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HOMOLOGY OF SYMMETRIC PRODUCTS AND OTHER FUNCTORS OF COMPLEXES

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Introduction

A main objective of this paper is to prove that the homology groups of symmetric products (or more generally Γ-products; Definition 7.1) of a polyhedron (a CW-complex) Y are determined by the homology groups of Y (Theorem 7.2). This is done in the algebraic frame work of FD-modules as defined by Eilenberg-MacLane [4].

We begin with a comparison between FD-modules and chain-modules; we show that these notions are equivalent. More precisely, there is a functor $\mathfrak{N}$ (Definition 1.2) from FD-modules to chain-modules and a functor $\mathfrak{K}$ (Definition 1.8) the other way such that the composite functors $\mathfrak{N}\mathfrak{K}$ and $\mathfrak{K}\mathfrak{N}$ are naturally equivalent to the respective identity functors (Theorem 1.9). These functors preserve homotopy, i.e., the relations of FD-homotopy (Definition 2.1) and chain-homotopy are transformed into each other under the functors $\mathfrak{N}$ resp. $\mathfrak{K}$ (Theorem 2.6). As consequences, we obtain for free FD-modules K, K' (over a principal ideal domain) that every homomorphism of homology modules can be realized by an FD-map $K \to K'$ (Proposition 3.5), and that K, K' are of the same homotopy type if and only if they have isomorphic homology modules (Theorem 3.4).

Then we ask for which functors T from FD-modules to FD-modules the homology of $T(K)$ is determined by the homology of K. From § 1—3 we easily see (Proposition 4.2) that T has this property, if it preserves homotopy (Definition 4.1). A large class of homotopy preserving functors T is obtained by prolongation (Definition 5.1 and 5.1′) of functors defined on modules (Theorem 5.6). For these T we show that $H_q(T(K)) \cong H_q(T(K'))$ if K, K' are free FD-modules (over a principal ideal domain) with $H(K) \cong H(K')$ for $i \leqq q$ (Theorem 5.11).

Examples are given in § 6, including symmetric products (Γ-products) of FD-modules (6.2). Using the geometric realization of semi-simplicial complexes [9] we translate these examples into geometry (§ 7; also §§ 10 — 11), and obtain the result stated in the beginning of this introduction (Theorem 7.2; also 7.6).

There is an appendix (§§ 8—11) which is independent (except for the notation) of the preceding paragraphs. We describe a splitting property of symmetric products (Γ-products) of direct sums of FD-modules (8.7, 8.8), and thereby obtain the splitting formula of Steenrod for symmetric products of FD-modules with base point (9.3, 10.3). Finally we discuss infinite symmetric products (§ 10) resp. reduced products (§ 11) of FD-modules with base point and their relation to symmetric algebras resp. tensor algebras of (FD-) modules.

I wish to thank J. C. Moore and D. M. Kan for very helpful suggestions which led to generalizations of the results as well as simplifications of the proofs. Some of the methods of proof in § 1 are implicitly contained in seminar notes of J. C. Moore [11]. The main results of §§ 1—2 were obtained by D. M. Kan before the author found them.

1. Equivalence between chain-modules and FD-modules

Let Λ be a commutative ring with unit; by a module we always mean a unitary module over Λ.

(1.1). Definition [4]. An FD-module is a sequence of modules K_q, $q = 0, 1, \cdots$ together with *face-operators* $\partial_i : K_q \to K_{q-1}$ and *degeneracy-operators* $s_i : K_q \to K_{q+1}$ for $i = 0, 1, \cdots$. The ∂_i and s_i are module-homomorphisms and satisfy the FD-*identities*

$$\begin{array}{lll} \partial_i = 0,\ s_i = 0 & & \text{for } i > q \text{ (the trivial operators)} \\ \partial_i\partial_j = \partial_{j-1}\partial_i & & i < j \\ s_i s_j = s_{j+1}s_i & & i \leqq j \\ \partial_i s_j = s_{j-1}\partial_i & & i < j \\ \partial_i s_i = \partial_{i+1}s_i = \text{identity} & & i \leqq q \ (= 0 \text{ for } i > q) \\ \partial_i s_j = s_j\partial_{i-1} & & i > j+1. \end{array}$$

Let K, K' be FD-modules. An FD-map $F : K \to K'$ is a sequence of module homomorphisms $F_q : K_q \to K'_q$ such that

$$\partial_i F_q = F_{q-1}\partial_i, \quad s_i F_q = F_{q+1}s_i \qquad \text{for all } i \text{ and } q.$$

FD-modules arise naturally in the theory of (complete) semi-simplicial complexes [3]. The module of q-chains $K(X, \Lambda)_q$ of such a complex X (with coefficients in Λ) is freely generated by the q-simplices of X. Therefore the face- and degeneracy-operators in X extend in a unique way to homomorphisms $\partial_i :\ K(X, \Lambda)_q \to K(X, \Lambda)_{q-1}$, $s_i :\ K(X, \Lambda)_q \to K(X, \Lambda)_{q+1}$, thus turning the sequence $K(X, \Lambda)_q$ into an FD-module $K(X, \Lambda)$, the FD-*module of* X.

There are several chain-modules associated with an FD-module K. One is obtained by introducing the boundary homomorphism $\partial =$

$\partial_0 - \partial_1 + \partial_2 \cdots$ in K, another one is the normalized chain-module of K as defined by Eilenberg-MacLane in [4]. The following definition is due to J. C. Moore [11]; we shall see (1.12) that, up to an isomorphism, it leads to the normalized chain-module.

(1.2). DEFINITION. Let K be an FD-module[1]. Define

$$\mathfrak{N}(K)_q = \cap_{i<q} \ker (\partial_i : K_q \to K_{q-1}) , \qquad q = 0, 1, \cdots .$$

It follows from $\partial_i \partial_q = \partial_{q-1}\partial_i$, $i < q$, that

$$\partial_q(\mathfrak{N}(K)_q) \subset \mathfrak{N}(K)_{q-1}$$

and

$$\partial_{q-1}\partial_q \mid \mathfrak{N}(K)_q = 0 ;$$

i. e., the modules $\mathfrak{N}(K)_q$ together with the homomorphisms $\partial = \partial_q \mid \mathfrak{N}(K)_q$ form a chain-module. We denote it by $\mathfrak{N}(K)$ and call it the *normal chain-module* of K.

Every FD-map $F: K \to K'$ defines by restriction a chain-map

$$\mathfrak{N}(F): \quad \mathfrak{N}(K) \to \mathfrak{N}(K') .$$

It is clear that $\mathfrak{N}$ is an *additive functor* from FD-modules to chain-modules. (Additive means : $\mathfrak{N}(F + F') = \mathfrak{N}(F) + \mathfrak{N}(F')$).

We want to prove that the FD-module K is entirely determined by its normal chain-module $\mathfrak{N}(K)$. More generally we shall exhibit a functor $\mathfrak{K}$ from chain-modules to FD-modules which is " inverse " to the functor $\mathfrak{N}$ (Theorem 1.9). The following considerations are to prepare and motivate the definition of $\mathfrak{K}$.

Let $K(q)$ be the FD-module of the *standard q-simplex*, i.e., $K(q)_r$ is freely generated by the $(r + 1)$-tuples $(a_0, a_1, \cdots a_r)$ of integers such that $0 \leqq a_0 \leqq a_1 \leqq \cdots \leqq a_r \leqq q$ (" the r-simplices "), and

$$\partial_i(a_0, a_1, \cdots a_r) = (a_0, \cdots a_{i-1}, a_{i+1}, \cdots a_r)$$
$$s_i(a_0, a_1, \cdots a_r) = (a_0, \cdots a_{i-1}, a_i, a_i, a_{i+1}, \cdots a_r) .$$

We denote by Δ_r the *basic element* $(0, 1, \cdots q) \in K(q)_q$, and by $\mathrm{N}(q)$ the normal chain-module of $K(q)$.

If K is an FD-module and $a \in K_q$ there is a unique FD-map $F: K(q) \to K$ such that $F(\Delta_q) = a$ [4, 3]; i. e., if we associate with every $F: K(q) \to K$ its value $F(\Delta_q) \in K_q$ we obtain an isomorphism

(1.3) $$K_q \cong \text{FD-Hom}(K(q), K) .$$

In order to describe what happens to faces and degeneracies under these isomorphisms define FD-maps

$$\varepsilon^i : K(q-1) \to K(q), \quad \eta^i : K(q+1) \to K(q)$$

[1] ker = kernel, im = image.

by

(1.4) $$\varepsilon^i \Delta_{q-1} = \partial_i \Delta_q$$

(1.5) $$\eta^i \Delta_{q+1} = s_j \Delta_q \; .$$

Then the isomorphisms (1.3) transform ∂_i, s_i into composition with ε^i, η^i.

Assume now there is an inverse functor to $\mathfrak{N}$. This would imply (cf. 1.23) that for every two FD-modules K, K' we have

(1.6) $$\mathfrak{N} : \text{FD-Hom}(K, K') \cong \text{Chain-Hom}(\mathfrak{N}(K), \mathfrak{N}(K')) \; .$$

Combining (1.3) and (1.6) gives

(1.7) $$K_q \cong \text{Chain-Hom}(N(q), \mathfrak{N}(K)) \; ,$$

and these isomorphisms transform ∂_i, s_i into composition with $\mathfrak{N}(\varepsilon^i)$, $\mathfrak{N}(\eta^i)$. This suggests the following definition which is due to D. M. Kan.

(1.8). DEFINITION. Let C be a chain-module. Define an FD-module $\mathfrak{K}(C)$ by

$$\mathfrak{K}(C)_q = \text{Chain-Hom}(N(q), C)$$

$$\partial_i \varphi = \varphi \circ \mathfrak{N}(\varepsilon^i),^{2} \quad s_i \varphi = \varphi \circ \mathfrak{N}(\eta^i), \quad \varphi \in \mathfrak{K}(C)_q, i = 0, 1, \cdots .$$

(ε^i and η^i are defined by (1.4), (1.5)).

The FD-identities for the homomorphisms ∂_i and s_i follow from the dual identities which hold for ε^i and η^i [4, (2.3′)-(2.5′)]. For instance $\partial_i \partial_j = \partial_{j-1} \partial$ for $i < j$ follows from $\varepsilon^j \varepsilon^i = \varepsilon^i \varepsilon^{j-1}$.

If $f : C \to C'$ is a chain-map then by composition with f we obtain an FD-map $\mathfrak{K}(f) : \mathfrak{K}(C) \to \mathfrak{K}(C')$, $\mathfrak{K}(f)\varphi = f \circ \varphi$.

It is clear that $\mathfrak{K}$ is an *additive functor* from chain-modules to FD-modules. It is inverse to $\mathfrak{N}$ in the following sense.

(1.9). THEOREM. *The composite functors $\mathfrak{N}\mathfrak{K}$ and $\mathfrak{K}\mathfrak{N}$ are naturally equivalent to the respective identity functors.*

A general argument on functors will give the following consequence of (1.9).

(1.10). COROLLARY. (a) *Let K, K' be* FD-*modules. Then*

$$\mathfrak{N} : \text{FD-Hom}(K, K') \cong \text{Chain-Hom}(\mathfrak{N}(K), \mathfrak{N}(K')) \; ,$$

i. e., every chain-map $f : \mathfrak{N}(K) \to \mathfrak{N}(K')$ has a unique FD-*extension $F : K \to K'$.*

(b) *Let C, C' be chain-modules. Then*

$$\mathfrak{K} : \text{Chain-Hom}(C, C') \cong \text{FD-Hom}(\mathfrak{K}(C), \mathfrak{K}(C')) \; .$$

For the proof of 1.9 we need the following

(1.11). LEMMA. *Let K be an* FD-*module. Define*

[2] The sign $\circ$ denotes composition, but is used only if it is considered as helpful.

$$\mathfrak{N}^k = \cap_{i \leqq k} \ker(s_i\partial_i)^{1)} \qquad \textit{(in all dimensions)}$$

$$\mathfrak{D}^k = \{\cup_{i \leqq k} \mathrm{im}(s_i)\} = \text{submodule generated by } \cup_{i \leqq k} \mathrm{im}(s_i).^{1)}$$

Then

$$K = \mathfrak{N}^k + \mathfrak{D}^k \qquad \textit{(direct sum)}.$$

If we observe that in K_q we have

$$\ker(s_i\partial_i) = \ker \partial_i \text{ for } i < q \quad \text{and} \quad \ker(s_i\partial_i) = K_q \text{ for } i \geqq q$$

this implies (put $k = \infty$)

(1.12). COROLLARY. *For every* FD-*module K we have*

$$K_q = \mathfrak{N}(K)_q + \mathfrak{D}(K)_q \qquad q = 0, 1, \cdots$$

where $\mathfrak{D}(K)_q \subset K_q$ is the module which is generated by the degenerate elements $s_i x$, $x \in K_{q-1}$, $i = 0, 1, \cdots$.

This shows that $\mathfrak{N}(K)$ *is naturally isomorphic with the normalized chain-module $K/\mathfrak{D}$* [4, 4].

PROOF OF (1.11). The following relations follow immediately from the FD-identities.

(1.13) $$(s_k\partial_k)s_k = s_k, \quad (s_k\partial_k)(s_k\partial_k) = s_k\partial_k \,.$$

(1.14) $$\partial_k(\mathfrak{N}^{k-1}) \subset \mathfrak{N}^{k-1} \,.$$

(1.15) $$s_k(\mathfrak{N}^{k-1}) \subset \mathfrak{N}^{k-1}$$

(1.16) $$s_k(\mathfrak{D}^{k-1}) \subset \mathfrak{D}^{k-1}$$

(for instance, (1.15) follows from $s_i\partial_i s_k = s_i s_{k-1}\partial_i = s_k s_i \partial_i$, $i < k$).

From (1.13) we get.

(1.17) $$K = \ker(s_k\partial_k) + \mathrm{im}(s_k\partial_k) = \ker(s_k\partial_k) + \mathrm{im}(s_k)\,,$$

and since $s_k\partial_k(\mathfrak{N}^{k-1}) \subset \mathfrak{N}^{k-1}$ (1.14) and (1.15).

(1.18) $$\mathfrak{N}^{k-1} = \ker(s_k\partial_k) \cap \mathfrak{N}^{k-1} + \mathrm{im}(s_k\partial_k) \cap \mathfrak{N}^{k-1} = \mathfrak{N}^k + \mathrm{im}(s_k) \cap \mathfrak{N}^{k-1}.$$

Now proceed by induction on k. The case $k = 0$ of (1.11) is contained in (1.17). The inductive hypothesis is

$$K = \mathfrak{N}^{k-1} + \mathfrak{D}^{k-1} \,.$$

Using (1.15) and (1.16) it gives

(1.19) $$\mathrm{im}(s_k) = \mathrm{im}(s_k) \cap \mathfrak{N}^{k-1} + \mathrm{im}(s_k) \cap \mathfrak{D}^{k-1}\,,$$

hence

$$\begin{aligned} K &= \mathfrak{N}^{k-1} + \mathfrak{D}^{k-1} \\ &= \mathfrak{N}^k + \mathrm{im}(s_k) \cap \mathfrak{N}^{k-1} + \mathfrak{D}^{k-1} && \text{(by 1.18)} \\ &= \mathfrak{N}^k + \mathfrak{D}^k && \text{(by 1.19)} \end{aligned}$$

PROOF OF THEOREM (1.9). We first construct the natural equivalence $\Phi : \mathfrak{N}\mathfrak{K} \sim \mathrm{Id}$, i. e., for every chain-module C we construct an isomorphism $\Phi(C) : \mathfrak{N}\mathfrak{K}(C) \cong C$ such that commutativity holds in

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & C' \\ {\scriptstyle \Phi(C)}\uparrow & & \uparrow{\scriptstyle \Phi(C')} \\ \mathfrak{N}\mathfrak{K}(C) & \xrightarrow[\mathfrak{N}\mathfrak{K}(f)]{\qquad} & \mathfrak{N}\mathfrak{K}(C') \end{array}$$

for every chain-map $f : C \to C'$ (this is the naturality condition for Φ).

Consider the chain-module $N(q)$. By (1.12) it is isomorphic with the normalized chain-module $K(q)/\mathfrak{D}(K(q))$. The structure of the latter is well known. In particular we shall use the following properties without proof.

The chain-submodule $D(q) \subset N(q)$ which is generated by the images of $\mathfrak{N}(\varepsilon^i) : N(q-1) \to N(q)$ for $i < q$ covers all of $N(q)$ except in dimensions $(q-1)$ and q. More precisely, if $\overline{\Delta}_q \in N(q)_q$ is the component of the basic element $\Delta_q \in K(q)_q$ with respect to the direct sum decomposition (1.12) then

$$\begin{array}{ll} (N(q)/D(q))_r = 0 & \text{for } r \neq q-1,\, q \\ (N(q)/D(q))_q = N(q)_q \cong \Lambda, & \text{generated by } \overline{\Delta}_q \\ (N(q)/D(q))_{q-1} \cong \Lambda, \text{ generated by } \partial\overline{\Delta}_q = \text{class of } \mathfrak{N}(\varepsilon^q)\overline{\Delta}_{q-1}. & \end{array}$$

Now let C be a chain-module. Then $\mathfrak{K}(C)_q = \text{Chain-Hom}\ (N(q), C)$, and $\mathfrak{N}\mathfrak{K}(C)_q$ consists of all those chain-maps $\varphi : N(q) \to C$ which vanish on $D(q)$, i. e.,

$$\mathfrak{N}\mathfrak{K}(C)_q \cong \text{Chain-Hom}\ (N(q)/D(q), C)\ .$$

It is clear that for every $c \in C_q$ there is exactly one chain-map $\varphi : N(q)/D(q) \to C$ such that $\varphi(\overline{\Delta}_q) = c$, i. e., if we associate with each φ its value on $\overline{\Delta}_q$ we obtain an isomorphism

(1.20) $$\Phi(C)_q : \mathfrak{N}\mathfrak{K}(C)_q \cong C_q\ ;\ \Phi(C)_q\varphi = \varphi(\overline{\Delta}_q)\ .$$

We show that $\{\Phi_q\} = \{\Phi(C)_q\}$ is a chain-map:

$$\partial \circ \Phi_q(\varphi) = \partial \circ \varphi(\overline{\Delta}_q) = \varphi \circ \partial(\overline{\Delta}_q) = \varphi \circ \mathfrak{N}(\varepsilon^q)\overline{\Delta}_{q-1} = (\partial\varphi)\overline{\Delta}_{q-1} = \Phi_{q-1}(\partial\varphi).$$

The proof of the naturality of Φ is straightforward.

Now we establish an equivalence $\Psi' : \text{Id} \sim \mathfrak{K}\mathfrak{N}$. For every FD-module K let $\Psi'(K) : K \to \mathfrak{K}\mathfrak{N}(K)$ be the FD-map which is given by composition

$$K_q \overset{1.3}{\cong} \text{FD-Hom}\ (K(q), K) \xrightarrow{\mathfrak{N}} \text{Chain-Hom}\ (N(q), \mathfrak{N}(K)) = \mathfrak{K}\mathfrak{N}(K)_q\ .$$

Ψ' is clearly a natural transformation $\text{Id} \to \mathfrak{K}\mathfrak{N}$. We have to show that each $\Psi'(K)$ is an isomorphism.

It follows from the definition that $\Psi'(K)$ maps $a \in K_q$ into a chain-map

$\varphi : N(q) \to \mathfrak{N}(K)$ such that $\varphi(\overline{\Delta}_q) = \bar{a}$, where $\bar{a} \in \mathfrak{N}(K)$ denotes the component of a with respect to the decomposition (1.12); similarly for $\overline{\Delta}_q$. If $a \in \mathfrak{N}(K)$ then $\varphi(\overline{\Delta}_q) = a$, i. e., the restriction of $\Psi'(K)$ to $\mathfrak{N}(K)$ is nothing but (cf. 1.20)

$$\mathfrak{N}(\Psi'(K)) = (\Phi(\mathfrak{N}(K))^{-1}\,, \tag{1.21}$$

and is therefore an isomorphism. The theorem now follows from

(1.22). Lemma. *Let* $F: K \to K'$ *be an* FD-*map such that* $\mathfrak{N}(F) : \mathfrak{N}(K) \cong \mathfrak{N}(K')$. *Then* $F: K \cong K'$.

Proof. We prove by induction on q that $F_q : K_q \to K'_q$ is monomorphic (epimorphic) if $\mathfrak{N}(F)_i$ is monomorphic (epimorphic) for $i \leqq q$. This is clear for $q = 0$ since $\mathfrak{N}(F)_0 = F_0$.

Now let $q > 0$, let $\mathfrak{N}(F)_i$ be monomorphic for $i \leqq q$, and let $a \in K_q$ such that $F_q(a) = 0$. Then $\partial_i F_q(a) = F_{q-1}(\partial_i a) = 0$ for all i, hence $\partial_i a = 0$ by induction, hence $a \in \bigcap_i \ker \partial_i \subset \mathfrak{N}(K)$, hence $a \in \ker(\mathfrak{N}(F)_q) = 0$.

If $\mathfrak{N}(F)_i$ is epimorphic for $i \leqq q$ then $\mathfrak{N}(K')_q \subset \operatorname{im}(F_q)$, and by (1.12) it is sufficient to show that every degenerate element lies in the image of F_q. This is clear since F_{q-1} is epimorphic (by induction) and F commutes with degeneracies (i. e., $s_i F_{q-1} = F_q s_i$).

Proof of (1.10). We prove quite generally

(1.23). *Let* $S: \mathfrak{A} \to \mathfrak{B}$ *and* $T: \mathfrak{B} \to \mathfrak{A}$ *be (covariant) functors* ($\mathfrak{A}$ *and* $\mathfrak{B}$ *are categories) such that both compositions are naturally equivalent to the respective identity functors*: $ST \sim \mathrm{Id}$, $TS \sim \mathrm{Id}$. *Then*

$$S : H(A, A') \doteq H(S(A')), S(A')) \qquad A, A' \in \mathfrak{A} \tag{1.24}$$

and

$$T : H(B, B') \doteq H(T(B), T(B')) \qquad B, B' \in \mathfrak{B}\,. \tag{1.25}$$

$H(\ \ )$ *denotes the set of maps in the corresponding category, and the sign* $\doteq$ *simply means that we have a* 1–1 *correspondence.*

Proof: Let $\Phi : TS \sim \mathrm{Id}$ be a natural equivalence. Then for every map $f: A \to A'$ in $\mathfrak{A}$ we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & A' \\ {\scriptstyle \Phi(A)}\uparrow & & \uparrow{\scriptstyle \Phi(A')} \\ TS(A) & \xrightarrow[\quad TS(f) \quad]{} & TS(A') \end{array}$$

i. e., $TS(f) = \Phi(A')^{-1} f \Phi(A)$. This shows

$$TS : H(A, A') \doteq H(TS(A), TS(A'))\,. \tag{1.26}$$

Similarly

(1.27) $$ST : H(S(A), S(A')) \doteq H(STS(A), STS(A')) .$$

Combining (1.26) and (1.27) we obtain

(1.28) $$T : H(S(A), S(A')) \doteq H(TS(A), TS(A')) .$$

Now, the composition

$$H(A, A') \overset{TS}{\doteq} H(TS(A), TS(A')) \overset{T^{-1}}{\doteq} H(S(A), S(A'))$$

gives (1.24). The relation (1.25) follows similarly.

(1.29). REMARK. Let $\Psi : \mathfrak{K}\mathfrak{N} \sim \mathrm{Id}$ be the natural equivalence given by $\Psi(K) = \Psi'(K)^{-1}$ (Proof of 1.9). Then (1.21) shows

$$\mathfrak{N}\Psi = \Phi\mathfrak{N} .$$

One may also show

$$\mathfrak{K}\Phi = \Psi\mathfrak{K} .$$

In general, if S and T are functors as in Proposition (1.23) *we can always find natural equivalences* $\Phi : TS \sim \mathrm{Id}$ and $\Psi : ST \sim \mathrm{Id}$ *such that*

(1.30) $$T\Psi = \Phi T, \quad S\Phi = \Psi S .$$

To prove this let $\Phi : TS \sim \mathrm{Id}$ be any natural equivalence and define Ψ by

$$T(\Psi(B)) = \Phi(T(B)) ,$$

This is possible since (1.25) holds.

One has to prove naturality of Ψ, and the second equation (1.30) (the first holds by definition of Ψ). This can be done without difficulty using the naturality of Φ, and the relation (1.25). We omit the proof because the result is not needed in the sequence.

2. Chain-homotopy and FD-homotopy

We show that chain-homotopy and FD-homotopy are corresponding notions under the " isomorphisms " $\mathfrak{N}$ and $\mathfrak{K}$ of § 1.

We begin with a definition of chain-homotopy which uses tensor products of chain-modules. Recall that the *tensor product* of chain-modules C^1 and C^2 is the chain-module $C^1 \otimes C^2$ defined by

$$(C^1 \otimes C^2)_q = \textstyle\sum_{i+j=q} C^1_i \otimes C^2_j$$
$$\partial(c^1_i \otimes c^2_j) = (\partial c^1_i) \otimes c^2_j + (-1)^i c^1_i \otimes (\partial c^2_j) , \quad c^\nu_r \in C^\nu_r .$$

(2.1). DEFINITION. Let $N(1)$ be the normal chain-module of the standard 1-simplex (§ 1). Denote by e_0, e_1 the generators (vertices) of $N(1)_0$ and by e the generator of $N(1)_1$ with $\partial e = e_1 - e_0$.

Let $f^i : C \to C'$ be chain maps, $i = 0, 1$. A chain-homotopy between

f^0 and f^1 is a chain-map $D : N(1) \otimes C \to C'$ such that

$$D(e_i \otimes c) = f^i(c) , \qquad i = 0, 1 . \tag{2.2}$$

If such a D exists f^0 and f^1 are called *(chain-) homotopic*, in symbols $f^0 \simeq f^1$. We say C and C' are *chain homotopy equivalent*, $C \simeq C'$, if there are chain-maps $f : C \to C'$, $f^- : C' \to C$ such that $f^- f \simeq \mathrm{Id}$, $f^- f \simeq \mathrm{Id}$. Then f, f^- are called *reciprocal chain homotopy equivalences*.

Usually chain-homotopy is defined as follows [6, Ch. V, 4.1]: A chain-homotopy between f^0 and f^1 is a sequence of homomorphisms

$$D'_q : C_q \to C'_{q+1} , \qquad q = 0, 1, \cdots$$

such that

$$\partial D' + D' \partial = f^1 - f^0 . \tag{2.3}$$

The two definitions are equivalent: Given D define D' by $D'(c) = D(e \otimes c)$, and vice versa.

The notion of FD-homotopy is analogous to that of chain-homotopy, using cartesian products instead of tensor products. The *cartesian product* [4, 5] of FD-modules K^1 and K^2 is the FD-module $K^1 \times K^2$ which is defined by

$$(K^1 \times K^2)_q = K^1_q \otimes K^2_q$$

$$\partial_i(a^1_q \times a^2_q) = \partial_i a^1_q \times \partial_i a^2_q, \; s_i(a^1_q \times a^2_q) = s_i a^1_q \times s_i a^2_q, \; a^\nu_q \in K^\nu_q .$$

(We write $a^1 \times a^2$ instead of $a^1 \otimes a^2$ in order to avoid confusion with the tensor product of chain-modules).

(2.4). DEFINITION. Let $K(1)$ be the FD-module of the standard 1-simplex (§ 1) and let $e_0, e_1 \in K(1)_0$ be the "vertices" as in (2.1).

If $F^i : K \to K'$, $i = 0, 1$, are FD-maps then a FD-homotopy between F^0 and F^1 is an FD-map $\Theta : K(1) \times K \to K'$ such that

$$\Theta(s_0^q(e_i) \times a_q) = F^i(a_q) , \qquad a_q \in K_q, \; i = 0, 1. \tag{2.5}$$

If such a Θ exists F^0 and F^1 are called FD-homotopic, $F^0 \simeq F^1$. Define also FD-*homotopy equivalence* etc., in analogy to chain homotopy equivalence$\cdots$.

We can now formulate the main result of this paragraph.

(2.6). THEOREM. *The functors $\mathfrak{N}$ and $\mathfrak{K}$ of* § 1 *preserve homotopy*, i.e.,

(a) *Two* FD-*maps $F^i : K \to K'$, $i = 0, 1$, are* FD-*homotopic if and only if their normal chain-maps $\mathfrak{N}(F^i) : \mathfrak{N}(K) \to \mathfrak{N}(K')$ are chain-homotopic.*

(b) *Two chain-maps $f^i : C \to C'$, $i = 0, 1$, are homotopic if and only if $\mathfrak{K}(f^0) \simeq \mathfrak{K}(f^1)$.*

(2.7). COROLLARY. *Two* FD-*modules K and K' are* FD-*homotopy equiva-*

lent if and only if $\mathfrak{N}(K)$ *and* $\mathfrak{N}(K')$ *are chain homotopy equivalent* (similarly for the functor $\mathfrak{K}$)

The corollary follows from (1.10) and (2.6): If $K \underset{F^-}{\overset{F}{\rightleftarrows}} K'$ are reciprocal FD-homotopy equivalences then $\mathfrak{N}(K) \underset{\mathfrak{N}(F^-)}{\overset{\mathfrak{N}(F)}{\rightleftarrows}} \mathfrak{N}(K')$ are reciprocal chain-homotopy-equivalences, and vice versa.

PROOF OF (2.6). Part (b) of (2.6) can be obtained from (a) as follows: Use the natural equivalence $\mathfrak{N}\mathfrak{K} \sim \mathrm{Id}$ (1.9) to get $f^0 \simeq f^1 \Longleftrightarrow \mathfrak{N}\mathfrak{K}(f^0) \simeq \mathfrak{N}\mathfrak{K}(f^1)$. Now apply (a) to $\mathfrak{K}(f^i)$ and obtain $\mathfrak{K}(f^0) \simeq \mathfrak{K}(f^1) \Longleftrightarrow \mathfrak{N}\mathfrak{K}(f^0) \simeq \mathfrak{N}\mathfrak{K}(f^1)$.

In order to prove (a) we consider chain-maps

(2.8) $$N(1) \otimes \mathfrak{N}(K) \xrightarrow{\nabla} \mathfrak{N}(K(1) \times K) \xrightarrow{g} N(1) \otimes \mathfrak{N}(K)$$

as defined in the proof of the Eilenberg-Zilber theorem. (∇ is given by the "shuffle-formula" [4, 5.3], g by the map [5, 2.9].) They have the properties

(2.9) $$\begin{aligned} \nabla(e_i \otimes a_q) &= s_0^q(e_i) \times a_q \\ g(s_0^q(e_i) \times a_q) &= e_i \otimes a_q \,. \end{aligned} \qquad a_q \in \mathfrak{N}(K)_q \,.$$

Now for every FD-homotopy $\Theta : K(1) \times K \to K'$ define a chain-homotopy $D : N(1) \otimes \mathfrak{N}(K) \to \mathfrak{N}(K')$ by

(2.10) $$D = \mathfrak{N}(\Theta) \circ \nabla$$

Conversely for every chain-homotopy $D : N(1) \otimes \mathfrak{N}(K) \to \mathfrak{N}(K')$ let

(2.11) $$\theta = D \circ g : \mathfrak{N}(K(1) \times K) \to \mathfrak{N}(K')$$

and let $\Theta : K(1) \times K \to K'$ the unique FD-extension of θ (1.10a).

The equations (2.9) show: If Θ is a FD-homotopy between F^0 and $F^1 : K \to K'$ then D is a chain-homotopy between $\mathfrak{N}(F^0)$ and $\mathfrak{N}(F^1) : \mathfrak{N}(K) \to \mathfrak{N}(K')$, and vice versa. This proves the theorem.

3. Homotopy and homology

We show that under certain restrictions the homotopy type of an FD-module is determined by its homology modules.

(3.1). DEFINITION. Let K be an FD-module. We define its homology to be the homology of its normal chain-module, i. e.,

(a) $H(K) = H(\mathfrak{N}(K))$

(we write $H(K)$ for the sequence $H_q(K)$)

(b) If $F : K \to K'$ is an FD-map then the induced homomorphism F_* is given by $F_* : H(K) \to H(K') = \mathfrak{N}(F)_* : H(\mathfrak{N}(K)) \to H(\mathfrak{N}(K'))$.

REMARK. Usually $H(K)$ is defined by introducing the boundary operator $\partial = \partial_0 - \partial_1 + \partial_2 - + \cdots$ *in* K. However, the normalization theorem of Eilenberg-MacLane [4, 4] shows that this homology is naturally isomorphic with the homology of the normalized chain module of K; this chain-module in turn is isomorphic with $\mathfrak{N}(K)$ (1.12).

We recall the fact that homotopy implies homology, more precisely.

(3.2.) PROPOSITION. *Let* $F^0 \simeq F^1 : K \to K'$ *be homotopic* FD-*maps. Then*

$$F^0_* = F^1_* : H(K) \to H(K') .$$

PROOF. By (2.6) we have $\mathfrak{N}(F^0) \simeq \mathfrak{N}(F^1)$, hence $\mathfrak{N}(F^0)_* = \mathfrak{N}(F^1)_*$ [6, Ch. V, 4.4].

(3.3). COROLLARY. *If* K *and* K' *are homotopy equivalent* FD-*modules then*

$$H(K) \cong H(K') .$$

PROOF. Let $K \underset{F^-}{\overset{F}{\rightleftarrows}} K'$ be reciprocal homotopy equivalences. Then $F_* F^-_* = \mathrm{Id}$, $F^-_* F_* = \mathrm{Id}$, i.e., F_* and F^-_* are reciprocal isomorphisms.

The main result of this paragraph is a partial inverse of (3.3), namely.

(3.4). THEOREM. *Let* Λ *be a principal ideal domain and let* K, K' *be free* FD-*modules over* Λ (i. e., each K_q, K'_q is free). *Then* K *and* K' *are* FD-*homotopy equivalent if and only if* $H(K) \cong H(K')$.

The "if-part" of (3.4) will follow from Proposition (3.5) below and earlier results.

(3.5). PROPOSITION. *Let* Λ *be a principal ideal domain,* K *a free and* K' *an arbitrary* FD-*module over* Λ. *Then for every sequence of homomorphisms* $h_q : H_q(K) \to H_q(K')$ *there is an* FD-*map* $F \colon K \to K'$ *such that* $F_* = h = \{h_q\}$, i. e., F *induces the homomorphisms* h_q.

PROOF of (3.4) (using 3.5). The " only-if-part " is contained in (3.3.)

If $H(K) \cong H(K')$ then by (3.5) there is an FD-map $F : K \to K'$ such that $\mathfrak{N}(F)_* : H(K) \cong H(K')$. Since $\mathfrak{N}(K)$ and $\mathfrak{N}(K')$ are free (as submodules of free modules) this implies that $\mathfrak{N}(F)$ is a homotopy equivalence [6, Ch. V, 13.3], hence $\mathfrak{N}(K) \simeq \mathfrak{N}(K')$. Therefore (2.7), $K \simeq K'$ (F is a homotopy equivalence).

PROOF of (3.5). By (1.10) it is sufficient to find a chain-map $f : \mathfrak{N}(K) \to \mathfrak{N}(K')$ such that $f_* = h$. This is proved in the standard way: Since $\mathfrak{N}(K)$ is free we have $\mathfrak{N}(K)_q = Z_q + C_q$ where Z_q is the submodule of cycles and C_q is a complementary module. Consider the diagram

(3.6)
$$\begin{array}{ccccccc} C_{q+1} & \xrightarrow{\partial} & Z_q & \xrightarrow{\eta} & H_q(K) & & \\ \Big\downarrow{\scriptstyle f_{q+1}} & & \Big\downarrow{\scriptstyle f_q} & & \Big\downarrow{\scriptstyle h_q} & & \\ \mathfrak{N}(K')_{q+1} & \xrightarrow[\partial]{} & Z'_q & \xrightarrow[\eta']{} & H_q(K') & \longrightarrow & 0 \end{array}$$

where η, η' denote the natural projections. The horizontal lines are exact. Therefore, since Z_q and C_{q+1} are free it is possible to complete (3.6) by homomorphisms f_q and f_{q+1} (dotted in 3.6) to a commutative diagram (define f_q and f_{q+1} on generators first). Doing this for all q gives the required chain-map f.

(3.7). REMARK. Assertion and proof of (3.4) and (3.5) remain valid if "free over a principal ideal domain" is replaced by the weaker condition "projective over a hereditary ring".

(3.8). REMARK. Let K, K' be free FD-modules over a principal ideal domain. Theorem 3.4 asserts that $H(K) \cong H(K') \Rightarrow K \simeq K'$. However, a similar looking conclusion for FD-maps is false, namely there exists FD-maps $F^i : K \to K'$, $i = 0, 1$, such that $F^0_* = F^1_*$ and yet $F^0 \simeq F^1$ (Compare Cartan-Eilenberg: Homological Algebra, Ch. VI, Th. 3. 1a). For an example take an essential simplicial map of the real projective plane onto the 2-sphere and let F be the corresponding FD-map. Then $F_* = 0$, however $F \simeq 0$ because it induces non-trivial homomorphisms of the mod 2 homology.

4. Homotopy preserving functors

In the introduction we asserted that for certain functors T on FD-modules K the homology of $T(K)$ is determined by the homology of K. Not all functors T have this property. If we define $T(K)$ to be the n-skeleton of K(i. e., $T(K) \subset K$ is the sub-FD-module generated by $K_0, K_1, \cdots K_n$) then $H_n(T(K))$ is not determined by $H(K)$ (the proof is left to the reader). The problem arises to find general conditions on T which insure that the property holds. Such a condition is described in the following:

(4.1). DEFINITION. A (covariant) functor T from FD-modules (over Λ_1) to FD-modules (over Λ_2) *preserves homotopy* if for every pair of homotopic FD-maps $F^0 \simeq F^1 : K \to K'$ (over Λ_1) we have $T(F^0) \simeq T(F^1) : T(K) \to T(K')$.

(4.2). PROPOSITION. *Let T be a homotopy preserving functor from* FD-*modules over a principal ideal domain Λ_1 to* FD-*modules over an arbitrary ring Λ_2. If K, K' are free* FD-*modules over Λ_1 such that $H(K) \cong H(K')$*

then $H(T(K)) \cong H(T(K'))$. *More precisely, there is an* FD-*map* $F: K \to K'$ *such that* $T(F)_* : H(T(K)) \cong H(T(K'))$.

PROOF. By (3.4) there are reciprocal homotopy equivalences $K \underset{F'}{\overset{F}{\rightleftarrows}} K'$. Since T preserves homotopy we obtain reciprocal homotopy equivalences $T(K') \underset{T(F')}{\overset{T(F)}{\rightleftarrows}} T(K)$. By (3.2) this implies that $T(F)_*$ isa n isomorphism.

In order to prove that a functor preserves homotopy we shall use the following criterion.

(4.3). LEMMA. *A functor T from* FD-*modules over* Λ_1 *to* FD-*modules over* Λ_2 *preserves homotopy if and only if for every* FD-*module K over* Λ_1 *there exists an* FD-*map.*

$$\tau : {}^2K(1) \times T(K) \to T({}^1K(1) \times K)$$

such that

$$\tau(s_0^q e_i \times b_q) = T(I^i)b_q\,, \qquad b_q \in T(K)_q,\ i = 0, 1. \tag{4.4}$$

${}^{\nu}K(1)$ *denotes the* FD-*module of the standard* 1-*simplex*; *the index* $\nu = 1, 2$ *indicates whether it is taken over* Λ_1 *or* Λ_2; $e_0, e_1 \in K(1)_0$ *are the vertices. The* FD-*maps*

$$I^i : K \to {}^1K(1) \times K\,, \qquad i = 0, 1$$

are defined by

$$I^i(a_q) = s_0^q e_i \times a_q\,, \qquad a_q \in K_q\,.$$

PROOF. The maps I^i are homotopic (the identity of ${}^1K(1) \times K$ is a homotopy), therefore if T preserves homotopy $T(I^0)$ and $T(I^1)$ are homotopic, i, e., there exists a homotopy τ: ${}^2K(1) \times T(K) \to T({}^1K(1) \times K)$ satisfying (4.4).

On the other hand if such a τ exist and if we have a homotopy $\Theta : {}^1K(1) \times K \to K'$ between FD-maps $F^0, F^1 : K \to K'$ then we define a homotopy between $T(F^0)$ and $T(F^1)$ by composition

$${}^2K(1) \times T(K) \xrightarrow{\tau} T({}^1K(1) \times K) \xrightarrow{T(\Theta)} T(K')\,.$$

This proves the lemma.

(4.5). REMARK. If $F^0, F^1 : K \to K'$ are FD-maps (between FD-modules over a principal ideal domain) such that $F^0_* = F^1_* : H(K) \to H(K')$ then we *cannot conclude that* $T(F^0)_* = T(F^1)_* :\ H(T(K)) \to H(TK'))$. Consider the example in Remark (3.8). Reduction mod 2 (tensoring with Z_2) is a homotopy preserving functor T. In the example we have $F_* = 0$ but $T(F)_* \neq 0$.

Therefore. if we wish to construct a functor T_* which, when applied to the homology of K, gives the homology of $T(K)$ (i. e., $T_*(H(K)) = H(T(K))$), we cannot follow the obvious way. Without giving any detail we claim that T_* can be constructed by using "special" FD-modules (direct sums K^i of FD-modules K^i with non-vanishing homology only in dimension i) and mappings of those.

5. Prolongation of functors on modules

For every functor from modules to modules, or more generally from modules to FD-modules, we define a prolongation from FD-modules to FD-modules (5.1), (5.1′). We show that these prolongations preserve homotopy (5.6). We then apply and sharpen the results of § 4 (5.11).

(5.1). DEFINITION. Let t be a (covariant) functor from modules over Λ_1 to FD-modules over Λ_2. The prolongation of t is a functor T from FD-modules K over Λ_1 to FD-modules over Λ_2; it is defined as follows

(5.2) $$T(K)_q = t(K_q)_q .$$

The i^{th} face D_i or degeneracy $S_{i,i\leqq q}$, are given by composition

(5.3) $$D_i : t(K_q)_q \xrightarrow{t(\partial_i)} t(K_{q-1})_q \xrightarrow{\partial_i} t(K_{q-1})_{q-1}$$

(5.4) $$S_i : t(K_q)_q \xrightarrow{t(s_i)} t(K_{q+1})_q \xrightarrow{s_i} t(K_{q+1})_{q+1} .$$

We verify some of the FD-identities. Note first that $t(\partial_i)$ and $t(s_i)$ are FD-maps, i. e., $t(\partial_i)\partial_j = \partial_j t(\partial_i)$ and $t(\partial_i)\, s_j = s_j\, t(\partial_i)$; similarly for $t(s_i)$. Hence for $i < j$

$$\begin{aligned} D_iD_j &= \partial_i t(\partial_i)\partial_j t(\partial_j) = \partial_i\partial_j t(\partial_i)t(\partial_j) = \partial_i\partial_j t(\partial_i\partial_j) \\ &= \partial_{j-1}\partial_i t(\partial_{j-1}\partial_i) = \partial_{j-1}t(\partial_{j-1})\partial_i t(\partial_i) = D_{j-1}D_i ; \end{aligned}$$

or in $T(K)_q$, for $i \leqq q$,

$$D_iS_i = \partial_i t(\partial_i)s_i t(s_i) = \partial_i s_i t(\partial_i s_i) = \text{Id}.$$

The other FD-identities follow similarly.

If $F: K \to K'$ is an FD-map over Λ_1 put

(5.5) $$T(F)_q = t(F_q)_q : T(K)_q \to T(K')_q .$$

Then in $T(K)_q$ we have

$$\begin{aligned} D_iT(F)_q &= \partial_i t(\partial_i)_q t(F_q)_q = \partial_i t(\partial_i F_q)_q = \partial_i t(F_{q-1}\partial_i)_q \\ &= t(F_{q-1}\partial_i)_{q-1}\partial_i = t(F_{q-1})_{q-1}t(\partial_i)_{q-1}\partial_i = T(F)_{q-1}D_i . \end{aligned}$$

Similarly $S_i\, T(F) = T(F)S_i$, i.e., $T(F)$ is an FD-map.

The functor properties $T(FF') = T(F)T(F')$ and $T(\text{Id}) = \text{Id}$ are easy consequences of the corresponding properties of t. Therefore (5.2)–(5.5) define indeed a functor T from FD-modules to FD-modules.

(5.1′). DEFINITION. There is an embedding i of the category of modules into the category of FD-modules: For every module M we define $i(M)$ by

$$i(M)_q = M;$$

all non-trivial face- and degeneracy-operators are given by the identity map of M.
Similarly

$$i(\beta)_q = \beta$$

for every module homomorphism β.

Now if u is a functor from modules to modules then the composition $t = i \circ u$ is a functor from modules to FD-modules and its prolongation T is defined. We call it the *prolongation of* u; it is given by

(5.2′) $$T(K)_q = u(K_q)$$

(5.3′) $$i^{\text{th}} \text{ face} = u(\partial_i): T(K)_q \to T(K)_{q-1}$$

(5.4′) $$i^{\text{th}} \text{ degeneracy} = u(s_i): T(K)_q \to T(K)_{q+1}, \qquad i \leqq q,$$

and

(5.5′) $$T(F)_q = u(F_q)$$

for FD-modules K, and FD-maps F.

(5.6). THEOREM. *Functors from* FD-*modules over* Λ_1 *to* FD-*modules over* Λ_2 *which are obtained by prolongation* (5.1) *preserve homotopy* (4.1).

PROOF. By Lemma (4.3), it suffices to construct for every FD-module K over Λ_1 an FD-map $\tau: {}^2K(1) \times T(K) \to T({}^1K(1) \times K)$ with the property (4.4). This FD-map will be defined in a more general situation: Let ${}^\nu K(X)$ be the FD-module (over Λ_ν, $\nu = 1, 2$) of a semi-simplicial complex X. (Recall that ${}^\nu K(X)_q$ is freely generated by the q-simplices σ of X, and every face $\partial_i\sigma$ and degeneracy $s_i\sigma$, $i \leqq q$, of a simplex is again a simplex; see §1. Then we shall define in a natural way an FD-map

(5.7) $$\tau: {}^2K(X) \times T(K) \to T({}^1K(X) \times K)$$

which has the property 4.4 if X is the standard 1-simplex.

Let σ be a q-simplex of X. It defines a homomorphism

(5.8) $$[\sigma]: K_q \to ({}^1K(X) \times K)_q = {}^1K(X)_q \otimes K_q \text{ by } [\sigma]c = \sigma \otimes c, \quad c \in K_q.$$

Applying ∂_i or s_i gives

(5.9) $$\partial_i \circ [\sigma] = [\partial_i\sigma] \circ \partial_i, \quad s_i \circ [\sigma] = [s_i\sigma] \circ s_i .$$

Now let t be the functor from modules to FD-modules whose prolongation is T. Define

$$\tau_q: {}^2K(X)_q \otimes T(K)_q \to T({}^1K(X) \times K)_q = t(({}^1K(X) \times K)_q)_q$$

by

(5.10) $\tau_q(\sigma \times a) = t([\sigma])a, \quad a \in T(K)_q = t(K_q)_q, \quad \sigma$ a q-simplex of X.

Since the q-simplices form a basis of $K(X)_q$ this defines a unique homomorphism τ_q. We show that $\tau = \{\tau_q\}$ is an FD-map, i. e., commutes with face- and degeneracy-operators (denoted by D_i, S_i in $K(X) \times T(K)$, $T(K(X) \times K)$, and by ∂_i, s_i also.)

$$\begin{aligned} D_i\tau(\sigma \times a) &= \partial_i t(\partial_i)\tau(\sigma \times a) = \partial_i t(\partial_i)t([\sigma])\, a = \partial_i t(\partial_i[\sigma])a \\ &= \partial_i t([\partial_i\sigma]\partial_i)a = \partial_i t([\partial_i\sigma])t(\partial_i)a = t([\partial_i\sigma])\partial_i t(\partial_i)a \\ &= t([\partial_i\sigma])D_i a = \tau(\partial_i\sigma \times D_i a) = \tau D_i(\sigma \times a)\ . \end{aligned}$$

Similarly $S_i\tau = \tau S_i$.

If X is the standard 1-simplex, i. e., $K(X) = K(1)$, then

$$[s_0^q e_i] = I_q^i \ (4.3),$$

hence

$$\tau(s_0^q e_i \times a) = t[s_0^q e_i]_q a = T(I^i)_q a\ , \qquad a \in T(K)_q,$$

i. e., (4.4) holds.

We can now apply Proposition (4.2) to prolongations T. Using a special property of these functors (5.12) we shall be able to sharpen this proposition, and obtain

(5.11). THEOREM. *Let T be a functor from* FD-*modules over* Λ_1 *to* FD-*modules over* Λ_2 *which is obtained by prolongation* (5.1), *and assume* Λ_1 *is a principal ideal domain* (*a hereditary ring*). *If K, K' are free* (*projective*) FD-*modules over* Λ_1 *and if* $H_i(K) \cong H_i(K')$ *for* $i < q$ *then* $H_j(T(K)) \cong H_j(T(K'))$ for $j < q$. *More precisely, there is an* FD-*map* $F: K \to K'$ *such that*

$$T(F)_* : H_j(T(K)) \cong H_j(T(K'))\ , \qquad j < q.$$

PROOF. The case $q = \infty$ follows from (4.2) and (5.6). The proof for $q < \infty$ uses the following property which a prolongation T obviously possesses :

(5.12) *If* $F: L \to L$ *is an* FD-*map such that* $F_i = \mathrm{Id}$ *for* $i < q$ *then* $T(F)_j = \mathrm{Id}$ for i $< q$.

Let $0 < q < \infty$ and let K be a free FD-module over Λ_1. Its normal chain-module $\mathfrak{N}(K)$ is also free and $\mathfrak{N}(K)_q = Z_q + C_q$ where Z_q denotes the q-cycles and C_q a complementary summand. Define a chain-submodule $N^{(q)} \subset \mathfrak{N}(K)$ by

$$(5.13) \qquad N_\mu^{(q)} = \begin{cases} \mathfrak{N}(K)_\mu & \text{for } \mu < q \\ C_q & \text{for } \mu = q \\ 0 & \text{for } \mu > q\ . \end{cases}$$

Then $N^{(q)}$ is a direct summand of $\mathfrak{N}(K)$ (the complementary summand is

0 for $\mu < q$, Z_q for $\mu = q$, and $\mathfrak{N}(K)_\mu$ for $\mu > q$). If $N^{(q)} \xrightarrow{j} \mathfrak{N}(K) \xrightarrow{r} N^{(q)}$ are the corresponding injection and projection then $rj = \mathrm{Id}$, and $jr = \mathrm{Id}$ in dimensions less than q. Apply the functor $\mathfrak{K}$ of § 1 and obtain (using $\mathfrak{K}\mathfrak{N}(K) \cong K$) FD-maps

$$K^{(q)} \xrightarrow{J} K \xrightarrow{R} K^{(q)}, \quad RJ = \mathrm{Id}, \quad JR = \mathrm{Id} \tag{5.14}$$

in dimensions less than q. Therefore $K^{(q)}$ is a free FD-module and $J_* : H_i(K^{(q)}) \cong H_i(K)$ for $i < q$. Further, $H_i(K^{(q)}) = H_i(N^{(q)}) = 0$ for $i \geqq q$.

Applying the functor T to (5.14) and using (5.12) we obtain

$$T(K^{(q)}) \xrightarrow{T(J)} T(K) \xrightarrow{T(R)} T(K^{(q)}), \quad T(R)T(J) = \mathrm{Id}, \quad T(J)T(R) = \mathrm{Id}$$

n dimensions less than q. Therefore $T(J)$ or $T(R)$ induce isomorphisms $H_i(T(K^{(q)})) \cong H_i(T(K))$ for $i < q$.

Now under the assumptions of (5.11) we have $H(K^{(q)}) \cong H(K'^{(q)})$, hence $H(T(K^{(q)})) \cong H(T(K'^{(q)}))$, and finally

$$H_i(T(K)) \underset{i<q}{\cong} H_i(T(K^{(q)})) \underset{\text{all } i}{\cong} H_i(T(K'^{(q)})) \underset{i<q}{\cong} H_i(T(K')) ,$$

all isomorphisms being induced by FD-maps of the form $T(F)$. This proves the theorem.

(5.15). FD-ALGEBRAS. If in an FD-module K all the K_q are algebras with unit and the non-trivial face- and degeneracy-operators are homomorphisms of algebras with unit then K is called an FD-*algebra*. In this case the homology modules $H(K)$ form an algebra $H_*(K)$ with unit, the Pontrjagin-algebra of K [4, 6]. If $F: K \to K'$ is a map of FD-algebras (i. e., an FD-map with each F_q a homomorphism of algebras with unit) then $F_* : H_*(K) \to H_*(K')$ is a homomorphism of algebras with unit.

We clearly have the following.

(5.16). COROLLARY TO (5.11). *If under the assumption of* (5.11) *the values of T lie in the category of* FD-*algebras then the algebras $H_*(T(K))$ and $H_*(T(K'))$ are isomorphic up to dimension q.*

This applies to the prolongation T of a functor u (resp. t) from modules to algebras (resp. FD-algebras).

(5.17). FUNCTORS OF SEVERAL VARIABLES. The definitions and results of §§ 4–5 may be generalized to functors of several variables. For instance if t is a functor from pairs of modules to FD-modules we may define its prolongation T as in (5.1). T is a functor from pairs of FD-modules to FD-modules, it preserves homotopy (§ 4), and if K^1, K^2 are free FD-modules over principal ideal domains then $H_q(T(K^1, K^2))$ *is determined* by $H_i(K^1)$, $H_i(K^2)$, $i \leqq q$ (5.11).

Instead of two variables, we may admit arbitrarily many.

6. Examples for § 5

(6.1). CARTESIAN PRODUCT WITH A FIXED FD-MODULE. Let $\overline{K}$ be a fixed FD-module over Λ. Define a functor t from modules to FD-modules by

$$t(M) = M \otimes \overline{K}, \qquad \text{i. e., } t(M)_q = M \otimes \overline{K}_q,$$
$$\partial_i(m \otimes \overline{a}) = m \otimes \partial_i\overline{a}, \quad s_i(m \otimes \overline{a}) = m \otimes s_i\overline{a}$$
$$t(\beta) = \beta \otimes \mathrm{Id}\,,$$

for modules M and homomorphisms β. The prolongation T of t is given by

$$T(K)_q = t(K_q)_q = K_q \otimes \overline{K}_q\,,$$
$$\partial_i(a \otimes \overline{a}) = \partial_i a \otimes \partial_i\overline{a}, \quad s_i(a \otimes \overline{a}) = s_i a \otimes s_i\overline{a}, \quad a \in K_q, \; \overline{a} \in \overline{K}_q,$$

i. e., $T(K)$ is the cartesian product $K \times \overline{K}$ (§ 2). Theorem (5.11) applied to this T gives a well known result; if also $\overline{K}$ is free, then the Künneth formula gives an explicit description of $H(T(K))$ in terms of $H(K)$, $H(\overline{K})$.

(6.2). Γ-PRODUCTS (SYMMETRIC PRODUCTS) AND GENERALIZATIONS. If the group π operates in the module M (i. e., each $g \in \pi$ defines a module endomorphism $\tilde{g}$ of M and $\widetilde{gg'} = \tilde{g} \circ \tilde{g}\tilde{g}'$, $\tilde{e} = \mathrm{Id}$) we denote by M/π the quotient of M by the submodule which is generated by the elements $m - \tilde{g}(m)$, $m \in M$, $g \in \pi$. (The operation in question will always be clear and is, therefore, not indicated.)

Now let S_n be the symmetric group of degree n, i. e., the group of permutations of n elements, and let $\Gamma \subset S_n$ be a subgroup. For every module M let $\otimes_n M = M \otimes M \otimes \cdots \otimes M$ (n factors) be its n^{th} tensor power. The group S_n operates in $\otimes_n M$ by permuting the factors and so does the subgroup Γ. Define the Γ-product of M to be the quotient

(6.3) $$M^\Gamma = \otimes_n M/\Gamma$$

with respect to this operation, and let $\gamma : \otimes_n M \to M^\Gamma$ denote the natural projection.

If $\beta : M \to N$ is a module homomorphism then its n^{th} tensor power $\otimes_n \beta : \otimes_n M \to \otimes_n N$ is a module homomorphism which commutes with the operations of Γ. Therefore we have a unique homomorphism $\beta^\Gamma : M^\Gamma \to N^\Gamma$, the Γ-*product of* β, such that the diagram

$$\begin{array}{ccc} \otimes_n M & \xrightarrow{\otimes_n \beta} & \otimes_n N \\ \gamma \downarrow & & \downarrow \gamma \\ M^\Gamma & \xrightarrow[\beta^\Gamma]{} & N^\Gamma \end{array}$$

is commutative. For composite homomorphisms $M \xrightarrow{\beta} N \xrightarrow{\alpha} L$ we have $\otimes_n(\alpha \circ \beta) = \otimes_n\alpha \circ \otimes_n\beta$ hence $(\alpha \circ \beta)^\Gamma = \alpha^\Gamma \circ \beta^\Gamma$. Since also $(\mathrm{Id})^\Gamma = \mathrm{Id}$ it follows that the Γ-product is a functor u from modules to modules. Its prolongation (5.1′) to the FD-category defines the *Γ-product* $T(K) = K^\Gamma$ resp. $T(F) = F^\Gamma$ of FD-modules resp. FD-maps. By (5.2′)–(5.4′) we have $(K^\Gamma)_q = (K_q)^\Gamma$ with face- and degeneracy-operators ∂_i^Γ, s_i^Γ.

If $\Gamma = S_n$ is the full symmetric group then $K^\Gamma(F^\Gamma)$ is called the *symmetric product of order n of K (of F)*; it is denoted by $SP^n(K)$ resp. $SP^n(F)$. If $\Gamma = Z_n$ is cyclic of order n then K^Γ is the *cycle product* of order n of K.

A generalization of Γ-products is obtained as follows. Let $\overline{K}$ be a fixed FD-module over Λ in which the group Γ operates. Then Γ operates in $K^n \times \overline{K}$

$$\tilde{g}(a \times b) = \tilde{g}(a) \times \tilde{g}(b), \qquad g \in \Gamma,\ a \in \overline{K}_q^n,\ b \in K_q \tag{6.4}$$

(K^n is the n-fold cartesian product of K in which Γ permutes the factors), and we may form the corresponding quotient

$$T(K) = (K^n \times \overline{K})/\Gamma. \tag{6.5}$$

This functor is the prolongation of a functor from modules to FD-modules but not, in general, of the special type (5.1′). The Γ-product is obtained if $\overline{K} = i(\Lambda)$ (5.1′) in which Γ operates trivially.

(6.6.) TENSOR ALGEBRA, SYMMETRIC ALGEBRA, EXTERIOR ALGEBRA. For every module M we have the tensor algebra $\otimes M$, the symmetric algebra $S(M)$, and the exterior algebra ΛM [1, Ch. III for $\otimes M$ and ΛM]. These are functors from Λ-modules to Λ-algebras with unit. Their prolongations (5.1′) are functors from FD-modules K to FD-algebras (5.15), denoted by $\otimes K$, $S(K)$, ΛK. In § 11 we shall interpret $\otimes K$ as a "reduced product" in the sense of James [7], and $S(K)$ as an infinite symmetric product. I do not know a simple geometric interpretation for ΛK.

7. Geometric interpretations

Using geometric realizations of semi-simplicial complexes as defined by Milnor [9] we describe the connection between Γ-products of FD-modules (6.2) and the well known Γ-products of topological spaces (cf. for instance, Liao [8]). We recall the definition of the latter.

(7.1). DEFINITION. Let S_n be the symmetric group of degree n and let $\Gamma \cap S_n$ be a subgroup. If Y is a topological space form $Y^n = Y \times Y \times \cdots \times Y$, the n-fold product of Y with itself. The group S_n operates in Y^n by permuting the factors and so does the subgroup Γ. The orbit space with respect to this operation is the Γ-product of Y; we denote it by Y^Γ. It is obtained from Y^n by identifying points which can be transformed, one into each other, by elements of Γ.

From Theorem (5.11), applied to Γ-products of FD-modules, we shall obtain

(7.2). THEOREM. *Let Y and Y' be* CW-*complexes and let Λ be a principal ideal domain (a hereditary ring). If $H_i(Y, \Lambda) \cong H_i(Y', \Lambda)$ for $i < q$ then $H_j(Y^\Gamma, \Lambda) \cong H_j(Y'^\Gamma, \Lambda)$ for $j < q$.*

PROOF. The link between spaces and FD-modules are the semi-simplicial complexes. In this category there is also a Γ-product defined, in analogy to (6.2) and (7.1): If X is a semi-simplicial complex then its Γ-product X is the "orbit complex" with respect to the obvious operation of Γ in $X^n = X \times X \times \cdots \times X$.

Now, if $K = K(X, \Lambda)$ is the FD-module of X (1.1) then K^Γ is naturally isomorphic with the FD-module of X^Γ

(7.3) $$K(X^\Gamma, \Lambda) \cong K(X, \Lambda)^\Gamma = K^\Gamma.$$

On the other hand if $|X|$ is the geometric realization of X, (cf. [9]), we may consider K as the complex of simplicial chains of $|X|$ with coefficients in Λ; in particular

(7.4) $$H_i(|X|, \Lambda) \cong H_i(K) .$$

Similarly K^Γ may be considered as chain-module of the geometric realization $|X^\Gamma|$ which in turn is equivalent to $|X|^\Gamma$. More precisely, there is a natural continuous 1-1-map $|X^\Gamma| \to |X|^\Gamma$ which is a homeomorphism if X is countable, and which is still a homeomorphism between compact subsets if X is not countable [12, § 2.]. In particular

(7.5) $$H_i(|X|^\Gamma, \Lambda) \cong H_i(|X^\Gamma|, \Lambda) \cong H_i(K^\Gamma) .$$

The isomorphisms (7.4) and (7.5) show that Theorem (7.2) follows from (5.11) if Y and Y' are geometric realizations of semi-simplicial complexes. In any case Y is of the same homotopy type as some geometric realization $|X|$ [9, Theorem 4], and Y^Γ is then of the same homotopy type as $|X|^\Gamma$ [8, 1.2]; similarly for Y'. Therefore the theorem holds in general.

(7.6). REMARK. There is a similar geometric interpretation for generalized Γ-products. Without proof (it is analogous to the one for (7.2)) we state the result:

Let $\bar{Y}$ be a topological space in which Γ operates. For every space Y let $T(Y) = (Y^n \times \bar{Y})/\Gamma$ denote the orbit space with respect to the operation

$$g('y, \bar{y}) = (g'y, g\bar{y}), \qquad 'y \in Y^n, \bar{y} \in \bar{Y}, g \in \Gamma,$$

of Γ in $Y^n \times \bar{Y}$.

If $\overline{Y}$ is a simplicial complex (the geometric realization of a semi-simplicial complex) in which Γ operates simplicially, and if Y, Y' are CW-*complexes such that $H_i(Y, \Lambda) \cong H_i(Y', \Lambda)$ for $i < q$, then $H_j(T(Y), \Lambda) \cong H_j(TY'), \Lambda)$ for $j < q$; Λ a principal ideal domain.*

Appendix

This appendix is essentially independent of the preceding paragraphs. It contains some additional results on Γ-products (Γ-products of direct sums (8.7), (8.8); splitting properties of Γ-products (9.2); a formula of Steenrod (9.3)), infinite symmetric products (§ 10), and reduced products (§ 11).

8. Γ-products of direct sums

Let $M^1 + M^2$ be a direct sum of modules. The n^{th} tensor power of $M^1 + M^2$ splits into a direct sum

$$(8.1) \qquad \otimes_n(M^1 + M^2) = \textstyle\sum_{\nu_i=1,2} M^{\nu_1} \otimes M^{\nu_2} \otimes \cdots \otimes M^{\nu_n},$$

one summand for each n-tuple $(\nu_1, \nu_2, \cdots \nu_n)$ with $\nu_i = 1, 2$. Let $(M^1, M^2)^q$ denote the partial sum (8.1) consisting of those terms $M^{\nu_1} \otimes \cdots \otimes M^{\nu_n}$ with exactly q factors M^1 (and n-q factors M^2). Then

$$(8.2) \qquad \otimes_n(M^1 + M^2) = \textstyle\sum_{q=0}^{n} (M^1, M^2)^q ,$$

and each summand $(M^1, M^2)^q$ is *invariant* (as a whole) under the operations of the symmetric group (6.2). Therefore for every subgroup $\Gamma \subset S^n$, (cf. 6.3)

$$(8.3) \qquad (M^1 + M^2)^\Gamma = \otimes_n(M^1 + M^2)/\Gamma = \textstyle\sum_{q=0}^{n} (M^1, M^2)^q/\Gamma .$$

In many cases the summands $(M^1, M^2)^q/\Gamma$ can be expressed as direct sums of tensor products of certain Γ'-products of M^1 and M^2. For instance $(M^1, M^2)^0/\Gamma = (M^2)^\Gamma$ and $(M^1, M^2)^n/\Gamma = (M^1)^\Gamma$. If $\Gamma = S_n$ is the full symmetric group then any two terms $M^{\nu_1} \otimes \cdots \otimes M^{\nu_n}$ with the same number q of factors M^1 are equivalent under Γ. Therefore $(M^1, M^2)^q/\Gamma$ may be obtained as quotient of $(\otimes_q M^1) \otimes (\otimes_{n-q} M^2)$ by the subgroup of S_n which permutes the letters $1, 2, \cdots q$ among themselves; i. e., the subgroup $S_q \times S_{n-q} \subset S_n$. This gives [1, Ch. IV, 1, ex. 4]

$$(M^1, M^2)^q/S_n = (\otimes_q M^1/S_q) \otimes (\otimes_{n-q} M^2/S_{n-q}) = SP^q(M^1) \otimes SP^{n-q}(M^2)$$

(convention: $SP^0(M) = \Lambda$) and by (8.3)

$$(8.4) \qquad SP^n(M^1 + M^2) = \textstyle\sum_{q=0}^{n} SP^q(M^1) \otimes SP^{n-q}(M^2) .$$

If $\beta^i : M^i \to N^i$, $i = 1, 2$, are module homomorphisms then

$$\beta^1 + \beta^2 : M^1 + M^2 \to N^1 + N^2$$

is a module homomorphism and $(\beta^1+\beta^2)^\Gamma$ (6.2) splits corresponding to (8.3). We may write

(8.5) $$(\beta^1 + \beta^2)^\Gamma = \sum_{q=0}^n (\beta^1, \beta^2)^q/\Gamma$$

and

(8.6) $$SP^n(\beta^1 + \beta^2) = \sum_{q=0}^n SP^q(\beta^1) \otimes SP^{n-q}(\beta^2) .$$

It follows easily that the correspondence $(M^1, M^2) \to (M^1, M^2)^q/\Gamma$ and $(\beta^1, \beta^2) \to (\beta^1, \beta^2)^q/\Gamma$ defines a functor on the category of pairs (M^1, M^2) resp. (β^1, β^2). Therefore we have the same splitting for FD-modules K^1, K^2 and FD-maps F^1, F^2, i. e.

(8.7) $$(K^1 + K^2)^\Gamma = \sum_{q=0}^n (K^1, K^2)^q/\Gamma$$

with $$(K^1, K^2)^0/\Gamma = (K^2)^\Gamma,\ (K^1, K^2)^n/\Gamma = (K^1)^\Gamma ,$$

(8.8) $$SP^n(K^1 + K^2) = \sum_{q=0}^n SP^q(K^1) \times SP^{n-q}(K^2) ,$$

(8.9) $$SP^n(F^1 + F^2) = \sum_{q=0}^n SP^q(F^1) \times SP^{n-q}(F^2) .$$

We make the convention that $SP^0(K) = P$ is the FD-*module of a point*; i.e., $P = K(0)$ in the notation of § 1, or $P = i(\Lambda)$ in the notation of (5.1′). We also put $SP^0(F) = \mathrm{Id}$.

9. Steenrod's formula for symmetric products

As above, let P denote the FD-module of a point. An *augmentation* of the FD-module K is an FD-map $\varepsilon : K \to P$. A *base point* in K is an FD-map $\xi : P \to K$. We shall always require that $\varepsilon\xi = \mathrm{Id}$. This implies that ξ maps P isomorphically onto a direct summand of K:

(9.1) $$K = \ker \varepsilon + \operatorname{im} \xi \cong K^0 + P ,$$

where $K^0 = \varepsilon^{-1}(0) \subset K$ is the *sub*-FD-*module of augmentation zero.*

If K is an FD-algebra then an augmentation $\varepsilon : K \to P$ has to be a homomorphism of FD-algebras. Hence $\varepsilon(1_q) = 1_q$, and we define the base point ξ by $\xi(1_q) = 1_q$ (1_q denotes the unit in K_q as well as in P_q).

Assume now K is an FD-module with augmentation and base point. Applying (8.7) to the decomposition (9.1) gives

(9.2) $$K^\Gamma = \sum_{q=0}^n (K^0, P)^q/\Gamma .$$

and (8.8) becomes (using $SP^r(P) = P$ and $P \times K = K$

(9.3) $$SP^n(K) = P + SP^1(K^0) + SP^2(K^0) + \cdots + SP^n(K^0) = \sum_{q=0}^n SP^q(K^0)$$

(*formula of Steenrod,* [13, 22.3]).

For a geometric interpretation let X be a semi-simplicial complex with base point b, and let $K = K(X, \Lambda)$ be its FD-module (§ 1). The projection $X \to b$ and injection $b \to X$ define an augmentation $\varepsilon : K \to K(b) = P$

and a base point $\xi : P \to K$ in K.

$(K^0, P)^q$ is generated (in each dimension) by the elements $a_1 \times a_2 \times \cdots \times a_n$ such that n-q of the a_ν are "at the base point P", and q of them are in K^0. Therefore $K^{r,n} = \sum_{q=0}^{r} (K^0, P)^q$ is generated by the simplices $\sigma_1 \times \cdots \times \sigma_n$, $\sigma_\nu \in X$, such that at most r of the σ_ν are not at the base point b. These simplices form a subcomplex $X^{r,n}$ of X^n which is invariant under the operations of Γ. The formula (9.2) asserts that $K^{r,n}/\Gamma$ is a direct summand of $K^{r+1,n}/\Gamma$; i. e., *each of the inclusions*

$$b \in X = X^{1,n}/\Gamma \subset X^{2,n}/\Gamma \subset \cdots \subset X^{n-1,n}/\Gamma \subset X^{n,n}/\Gamma = X^\Gamma$$

maps the FD-*module* (*the homology module*) *of the subcomplex isomorphically onto a direct summand of the* FD-*module* (*the homology module*) *of the total complex* (*arbitrary coefficients*).

Passing to geometric realizations (§ 7) we obtain : *Let Y be a* CW-*complex with base point $p \in Y$. Let $Y^{r,n} \subset Y^n$ be the subspace of points $(y_1, y_2, \cdots y_n)$ with at most r components y_ν different from p. Then each of the inclusions.*

$$p \in Y = Y^{1,n}/\Gamma \subset Y^{2,n}/\Gamma \subset \cdots \subset Y^{n-1,n}/\Gamma \subset Y^{n,n}/\Gamma = Y^\Gamma$$

maps the homology of the subspace isomorphically onto a direct summand of the homology of the total space (*arbitrary coefficients*).

In the case of symmetric products ($\Gamma = S_n$) we may identify $Y^{q,n}/S_n$ with $SP^q(Y)$ by the inclusion map $SP^q(Y) \to SP^n(Y)$ which is defined by $[y_1, y_2, \cdots y_q] \to [y_1, y_2, \cdots y_q, p, p, \cdots p]$. The formula of Steenrod (9.3) implies that this inclusion maps $H(SP^q(Y))$ isomorphically onto a direct summand of $H(SP^n(Y))$.

10. Infinite symmetric product and symmetric algebra

Let K be an augmented FD-module with base point $\xi : P \to K$ (§ 9). Define an inclusion $i_n : K^n \to K^{n+1}$ by composition

$$(10.1) \qquad i_n : K^n \cong P \times K^n \xrightarrow{\xi \times \mathrm{Id}} K \times K^n = K^{n+1}.$$

This inclusion is compatible with the formation of symmetric product; i. e., there is a unique inclusion $j_n : SP^n(K) \to SP^{n+1}(K)$ (corresponding to the geometric inclusion at the end of § 9), such that the diagram

$$(10.2) \qquad \begin{array}{ccc} K^n & \xrightarrow{i_n} & K^{n+1} \\ s \downarrow & & \downarrow s \\ SP^n(K) & \xrightarrow[j_n]{} & SP^{n+1}(K) \end{array}$$

is commutative; s denotes the natural projection. The direct limit of the sequence $K = SP^1(K) \xrightarrow{j_1} SP^2(K) \xrightarrow{j_2} \cdots$ is denoted by $SP(K)$ and is called the *infinite symmetric product* of the (augmented) FD-module K with base point ξ. (No augmentation was needed for the definition but it is essential for the sequence.)

It is easy to see that the inclusion j_n maps the direct summand $SP^q(K^0)$ of $SP^n(K) = \sum_{q=0}^{n} SP^q(K^0)$ (9.3) identically onto the same summand in the decomposition of $SP^{n+1}(K)$. Therefore

(10.3) $\quad SP(K) = P + SP^1(K^0) + SP^2(K^0) + \cdots = \sum_{q=0}^{\infty} SP^q(K^0)$

(Steenrod).

This suggests the following

DEFINITION. For every module M over Λ put

(10.4) $\quad S(M) = \Lambda + SP^1(M) + SP^2(M) + \cdots = \sum_{q=0}^{\infty} SP^q(M)$.

For every module homomorphism $\beta : M \to M'$ define

(10.5) $S(\beta) : S(M) \to S(M')$ by $S(\beta) = \mathrm{Id} + SP^1(\beta) + SP^2(\beta) + \cdots$.

Then S is a functor from modules to modules. Its prolongation to the FD-category (5.1′) is also denoted by S. We have

(10.6) $\quad S(K) = P + SP^1(K) + SP^2(K) + \cdots = \sum_{q=0}^{\infty} SP^q(K)$

(10.7) $\quad S(F) = \mathrm{Id} + SP^1(F) + SP^2(F) + \cdots = \sum_{q=0}^{\infty} SP^q(F)$

for FD-modules K, and FD-maps F. Formulas (8.4), (8.6), (8.8), and (8.9) give

(10.8) $\quad S(M^1 + M^2) = S(M^1) \otimes S(M^2),\ S(\beta^1 + \beta^2) = S(\beta^1) \otimes S(\beta^2)$,

(10.9) $\quad S(K^1 + K^2) = S(K^1) \times S(K^2),\ S(F^1 + F^2) = S(F^1) \times S(F^2)$.

Comparing (10.3) and (10.6) shows

(10.10) $\quad SP(K) = S(K^0)$

for an augmented FD-module K with base point. If K is generated by sub-FD-modules K^1 and K^2 such that $K^1 \cap K^2 = \xi(P)$; i. e., $K^0 = (K^1)^0 + (K^2)^0$ then (10.9) and (10.10) give

(10.11) $\quad SP(K) = SP(K^1) \times SP(K^2)$.

This is the algebraic analogue to (3.14) in [2].

There is a natural multiplication in $S(M)$: Let $\alpha : M + M \to M$ be the homomorphism which is the identity on each summand. Then

$$S(M) \otimes S(M) \cong S(M + M) \xrightarrow{S(\alpha)} S(M)$$

defines a multiplication in $S(M)$ which is abelian and associative and has a unit, namely the unit of $\Lambda \subset S(M)$. (α does not change if we interchange summands in $M + M$, therefore $S(\alpha)$ does not change if we interchange factors in $S(M) \otimes S(M)$. Similarly the associativity of α implies associativity of $S(\alpha)$.)

We call $S(M)$ (resp. $S(K)$) the *symmetric algebra* over M (over K). It may be characterized by a universal property (Chevalley, Fundamental concepts of algebra, New York, 1956, Ch. V, 18), namely as " free commutative algebra with unit over M". For instance $S(\Lambda) = \Lambda[x] =$ polynomial ring over Λ in one variable x.

Let $\beta: \ M \to M'$ be a module homomorphism. If we apply the functor S to the commutative diagram

$$\begin{array}{ccc} M + M & \xrightarrow{\alpha} & M \\ {\scriptstyle \beta+\beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ M' + M' & \xrightarrow[\alpha]{} & M' \end{array}$$

where α denotes the addition we obtain a commutative diagram

$$\begin{array}{ccc} S(M) \otimes S(M) & \xrightarrow{S(\alpha)} & S(M) \\ {\scriptstyle S(\beta) \otimes S(\beta)}\downarrow & & \downarrow{\scriptstyle S(\beta)} \\ S(M') \otimes S(M') & \xrightarrow[S(\alpha)]{} & S(M') \end{array}$$

in which $S(\alpha)$ is the multiplication. This shows that $S(\beta)$ is a homomorphism of algebras (with unit since obviously $S(\beta)(1) = 1$). It follows that $S(K)$ is a commutative FD-algebra for every FD-module K.

There is a natural augmentation and base point in $S(K)$: Let 0 be the zero-FD-module, then $S(0) = P =$ FD-algebra of a point. Applying the functor S to the zero maps $K \to 0$ and $0 \to K$ gives augmentation $S(K) \to P$ and base point $P \to S(K)$ in $S(K)$. The augmentation is nothing but the projection onto the first summand of (10.6).

The connection between infinite symmetric products of FD-modules and infinite symmetric products of semi-simplicial complexes or topological spaces is the same as in the case of Γ-products (§ 7). In particular it follows from (5.11) and [2] that for a free FD-module L (over the integers) with $H_0(L) = 0$ the Pontrjagin algebra $H_*(S(L))$ is the same as in a (weak) product of Eilenberg-MacLane complexes $\Pi_q K(H_q(L), q)$.

11. Reduced product space and tensor algebra

The semi-simplicial equivalent of the reduced product space of James [7] is a s.s. monoid complex F^+X which has been defined by Milnor [10] as follows: Let X be a s.s. complex with base point $b \in X_0$. Let $(F^+X)_n$

be the free (associative) monoid generated by X_n with the single relation $b_n = s_0^n\, b = 1$, and let $\partial_i : (F^+X)_n \to (F^+X)_{n-1}$ be the homomorphic extensions of the face operators given on the generators X_n ; similarly for s_i.

This definition generalizes to augmented FD-modules K with base point $\xi : \ P \to K$. Let $(F^+K)_n$ be the free associative algebra over K_n (i. e., the tensor algebra $\otimes K_n$) with the single relation $b_n = 1 = 1_n$ (b_n the unit in $\xi(P)_n \subset K_n$). Let $\partial_i : (F^+K)_n \to (F^+K)_{n-1}$, $i \leqq n$, be the unique homomorphisms of algebras with unit which extend the given face-operators on K_n, and define s_i similarly.

The single relation $b_n = 1$ shows that $(F^+K)_n$ is simply the free algebra over K_n^0, i.e., $\otimes K_n^0$, and F^+K is nothing but the tensor algebra $\otimes (K^0)$ (6.6) over the FD-module $K^0 \subset K$ of augmentation zero.

(11.1) $F^+K = \otimes (K^0) = P + K^0 + K^0 \times K^0 + K^0 \times K^0 \times K^0 + \cdots$.

The connection between this and Milnor's construction is formulated in

(11.2). PROPOSITION. *Let X be a semi-simplicial complex with base point b, and let K be its* FD-*module* (§ 1) *with augmentation and base point induced by $X \to b$, $b \to X$* (§ 9). *Then the* FD-*module of F^+X is*

$$K(F^+X) \cong F^+K \cong \otimes K^0 = P + K^0 + K^0 \times K^0 + K^0 \times K^0 \times K^0 + \cdots$$

PROOF. For every set I let $F'I$ be the free monoid with unit over I, and let $\Lambda\ F'I$ be the algebra (over Λ) of the monoid $F'I$. On the other hand form ΛI, the free module (over Λ) with generators I, and take its tensor algebra $\otimes\ \Lambda I$. According to [1, Ch. III, 7, Ex. 7], these algebras are isomorphic, $\Lambda F'I \cong \otimes \Lambda I$. The Proposition (11.2) is a slight generalization of this : the set is replaced by a s. s. complex (a graded set with operators ∂_i, s_i), and instead of modules we take FD-modules (graded modules with operators ∂_i, s_i). The details of the proof are left to the reader.

By Theorem (5.11), the homology of F^+K is determined by the homology of K^0 (assuming K free over a principal ideal domain), which is essentially the same as for K ($H_i(K) = H_i(K^0)$ for $i > 0$; $H_0(K) = \Lambda + H_0(K^0)$). This result of course follows much easier and in a more explicit form from (11.1) (cf., also [10, Theorem 5]), since one knows very well how to compute the homology of direct sums and cartesian products (the latter being equivalent to tensor products [5, 2]). The ring structure of $H_*(F^+K)$ is also clear from (11.1).

Using (11.2) this applies to the free monoid F^+X over a s. s. complex and by geometric realization to the reduced product space of James

[10, Lemma 4], hence to the loop space ΩEY of the suspension EY of a CW-complex Y. For instance, if Y is a wedge of spheres we easily recover the result of Bott and Samelson on $H_*(\Omega Y)$ (Commentarii Mathem. Helv. 27, (1953), III, I. B).

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