' = \Lambda \cup \{0\}$. Then $\{\pi'_\lambda\}_{\lambda \in \Lambda'}$ is a locally finite partition of unity, and p has the SEP over every $\pi'^{-1}_\lambda(0, 1]$ with $\lambda \neq 0$. From now on we write π_λ instead of π'_λ, and Λ instead of Λ'.

For every $\Gamma \subset \Lambda$ put $\pi_\Gamma = \Sigma_{\lambda \in \Gamma}\pi_\lambda\colon B \to [0, 1]$. Consider all pairs (Γ, S_Γ) where $0 \in \Gamma \subset \Lambda$, and $S_\Gamma\colon \pi_\Gamma^{-1}(0, 1] \to E$ is a section with $S_\Gamma \mid \tau^{-1}(1) = s \mid \tau^{-1}(1)$; e.g., $\Gamma = \{0\}$, $S_\Gamma = s$. Define a partial ordering $\leqq$ as follows: $(\Gamma, S_\Gamma) \leqq (\Gamma', S_{\Gamma'})$ if and only if $\Gamma \subset \Gamma'$ and $S_\Gamma(b) = S_{\Gamma'}(b)$ for $\pi_\Gamma(b) = \pi_{\Gamma'}(b) > 0$, $b \in B$ (i.e., $S_\Gamma(b) \neq S_{\Gamma'}(b) \Rightarrow \pi_\mu(b) \neq 0$ for some $\mu \in \Gamma' - \Gamma$).

If $\Sigma = \{(\Gamma^\sigma, S_{\Gamma^\sigma})\}$ is a strictly ordered system of such pairs, put $\Gamma = \bigcup_{\sigma \in \Sigma}\Gamma^\sigma$; we want to construct S_Γ such that $(\Gamma, S_\Gamma) \geqq (\Gamma^\sigma, S_{\Gamma^\sigma})$ for all $\sigma \in \Sigma$. Let $b \in \pi_\Gamma^{-1}(0, 1]$. It has a neighborhood W which intersects only finitely many of the sets $\pi_\gamma^{-1}(0, 1]$, $\gamma \in \Gamma$, say those with index $\gamma_1, \gamma_2, \cdots, \gamma_r$. Pick $\rho \in \Sigma$ such that $\gamma_1, \cdots, \gamma_r \in \Gamma^\rho$. Then, if $\sigma \geqq \rho$, all functions π_μ with $\mu \in \Gamma^\sigma - \Gamma^\rho$ vanish on W, hence $S_{\Gamma^\sigma} \mid W = S_{\Gamma^\rho} \mid W$, and we define $S_\Gamma\colon \pi_\Gamma^{-1}(0, 1] \to E$ by $S_\Gamma \mid W = S_{\Gamma^\rho} \mid W$. Suppose now $S_\Gamma(b) \neq S_{\Gamma^{\sigma'}}(b)$ for some $\sigma' \in \Sigma$ and $b \in \pi_{\Gamma^{\sigma'}}^{-1}(0, 1]$, and pick ρ as above. Then $S_\Gamma(b) = S_{\Gamma^\rho}(b)$, hence $S_{\Gamma^{\sigma'}}(b) \neq S_{\Gamma^\rho}(b)$, hence $\rho > \sigma'$ and $\pi_\mu(b) \neq 0$ for some $\mu \in \Gamma^\rho - \Gamma^{\sigma'} \subset$

$\Gamma - \Gamma^{\sigma'}$. This proves $(\Gamma, S_\Gamma) \geqq (\Gamma^{\sigma'}, S_{\Gamma^{\sigma'}})$ for all $\sigma' \in \Sigma$.

We now apply Zorn's lemma and obtain a maximal (Γ, S_Γ). We claim $\Gamma = \Lambda$; since $\pi_\Lambda^{-1}(0, 1] = B$, this will finish the proof. Suppose $\Gamma \neq \Lambda$; then pick $\mu \in \Lambda - \Gamma$, and put $\Gamma' = \Gamma \cup \{\mu\}$. Define $\varphi \colon \pi_{\Gamma'}^{-1}(0, 1] \to [0, 1]$ by

$$\varphi(b) = \begin{cases} 1 & \text{if } \pi_\mu(b) \leqq \pi_\Gamma(b) \text{ (hence } \pi_\Gamma(b) > 0) \\ \pi_\Gamma(b)/\pi_\mu(b) & \text{if } \pi_\mu(b) \geqq \pi_\Gamma(b) \text{ (hence } \pi_\mu(b) > 0) . \end{cases}$$

Then $\varphi(b) > 0 \iff \pi_\Gamma(b) > 0$, so that S_Γ is defined on $\varphi^{-1}(0, 1]$. Since p has the SEP over $\pi_\mu^{-1}(0, 1]$, there exists a section $S_\mu \colon \pi_\mu^{-1}(0, 1] \to E$ with $S_\mu \mid \varphi^{-1}(1) \cap \pi_\mu^{-1}(0, 1] = S_\Gamma \mid \varphi^{-1}(1) \cap \pi_\mu^{-1}(0, 1]$. Now define

$$S_{\Gamma'} \colon \pi_{\Gamma'}^{-1}(0, 1] \to E , \qquad S_{\Gamma'}(b) = \begin{cases} S_\Gamma(b) & \text{if } \pi_\mu(b) \leqq \pi_\Gamma(b) \\ S_\mu(b) & \text{if } \pi_\mu(b) \geqq \pi_\Gamma(b) . \end{cases}$$

This is compatible at $\pi_\mu = \pi_\Gamma$, because there $\varphi = 1$.

If $S_\Gamma(b) \neq S_{\Gamma'}(b)$ for some $b \in \pi_\Gamma^{-1}(0, 1]$, then $\varphi(b) < 1$, hence $\pi_\mu(b) > \pi_\Gamma(b) \geqq 0$. This shows $(\Gamma', S_{\Gamma'}) > (\Gamma, S_\Gamma)$, and contradicts the maximality of (Γ, S_Γ), q.e.d.

3. Hereditary SEP and fibre homotopy equivalence

The SEP is not preserved under taking induced spaces; we show, in fact, that such a hereditary SEP characterizes shrinkable spaces (3. 1a, d). Together with Theorem 2.7 (and an auxiliary construction 3.4) this implies that a map over B which is locally a fibre homotopy equivalence is globally so (3.3).

3.1. PROPOSITION. *The following properties of $p \colon E \to B$ are equivalent.*

(a) *Every induced space p_α has the* SEP (*in particular $p = p_{\mathrm{id}}$*).

(b) *Given $\bar{F} \colon X \to B$, a halo V around $A \subset X$, and $f \colon V \to E$ with* $pf = \bar{F} \mid V$ (a partial lift of $\bar{F}$), *there exists $F \colon X \to E$ with $F \mid A = f \mid A$, and* $pF = \bar{F}$ (a lift of F).

(c) *Given $\bar{F} \colon X \to B$, $A \subset X$, $f \colon A \to E$ with $pf = \bar{F}/A$, there exists a lift $F \colon X \to E$ of $\bar{F}$ with $F \mid A \simeq_B f$.*

(d) *$p \colon E \to B$ is shrinkable* (i.e., *p is a fibre homotopy equivalence*).

(e) *p is fibre homotopy equivalent to a trivial space $B \times Y \to B$ with contractible Y.*

(f) *p is dominated by a trivial space $B \times Y \to B$ with contractible Y.*

PROOF. (a) and (b) are equivalent because lifts of $\bar{F}$ correspond to sections in the induced space $p_{\bar{F}}$ (see 1.6); also (d) $\iff$ (e) $\iff$ (f) obviously (1.5).

(d) $\Rightarrow$ (c): By assumption there exists a section $S \colon B \to E$ and a vertical homotopy Θ between $\Theta_0 = \mathrm{id}_E$ and $\Theta_1 = Sp$. Now defined $F = S\bar{F}$, and use $\Theta(f(x), t)$ as vertical homotopy between f and $F \mid A$.

(c) $\Rightarrow$ (b): Given $\bar{F}, A \subset V, f$, as in (b), and assuming (c), there exists a lift $F'\colon X \to E$ of $\bar{F}$ and a vertical homotopy $D\colon F' \mid V \simeq_V f$. Choose $\tau\colon X \to [0, 1]$ with $\tau \mid A = 1$, $\tau \mid CV = 0$, and define

$$F(x) = \begin{cases} F'(x) & \text{for } \tau(x) \leqq \frac{1}{2} \\ D(x, 2\tau(x) - 1) & \text{for } \tau(x) \geqq \frac{1}{2} \, . \end{cases}$$

(b) $\Rightarrow$ (d): By assumption a lift always exists. In particular, id_B lifts to a section $S\colon B \to E$ (take $A = \varnothing = V$). Define $\bar{F}\colon E \times [0, 1] \to B$, $\bar{F}(x, t) = p(x)$, $A = E \times \{0\} \cup E \times \{1\}$, $V = E \times [0, \frac{1}{2}) \cup E \times (\frac{1}{2}, 1]$,

$$f\colon V \to E\,, \qquad f(x, t) = \begin{cases} x & \text{for } t < \frac{1}{2} \\ Sp(x) & \text{for } t > \frac{1}{2} \, . \end{cases}$$

Then V is a halo around A, and a lift F of $\bar{F}$ with $F \mid A = f \mid A$ (which exists by assumption) will be a vertical homotopy $\mathrm{id}_E \simeq_B Sp$. Hence p is shrinkable.

3.2. Corollary. *If $p\colon E \to B$ is shrinkable over each set V_λ of a numerable covering $\{V_\lambda\}$ of B, then p is shrinkable.*

Proof. Let $\alpha\colon X \to B$; we have to show that the induced space $p_\alpha\colon E_\alpha \to X$ has the SEP (3.1, (a) $\Longleftrightarrow$ (d)). Now, $\{\alpha^{-1}(V_\lambda)\}$ is a numerable covering of X, and p_α is shrinkable over $\alpha^{-1}(V_\lambda)$ (because p is over V_λ), hence p_α has the SEP over $\alpha^{-1}(V_\lambda)$ (3.1, (d) $\Longleftrightarrow$ (a)), hence the result by Theorem 2.7.

Roughly speaking, the corollary says that p is a fibre homotopy equivalence provided this is locally the case. The following theorem asserts the same for arbitrary maps over B.

3.3. Theorem. *Let $f\colon E' \to E$ be a map over B. If f is a fibre homotopy equivalence over each set V_λ of a numerable covering $\{V_\lambda\}$ of B, then f is a fibre homotopy equivalence.*

More generally, if (under the same assumptions on f) a partial homotopy inverse $f_V^-\colon p^{-1}(V) \to p'^{-1}(V)$ of f, and a vertical homotopy $D_V\colon \mathrm{id}(p^{-1}(V)) \simeq_V f_V f_V^-$ are given over a halo V around $A \subset B$, then f_A^-, D_A can be extended over all of B.

It is remarkable that one can not, in addition to f_A^-, D_A, prescribe $D'_A\colon \mathrm{id}(p'^{-1}(A)) \simeq_A f_A^- f_A$, as the following example shows. Let $B = [0, 1]$, $E = B \times S^1 = E'$ (S^1 = circle), $f = \mathrm{id}$, $A = \{0\} \cup \{1\}$. Prescribing $f_A^- = \mathrm{id}$, $D(0, z, t) = D'(0, z, t) = (0, z)$, $D(1, z, t) = (1, e^{2\pi i t} z)$, $D'(1, z, t) = (1, z)$ then leads to an unsolvable extension problem.

Proof of 3.3. Let E^I denote the space of all paths in E (with the compact-open topology), and define

$$R = \{(y, w) \in E' \times E^I \mid p'(y) = pw(I) \text{ and } w(1) = f(y)\} ,$$

i.e., a point $(y, w) \in R$ is a pair consisting of a point $y \in E'$ and a path w completely contained in $p^{-1}(p'(y))$ and ending in $f(y)$. Make it a space over E by

$$q\colon R \longrightarrow E , \qquad q(y, w) = w(0) .$$

We show below

3.4. LEMMA. *If f is a fibre homotopy equivalence, then $q\colon R \to E$ is shrinkable.*

Applying 3.4 to f_{V_λ}, the part of f over V_λ, we see that in our case q is shrinkable over $p^{-1}(V_\lambda)$ for all λ, so by 3.2, q itself is shrinkable, hence has the SEP. Now, a section $S\colon E \to R$ is a pair, $S = (f', \theta)$, where $f'\colon E \to E'$ is a map over B, and $\Theta\colon E \times I \to E$, $\Theta(z, t) = \theta(z)(t)$, is a vertical homotopy $\mathrm{id}_E \simeq_B ff'$. Because of the SEP, we can choose S such that f', Θ agree over A with the given f_A^-, D_A. It remains to be shown that $f'f \simeq_B \mathrm{id}_{E'}$.

The homotopy $ff' \simeq_B \mathrm{id}_E$ implies that, *over* V_λ, the map f' *is* fibre homotopy inverse to f; in particular, it is a fibre homotopy equivalence there. We can then apply our argument to f' instead of f, and find $f''\colon E' \to E$ with $f'f'' \simeq_B \mathrm{id}_{E'}$, hence

$$f'f \simeq_B (f'f)(f'f'') = f'(ff')f'' \simeq_B f'f'' \simeq_B \mathrm{id}_{E'} , \qquad \text{q.e.d} .$$

PROOF OF LEMMA 3.4. Let $f'\colon E \to E'$ be a fibre homotopy inverse of f, and $\varphi\colon \mathrm{id}_{E'} \simeq_B f'f$, $\psi\colon \mathrm{id}_E \simeq_B ff'$ vertical homotopies. For fixed $t \in I$ (resp. $y \in E'$) the homotopy φ defines a map $\varphi_t\colon E' \to E'$ (resp. a path $\varphi_y\colon I \to E'$); similarly $\psi_t\colon E \to E$ (resp. $\psi_z\colon I \to E$ for $z \in E$).

If w, w' are paths with $w(1) = w'(0)$, and $\tau \in I$, we denote by

$w \cdot w'$, the product path;

${}_\tau w$ (resp. ${}^\tau w$), the path ${}_\tau w(t) = w(t\tau)$ (resp. ${}^\tau w(t) = w(1 - \tau + t\tau)$);

w^-, the inverse path, $w^-(t) = w(1 - t)$; and

c, any constant path.

In particular, ${}_1w = w = {}^1w$, ${}_0w = c$, ${}^0w = c$.

We now construct a vertical homotopy $D\colon \mathrm{id}_R \simeq_E \sigma q$, where $\sigma\colon E \to R$ is the section given by $\sigma(z) = (f'(z), \psi_z)$. Consider first

$$u\colon R \longrightarrow E'^I , \qquad u(y, w) = \varphi_y \cdot f'[w^- \cdot \psi_{w(0)} \cdot ff'w \cdot f\varphi_y^- \cdot w^-] .$$

Then $u(y, w)$ is a vertical path from y to $f'w(0)$, and we can define a vertical deformation of id_R first into the map $(y, w) \to (y, w \cdot c)$ (in an obvious way), and then by

$$d\{(y, w), \tau\} = [u(y, w)(\tau), w\cdot f({}_{\tau}u(y, w))] , \qquad \tau \in I ,$$

into

$$K: R \longrightarrow R , \qquad K(y, w) = [f'w(0), w\cdot fu(y, w)] .$$

Now

$$\begin{aligned} w\cdot fu(y, w) &= (w\cdot f\varphi_y\cdot ff'w^-)\cdot ff'\psi_{w(0)}\cdot ff'(ff'w\cdot f\varphi_y^-\cdot w^-) \\ &= v\cdot ff'\psi_{w(0)}\cdot ff'v^- = v\cdot ff'(\psi_{w(0)}\cdot v^-) \end{aligned}$$

with $v = w\cdot f\varphi_y\cdot ff'w^-$. We define a vertical deformation of K first into

$$(y, w) \longrightarrow [f'w(0), v\cdot c\cdot ff'(\psi_{w(0)}\cdot v^-)\cdot c]$$

(in an obvious way), and then by

$$d'\{(y, w), \tau\} = [f'w(0), v\cdot{}_{\tau}(\psi^-_{\bar{w}(0)})\cdot\psi_{1-\tau}(\psi_{w(0)}\cdot v^-)\cdot{}^{\tau}\psi_{w(0)}]$$

into

$$\begin{aligned} K': R \longrightarrow R, \quad K'(y, w) &= [f'w(0), v\cdot\psi^-_{\bar{w}(0)}\cdot\psi_{w(0)}\cdot v^-\cdot\psi_{w(0)}] \\ &= [f'w(0), (v\cdot\psi^-_{\bar{w}(0)})\cdot(v\cdot\psi^-_{\bar{w}(0)})^-\cdot\psi_{w(0)}] . \end{aligned}$$

Finally, an obvious homotopy will deform K' into

$$(y, w) \longrightarrow [f'w(0), \psi_{w(0)}] = \sigma q(y, w) , \qquad \text{q.e.d} .$$

4. The covering homotopy property (CHP)

A space over B has the CHP if every continuous family of paths in B whose initial points have been lifted can be lifted completely (4.2). Hurewicz and Huebsch proved that (under mild restrictions) $p: E \to B$ has the CHP if this is locally the case. We obtain the same results here as a consequence of the *section extension theorem* 2.7; our procedure is similar to Hurewicz's.

4.1. DEFINITION. Let $p: E \to B$ be a space over B and $\bar{H}: X \times I \to B$ a homotopy. We say p has *the* CHP for $\bar{H}$ if the following holds. Given $h: X \to E$ with $ph(x) = \bar{H}(x, 0)$, given further $\tau: X \to I$ and $H': \tau^{-1}(0, 1] \times I \to E$ with $pH'(x, t) = \bar{H}(x, t)$, $H'(x, 0) = h(x)$, $x \in \tau^{-1}(0, 1]$, $t \in I$, there exists $H: X \times I \to E$ with

$$pH = \bar{H}, \qquad H \,|\, \tau^{-1}(1) = H' \,|\, \tau^{-1}(1) , \qquad H(x, 0) = h(x) , \qquad x \in X .$$

We also use an analogous terminology if I is replaced by an arbitrary interval $[a, b]$, $a < b$.

We say p has the CHP for X if it has the CHP for all homotopies $\bar{H}$ with range $X \times I$. If it has the CHP for all spaces, then we say it has the CHP.

The CHP for an individual $\bar{H}$ should be considered as an auxiliary notion, more important is the CHP for spaces or classes of spaces. In this case we have the following familiar characterization.

4.2. PROPOSITION. *p: $E \to B$ has the* CHP *for X if and only if for every $\bar{G}$: $X \times I \to B$ and g: $X \to E$ with $pg(x) = \bar{G}(x, 0)$, there exists G: $X \times I \to E$ with $pG = \bar{G}$ and $G(x, 0) = g(x)$.*

This means: It suffices to have covering homotopies in case $\tau = 0$. Note that $\bar{G}$ is not fixed now.

PROOF. Given $\bar{H}$, h, τ, H' as in 4.1; we have to construct H. Replacing τ by $\text{Max}\,(0, 2\tau(x) - 1)$, if necessary, we see that we can assume that $H'(x, t)$ is defined and continuous for all (x, t) with $t \leqq \tau(x)$ (not only for $\tau(x) > 0$). Put

$$\bar{G}: X \times I \longrightarrow B\,, \qquad \bar{G}(x, t) = \bar{H}(x, \text{Min}\,(1, \tau(x) + t))\,,$$
$$g: X \longrightarrow E\,, \qquad g(x) = H'(x, \tau(x))\,.$$

By assumption there is a G: $X \times I \to E$ with $pG = \bar{G}$ and $G(x, 0) = g(x)$. Then

$$H(x, t) = \begin{cases} H'(x, t) & \text{for } t \leqq \tau(x) \\ G(x, t - \tau(x)) & \text{for } t \geqq \tau(x) \end{cases}$$

is a map as required, q.e.d.

4.3. PROPOSITION. *If p: $E \to B$ has the* CHP *for X, then also every induced space p_α: $E_\alpha \to B_\alpha$ (where α: $B_\alpha \to B$; see* 1.6).

PROOF. Given $\bar{G}$: $X \times I \to B_\alpha$, g: $X \to E_\alpha$ as in 4.2, we have to construct G. Now $\alpha\bar{G}$ is a homotopy which can be lifted to E with initial position $\hat{\alpha}g$ (see 1.6 for $\hat{\alpha}$). If G': $X \times I \to E$ is such a lifting, then the pair $G = (\bar{G}, G')$ gives the required map.

4.4. EXAMPLE. *Every trivial space $B \times Y \to B$ has the* CHP.

PROOF. Given $\bar{G}$, g as in 4.2, define G by $G(x, t) = (\bar{G}(x, t), g(x))$, q.e.d.

We now proceed to localize the CHP. For every $\bar{H}$: $X \times I \to B$, h: $X \to E$ with $ph(x) = \bar{H}(x, 0)$, we define a space q: $R \to X$ over X as follows

$$R = \{(x, w) \in X \times E^I \mid h(x) = w(0) \text{ and } pw(t) = \bar{H}(x, t)\}\,, \quad q(x, w) = x\,.$$

A covering homotopy H gives a section S for q by $S(x) = (x, H_x)$ where $H_x(t) = H(x, t)$. In fact, it is obvious from the definitions 2.2, 4.1 that:

4.5. LEMMA. *p has the* CHP *for $\bar{H}$ if and only if $q = q_h$ has the* SEP *for all* $h: X \to E$ *with* $ph(x) = \bar{H}(x, 0)$.

We can therefore apply 2.7; in order to get a more applicable result (4.7) we show first

4.6. LEMMA. *Let* $a < b < c$ *be real numbers and* $\bar{H}: X \times [a, c] \to B$. *If* p *has the* CHP *for* $\bar{H} \mid X \times [a, b]$ *and* $\bar{H} \mid X \times [b, c]$, *then for* $\bar{H}$ *itself.*

PROOF. Let $\bar{H}$, h, τ, H' be as in 4.1 with I replaced by $[a, c]$; we have to construct H. Starting with $\bar{H}_1 = \bar{H} \mid X \times [a, b]$, $h_1 = h$, $\tau_1 = \text{Min}(b, \tau)$, $H_1' = H' \mid \tau^{-1}(a, c] \times [a, b]$ we first find $H_1: X \times [a, b] \to E$ with

$$H_1 \mid \tau^{-1}[b, c] \times [a, b] = H_1' \mid \tau^{-1}[b, c] \times [a, b] .$$

Then from $\bar{H}_2 = \bar{H} \mid X \times [b, c]$, $h_2(x) = H_1(x, b)$, $\tau_2 = \text{Max}(b, \tau)$, $H_2' = H' \mid \tau^{-1}(b, c] \times [b, c]$, we get $H_2: X \times [b, c] \to E$, and finally H as $H \mid X \times [a, b] = H_1$, $H \mid X \times [b, c] = H_2$, q.e.d.

4.7. THEOREM. *Let* $p: E \to B$ *be a space over* B, *and* $\bar{H}: X \times I \to B$ *a homotopy. If there exists a numerable covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *of* X, *and for every* $\lambda \in \Lambda$ *real numbers* $0 = t_0^\lambda < t_1^\lambda < \cdots < t_{r_\lambda}^\lambda = 1$ *such that* p *has the* CHP *for* $\bar{H} \mid V_\lambda \times [t_i^\lambda, t_{i+1}^\lambda]$ (*for all* λ, i) *then* p *has the* CHP *for* $\bar{H}$.

PROOF. Lemma 4.6 shows that p has the CHP for $\bar{H} \mid V_\lambda \times I$ and all λ. Then $q: R \to X$ has the SEP over each V_λ (4.5), hence q itself has the SEP (2.7), and again, by 4.5, p has the CHP for $\bar{H}$, q.e.d.

4.8. THEOREM. *If* $p: E \to B$ *has the* CHP *over every set* V_λ (e.g., if p is trivial over V_λ; see 4.4) *of* (a) *a numberable covering resp.* (b) *an open covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *of* B, *then* p *has the* CHP *for* (a) *all spaces* X *resp.* (b) *all paracompact spaces* X.

PROOF. Let $\bar{H}: X \times I \to B$ be a homotopy; we have to show that p has the CHP for $\bar{H}$.

(a) We can assume that $\{V_\lambda\}$ is given by a locally finite partition of unity, say $\{\pi_\lambda: B \to [0, 1]\}$, so $V_\lambda = \pi_\lambda^{-1}(0, 1]$, $\lambda \in \Lambda$. For every r-tuple $\lambda_1, \lambda_2, \cdots, \lambda_r \in \Lambda$, define

$$\pi_{\lambda_1 \cdots \lambda_r}: X \longrightarrow [0, 1] ,$$

$$\pi_{\lambda_1 \cdots \lambda_r}(x) = \prod_{i=1}^r \text{Min} \left\{ \pi_{\lambda_i} \bar{H}(x, t) \mid t \in \left[\frac{i-1}{r}, \frac{i}{r} \right] \right\} .$$

This is easily verified to be continuous. Furthermore, $\pi_{\lambda_1 \cdots \lambda_r}(x) \neq 0$ if and only if $\bar{H}(x \times [(i-1)/r, i/r]) \subset V_{\lambda_i}$ for all i. If we can show that $\{W_{\lambda_1 \cdots \lambda_r} = \pi_{\lambda_1 \cdots \lambda_r}^{-1}(0, 1]\}$ is a numerable covering of X, then the result follows from 4.7 (because p has the CHP for all $\bar{H} \mid W_{\lambda_1 \cdots \lambda_r} \times [(i-1)/r, i/r]$).

Every $(x, t) \in X \times I$ has a neighborhood which is contained in one of the sets $\bar{H}^{-1}(V_\lambda)$ and which meets only a finite number of these sets. By compactness of I, it follows that for every $x \in X$ there exists a neighborhood U and a natural number r such that

(α) $U \times [(i-1)/r, i/r] \subset \bar{H}^{-1}(V_{\gamma_i})$ for some $\gamma_i \in \Lambda$ $(i = 1, \cdots, r)$, and

(β) $U \times I$ meets only finitely many $\bar{H}^{-1}(V_\lambda)$.

Property (α) shows that $\{W_{\lambda_1 \cdots \lambda_r}\}$ is indeed a covering of X, and (β) implies that *for fixed* r the family $\{\pi_{\lambda_1 \cdots \lambda_r}\}$ is locally finite. If we let r vary, too, the system is no longer locally finite, but a suitable refinement is, as we shall see.

Let q_r denote the sum of all functions $\pi_{\lambda_1 \cdots \lambda_i}$ with $i < r$, and define $\pi'_{\lambda_1 \cdots \lambda_r}(x) = \operatorname{Max}(0, \pi_{\lambda_1 \cdots \lambda_r}(x) - rq_r(x))$. Let $x \in X$, and pick the minimal k such that $\pi_{\lambda_1 \cdots \lambda_k}(x) \neq 0$ for some $(\lambda_1 \cdots \lambda_k)$. Then $q_k(x) = 0$, hence $\pi'_{\lambda_1 \cdots \lambda_k}(x) = \pi_{\lambda_1 \cdots \lambda_k}(x) \neq 0$, which shows that the sets $\pi'^{-1}_{\lambda_1 \cdots \lambda_r}(0, 1]$ still cover X. Further, if we choose $N > k$ such that $\pi_{\lambda_1 \cdots \lambda_k}(x) > 1/N$ then $q_N(x) > 1/N$, hence $Nq_N(y) > 1$ for all y in a neighborhood of x. In this neighborhood all $\pi'_{\lambda_1 \cdots \lambda_m}$ with $m \geqq N$ vanish, which shows that the system $\pi'_{\lambda_1 \cdots \lambda_r}$ is locally finite. It clearly refines $\{W_{\lambda_1 \cdots \lambda_r}\}$. To make it a partition of unity, simply divide each π' by the sum of all these functions.

(b) As above the compactness of I yields for every $x \in X$ a neighborhood $U = U_x$ and a natural number $r = r_x$ such that every $U \times [(i-1)/r, i/r]$, $i = 1, \cdots, r$, is contained in some $\bar{H}^{-1}(V_\lambda)$. Because X is paracompact, the covering $\{U_x\}_{x \in X}$ is numerable, and 4.7 applies, q.e.d.

We can modify Theorem 4.8 by restricting the assumption and conclusion to certain classes of spaces, e.g.,

4.9. THEOREM. *If $p: E \to B$ has the local* CHP *for* CW*-complexes of dimension $\leqq m$, then p has the (global)* CHP *for all paracompact spaces X which are locally* CW*-complexes of dimension $\leqq m$ (or retracts of such; see* 2.4).

The proof is as in case 4.8 (b). The theorem is strengthened by a result of James-Whitehead [11, 5] according to which the CHP for *simplices* of dimension $\leqq m$ implies the CHP for CW-complexes of dimension $\leqq m$.

4.10. REMARK. A homotopy $\bar{H}: X \times I \to B$ is called *stationary* at x for $t \in [t_1, t_2]$ if $\bar{H}(x \times [t_1, t_2]) = \bar{H}(x, t_1)$. If we require all covering homotopies H of $\bar{H}$ to be stationary with $\bar{H}$ (compare Steenrod [15, 11.7]), we get a variation of the CHP which we denote by CHPS. All results and proofs above remain valid if we replace CHP by CHPS throughout, and $q: R \to X$ by $q_S = q \mid R_S: R_S \to X$ where

$$R_S = \{(x, w) \in X \times E^I \mid h(x) = w(0), pw(t) = \bar{H}(x, t), \text{ and }$$
$$\bar{H}(x \times [t_1, t_2]) = \bar{H}(x, t_1) \Rightarrow w[t_1, t_2] = w(t_1)\} .$$

A similar remark applies to *regular* covering homotopies in the sense of Hurewicz [10, 3].

5. The weak covering homotopy property (WCHP)

Given $p\colon E \to B$, a homotopy $\bar{H}\colon X \times I \to B$, and a lifting $h\colon X \to E$ of the initial position of $\bar{H}$, the WCHP for p requires that $\bar{H}$ can be covered by a homotopy $H\colon X \times I \to E$ whose initial position is vertically homotopic to h (5.13). This notion is particularly adequate in questions dealing with fibre homotopy equivalence (see § 6); it is invariant under fibre homotopy equivalence whereas the ordinary CHP is not (see 5.2 and the counter-example after 5.3). The usual conclusions from the CHP (exact homotopy sequence, spectral sequence) can already be drawn from the WCHP. The main results of this section (5.11, 5.12) show that the WCHP is essentially a local property, quite as the CHP. The ideas of the proof are the same as in § 4, but the details are more complicated.

The WCHP was already considered by Fuchs who proved the assertion of Theorem 5.12 for compact X if p is locally dominated by trivial spaces. The same result was found by Fadell [5, 5.1] for a somewhat weaker WCHP. A further weakening of this property came up in Dold-Thom ("Relèvement des homotopies homotopes").

5.1. DEFINITION. Let $p\colon E \to B$ be a space over B, and $\bar{H}\colon X \times [0, 1] \to B$ a homotopy. We say p has the WCHP for $\bar{H}$ if it has the ordinary CHP for the following:

$$\hat{H}\colon X \times [-1, 1] \longrightarrow B\ , \quad \hat{H}(x \times [-1, 0]) = \bar{H}(x, 0)\ , \quad \hat{H} \mid X \times [0, 1] = \bar{H}\ .$$

Every map $H\colon X \times [-1, 1] \to E$ with $pH = \hat{H}$ will be called a *weak covering homotopy* of $\bar{H}$.

As in § 4 we use the same terminology if $[0, 1]$, $[-1, 1]$ are replaced by other intervals $[b, c]$, $[a, c]$ $(a < b < c)$.

We say p has the WCHP (for X) if it has the WCHP for all $\bar{H}$ (with range $X \times I$). Equivalently, p has the WCHP if and only if it has the CHP for all homotopies $\bar{H}\colon X \times I \to B$ which are stationary in $[0, 1/2]$.

5.2. PROPOSITION. *If $p\colon E \to B$ is dominated* (1.3) *by $p'\colon E' \to B$, and p' has the* WCHP *for $\bar{H}$, then the same holds for p.*

5.3. COROLLARY. *If $p\colon E \to B$ is dominated by a trivial space $B \times Y \to B$ then p has the* WCHP.

This follows from 4.4 and 5.2. An example which satisfies the WCHP but not the CHP is as follows. Let $E = \{(x, y) \in \mathbf{R}^2 \mid xy \geqq 0\}$, $B = \mathbf{R}$, $p(x, y) = x$. The interval $[0, 1] \subset B$ can not be lifted with initial position

(0, -1), but p: $E \to B$ is shrinkable (to $y = 0$), hence has the WCHP by 5.3.

PROOF OF 5.2. We are given $\hat{H}$ as in 5.1, h: $X \to E$ with $h(x) = \hat{H}(x, -1)$, τ: $X \to [-1, 1]$, H': $\tau^{-1}(-1, 1] \times [-1, 1] \to E$ with $pH'(x, t) = \hat{H}(x, t)$, $H'(x, -1) = h(x)$, and we have to construct H: $X \times [-1, 1] \to E$ with $pH = \hat{H}$, $H | \tau^{-1}(1) \times [-1, 1] = H' | \tau^{-1}(1) \times [-1, 1]$, $H(x, -1) = h(x)$.

We first replace H' by H'': $\tau^{-1}(-1, 1] \times [-1, 1] \to E$,

$$H''(x, t) = \begin{cases} H'(x, t) & \text{if } t \geqq 0 , \\ H'\left(x, \dfrac{2t}{1 + \tau'(x)}\right) & \text{if } -(1 + \tau'(x)) \leqq 2t \leqq 0 , \\ h(x) & \text{if } 2t \leqq -(1 + \tau'(x)) , \end{cases}$$

where $\tau'(x) = \operatorname{Max}(0, \tau(x))$. This H'' satisfies the same conditions as H', agrees with H' where $\tau(x) = 1$, but in addition $H''(x, t) = h(x)$ for $t \leqq -\frac{1}{2}(1 + \tau'(x))$.

Now choose maps f: $E \to E'$, g: $E' \to E$ over B and a vertical homotopy Θ: $\mathrm{id}_E \simeq_B gf$. Because p' has the WCHP for $\bar{H}$, we can find a map $\mathring{H}$: $X \times [-1, 1] \to E'$ such that

$$p\mathring{H} = \hat{H},\ \mathring{H} | \tau^{-1}[0, 1] \times [-1, 1] = fH'' | \tau^{-1}[0, 1] \times [-1, 1] ,$$

and $\mathring{H}(x, t) = fH''(x, t)$ for $t \leqq -\frac{1}{2}$ (we apply the WCHP for the new interval $[-\frac{1}{2}, 1]$ and the function $\tau''(x) = \operatorname{Min}(1, \frac{3}{2}\tau(x) + 1)$). Then the following is a map H as required

$$H(x, t) = \begin{cases} g\mathring{H}(x, t) & \text{if } \tau(x) \leqq 0 \text{ and } 2t \geqq -1 , \\ \Theta(H''(x, t), 1 - \tau(x)) & \text{if } \tau(x) \geqq 0 \text{ and } 2t \geqq -(1 + \tau(x)) , \\ \Theta(h(x), 2t + 2) & \text{if } 2t \leqq -1 \text{ and } 2t \leqq -(1 + \tau(x)) , \\ & \quad \text{i.e., if } 2t \leqq -(1 + \tau'(x)) . \end{cases}$$

All verifications are left to the reader; the figure indicates where the three pieces of H are defined.

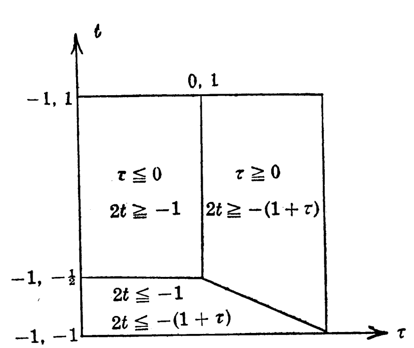

Fig. 2

The proof of the analogue of 4.6 (see 5.10) requires some preliminaries.

5.4. Define φ: $[-1, 2] \times [0, 1] \to [-1, 2]$ by

$$\varphi(t_1, t_2) = \begin{cases} t_1 & \text{for } t_1 \leqq 0 \\ 0 & \text{for } 0 \leqq t_1 \leqq 1 - t_2 \\ 2\dfrac{t_1 + t_2 - 1}{t_2 + 1} & \text{for } t_1 \geqq 1 - t_2 \end{cases}$$

(φ_{t_2} maps identically for $t_2 = 1$, and shrinks $[0, 1]$ to a point for $t_2 = 0$), and

$$\hat{\varphi}\colon [-1, 2] \times [-1, 1] \longrightarrow [-1, 2] , \qquad \hat{\varphi}(t_1, t_2) = \varphi(t_1, \mathrm{Max}(0, t_2)) .$$

Further, let

$$\psi\colon [-1, 2] \longrightarrow [-1, 2] \times [-1, 1] , \qquad \psi(t) = (t, \mathrm{Min}(1, 2t + 1)) .$$

Clearly $\hat{\varphi}\psi = \mathrm{id}$.

Given

$$\bar{H}\colon X \times [-1, 2] \longrightarrow B , \qquad H\colon X \times [-1, 2] \longrightarrow E ,$$
$$\bar{G}\colon X \times [-1, 2] \times [-1, 1] \longrightarrow B , \qquad G\colon X \times [-1, 2] \times [-1, 1] \longrightarrow E ,$$

define

$$\bar{H}^{\hat{\varphi}}\colon X \times [-1, 2] \times [-1, 1] \longrightarrow B , \qquad \bar{H}^{\hat{\varphi}}(x, t_1, t_2) = \bar{H}(x, \hat{\varphi}(t_1, t_2)) ,$$
$$\bar{H}^{\varphi} = \bar{H}^{\hat{\varphi}} \mid X \times [-1, 2] \times [0, 1] ,$$
$$\bar{G}^{\psi}\colon X \times [-1, 2] \longrightarrow B , \qquad \bar{G}^{\psi}(x, t) = \bar{G}(x, \psi(t)) ,$$

and similarly for H^{φ}, $H^{\hat{\varphi}}$, G^{ψ}.

The following properties are obvious:

(5.5) $$pH = \bar{H} \Longrightarrow pH^{\hat{\varphi}} = \bar{H}^{\hat{\varphi}} ,$$

(5.6) $$pG = \bar{G} \Longrightarrow pG^{\psi} = \bar{G}^{\psi} ,$$

(5.7) $$H^{\hat{\varphi}\psi} = H .$$

If $[b, d]$ is an arbitrary interval, $\bar{H}\colon X \times [b, d] \to B$ a homotopy, and $b < c < d$, we denote by $\bar{H}^{c}\colon X \times [b, d] \times [0, 1] \to B$ the map which corresponds to $\bar{H}^{\varphi}$ above under the homeomorphism $[b, d] \approx [-1, 2]$ which takes $[b, c]$ linearly onto $[-1, 0]$, and $[c, d]$ linearly onto $[0, 2]$.

5.8. LEMMA. *Let $b < c < d$ be real numbers, $\bar{H}\colon X \times [b, d] \to B$, $\mathring{H}\colon X \times [b, c] \to E$ with $p\mathring{H}(x, t) = \bar{H}(x, t)$, $\tau\colon X \to [b, d]$, $H'\colon \tau^{-1}(b, d] \times [b, d] \to E$ with $pH'(x, t) = \bar{H}(x, t)$ for $\tau(x) > b$, and $\mathring{H}(x, t) = H'(x, t)$ for $t \leqq c$, $\tau(x) > b$. Assume $p\colon E \to B$ has the* WCHP *for $\bar{H} \mid X \times [c, d]$ and for*

$\bar{H}^c$ *(see remark after* 5.7). *Then there exists* $H\colon X \times [b, d] \to E$ *with* $pH = \bar{H}$, $H(x, b) = \mathring{H}(x, b)$, $H \mid \tau^{-1}(d) \times [b, d] = H' \mid \tau^{-1}(d) \times [b, d]$.

Roughly speaking this means: If we already have a covering homotopy for $\bar{H} \mid X \times [b, c]$, then we can find one for $\bar{H}$ itself with the same initial condition.

PROOF. We can assume $(b, c, d) = (-1, 0, 2)$. Because p has the WCHP for $\bar{H} \mid X \times [0, 2]$, there is an $H''\colon X \times [-1, 2] \to E$ with $pH''(x, t) = \bar{H}(x, \text{Max}(0, t))$, $H''(x, -1) = \mathring{H}(x, 0)$, $H''(x, t) = H'(x, \text{Max}(0, t))$ for $\tau(x) \geqq 1$. Because p has the WCHP for $\bar{H}^c = \bar{H}^\varphi$, we can then find $G\colon X \times [-1, 2] \times [-1, 1] \to E$ with $pG = \bar{H}^{\hat{\varphi}}$,

$$G(x, t_1, -1) = \begin{cases} \dot{H}(x, t_1) & \text{for } t_1 \leqq 0 \\ H''(x, t_1 - 1) & \text{for } 0 \leqq t_1 \leqq 1 \\ H''(x, 2t_1 - 2) & \text{for } 1 \leqq t_1 \leqq 2\,, \end{cases} \tag{5.9}$$

and $G(x, t_1, t_2) = H'^{\hat{\varphi}}(x, t_1, t_2)$ for $\tau(x) = 2$. (Note that $H'^{\hat{\varphi}}(x, t_1, t_2)$ is defined, and $H'^{\hat{\varphi}}(x, t_1, -1)$ equals the right side of 5.9 whenever $\tau(x) \geqq 1$; also $pH'^{\hat{\varphi}}(x, t_1, t_2) = \bar{H}^{\hat{\varphi}}(x, t_1, t_2)$ by 5.5.)

Now let $H = G^\psi$. By 5.6, 5.7 we have $pH = pG^\psi = (\bar{H}^{\hat{\varphi}})^\psi = \bar{H}$. On $\tau^{-1}(2) \times [-1, 2]$, we have $G = H'^{\hat{\varphi}}$, hence $H = G^\psi = H'^{\hat{\varphi}\psi} = H'$ there. Finally $H(x, -1) = G^\psi(x, -1) = G(x, -1, -1) = \mathring{H}(x, -1)$, which finishes the proof.

5.10. LEMMA (compare 4.6). *Let* $\bar{H}\colon X \times [a, d] \to B$ *be a homotopy, and* $a < b < c < d$. *If* $p\colon E \to B$ *has the* WCHP *for* $\bar{H} \mid X \times [a, c]$, $\bar{H} \mid X \times [c, d]$, *and* $(\bar{H} \mid X \times [b, d])^c$ *(see remark after* 5.7), *then* p *has the* WCHP *for* $\bar{H}$.

PROOF. We can first find a weak covering homotopy $H_1\colon X \times [a - 1, c] \to E$ of $\bar{H} \mid X \times [a, c]$ with given initial conditions on $X \times \{a - 1\}$ and $\tau^{-1}[b, d] \times [a - 1, c]$. By Lemma 5.8, we can then find $H_2\colon X \times [b, d] \to E$ which covers $\bar{H} \mid X \times [b, d]$, and which agrees with H_1 on $X \times \{b\}$ (note that H_1 is defined on $X \times [a, c]$) and with an initially given H' on $\tau^{-1}(d) \times [b, d]$. Therefore $H_1 \mid X \times [a, b]$ and H_2 fit together to yield the required homotopy $H\colon X \times [a - 1, d] \to E$, q.e.d.

5.11. THEOREM (compare 4.7). *Let* $p\colon E \to B$ *be a space over* B, *and* $\bar{H}\colon X \times I \to B$ *a homotopy. If there exists a numerable covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *of* X *and for every* $\lambda \in \Lambda$ *real numbers* $0 = t_0^\lambda < t_1^\lambda < \cdots < t_{r_\lambda}^\lambda = 1$ *such that* p *has the* WCHP *for* $\bar{H} \mid V_\lambda \times [t_i^\lambda, t_{i+1}^\lambda]$ *and* $(\bar{H} \mid V_\lambda \times [t_{i-1}^\lambda, t_{i+1}^\lambda])^{t_i^\lambda}$ *(see remark after* 5.7), *all* λ, i, *then* p *has the* WCHP *for* $\bar{H}$.

PROOF. Iterated application of 5.10 shows that p has the WCHP for each $\bar{H} \mid V_\lambda \times I$, i.e., (5.1), the CHP for each $\hat{H} \mid V_\lambda \times [-1, 1]$, therefore the CHP for $\hat{H}$ (4.5, 2.7), i.e., the WCHP for $\bar{H}$, q.e.d.

5.12. THEOREM (compare 4.8). *If* $p: E \to B$ *has the* WCHP *over every set* V_λ (e.g., *if* p *is dominated by a trivial space over each* V_λ; *see* 5.3) *of* (a) *a numerable covering* resp. (b) *an open covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *of* B, *then* p *has the* WCHP *for* (a) *all spaces* X resp. (b) *all paracompact spaces* X.

This follows from 5.11 as 4.8 does from 4.7: We construct a numerable covering $\{W_\mu\}$ of X, and for every μ a number $r = r_\mu$ such that $\bar{H}(W_\mu \times [(i-1)/r, (i+1)/r])$ is contained in some V_λ. Then the numbers $t_i = i/r$ satisfy the conditions of 5.11.

We leave it to the reader to formulate the analogue of 4.9 (it is not literally the same as 4.9 because homotopies $\bar{H}^c$ have to be considered). The following is an easy-to-remember characterization of the WCHP, and corresponds to 4.2.

5.13. PROPOSITION. $p: E \to B$ *has the* WCHP (*for all spaces, for all paracompact spaces, for all* CW-*complexes*, $\cdots$) *if and only if for every* $\bar{G}: X \times I \to B$ *and* $g: X \to E$ *with* $pg(x) = \bar{G}(x, 0)$ (X *paracompact*, X *a* CW-*complex*, $\cdots$) *there exists* $G: X \times I \to E$ *with* $pG = \bar{G}$, *and such that* $G_0: X \to E$, $G_0(x) = G(x, 0)$, *is vertically homotopic to* g, $G_0 \simeq_B g$.

PROOF. If p has the WCHP, and $G': X \times [-1, 1] \to E$ is a weak covering homotopy of $\bar{G}$, then we take $G = G' \mid X \times [0, 1]$; a vertical homotopy $g \simeq_B G_0$ is provided by $G' \mid X \times [-1, 0]$.

Conversely, assume G always exists, and let $\hat{H}$, h, τ, H' be the given data (as in the proof of 5.2) for which a covering homotopy H has to be constructed. As in the proof of 4.2, we can assume that H' is defined and continuous for all (x, t) such that $t \leqq \tau(x)$ or $\tau(x) > -1$. Put $\rho(x) = \mathrm{Max}(0, 2\tau(x))$, and define $H''(x, t)$ for all $(x, t) \in X \times [-1, 1]$ with $t \leqq \rho(x)$ by

$$H''(x, t) = \begin{cases} H'(x, t) & \text{if } t \leqq \tau(x) \text{ or } \tau(x) \geqq 0 , \\ H'(x, \tau(x)) & \text{if } \tau(x) \leqq t \leqq 0 . \end{cases}$$

Now let

$$\bar{G}: X \times [-1, 1] \longrightarrow B , \qquad \bar{G}(x, t) = \hat{H}\big(x, \mathrm{Min}(1, \rho(x) + t)\big) ,$$
$$G': X \times [-1, 0] \longrightarrow E , \qquad G'(x, t) = H''\big(x, \mathrm{Min}(1, \rho(x) + t)\big) .$$

Clearly $pG'(x, t) = \bar{G}(x, t)$.

By assumption we can find $G: X \times [-1, 1] \to E$ with $pG = \bar{G}$ and $G_{-1} \simeq_B G'_{-1}$. But because G' is defined over $X \times [-1, 0]$ (not just on $X \times \{-1\}$), we can even achieve $G_{-1} = G'_{-1}$; this follows as in Lemma 5.8 because we can apply our assumption not only to $\bar{G}$ but also to

$\bar{G}^0$: $X \times [-1, 1] \times [0, 1] \to B$ (see remark after 5.7); the function τ which occurs in the proof of 5.8 is now $= -1$.

Taking this for granted (i.e., $G(x, -1) = G'(x, -1) = H'(x, \rho(x) - 1)$), we can then define

$$H(x, t) = \begin{cases} H'(x, t) & \text{for } t \leqq \rho(x) - 1 \\ G(x, t - \rho(x)) & \text{for } t \geqq \rho(x) - 1 \,. \end{cases}$$

The relations $pH = \hat{H}$ and $H(x, -1) = H'(x, -1) = h(x)$ follow easily. Further, if $\tau(x) = 1$, then $\rho(x) = 2$, hence $t \leqq \rho(x) - 1$, hence $H(x, t) = H'(x, t)$, q.e.d.

5.14. PROPOSITION. *If* p: $E \to B$ *has the* WCHP, *then also every induced space* p_α: $E_\alpha \to B_\alpha$ (*where* α: $B_\alpha \to B$; *see* 1.6).

Same proof as for 4.3.

6. Weak covering homotopy property (WCHP) and fibre homotopy equivalence

Under a weak local contractibility condition for B, we show that a space over B has the WCHP if and only if it is locally fibre-homotopically trivial (or locally dominated by trivial spaces, 6.4, 5.12). Further, if p: $E \to B$, p': $E' \to B$ are spaces over this B which have the WCHP, then a map over B, f: $E \to E'$, is a fibre homotopy equivalence provided

(a) f is an ordinary homotopy equivalence, or

(b) f_b: $p^{-1}(b) \to p'^{-1}(b)$ is a (ordinary) homotopy equivalence for each $b \in B$ (6.1, 6.3).

Special cases of (b) had already been proved by Dold [3, Satz 1], Fadell (Duke Math. J., 26 (1954), 699–706), and Fuchs [6].

6.1. THEOREM. *Let* p: $E \to B$, p': $E' \to B$ *be spaces over* B *which have the* WCHP (e.g., *if they are "locally" dominated by trivial spaces in the sense of* 5.12). *Then a map* f: $E \to E'$ *over* B *is a fibre homotopy equivalence if and only if it is an ordinary homotopy equivalence.*

6.2. COROLLARY. *If* p: $E \to B$ *has the* WCHP, *then* p *is shrinkable* (1.5, 3.1) *if and only if* p *is a homotopy equivalence.*

6.3. THEOREM. *Let* B *be a topological space which admits a numerable covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *such that the inclusion map* $V_\lambda \to B$ *is nulhomotopic for every* λ (e.g., *a* CW-*complex* (6.7) *or, more generally, a locally contractible paracompact space, or a classifying space* B_G (*see last sentence in the proof of* 9.1)). *Let* p: $E \to B$, p': $E' \to B$ *be spaces over* B *which have the* WCHP (*compare* 6.4). *Then a map* f: $E \to E'$ *over* B *is a fibre* *homotopy equivalence if and only if the restriction of f to every fibre, $f_b\colon p^{-1}(b) \to p'^{-1}(b)$, $b \in B$, is a (ordinary) homotopy equivalence.*

Examples show (Dold [3, p. 123]) that the assumption about B cannot be omitted in 6.3. It is easy to see that this assumption is invariant under homotopy equivalence (compare Fadell [5, 2]). It is also easy to see that the last condition in 6.3 is fulfilled if f_b is a homotopy equivalence for *one* b in every arc component of B; in fact, this is all we use in our proof.

6.4. THEOREM. *Let B be a topological space which admits a numerable covering $\{V_\lambda\}_{\lambda\in\Lambda}$ such that the inclusion map $V_\lambda \to B$ is nulhomotopic for every λ. Then $p\colon E \to B$ has the* WCHP *if and only if p is fibre homotopy equivalent over each V_λ to a trivial space.*

For a partial result in this direction see Fadell [5, 3.4].

PROOF OF 6.1. Only the "if-part" has to be proved. Assume then $f\colon E \to E'$ is a (ordinary) homotopy equivalence; let $f'\colon E' \to E$ be a homotopy inverse, and $d\colon E' \times [0, 1] \to E'$ a homotopy between $d_0 = ff'$ and $d_1 = \mathrm{id}_{E'}$. Because p has the WCHP, there exists a weak covering homotopy $G\colon E' \times [-1, 1] \to E$ of $\bar{G} = p'd$ with $G(e', -1) = f'(e')$, $e' \in E'$ (note that $pf' = p'ff' = p'd_0$). Define

$$f''\colon E' \to E, \qquad f''(e') = G(e', 1)\ .$$

Then $pf''(e') = \bar{G}(e', 1) = p'd(e', 1) = p'(e')$, so f'' is a map over B. Further, we have the following homotopy between ff'' and $\mathrm{id}_{E'}$:

$$h\colon E' \times [0, 3] \longrightarrow E'\ , \qquad h(e', t) = \begin{cases} fG(e', 1-t)\ , & 0 \leqq t \leqq 2\ , \\ d(e', t-2)\ , & 2 \leqq t \leqq 3\ . \end{cases}$$

h is not a vertical homotopy, but it satisfies $p'h(e', t) = p'h(e', 3-t)$, and we shall use this to deform h into a vertical homotopy.

Define $\bar{H}\colon E' \times [0, 3] \times [0, 1] \to B$ by

$$\bar{H}(e', t_1, t_2) = \begin{cases} p'h(e', t_1) & \text{for } t_1 \leqq \frac{3}{2}(1-t_2)\ . \\ p'h\left(e', \frac{3}{2}(1-t_2)\right) = p'h\left(e', \frac{3}{2}(1+t_2)\right) & \\ & \text{for } \frac{3}{2}(1-t_2) \leqq t_1 \leqq \frac{3}{2}(1+t_2)\ , \\ p'h(e', t_1) & \text{for } t_1 \geqq \frac{3}{2}(1+t_2)\ , \end{cases}$$

and choose a weak covering homotopy $H\colon E' \times [0, 3] \times [-1, 1] \to E'$ of $\bar{H}$ with $H(e', t_1, -1) = h(e', t_1)$. Then the following $D\colon E' \times [0, 7] \to E'$

is a vertical homotopy $ff'' \simeq_B \mathrm{id}_{E'}$ (verifications are left to the reder):

$$D(e', t) = \begin{cases} H(e', 0, t-1) & \text{if } t \leqq 2 \\ H(e', t-2, 1) & \text{if } 2 \leqq t \leqq 5 \\ H(e', 3, 6-t) & \text{if } t \geqq 5 . \end{cases}$$

We can now apply the same argument to f'' (which is right-inverse to f, hence a homotopy equivalence), and find $f'''\colon E \to E'$ with $f''f''' \simeq_B \mathrm{id}_E$, hence

$$f''f \simeq_B (f''f)(f''f''') = f''(ff'')f''' \simeq_B f''f''' \simeq_B \mathrm{id}_E ,$$

which proves the theorem.

In the proofs of 6.3, 6.4 we use the following

6.5. LEMMA. *If $p\colon E \to B \times [0, 1]$ has the* WCHP, *then there exists a map $R\colon E \times [0,1] \to E$ such that*

(i) $pR(e, t) = (\pi(e), t)$,

(ii) $r \simeq_{B\times[0,1]} \mathrm{id}_E$,

where $\pi\colon E \to B$, $\rho\colon E \to [0, 1]$, $r\colon E \to E$ are defined by

$$p(e) = (\pi(e), \rho(e)) , \qquad r(e) = R(e, \rho(e)) , \qquad e \in E .$$

PROOF. Define $\bar{H}\colon E \times [0, 1] \times [0, 1] \to B \times [0, 1]$ by $\bar{H}(e, t_1, t_2) = (\pi(e), (1 - t_2)\rho(e) + t_1t_2)$, and choose a weak covering homotopy $H\colon E \times [0, 1] \times [-1, 1] \to E$ of $\bar{H}$ with $H(e, t_1, -1) = e$ (note that $\bar{H}(e, t_1, t_2) = p(e)$ for $t_2 = 0$, and $= (\pi(e), t_1)$ for $t_2 = 1$). Then $R(e, t) = H(e, t, 1)$ clearly satisfies (i), and (ii) follows from

$$r(e) = R(e, \rho(e)) = H(e, \rho(e), 1) \simeq_{B\times[0,1]} H(e, \rho(e), -1) = e ,$$

where the vertical homotopy $\simeq_{B\times[0,1]}$ is achieved by $H(e, \rho(e), \tau)$ as τ goes from 1 to -1; note that $\bar{H}(e, \rho(e), \tau) = (\pi(e), \rho(e)) = p(e)$, q.e.d.

6.6. COROLLARY. *In the notation of* 6.5, *let $p^t\colon E^t \to B$ be the part of $p\colon E \to B \times I$ over $B \times \{t\} \approx B$. Then the maps*

$$h^1\colon E^0 \longrightarrow E^1 , \qquad h^1(x) = R(x, 1) ,$$
$$h^0\colon E^1 \longrightarrow E^0 , \qquad h^0(y) = R(y, 0)$$

are reciprocal fibre homotopy equivalences.

PROOF. $h^1h^0(y) = R(R(y, 0), 1) \simeq_B R(R(y, 1), 1) = r(r(y)) \simeq_B y$ where the first $\simeq_B$ is achieved by $R(R(y, \tau), 1)$, $0 \leqq \tau \leqq 1$, the second by 6.5 (ii). Similarly $h^0h^1 \simeq_B \mathrm{id}$ follows, q.e.d.

PROOF OF 6.4. The "if-part" of 6.4 is contained in 5.12. Assume then p has the WCHP, let $\alpha\colon V \times [0, 1] \to B$ be a contraction of $V = V_\lambda$ (i.e., $\alpha_0 =$ inclusion, $\alpha_1 =$ constant), and let $p_\alpha\colon E_\alpha \to V \times [0, 1]$ be the induced space (1.6). The part of p_α over $V \times \{0\} \approx V$ agrees with $p_V: E_V = p^{-1}(V) \to V$ because $\alpha_0 =$ inclusion, and the part over $V \times \{1\}$ is a trivial space because α_1 is constant. Therefore p_V is fibre homotopy equivalent to a trivial space by 6.6, q.e.d.

PROOF OF 6.3. If f is a fibre homotopy equivalence then f_b is a homotopy equivalence for every $b \in B$; we have to show the converse. By 3.3 it is enough to show that $f_V: E_V \to E'_V$ is a fibre homotopy equivalence for every $V = V_\lambda$. Let $\alpha: V \times [0, 1] \to B$ be a contraction ($\alpha_0 =$ inclusion, $\alpha_1(V) = b \in B$), and let $f_\alpha: E_\alpha \to E'_\alpha$ the induced map over $V \times [0, 1]$ (1.6). For every $t \in [0, 1]$ we denote by $f^t: E^t \to E'^t$ the part of f_α over $V \times \{t\} \approx V$. Then $f^0 = f_V$ because α_0 is the inclusion $V \subset B$. Using the notation of 6.5, 6.6 (primes refer to $p'_\alpha: E'_\alpha \to V \times [0, 1]$) we have

$$\begin{aligned} f_V(x) = f^0(x) &\simeq_V r'f^0r(x) = R'\big(f^0R(x, 0), 0\big) \\ &\simeq_V R'\big(f^1R(x, 1), 0\big) = h'^0f^1h^1(x) \ , \end{aligned}$$

where the first $\simeq_V$ comes from 6.5 (ii), and the second is given by $R'(f^\tau R(x, \tau), 0)$, $0 \leqq \tau \leqq 1$. By 6.6, the maps h'^0, h^1 are fibre homotopy equivalences, so that it suffices to show that f^1 is a fibre homotopy equivalence. But $\alpha_1(V) = b$, therefore $E^1 = V \times p^{-1}(b)$, $E'^1 = V \times p'^{-1}(b)$, and $f'^1 = \mathrm{id} \times f_b$. Because f_b is a homotopy equivalence by assumption, the assertion follows, q.e.d.

CW-complexes are paracompact (Miyazaki [14]) and locally contractible, and therefore satisfy the assumptions of 6.3, 6.4. The following is a direct (and rather simple) proof of this fact; part of it is due to D. Puppe.

6.7. PROPOSITION. *If B is a connected* CW-*complex, then there exists a numerable covering $\{V_i\}$, $i = 0, 1, \cdots$ of B such that the inclusion map $V_i \to B$ is nulhomotopic for every i.*

If B is not connected, we can argue for each component separately, and again get $\{V_\lambda\}$, which now, of course, need not be countable.

PROOF. Let B^i denote the i-skeleton of B. If we remove from B^{i+1} the center of every $(i + 1)$-cell we get a set $\check{B}^{i+1}$ which is open in B^{i+1}, and of which B^i is a strong deformation retract. Let $d^{i+1}: \check{B}^{i+1} \times [0, 1] \to \check{B}^{i+1}$ be such a deformation retraction, i.e.,

$$d_1^{i+1} = \mathrm{id}\,(\check{B}^{i+1}), \qquad d_0^{i+1}(\check{B}^{i+1}) \subset B^i \ , \qquad d_t^{i+1} \mid B^i = \mathrm{id}\,(B^i)$$

for all $t \in [0, 1]$. Define open sets V_i^j of B^{i+j} by induction on j as follows:

$$V_i^0 = B^i - B^{i-1} \ , \qquad V_i^{j+1} = (d_0^{i+j+1})^{-1}(V_i^j) \ , \qquad i, j = 0, 1, \cdots .$$

Then put $V_i = \bigcup_{j=0}^{\infty} V_i^j$. We claim this is a covering as required.

Since $B^i - B^{i-1} \subset V_i$, we have $B^n \subset \bigcup_{i=0}^{\infty} V_i$, so $\{V_i\}$ is a covering. We now show that $V_i \xrightarrow{\subset} B$ is nulhomotopic, then that $\{V_i\}$ is numerable.

By induction on j, we define deformations d_i^j: $V_i^j \times [0, j] \to V_i^j$, $j = 1, 2, \cdots$ with the following properties

(a) $d_i^j(x, t) = d_i^k(x, t)$ for $x \in V_i^k$, $t \leqq k \leqq j$,

(b) $d_i^j(x, t) = x$ for $x \in V_i^k$, $k \leqq t \leqq j$,

in particular, $d_i^j(x, j) = x$ for all $x \in V_i^j$,

(c) $d_i^j(x, 0) \in V_i^0$ for all $x \in V_i^j$.

Namely

$$d_i^1 = d^{i+1} \mid V_i^1 \times [0, 1] ,$$

$$d_i^{j+1}(x, t) = \begin{cases} d^{i+j+1}(x, t - j) & \text{for } t \geqq j \\ d_i^j(d^{i+j+1}(x, 0), t) & \text{for } t \leqq j . \end{cases}$$

Put

$$d_i\colon V_i \times [0, +\infty) \longrightarrow V_i , \qquad d_i \mid V_i^j \times [0, j] = d_i^j ,$$

and

$$D_i\colon V_i \times \left[0, \frac{\pi}{2}\right] \longrightarrow V_i , \qquad D_i(x, t) = \begin{cases} d_i(x, \tan(t)) & \text{if } t < \frac{\pi}{2} \\ x & \text{if } t = \frac{\pi}{2} . \end{cases}$$

D_i is continuous because the restrictions $D_i \mid V_i^j \times [0, \pi/2]$ are continuous, and B has the weak topology. Further, D_i deforms V_i into $V_i^0 = B^i - B^{i-1}$, and $B^i - B^{i-1}$, in turn, can be deformed into a discrete set (the set of centers of all i-cells). Because B is arcwise connected every discrete set can be deformed into a single point. This shows that $V_i \xrightarrow{\subset} B$ is nulhomotopic.

In order to show that $\{V_i\}$ is numerable, choose a function γ_i: $B^i \to [0, 1]$, $i = 0, 1, \cdots$, such that $\gamma_i^{-1}(0, 1] = B^i - B^{i-1}$ and $\gamma_i(c) = 1$ if and only if c is a center of an i-cell (i.e., $\gamma_i^{-1}(1) = B^i - \check{B}^i$; for example, we can let γ_i increase linearly from 0 to 1 on each radius in each i-cell). By induction on j define π_i^j: $B^{i+j} \to [0, 1]$,

$$\pi_i^0 = \gamma_i , \qquad \pi_i^{j+1}(x) = \begin{cases} \pi_i^j(d_0^{i+j+1}(x)) \cdot (1 - \gamma_{i+j+1}(x)) & \text{for } x \in \check{B}^{i+j+1} \\ 0 & \text{for } x \notin \check{B}^{i+j+1} . \end{cases}$$

One easily verifies that π_i^j is continuous, and $\pi_i^j \mid B^{k+i} = \pi_i^k$ for $k \leqq j$, $(\pi_i^j)^{-1}(0, 1] = V_i^j$. We can therefore define a continuous function π_i: $B \to [0, 1]$ by $\pi_i \mid B^{i+j} = \pi_i^j$, and have $\pi_i^{-1}(0, 1] = V_i$.

The family $\{\pi_i\}$ is not locally finite, but if we put

$$\pi'_i(x) = \text{Max}\,[0,\, \pi_i(x) - i \cdot \textstyle\sum_{\mu<i} \pi_\mu(x)]\ ,$$

then it follows as in the last part of the proof of 4.8 (a) that $\{\pi'_i\}$ is locally finite, $\bigcup \pi'^{-1}_i(0, 1] = B$, and $\pi'^{-1}_i(0, 1] \subset V_i$. To get a partition of unity, divide each π'_i by $\sum_\mu \pi'_\mu$.

7. Classification of numerable bundles

7.1. DEFINITION. A fibre bundle ζ [15] is called *numerable* if B_ζ, the base of ζ, admits a numerable covering $\{V_\lambda\}_{\lambda\in\Lambda}$ such that $\zeta \mid V_\lambda$, the part of ζ over V_λ, is trivial for every λ.

For example, ζ is always numerable if B_ζ is paracompact.

We use the *section extension theorem* 2.7 to prove a classification theorem for numerable bundles which makes no assumptions on the base space (7.5).

The class of numerable bundles has other agreeable properties: The *covering homotopy theorem* holds without assumptions on the base (4.8 a, 7.8), as does the *section extension theorem* for bundles with contractible fibre (2.8 α). If ζ is numerable, then all bundles which are induced from or associated with ζ are also numerable (this is obvious). If H is a subgroup of $G = G_\zeta$ (structure group of ζ), then every H-bundle which is obtained from ζ by reduction of the structure group is numerable provided the coset bundle $G \to G/H$ [15, §7] is numerable (we omit the proof).

Let $k_G X$ be the set of equivalence classes of numerable (principal) G-bundles with base X. If $f\colon X \to Y$ is a continuous map one defines $k_G f\colon k_G Y \to k_G X$ by taking induced bundles; thus k_G becomes a contravariant functor from topological spaces to sets. If $f_0, f_1\colon X \to Y$ are homotopic maps then $k_G(f_0) = k_G(f_1)$ (see 7.10). Therefore k_G can be viewed as a functor on the category $\mathcal{H}$ whose morphisms are *homotopy classes* of continuous maps.

7.2. DEFINITION. If X, B are topological spaces let $[X, B]$ denote the set of homotopy classes of maps $X \to B$. Keeping B fixed, this is a functor from the category $\mathcal{H}$ to the category of sets.

Let G be a topological group. A space B is called *classifying for* G if there exists a natural equivalence

$$T\colon [X, B] \approx k_G X\ ,$$

i.e., if B *represents the functor* k_G (Grothendieck [8, A. 1]). The bundle $\eta \in k_G B$, which under T corresponds to $\mathrm{id}_B \in [B, B]$, is called the *universal* G-bundle.

General properties of representable functors give the following:

(7.3) $T[f] = (k_G f)(\eta)$, where $f\colon X \to B$, and [ ] denotes the homotopy class.

(7.4) If $T\colon [X, B] \approx k_G X$ and $T'\colon [X, B'] \approx k_G X$ are natural equivalences, then there exist unique (up to homotopy) reciprocal homotopy equivalences $B \underset{h'}{\overset{h}{\rightleftarrows}} B'$ with $(k_G h)(\eta') = \eta$, $(k_G h')(\eta) = \eta'$ *(uniqueness of the classifying space and the universal bundle).*

7.5. CLASSIFICATION THEOREM (compare Steenrod [15, § 19]). *A numerable principal G-bundle η is universal if and only if E_η, its bundle space, is contractible.*

PROOF. Assume E_η is contractible. Let ζ be an arbitrary principal G-bundle, and define a new bundle (ζ, η) over $X = B_\zeta$ as follows. The fibre $(\zeta, \eta)_x$ of (ζ, η) over $x \in X$ consists of all admissible maps (bundle maps) of ζ_x into η; clearly $(\zeta, \eta)_x \approx E_\eta$ (these maps are determined by the image of one point). The local product structure in ζ gives a local product structure in (ζ, η). In fact, (ζ, η) is the associated bundle of ζ with fibre E_η on which G operates by $(g, e) \to eg^{-1}$, $e \in E_\eta$, $g \in G$. In particular, (ζ, η) is numerable if ζ is numerable.

A section s in (ζ, η) over $V \subset X$ associates, in a continuous fashion, with every $v \in V$ a map $s_v\colon \zeta_v \to \eta$, i.e., a section s over V is the same as a bundle map $\zeta \mid V \to \eta$.

If E_η is contractible and ζ numerable then $p_{(\zeta,\eta)}\colon E_{(\zeta,\eta)} \to X$ has the section extension property (2.7, 2.8. α). In particular, (ζ, η) admits a section, i.e., ζ admits a bundle map $\zeta \to \eta$. This shows T_η is surjective.

If $f_0, f_1\colon X \to B$ induce equivalent bundles $\zeta_i = f_i^{-1}(\eta) = T_\eta[f_i]$, $i = 0$, 1, let $s_i\colon \zeta_i \to \eta$ be the induced bundle maps, and $h\colon \zeta_0 \to \zeta_1$ an equivalence. Define $\zeta = \zeta_0 \times [0, 1]$ (this is a bundle over $X \times [0, 1]$; cf. Steenrod 11.1), and a partial bundle map of ζ,

$$s\colon \zeta \mid X \times ([0, \tfrac{1}{2} \cup \tfrac{1}{2}, 1]) \to \eta ,$$

$$s(z, t) = \begin{cases} s_0(z) & \text{for } t < \tfrac{1}{2} \\ s_1 h(z) & \text{for } t > \tfrac{1}{2},\ z \in E_{\zeta_0} , \end{cases}$$

View s as a section of (ζ, η) over $X \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1])$. Since this set is a halo around $X \times (\{0\} \cup \{1\})$ (cf. 2.1; take $\tau(x, t) = |2t - 1|$), there exists a global section S in (ξ, η) which agrees with s over $X \times (\{0\} \cup \{1\})$, i.e., there exists a bundle map $S\colon \zeta \to \eta$ which over $X \times (\{0\} \cup \{1\})$ agrees with s. On the base, we then have $B_S\colon X \times [0, 1] = B_\zeta \to B_\eta$, a homotopy between f_0 and f_1. Therefore T_η is injective.

We now assume T_η is bijective. As we shall see in § 8 there exists a numerable G-bundle ζ with contractible bundle space E_ζ. We just proved that T_ζ is also bijective. Surjectivity of T_ζ and T_η yields bundle maps $\eta \xrightarrow{\varphi} \xi \xrightarrow{\psi} \eta$, and injectivity of T_η implies $B_\psi B_\varphi \simeq \mathrm{id}\,(B_\eta)$. The covering homotopy theorem for bundle maps 7.8 then shows that $\psi\varphi$ is homotopic to a bundle map $h: \eta \to \eta$ with $B_h = \mathrm{id}\,(B_\eta)$; in particular, h is an automorphism. But also $h \simeq \psi\varphi \simeq 0$ (because $E_\zeta \simeq 0$), hence $E_\eta \simeq 0$, q.e.d.

7.6. REMARK. The first half of the proof of 7.5 shows: *If η and X are such that every numerable bundle over X and $X \times [0, 1]$ with fibre E_η has the* SEP, *then T_η: $[X, B_\eta] \to k_G X$ is bijective for this particular X.* This applies if $\pi_i(E_\eta) = 0$ for $i \leqq n$, X is paracompact, and every point of X has a neighborhood which is a retract of a CW-complex of dimension $\leqq n$ (2.8 β); it leads to the notion of *n-universality* [15, § 19].

7.7. More generally than G-bundles, one can consider *G-spaces*, i.e., topological spaces E on which G operates on the right, and *G-maps*. Call a G-space *free* if it comes from a numerable principal G-bundle. Then one has in analogy to Cartan-Eilenberg [2, V, 1.1]: *If E is a free G-space and E' a contractible G-space, then there exists a G-map $E \to E'$, and any two such maps are G-homotopic.* This contains the if-part of 7.5, and is proved in the same way.

The following was used above:

7.8. COVERING HOMOTOPY THEOREM FOR BUNDLE MAPS (compare Steenrod [15, 11.3]). *Let ζ, η be numerable (principal) G-bundles, φ: $\zeta \to \eta$ a bundle map, and D: $B_\zeta \times [0, 1] \to B_\eta$ a deformation of B_φ* (i.e., $D(b, 0) = B_\varphi(b)$). *Then there exists a bundle map Φ: $\zeta \times [0, 1] \to \eta$ such that $B_\Phi = D$ and $\Phi(z, 0) = \varphi(z)$, $z \in E_\zeta$.*

PROOF. We note first

7.9. LEMMA. *If p: $E \to X \times I$ has the* CHP, *then every section s of p over $X \times \{0\}$ has an extension over all of $X \times I$.*

Indeed, $\bar{H} = \mathrm{id}_{X \times I}$ is a deformation of ps, and a homotopy H which covers $\bar{H}$ and starts at s is a section as required.

The proof of 7.8 now follows from the covering homotopy theorem 4.8a by a "functional-bundle" argument similar to that in the proof of 7.5: Let ξ, η be (principal) G-bundles and f: $B_\xi \to B_\eta$ a map. Define a new bundle (ξ, η, f) over B_ξ whose fibre over $x \in B_\xi$ consists of all admissible maps $\xi_x \to \eta_{f(x)}$. The local product structures in ξ and η provide a local product structure in (ξ, η, f), and if ξ, η are numerable, then also (ξ, η, f). A section of (ξ, η, f) over $V \subset B$ is a bundle map $\xi \mid V \to \eta$ which, on the base, induces $f \mid V$.

With the data of 7.8, we then put $\xi = \zeta \times [0, 1]$ and consider the bundle (ξ, η, D). The map φ can be viewed as a section in (ξ, η, D) over $X \times \{0\}$. By Lemma 7.9 this section extends over all of $X \times I$, i.e., φ extends to a bundle map Φ as required, q.e.d.

7.10. COROLLARY. *If η is a numerable bundle and $f_0, f_1\colon X \to B_\eta$ are homotopic maps, then the induced bundles $f_0^{-1}(\eta)$, $f_1^{-1}(\eta)$ are equivalent.*

PROOF. Let $\varphi\colon f_0^{-1}(\eta) \to \eta$ be the induced bundle map. Since $B_\varphi = f_0 \simeq f_1$, there exists by 7.8 a (homotopic) bundle map $\varphi'\colon f_0^{-1}(\eta) \to \eta$ with $B_{\varphi'} = f_1$, hence $f_1^{-1}(\eta) \sim f_0^{-1}(\eta)$ (cf. Steenrod 10.3).

8. Existence of universal bundles (after Milnor [13])

From Milnor's construction we conclude

8.1. THEOREM. *There exists a functor η_G from topological groups to universal principal bundles.*

It is indeed clear from the construction that every continuous homomorphism $\gamma\colon G \to H$ induces a map $E_\gamma\colon E_G \to E_H$ (we write E_G instead of E_{η_G}, etc.) with $E_\gamma(e \cdot g) = E_\gamma(e) \cdot \gamma(g)$, $e \in E_G$, $g \in G$; in this sense η_G is a functor.

We have to show that η_G is numerable and E_G contractible (Milnor only proves $\pi_i(E_G) = 0$ for all i).

Recall first (cf. Milnor) that one has "coordinates"

$$t_j\colon E_G \to [0, 1]\,, \qquad g_j\colon t_j^{-1}(0, 1] \to G\,, \qquad\qquad j = 1, 2, \cdots .$$

For every $e \in E_G$, almost all $t_j(e)$ are zero, and $\sum_{j=1}^{\infty} t_j(e) = 1$; otherwise the coordinates can have arbitrary values. Two points $e, e' \in E_G$ are equal if and only if all of their coordinates are equal. The topology in E_G is the coarsest topology for which all t_j, g_j are continuous, i.e., a map into E_G is continuous if and only if the composite with every coordinate is continuous (where defined). This, of course, completely describes the space E_G. The operation of G on E_G is given by $t_j(eg) = t_j(e)$, $g_j(eg) = g_j(e)g$ (where defined). The projection onto the orbit space, i.e., the bundle projection, is denoted by $p_G\colon E_G \to B_G$.

Since t_j is unchanged under the operation of G, it passes to the quotient and defines $t_j\colon B_G \to [0, 1]$. According to Milnor, the bundle η_G is trivial over $V_j = t_j^{-1}(0, 1]$. We now show that $\{V_j\}$ is a numerable covering, hence η_G is a numerable bundle. Define

$$\pi_j\colon B_G \to [0, 1], \quad \pi_j(b) = \operatorname{Max}\,[0, t_j(b) - \textstyle\sum_{\mu<j} t_\mu(b)]\,.$$

For fixed $b_0 \in B_G$ let k be the smallest integer such that $t_k(b_0) \neq 0$, and choose N such that $\sum_{j=1}^{N} t_j(b_0) = 1$. Then $\pi_k(b_0) = t_k(b_0) \neq 0$, hence

$\bigcup_k \pi_k^{-1}(0, 1] = B_G$. Further, if $j > N$, then $\pi_j(b) = 0$ for all b with $(\sum_{i=1}^N t_i)(b) > \frac{1}{2}$, so $\{\pi_j\}$ is locally finite. The sequence $\pi_j / \sum_\mu \pi_\mu$, $j = 0, 1, \cdots$ is then a locally finite partition of unity which refines $\{V_j\}$.

In order to describe a contraction of E_G, we introduce some notation. Put

$$\sigma_k \colon E_G \longrightarrow [0, 1] , \qquad \sigma_k = \sum_{j \leqq k} t_j ,$$
$$E_k = \sigma_k^{-1}(1) , \qquad A_k = \sigma_k^{-1}(2/3, 1] , \qquad U_k = \sigma_k^{-1}(0, 1] ,$$
$$\Delta = \{e \in E_G \mid g_j(e) = \varepsilon = \text{neutral element of } G, \text{ for all } j \text{ with } t_j(e) > 0\} .$$

Define a deformation $d' \colon U_k \times [0, 1] \to U_k$ by

$$t_j(d'(e, \tau)) = \begin{cases} \dfrac{\tau + (1 - \tau)\sigma_k(e)}{\sigma_k(e)}\, t_j(e) & \text{for } j \leqq k \\ (1 - \tau) t_j(e) & \text{for } j > k,\ \tau \in [0, 1] \end{cases}$$

(note that $\sum_j t_j(d'(e, \tau)) = ((\tau + (1 - \tau)\sigma_k(e))/\sigma_k(e))\sigma_k(e) + (1 - \tau)(1 - \sigma_k(e)) = 1$), and $g_j(d'(e, \tau)) = g_j(e)$ if $t_j(d'(e, \tau)) > 0$. We clearly have

(8.2) $\qquad d'(e, 0) = e , \qquad d'(e, 1) \in E_k , \qquad e \in \Delta \Longrightarrow d'(e, \tau) \in \Delta .$

Next deform E_k as follows

$$d'' \colon E_k \times [0, 1] \longrightarrow E_{k+1} ,$$
$$t_j(d''(e, \tau)) = \begin{cases} (1 - \tau) t_j(e) & \text{for } j \leqq k \\ \tau & \text{for } j = k + 1 \\ 0 & \text{for } j > k + 1 , \end{cases}$$
$$g_j(d''(e, \tau)) = \begin{cases} g_j(e) & \text{for } j \leqq k \\ \varepsilon & \text{for } j = k + 1 . \end{cases}$$

We have

(8.3) $\qquad d''(e, 0) = e , \qquad d''(e, 1) \in \Delta , \qquad e \in \Delta \Longrightarrow d''(e, \tau) \in \Delta .$

Combining d' and d'' we obtain a deformation (in fact a nulhomotopy of the inclusion $U_k \xrightarrow{\subset} U_{k+1}$)

$$d^k \colon U_k \times [0, 1] \longrightarrow U_{k+1}$$

(8.4) with

$$d_k(e, 0) = e , \qquad d^k(e, 1) \in \Delta , \qquad e \in \Delta \Longrightarrow d^k(e, \tau) \in \Delta .$$

Using $\{d^k\}$ we shall construct deformations

$$\varphi^k \colon U_k \times [0, 1] \longrightarrow U_{k+1} , \qquad k = 1, 2, \cdots$$

with

(8.5) $\qquad \varphi^{k+1} \mid A_k \times [0, 1] = \varphi^k \mid A_k \times [0, 1] ,$

(8.6) $$\varphi^k(e, 0) = e\ ,\quad \varphi^k(e, 1) \in \Delta\ .$$

Assume this is done. Then we can define $\varphi\colon E_G \times [0, 1] \to E_G$ by $\varphi \mid A_k \times [0, 1] = \varphi^k \mid A_k \times [0, 1]$. By 8.5, φ is well-defined, and since $\{A_k\}$ is an open covering of E_G, φ is continuous. Further, $\varphi(e, 0) = e$, $\varphi(e, 1) \in \Delta$, so φ is a deformation of E_G into Δ. But Δ is contractible (it is a simplex!) as the deformation

$$\psi\colon \Delta \times [0, 1] \longrightarrow \Delta\ ,\qquad t_j\psi(e, \tau) = \begin{cases}(1 - \tau)t_1(e) + \tau\ , & j = 1\ ,\\ (1 - \tau)t_j(e)\ , & j > 1\ ,\end{cases}$$

shows. This proves contractibility of E_G, provided we can construct the deformations φ^k.

Put $\varphi^1 = d^1$. If φ^k is already constructed define φ^{k+1} as follows

(8.7) $$\varphi^{k+1}(e, \tau) = \begin{cases}\varphi_k(e, \tau) & \text{if } 2 \leqq 3\sigma_k(e)\\ d^{k+1}(\varphi^k(e, \tau), 2\tau[2 - 3\sigma_k(e)]) & \text{if } \frac{3}{2} \leqq 3\sigma_k(e) \leqq 2\\ d^{k+1}(\varphi^k[e, 2\tau(3\sigma_k(e) - 1)], \tau) & \text{if } 1 \leqq 3\sigma_k(e) \leqq \frac{3}{2}\\ d^{k+1}(e, \tau) & \text{if } 3\sigma_k(e) \leqq 1\ .\end{cases}$$

If $3\sigma_k(e) = 2$, the second expression becomes $d^{k+1}(\varphi^k(e, \tau), 0) = \varphi^k(e, \tau)$. For $3\sigma_k(e) = \frac{3}{2}$ the second and third expressions both equal $d^{k+1}(\varphi^k(e, \tau), \tau)$. For $3\sigma_k(e) = 1$, the third expression is $d^{k+1}(\varphi^k(e, 0), \tau) = d^{k+1}(e, \tau)$. Thus φ^{k+1} is well-defined.

Assume $e \in A_k$. Then $3\sigma^k(e) \geqq 2$, hence $\varphi^{k+1}(e, \tau) = \varphi^k(e, \tau)$, i.e., φ^{k+1} satisfies 8.5. The relation $\varphi^k(e, 0) = e$ is clear; from the inductive hypothesis 8.6 and from 8.4, one gets $\varphi^{k+1}(e, 1) \in \Delta$, q.e.d.

8.8. REMARK. In order to show that a bundle ζ is numerable if its base B_ζ is a CW-complex, it is not necessary to invoke Miyazaki's theorem on the paracompactness of B_ζ. One simply constructs a bundle map $\zeta \to \eta_G$ in the usual skeleton-after-skeleton fashion; ζ is then induced from η_G and therefore numerable.

9. Application to associated bundles

A continuous homomorphism $\gamma\colon G \to H$ between topological groups induces a natural transformation $k_\gamma\colon k_G \to k_H$ (see § 7 for the definition of k_G) by taking associated bundles (weakly associated in Steenrod [15, 9.1]). Under the equivalence $k_G \approx [-, B_G]$ (7.5), this transformation corresponds to composition with $[B_\gamma] \in [B_G, B_H]$. As an application of 7.5 and 8.1 we show

9.1. THEOREM. *A continuous homomorphism $\gamma\colon G \to H$ induces an equivalence $k_\gamma\colon k_G \approx k_H$ if and only if γ is an (ordinary) homotopy equivalence.*

9.2. REMARK. One should be careful *not to conclude* from this that homomorphisms $\gamma_0, \gamma_1: G \to H$ which are homotopic *as maps* induce the same transformation $k_{\gamma_0}, k_{\gamma_1}$. For example, if $G = \mathbf{Z}_2$, $H = SO(2)$, and $\gamma: G \to H$ maps the generator into the antipodal map, then γ is nulhomotopic (because SO(2) is arcwise connected) but k_γ is not trivial: the twofold covering of the projective plane $P_2\mathbf{R}$ goes into a nontrivial SO(2)-bundle under k_γ.

If γ_0, γ_1 are homotopic *as homomorphisms*, i.e., if they are connected by a continuous family of homomorphisms $\gamma_t: G \to H$, $0 \leqq t \leqq 1$, then B_{γ_t} is a homotopy from B_{γ_0} to B_{γ_1}, hence $k_{\gamma_0} = k_{\gamma_1}$. This result can be expressed more generally by saying that B_G (or η_G) is a *continuous functor* (i.e., takes continuous families of homomorphisms into continuous families of maps).

PROOF OF 9.1. The homomorphism $\gamma: G \to H$ induces $E_\gamma: E_G \to E_H$ such that $E_\gamma(eg) = E_\gamma(e)\gamma(g)$, and E_γ in turn induces $B_\gamma: B_G \to B_H$ by passage to quotients. We have to show that γ is a homotopy equivalence if and only if B_γ is a homotopy equivalence.

Let $\eta'_H = B_\gamma^{-1}(\eta_H)$ be the induced H-bundle over B_G, and $p'_H: E'_H \to B'_H = B_G$ its bundle projection. Then B_γ is a homotopy equivalence if and only if η'_H is a universal H-bundle (from the definition of universal bundles, § 7), i.e., if and only if E'_H is contractible (7.5). Now E_γ induces $E'_\gamma: E_G \to E'_H$ with $E'_\gamma(eg) = E'_\gamma(e)\gamma(g)$, and we have to show that γ is a homotopy equivalence if and only if E'_γ is a homotopy equivalence (since E_G is contractible). This follows from 6.1 and 3.3 or 6.3:

If E'_γ is a homotopy equivalence, then it is a fibre homotopy equivalence by 6.1, hence a homotopy equivalence on each fibre. But the fibre of η_G (resp. η'_H) over $b \in B_G$ can be identified with G (resp. H) and $E'_\gamma \mid p_G^{-1}(b)$ with γ.

Conversely, let γ be a homotopy equivalence. Choose a section s of η_G over $V_j = t_j^{-1}(0, 1]$ (see proof of 8.1); then $s' = E'_\gamma s$ is a section of η'_H over V_j. Using s, s' one can identify $p_G^{-1}(V_j)$ (resp. $p'^{-1}_H(V_j)$) with $V_j \times G$ (resp. $V_j \times H$) because we have principal bundles; Steenrod [15, 8.3] and $E'_\gamma \mid p_G^{-1}(V_j)$ with $\mathrm{id} \times \gamma$, which is clearly a fibre homotopy equivalence. Therefore E'_γ is itself a fibre homotopy equivalence by 3.3, in particular a homotopy equivalence. We could also apply 6.3 because V_j is contractible in B_G (it lifts to E_G, and E_G is contractible).

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