Then

$$|\{k: q_{k,N+n} \in \mathscr{I} \times [0, \varepsilon]\}| \geqslant 2^n,$$

$n = 0, 1, \ldots,$ and thus

$$\mu(G) \geqslant \sum_{n=0}^{\infty} 2^{-(N+n)}(2^n r) = +\infty.$$

Since $\mu([0,1]) = 0$, μ is not regular.

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Труды Матем. Инст. Стеклов.
154(1983)

Proceedings of the Stekl
Institute of Mathemati
1984, Issue

UDC 513.83

DUALITY, TRACE AND TRANSFER

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ABSTRACT. The notions of "strong duality", "trace" and "transfer" in monoidal categories are discussed.
Bibliography: 22 titles.

Introduction

Motivated by topological applications we discuss the notion of "strong duality" (§§1 and 2), of "trace" (§4) and of "transfer" (§5) in monoidal categories. Primarily we have in mind the following examples of monoidal categories:

1. Mod_R— Modules over some commutative ring R (1.4),
2. ∂-Mod_R—Chain complexes (1.5, 1.8),
3. Stab—Stable homotopy (§3),
4. Stab_B—Parametrized stable homotopy over some parameter space B (§6),
5. Stab^G—G-equivariant stable homotopy for some group G (§7), and
6. Stable shape theory (§8).

Strong duality in Stab is S-duality in the sense of Spanier and Whitehead [19]. We give a direct geometric proof for the S-duality between a compact neighborhood retract in $\mathbf{R}^n$ and its complement (3.1), which can easily be generalized to Stab_B (6.1), Stab^G (7.1) and to stable shape theory (8.1). Alexander duality is not used for the proof of Theorem 3.1, but is a corollary (3.3). Another corollary is the Lefschetz-Hopf fixed point theorem (4.6).

What one gets in Stab_B (§6) is closely related to Dold [2], [3], [4], and to Becker-Gottlied [1]. Studying the latter paper we realized that some simple abstract notions are very powerful in this context. This not only enabled us to replace some *ad hoc* computations by conceptual proofs (cf. e.g. 4.4, 4.5), but also to prove stronger results: Extending Stab_B to a category of spectra one can do without Becker-Gottlied's hypothesis that the base space B has finite dimension. In particular we get a stronger theorem (6.2) than Theorem 1.1 in Becker-Gottlieb [1].

Some proofs are omitted in this paper. A complete and detailed exposition will be published elsewhere.

Cf. also the remarks added in proof at the end of the paper.

§1. Duality in monoidal categories

Let $\mathscr{C}$ be a (symmetric) monoidal category with multiplication $\otimes$ and neutral object I. That means that $\otimes$ is a bifunctor $(A, B) \mapsto A \otimes B$ of $\mathscr{C}$ into itself and we have given coherent natural equivalences

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C,$$
$$I \otimes A \cong A \cong A \otimes I,$$
$$\gamma = \gamma_{AB}\colon A \otimes B \overset{\cong}{\to} B \otimes A.$$

One knows by experience that it will do no harm to replace the three equivalences in the first two lines by equalities. In the third line one has to be a little more careful because $\gamma_{AA}\colon A \otimes A \to A \otimes A$ is not the identity in general.

An object B of $\mathscr{C}$ is called a (*weak*) *dual* of A if it represents the functor $X \mapsto \mathscr{C}(X \otimes A, I)$, i.e. if there is a bijection

(1.1) $$\mathscr{C}(X \otimes A, I) \cong \mathscr{C}(X, B)$$

which is natural with respect to X. Letting $X = B$, there is a definite morphism, called *evaluation*,

$$\varepsilon = \varepsilon_A\colon B \otimes A \to I$$

which corresponds to $\mathrm{id}_B \in \mathscr{C}(B, B)$. Let $\mathscr{C}^*$ be the full subcategory of $\mathscr{C}$ whose objects have duals. By picking a particular dual[1] DA for each object A of $\mathscr{C}^*$ we get a contravariant functor $D\colon \mathscr{C}^* \to \mathscr{C}$. For $f \in \mathscr{C}^*(A, A')$ the *dual* (or *transposed*) *morphism* $Df \in \mathscr{C}(DA', DA)$ is characterized by the commutativity of the diagram

$$\begin{array}{ccc} DA' \otimes A & \xrightarrow{\mathrm{id} \otimes f} & DA' \otimes A' \\ {\scriptstyle Df \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \varepsilon_{A'}} \\ DA \otimes A & \xrightarrow{\varepsilon_A} & I \end{array}$$

The object I is obviously a dual of itself. Hence we may assume $DI = I$.

By (1.1) a morphism $X \to DA$ may be defined by giving the corresponding $X \otimes A \to I$. If DA and DDA exist we define in this way $\delta = \delta_A\colon A \to DDA$ by

$$A \otimes DA \underset{\cong}{\overset{\gamma}{\to}} DA \otimes A \overset{\varepsilon}{\to} I.$$

If δ_A is an isomorphism, A is called *reflexive*. Now assume that duals of A, B and $B \otimes A$ exist. Then we define

$$\mu = \mu_{AB}\colon DA \otimes DB \to D(B \otimes A)$$

by

$$DA \otimes DB \otimes B \otimes A \xrightarrow{\mathrm{id} \otimes \varepsilon_B \otimes \mathrm{id}} DA \otimes I \otimes A = DA \otimes A \overset{\varepsilon_A}{\to} I.$$

1.2. Definition. A is called *strongly dualizable* if it is reflexive and $\mu_{A,DA}$ or equivalently the composition

$$DA \otimes A \xrightarrow{\mathrm{id} \otimes \delta} DA \otimes DDA \overset{\mu}{\to} D(DA \otimes A)$$

is an isomorphism. The latter means that $DA \otimes A$ is (canonically) self-dual.

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[1] *Russian translator's note.* Note that a dual object is defined uniquely up to isomorphism.

That A is strongly dualizable will also be expressed by saying that A and DA are *stong duals* or that DA is a *strong dual* of A (which obviously are symmetric relations).

If A is strongly dualizable the *coevaluation* $\eta = \eta_A\colon I \to A \otimes DA$ is defined to be the composition

$$I = DI \overset{D\varepsilon}{\to} D(DA \otimes A) \underset{\cong}{\overset{\mu^{-1}}{\to}} DA \otimes DDA \underset{\cong}{\xrightarrow{\mathrm{id} \otimes \delta^{-1}}} DA \otimes A \underset{\cong}{\overset{\gamma}{\to}} A \otimes DA.$$

1.3. Theorem. *Let A and B be objects of a (symmetric) monoidal category $\mathscr{C}$ and let $\varepsilon\colon B \otimes A \to I$. Then the following are equivalent:*

(a) *B is a strong dual of A with evaluation ε.*

(b) *There exists $\eta\colon I \to A \otimes B$ such that the following compositions are the identity morphisms of A and B respectively:*

$$\mathrm{id}_A\colon A = I \otimes A \xrightarrow{\eta \otimes \mathrm{id}_A} A \otimes B \otimes A \xrightarrow{\mathrm{id}_A \otimes \varepsilon} A \otimes I = A,$$
$$\mathrm{id}_B\colon B = B \otimes I \xrightarrow{\mathrm{id}_B \otimes \eta} B \otimes A \otimes B \xrightarrow{\varepsilon \otimes \mathrm{id}_B} I \otimes B = B.$$

(c) *The map*

$$\varphi_{XY}\colon \mathscr{C}(X, Y \otimes B) \to \mathscr{C}(X \otimes A, Y)$$

which sends $f\colon X \to Y \otimes B$ into the composition

$$X \otimes A \xrightarrow{f \otimes \mathrm{id}_A} Y \otimes B \otimes A \xrightarrow{\mathrm{id}_Y \otimes \varepsilon} Y \otimes I = Y$$

is a bijection for all objects X, Y of $\mathscr{C}$.

Furthermore, if one and hence all of these conditions are satisfied then the morphism η in (b) *is necessarily the coevaluation, and the bijection φ_{IA} of* (c) *sends it into*

$$I \otimes A = A \overset{\mathrm{id}_A}{\to} A.$$

Proof. (b) and (c) are equivalent because they are two well-known ways of expressing that the functors $X \mapsto X \otimes A$ and $Y \mapsto Y \otimes B$ are adjoint with counit

$$Y \otimes B \otimes A \xrightarrow{\mathrm{id}_Y \otimes \varepsilon} Y \otimes I = Y$$

(and unit

$$\left.X = X \otimes I \xrightarrow{\mathrm{id}_X \otimes \eta} X \otimes A \otimes B\right).$$

(Mac Lane [11], §IV, 1). Since (b) is obviously symmetric in A and B, the same follow for (c). Knowing that, the implication (c) ⇒ (a) is almost trivial. It is not hard to show (a) ⇒ (b) by writing down the appropriate diagrams, but we omit the details here.

We owe the idea of this theorem to Lindner [10], where we have seen formulation (b) for the first time. Strongly dualizable objects in monoidal categories have also been studied by Pareigis [13] and Ligon [9], where they are called "finite objects". But the aims of these papers are quite different from ours.

1.4. EXAMPLE. *Modules*. Let R be a commutative ring (with unit) and consider the category $\mathscr{C} = \mathrm{Mod}_R$ of (unitary) modules over R. The ordinary tensor product defines a monoidal structure for which $I = R$, considered as a module over itself, is a unit object. Every module A has a (weak) dual $DA = \mathrm{Hom}_R(A, R)$, and ε_A is the usual evaluation map. It is clear that every finitely generated projective module is strongly dualizable. The converse is also true.(2)

PROOF. Let

$$\eta(1) = \sum_{i=1}^{n} a_i \otimes b_i$$

where $\eta\colon R \to A \otimes \mathrm{Hom}_R(A, R)$ is the coevaluation. Then the map

$$A \to R^n, \qquad R^n \to A,$$
$$a \mapsto (\varepsilon(b_i \otimes a) | i = 1,\dots,n), \qquad x \mapsto \sum_{i=1}^{n} x_i a_i$$

show that A is isomorphic to a direct summand of R^n.

If R is a field then A is strongly dualizable if and only if it is reflexive. In general this is not true: If $R = \mathbf{Z}$ then the countable infinite product and the countable infinite coproduct (= direct sum) of copies of $\mathbf{Z}$ are duals of each other. Hence both are reflexive. But they are not strongly dualizable.

If A has a finite base $a_1,\dots,a_n$ then

$$\eta(1) = \sum_{i=1}^{n} a_i \otimes a_i',$$

where $a_1',\dots,a_n'$ is the dual base of $\mathrm{Hom}_R(A, R)$.

1.5. EXAMPLE. *Chain complexes and chain maps*. With R as above consider the category $\mathscr{C} = \partial\text{-}\mathrm{Mod}_R$ of chain complexes

$$A = (A_q, \partial\colon A_q \to A_{q-1} | q \in \mathbf{Z})$$

consisting of modules over R and "boundary operators" ∂. Morphisms are the chain maps $f = (f_q\colon A_q \to B_q | q \in \mathbf{Z})$ consisting of linear maps f_q which commute with ∂.

The tensor product of chain complexes giving a monoidal structure on $\partial\text{-}\mathrm{Mod}_R$ is defined by

$$(A \otimes B)_n = \bigotimes_{p+1=n} A_p \otimes B_q,$$

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^{|a|} a \otimes \partial b$$

where $|a| = p$ if $a \in A_p$. A neutral object $I = (I_q | q \in \mathbf{Z})$ is given by

$$I_q = \begin{cases} R, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

The equivalences showing that $\otimes$ is associative and that I is neutral are the obvious ones, whereas the commutativity of $\otimes$ is given by the equivalence

$$\gamma\colon A \otimes B \to B \otimes A$$
$$a \otimes b \mapsto (-1)^{|a||b|} b \otimes a.$$

(2) *Russian translator's note*. I.e. a finitely generated strongly dualizable module is projective.

Every chain complex A has a (weak) dual DA defined by

$$(DA)_q = \mathrm{Hom}_R(A_{-q}, R),$$

$$(DA)_q \xrightarrow{\partial} (DA)_{q-1},$$

$$\begin{array}{ccc} A_{-q} & & A_{-q+1} \\ \downarrow & b \mapsto (-1)^{q-1} b\partial & \downarrow \\ R & & R \end{array}$$

and the evaluation map is the obvious one.

A chain complex A will be called *finitely generated* if A_q is a finitely generated module for all q and $A_q \neq 0$ only for finitely many q. It will be called *projective* if A_q is a projective module for all q, although this does not imply that A is a projective object of $\partial\text{-}\mathrm{Mod}_R$.

1.6. PROPOSITION. *A chain complex A is strongly dualizable in $\partial\text{-}\mathrm{Mod}_R$ if and only if it is finitely generated and projective.*

We postpone the (easy) proof until after 2.4, where it will serve at the same time as an illustration for the interplay between duality and monoidal functors.

Even in the case that R is a field, a reflexive chain complex A need not be strongly dualizable, because A is reflexive in that case if and only if A_q is a finite-dimensional vector space for every q. There may be infinitely many q such that $A_q \neq 0$.

If the chain complex A has a finite base $a_1,\dots,a_n$ (by which we mean the union over $q \in \mathbf{Z}$ of bases of A_q) then just as in the case of modules the coevaluation η: $I \to A \otimes DA$ is given by

$$\eta(1) = \sum_{i=1}^{n} a_i \otimes a_i' \tag{1.7}$$

where $a_1',\dots,a_n'$ is the dual base of DA which is characterized by

$$|a_i'| = -|a_i|, \quad a_i'(a_i) = 1, \quad a_i'(a_j) = 0$$

for all i and for all those $j \neq i$ for which $|a_j| = |a_i|$.

The formula (1.7) can either be proved by calculating the dual of the evaluation map ε or, more simply, by verifying that condition (b) of 1.3 is satisfied if one defined η by (1.7).

1.8. EXAMPLE. *Chain complexes and homotopy classes of chain maps*. This is the same monoidal category as in the preceding example except that the morphisms are not the chain maps themselves but homotopy classes of chain maps. We denote it by $\mathrm{Ho}(\partial\text{-}\mathrm{Mod}_R)$.

1.9. PROPOSITION. *A chain complex A is strongly dualizable in $\mathrm{Ho}(\partial\text{-}\mathrm{Mod}_R)$ if and only if it has the homotopy type of a chain complex which is strongly dualizable in $\partial\text{-}\mathrm{Mod}_R$.*

PROOF. Sufficiency is obvious. We omit the proof of necessity, which will not be used in this paper.

§2. Monoidal functors

Let $\mathscr{C}$ and $\mathscr{C}'$ be two monoidal categories with neutral objects I and I' respectively. The multiplication will be denoted by $\otimes$ in both cases. A *monoidal functor* T: $\mathscr{C} - \mathscr{C}'$ is a functor together with natural transformations

$$\tau = \tau_{AB}\colon TA \otimes TB \to T(A \otimes B), \qquad I' \to TI,$$

which are compatible with the natural transformations involved in the associativity and commutativity of $\otimes$ and in the "neutrality" of I and I'. Here, in addition, we shall always assume that $I' \to TI$ is an equivalence, because this is the case in all our applications. Hence we may even write $I' = TI$.

2.1. EXAMPLE OF MONOIDAL FUNCTORS. (a) Let $\mathscr{C}$ be any monoidal category in which every object A has a weak dual DA. Then D is a (contravariant) monoidal functor of $\mathscr{C}$ into itself (cf. §1).

(b) The canonical functor $\partial\text{-Mod}_R \to \text{Ho}(\partial\text{-Mod}_R)$ (1.5, 1.8).

(c) Let Gr-Mod_R be the category of graded R-modules, i.e. the full subcategory of $\partial\text{-Mod}_R$ consisting of all chain complexes with zero boundary operator. Then we have monoidal functors

$$\text{Gr-Mod}_R \subset \partial\text{-Mod}_R \to \text{Gr-Mod}_R$$

where the latter one just replaces ∂ by zero.

(d) Homology H: $\text{Ho}(\partial\text{-Mod}_R) \to \text{Gr-Mod}_R$.

(e) The functor $\text{Gr-Mod}_R \to \text{Mod}_R$ defined by $A \mapsto \bigoplus_{q\in\mathbf{Z}} A_q$.

In §3 we shall consider monoidal functors involving the category of stable homotopy which will be of particular interest for the applications we have in mind.

We shall say that a monoidal functor T is *compatible with* or *preserves tensor products* if the natural transformation τ is an equivalence. This is the case in examples (b), (c) and (e) but not (in general) in examples (a) and (d). It it is not the case then one has to ask sometimes whether for some particular objects A and B of $\mathscr{C}$ the morphism τ_{AB} is an isomorphism.

2.2. THEOREM. *Let T: $\mathscr{C} \to \mathscr{C}'$ be a monoidal functor and A an object of $\mathscr{C}$ which is strongly dualizable. Then the following conditions are equivalent*:

(a) τ: $TA \otimes TX \cong T(A \otimes X)$ *for all objects X of $\mathscr{C}$.*

(b) τ: $TA \otimes TDA \cong T(A \otimes DA)$.

(c) *There exists a morphism f such that the diagram*

$$\begin{array}{ccc} TI & \xrightarrow{T\eta} & T(A \otimes DA) \\ \| & & \uparrow \tau \\ I' & \xrightarrow{f} & TA \otimes TDA \end{array}$$

commutes.

PROOF. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. To prove (c) $\Rightarrow$ (a) we assume (c) and form the composition

$$T(A \otimes X) = I' \otimes T(A \otimes X) \xrightarrow{f\otimes \text{id}} TA \otimes TDA \otimes T(A \otimes X)$$

$$\xrightarrow{\text{id}\otimes\tau} TA \otimes T(DA \otimes A \otimes X) \xrightarrow{\text{id}\otimes T(\varepsilon\otimes\text{id})} TA \otimes T(I \otimes X) = TA \otimes TX.$$

Using 1.3(b) it is not hard to show that this is an inverse of τ: $TA \otimes TX \to T(A \otimes X)$.

2.3. DEFINITION. Let T: $\mathscr{C} \to \mathscr{C}'$ be a monoidal functor. An object A of $\mathscr{C}$ is called *T-flat* if it is strongly dualizable and satisfies one of the equivalent conditions (a)–(c) of 2.2.

If T preserves tensor products then the notions "T-flat" and "strongly dualizable" coincide.

2.4. COROLLARY. *If A is T-flat then TA and TDA are strongly dual with evaluation*

$$TDA \otimes TA \xrightarrow{\tau} T(DA \otimes A) \xrightarrow{T\varepsilon} TI = I'$$

and coevaluation

$$I' = TI \xrightarrow{T\eta} T(A \otimes DA) \xrightarrow{\tau^{-1}} TA \otimes TDA.$$

PROOF. Apply t to condition (b) in 1.3 and use 2.2(a).

PROOF OF PROPOSITION 1.6. Let

$$T\colon \partial\text{-Mod}_R \to \text{Gr-Mod}_R \to \text{Mod}_R, \qquad A \mapsto \bigoplus_{q\in\mathbf{Z}} A_q$$

be the composition of the functors in Examples 2.1(c) and (e). This T preserves tensor products. If A is strongly dualizable, so is TA by Corollary 2.4. Hence TA is finitely generated and projective (1.4). But this is obviously equivalent to saying that A is finitely generated and projective in the sense of 1.5. Conversely, if A is assumed to be finitely generated then the canonical maps $A \to DDA$ and $DA \otimes A \to D(DA \otimes A)$ are transformed by T into the corresponding maps for TA instead of A. If, in addition, A is projective then these transformed maps are isomorphisms. But T reflects isomorphisms. Hence A is strongly dualizable by 1.3(a).

We conclude this section by investigating what T-flat means in Examples 2.1(a) and (d). First let $\mathscr{C}$ be any monoidal category and consider $\otimes$: $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$. With the obvious monoidal structure on $\mathscr{C} \times \mathscr{C}$ this functor is monoidal and preserves tensor products. Hence we obtain from Corollary 2.4 that the tensor product of strongly dualizable objects is strongly dualizable (which means that the full subcategory $\tilde{\mathscr{C}}$ of $\mathscr{C}$ consisting of all strongly dualizable objects is a monoidal subcategory of $\mathscr{C}$) and that D: $\tilde{\mathscr{C}} \to \mathscr{C}$ preserves tensor products. Hence the notions "D-flat" and "strongly dualizable" coincide. With respect to Example 2.1(d) we have

2.5. PROPOSITION. *Let H: $\partial\text{-Mod}_R \to \text{Gr-Mod}_R$ be the homology functor and let A be a chain complex such that A_q is a flat R-module for each $q \in \mathbf{Z}$. Consider the conditions*:

(a) *H_qA is a flat R-module for each $q \in \mathbf{Z}$.*

(b) τ: *$HA \otimes HX \cong H(A \otimes X)$ for all chain complexes X.*

Then (b) $\Rightarrow$ (a). *The converse* (a) $\Rightarrow$ (b) *is true if one of the following additional hypotheses is satisfied*:

(1) *A has the homotopy type of a chain complex B which is bounded below* (*i.e. $B_q = 0$ for all $q < q_0$ and some $q_0 \in \mathbf{Z}$*).

(2) *The global homological dimension of the ring R is finite.*

PROOF. (b) $\Rightarrow$ (a) is obvious. The converse (a) $\Rightarrow$ (b) may first be reduced to the case where X is just a single module M by using the exact sequence of chain complexes $0 \to ZX \to X \xrightarrow{\partial} BX \to 0$. Then one uses a projective resolution P of M and compares $A \otimes M$ to the double complex $A \otimes P$.

In general (a) $\Rightarrow$ (b) is not true without (1) or (2). Example: $R = \mathbf{Z}/4\mathbf{Z}$, $A_q = \mathbf{Z}/4\mathbf{Z}$ for all $q \in \mathbf{Z}$, ∂ = multiplication by 2, $X = X_0 = M = \mathbf{Z}/2\mathbf{Z}$.

§3. Stable homotopy

We define the stable homotopy category Stab as follows: Objects are the pairs (X, n), where X is a well-pointed compactly generated space (not necessarily Hausdorff) and $n \in \mathbf{Z}$. The set of morphisms from (X, n) to (Y, m) is

$$\mathrm{Stab}((X, n), (Y, m)) = \operatorname*{colim}_{k \to \infty} [S^{n+k} \wedge X, S^{m+k} \wedge Y],$$

where $\wedge$ is the smash product in the category of (pointed) compactly generated spaces, the brackets [,] denote the set of homotopy classes and the bonding maps are given by smashing with S^1. Composition of morphisms is the obvious one.

A monoidal structure on Stab is given by

$$(X, n) \otimes (Y, m) = (X \wedge Y, n + m).$$

In order to turn this into a functor we have to be a little careful and introduce a sign. If two morphisms $(X, n) \to (X', n')$ and $(Y, m) \to (Y', m')$ are represented by

$$f: S^{n+k} \wedge X \to S^{n'+k} \wedge X', g: S^{m+l} \wedge Y \to S^{m'+1} \wedge Y',$$

respectively, then their $\otimes$-product is formed by forming

$$f \wedge g: S^{n+k} \wedge X \wedge S^{m+l} \wedge Y \to S^{n'+k} \wedge X' \wedge S^{m'+l} \wedge Y',$$

transforming it into

$$S^{n+m+k+l} \wedge X \wedge Y \to S^{n'+m'+k+l} \wedge X' \wedge Y'$$

by interchanging the middle factors and multiplying (the homotopy class) by $(-1)^{(n+n')l}$. Then the construction will be compatible with the bonding maps.

A neutral object is $I = (S^0, 0)$. The equivalences which give the associativity of $\otimes$ and the neutrality of I are the obvious ones. Commutativity of $\otimes$ is given by

$$\gamma: (X, n) \otimes (Y, m) \to (Y, m) \otimes (X, n)$$

represented by $(-1)^{nm}$ times the stable homotopy class of the ordinary interchange map $X \wedge Y \to Y \wedge X$ (the sign is necessary because otherwise we would not obtain a natural transformation).

Forming the $\otimes$-product with (S^0, p) as a left factor defines a functor Σ^p of Stab into itself which sends (X, n) into $(S^0 \wedge X, n + p) = (X, n + p)$. These functors Σ^p are defined for all $p \in \mathbf{Z}$ and satisfy

$$\Sigma^p \cdot \Sigma^q = \Sigma^{p+1}, \qquad \Sigma^0 = \text{identity functor}.$$

We sometimes abbreviate $(X, 0)$ by X. Then we may write $\Sigma^n X$ instead of (X, n).

If $p \geqslant 0$ then there is an obvious isomorphism $(S^0, p) \cong (S^p, 0)$, represented by the identity of S^p. This shows that Σ^p is equivalent to smashing with S^p, i.e. to ordinary p-fold suspension.

It is sometimes convenient to represent pointed spaces by pairs of spaces. We make the convention that if $i: X' \to X$ is any map between (unpointed) compactly generated spaces then (X, X') denotes the object $C_i = (C_i, 0)$ of Stab, where C_i is the mapping cone of i with the vertex of the cone as base point. Which map i we mean will always be clear from the context. Usually i will be an inclusion $X' \subset X$. Note that $(X, \varnothing) = X^+$, i.e. X with an additional isolated point as base point.

3.1. THEOREM. *Let K be a compact subset of $\mathbf{R}^n$ and a neighborhood retract. Then $(K, \varnothing)$ and $\Sigma^{-n}(\mathbf{R}^n, \mathbf{R}^n \setminus K)$ are strongly dual in* Stab.

Strong duality in Stab is nothing else but S-duality in the sense of Spanier and Whitehead (Spanier-Whitehead [19], Spanier [18], Switzer [20], 14.20(³)).

If K is a subcomplex of a simplicial decomposition of $\mathbf{R}^n$ then the above theorem reduces to Lemma 5.1 of Spanier [18]. Apart from proving a somewhat more general theorem our point is that our proof is independent and quite different from former proofs. It consists of defining explicitly the evaluation and coevaluation morphisms in Stab and verifying condition (b) of 1.3 in a "geometrical" way. In particular we do not use Alexander duality. On the contrary, Alexander duality is a corollary of Theorem 3.1, as we are going to show now.

If M is a module over R let (M, n) be the chain complex such that

$$(M, n)_q = \begin{cases} M, & q = n, \\ 0, & q \neq n. \end{cases}$$

For a pointed space X with base point x_0 let $\tilde{S}X$ denote the singular complex of the pair (X, x_0) with coefficients in R. We "extend" this to a functor

$$\tilde{S}: \mathrm{Stab} \to \mathrm{Ho}(\partial\text{-}\mathrm{Mod}_R)$$

as follows: An object (X, n) of Stab goes to $(R, n) \otimes \tilde{S}X$. A morphism $f: (X, n) \to (Y, m)$ of Stab is represented by a pointed map

$$f_k: S^{n+k} \wedge X \to S^{m+k} \wedge Y$$

for some $k \in \mathbf{Z}$. Consider

$$\begin{array}{ccc} (R, n+k) \otimes \tilde{S}X & \to & (R, m+k) \otimes \tilde{S}Y \\ \downarrow & & \downarrow \\ \tilde{S}S^{n+k} \otimes \tilde{S}X & & \tilde{S}S^{m+k} \otimes \tilde{S}Y \\ \downarrow & & \downarrow \\ \tilde{S}(S^{n+k} \wedge X) & \xrightarrow{\tilde{S}f_k} & \tilde{S}(S^{m+k} \wedge Y). \end{array}$$

The vertical arrows are chain homotopy equivalences, the upper ones induced by $\tilde{S}S^l \simeq (R, l)$, while the lower ones are Eilenberg-Zilber maps. The upper horizontal arrow is the morphism in $\mathrm{Ho}(\partial\text{-}\mathrm{Mod}_R)$ which makes the diagram commutative. Shifting the grading by k, we obtain from it

$$\tilde{S}f: (R, n) \otimes \tilde{S}X \to (R, m) \otimes \tilde{S}Y.$$

By the Eilenberg-Zilber theorem there is a natural equivalence

$$\begin{aligned} \tilde{S}(X, n) \otimes \tilde{S}(Y, m) &= (R, n) \otimes \tilde{S}X \otimes (R, m) \otimes \tilde{S}Y \\ &\cong (R, n+m) \otimes \tilde{S}X \otimes \tilde{S}Y \to (R, n+m) \otimes \tilde{S}(X \wedge Y) \\ &= \tilde{S}(X \wedge Y, n+m) = \tilde{S}[(X, n) \otimes (Y, m)]. \end{aligned}$$

(³)Switzer defines an S-duality by asking that two maps which he calls D_μ and ${}_\mu D$ are bijective. Note that by our Theorem 1.3 one of these two maps is bijective if and only if the other is.

Thus we have

3.2. Proposition. *$\tilde{S}$: Stab $\to$ Ho(∂-Mod$_R$) is a monoidal functor which preserves tensor products.*

By applying Corollary 2.4 to this functor $\tilde{S}$ we get from Theorem 3.1

3.3. Corollary. *If K is a compact subset of $\mathbf{R}^n$ and a neighborhood retract then the chain complexes SK and*

$$(R,-n)\otimes S(\mathbf{R}^n,\mathbf{R}^n\setminus K)=(R,1-n)\otimes\tilde{S}(\mathbf{R}^n\setminus K)$$

are strong duals in Ho(∂-Mod$_R$).

This obviously implies Alexander duality for K and $\mathbf{R}^n\setminus K$. (SK denotes the singular complex of K and $S(\mathbf{R}^n,\mathbf{R}^n\setminus K)=S\mathbf{R}^n/S(\mathbf{R}^n\setminus K)$.)

The rest of this section is devoted to the proof of Theorem 3.1. First we have to say a little more about the representation of pointed spaces by pairs of spaces. A map of pairs $(f,f')\colon (X,X')\to(Y,Y')$, i.e., a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

induces a pointed map of the mapping cones $C_i\to C_j$, hence a morphism in S tab, which will also be denoted by (f,f') or sometimes just by f. If $f\colon X\to Y$ and $f'\colon X'\to Y'$ are both homotopy equivalences then so is the induced map $C_i\to C_j$, hence f is an isomorphism in Stab.

The next lemma says that excision gives an isomorphism in Stab. In order to avoid a discussion about the notion of subspace in the category of compactly generated spaces we assume that everything happens in a metrizable space, although the lemma holds much more generally if one formulates it properly.

3.4. Lemma. *Let X be metrizable and let X' and U be subspaces of X whose interiors cover X. Let $U'=U\cap X'$. Then the inclusion $(U,U')\subset(X,X')$ is an isomorphism in* Stab.

Proof. Let $v\colon X\to[0,1]$ be a continuous function such that

$$\mathrm{Cl}\{x\mid v(x)<1\}\subset X',\qquad \mathrm{Cl}\{x\mid v(x)>0\}\subset U$$

(where Cl means closure). For the mapping cone of the inclusion $X'\subset X$ we may write $X\cup CX'$. Consider the commutative diagram

$$\begin{array}{ccc} (U\setminus U')\cup\{(x,t)\mid x\in U',\,t\leqslant v(x)\} & \overset{\subset}{\to} & U\cup CU' \\ \downarrow h & & \downarrow g \\ (X\setminus X')\cup\{(x,t)\mid x\in X',\,t\leqslant v(x)\} & \overset{\subset}{\to} & X\cup CX', \end{array}$$

where g is induced by $(U,U')\subset(X,X')$. (g is not always a topological embedding.) The horizontal inclusions are homotopy equivalences because there are obvious deformation retractions to the subspaces. h is a homeomorphism, as one can easily check. Therefore g is a homotopy equivalence.

$$(X,X')\times(Y,Y')=(X\times Y,\,X'\times Y\cup X\times Y'),$$

although this is not the product in the sense of category theory. Again, for simplicity, we assume that X and Y are metric.

3.5. Lemma. *There is a canonical morphism*

$$(X,X')\otimes(Y,Y')\to(X,X')\times(Y,Y')$$

in Stab. *It is an isomorphism if X' and Y' are open in X and Y respectively.*

Proof. Let

$$Z=X'\times Y\times 0\cup X'\times Y'\times I\cup X\times Y'\times 1$$

be the double mapping cylinder of

$$X'\times Y\overset{\supset}{\leftarrow}X'\times Y'\overset{\subset}{\to}X\times Y'.$$

There is an obvious map

$$Z\overset{p}{\to}X'\times Y\cup X\times Y'\subset X\times Y.$$

It is not hard to check that

$$(X,X')\otimes(Y,Y')=(X\cup CX')\wedge(Y\cup CY')$$

is canonically homeomorphic to the mapping cone of $Z\to X\times Y$. p induces a map from this into the mapping cone of

$$X'\times Y\cup X\times Y'\overset{\subset}{\to}X\times Y,$$

i.e., into $(X,X')\times(Y,Y')$. This map is a homotopy equivalence if p is. But if X' and Y' are open, then $X'\times Y$ and $X\times Y'$ are open in $X\times Y$, and then it is well known that p is a homotopy equivalence.

Note also that in Stab we have

$$S^n\cong\mathbf{R}^n/\{x\mid \|x\|\geqslant 1\}\cong(\mathbf{R}^n,\{x\mid\|x\|\geqslant 1\})\cong(\mathbf{R}^n,\mathbf{R}^n\setminus 0).$$

Now we are able to construct the morphisms ε and η in Stab which will turn out to be the evaluation and coevaluation for the strongly dual pair $(K,\varnothing)$ and $\Sigma^{-n}(\mathbf{R}^n,\mathbf{R}^n\setminus K)$. In the case of

$$\varepsilon\colon\Sigma^{-n}(\mathbf{R}^n,\mathbf{R}^n\setminus K)\otimes(K,\varnothing)\to S^0$$

K may be an arbitrary subset of $\mathbf{R}^n$. It suffices to define $\Sigma^n\varepsilon$. This is done by

$$\begin{array}{ccc} (\mathbf{R}^n,\mathbf{R}^n\setminus K)\otimes(K,\varnothing) & \overset{\Sigma^n\varepsilon}{\dashrightarrow} & \Sigma^n S^0 \\ \downarrow & & \wr\| \\ (\mathbf{R}^n,\mathbf{R}^n\setminus K)\times(K,\varnothing) & & S^n \\ \| & & \wr\| \\ (\mathbf{R}^n\times K,(\mathbf{R}^n\setminus K)\times K) & \overset{\mathrm{diff}}{\to} & (\mathbf{R}^n,\mathbf{R}^n\setminus 0), \end{array}\tag{3.6}$$

$$(x,k)\mapsto x-k.$$

$$\eta\colon S^0 \to (K,\varnothing)\otimes\Sigma^{-n}(\mathbf{R}^n,\mathbf{R}^n\setminus K)$$

we do need the hypotheses that K is compact and that there is a neighborhood V of K in $\mathbf{R}^n$ and a retraction $r\colon V\to K$. We also choose a closed ball B in $\mathbf{R}^n$ (with center 0) such that $K\subset B$. Now we define $\Sigma^n\eta$ by

$$\begin{array}{ccc}
\Sigma^n S^0 & \overset{\Sigma^n\eta}{\dashrightarrow} & (K,\varnothing)\otimes(\mathbf{R}^n,\mathbf{R}^n\setminus K)\\
\| \wr & & \downarrow p\\
(\mathbf{R}^n,\mathbf{R}^n\setminus 0) & & (K,\varnothing)\times(\mathbf{R}^n,\mathbf{R}^n\setminus K)\\
i\uparrow\cup & & \|\\
(\mathbf{R}^n,\mathbf{R}^n\setminus B)\subset(\mathbf{R}^n,\mathbf{R}^n\setminus K) & & (K\times\mathbf{R}^n, K\times(\mathbf{R}^n\setminus K))\\
j\uparrow\cup & & \uparrow r\times\mathrm{id}\\
(V,V\setminus K) & \overset{\mathrm{diag}}{\to} & (V\times\mathbf{R}^n, V\times(\mathbf{R}^n\setminus K)),\\
v & \mapsto & (v,v);
\end{array} \tag{3.7}$$

i is an isomorphism in Stab because it is represneted by a pair of homotopy equivalences. j and p are isomorphisms in Stab by Lemmas 3.4 and 3.5 respectively.

In order to finish the proof it suffices now to verify the two identities in Theorem 1.3(b). For the first one we have to form $(\mathrm{id}_k\otimes\varepsilon)(\eta\otimes\mathrm{id}_k)$, where we abbreviate $(K,\varnothing)$ by K. Suspending n times gives the composition

$$(\mathbf{R}^n,\mathbf{R}^n\setminus 0)\times K\xrightarrow{\Sigma^n\eta\times\mathrm{id}}K\times(\mathbf{R}^n,\mathbf{R}^n\setminus K)\times K$$
$$\xrightarrow{\mathrm{id}\times\Sigma^n\varepsilon}K\times(\mathbf{R}^n,\mathbf{R}^n\setminus 0)\overset{\gamma}{\to}(\mathbf{R}^n,\mathbf{R}^n\setminus 0)\times K,$$

where we have already replaced $\otimes$ by $\times$ using Lemma 3.5. (The interchange map γ is due to the transition from $\Sigma^n(\mathrm{id}_K\otimes\varepsilon)$ to $\mathrm{id}_K\otimes\Sigma^n\varepsilon$.) Inserting $\Sigma^n\eta$ and $\Sigma^n\varepsilon$ according to their definitions (3.7) and (3.6) leads to the composite morphism

$$\begin{array}{ccc}
(\mathbf{R}^n,\mathbf{R}^n\setminus 0)\times K & & (\mathbf{R}^n,\mathbf{R}^n\setminus 0)\times K\\
i\times\mathrm{id}\uparrow\ \|\wr & & \uparrow\gamma\\
(\mathbf{R}^n,\mathbf{R}^n\setminus B)\times K\subset(\mathbf{R}^n,\mathbf{R}^n\setminus K)\times K & & K\times(\mathbf{R}^n,\mathbf{R}^n\setminus 0)\\
j\times\mathrm{id}\uparrow\ \|\wr & & \uparrow r\times\mathrm{id}\\
(V,V\setminus K)\times K & \to & V\times(\mathbf{R}^n,\mathbf{R}^n\setminus 0),\\
(v,k) & \mapsto & (v,v-k).
\end{array}$$

As in the proof of Lemma (6.12) in Dold [4], one shows that replacing $(v,k)\mapsto(v,v-k)$ by $(v,k)\mapsto(k,v-k)$ does not change the morphism in Stab. But after the replacement the composite morphism equals

$$(\mathbf{R}^n,\mathbf{R}^n\setminus 0)\times K\xrightarrow{i\times\mathrm{id}}(\mathbf{R}^n,\mathbf{R}^n\setminus B)\times K\to(\mathbf{R}^n,\mathbf{R}^n\setminus 0)\times K,$$
$$(x,k)\mapsto(x-k,k),$$

which is homotopic to the identity by $(x,k,t)\mapsto(x-tk,k)$. The proof of the second identity in Theorem 1.3(b) is similar.

4.1. Definition. Let $\mathscr{C}$ be a monoidal category, and A a strongly dualizable object of $\mathscr{C}$ with evalution ε and coevaluation η. Let f be an endomorphism of A. Then the *trace* σf of f is defined to be the composition

$$\sigma f\colon I\overset{\eta}{\to}A\otimes DA\underset{\cong}{\overset{\gamma}{\to}}DA\otimes A\xrightarrow{\mathrm{id}\times f}DA\otimes A\overset{\varepsilon}{\to}I. \tag{4.2}$$

4.3. Examples. (a) $\mathscr{C}=\mathrm{Mod}_R$. We have $I=R$; hence $\sigma f\colon R\to R$, and we identify σf with $(\sigma f)(1)$. σf is the usual trace. If A has a finite base, σf is the sum of the diagonal in the matrix corresponding to f.

(b) $\mathscr{C}=\partial\text{-}\mathrm{Mod}_R$ or $\mathrm{Ho}(\partial\text{-}\mathrm{Mod}_R)$. We identify $I=(R,0)$ with R and again σf with $(\sigma f)(1)\in R$. σf is called the *Lefschetz number* of f. If the chain complex A is finitely generated and projective (not only up to homotopy equivalence), then

$$\sigma f=\sum_{q\in\mathbf{Z}}(-1)^q\sigma f_q,$$

where $f_q\colon A_q\to A_q$. The sign comes from the interchange map γ in (4.2).

(c) $\mathscr{C}=\mathrm{Stab}$. We have $\sigma f\in\mathrm{Stab}(S^0,S^0)=\mathbf{Z}$, where the identification is such that the identity map of S^0 corresponds to $1\in\mathbf{Z}$.

4.4. Proposition. *Let $T\colon\mathscr{C}\to\mathscr{C}'$ be a monoidal functor and A an object of $\mathscr{C}$ which is T-flat* (2.3). *Then $\sigma Tf=T\sigma f$ for any endormorphism f of A.*

Proof. Apply T to the line (4.2) and use Corollary 2.4.

4.5. Corollary. *Let f be an endormorphism of A.*

(a) *If $H\colon\mathrm{Ho}(\partial\text{-}\mathrm{Mod}_R)\to\mathrm{Gr}\text{-}\mathrm{Mod}_R$ is homology and A has the homotopy type of a finitely generated projective chain complex such that H_qA is a flat R-module for each $q\in\mathbf{Z}$, then the Lefschetz numbers of f and Hf are equal.*

(b) *If $\tilde S\colon$* Stab $\to\mathrm{Ho}(\partial\text{-}\mathrm{Mod}_R)$ *is the singular chain complex functor defined after* 3.1 *and A is strongly dualizable in* Stab, *then the Lefschetz number $\sigma\tilde Sf\in R$ is the canonical image of $\sigma f\in\mathbf{Z}$.*

(c) *If $\tilde H\colon$* Stab $\to\mathrm{Gr}\text{-}\mathrm{Mod}_R$ *is reduced homology (i.e. $\tilde H=H\circ\tilde S$ with $\tilde S$ as in* (b) *and H as in* (a)), *A is strongly dualizable in* Stab *and $\tilde H_qA$ is a flat R-module for each $q\in\mathbf{Z}$, then the Lefschetz number $\sigma Hf\in R$ is the canonical image of $\sigma f\in\mathbf{Z}$.*

Proof. (a) follows from 4.4 and 2.5. Note that HA is *a fortiori* finitely generated and projective (by 2.4 and 1.6).

(b) follows from 4.4 and 3.2.

(c) follows from (a) and (b). (Compare this proof to the proof of Lemma (2.1) in Becker-Gottlieb [1].)

One can now derive the Lefschetz-Hopf fixed point theorem as a corollary of Theorem 3.1 and Proposition 4.4. Let K be a compact subset of $\mathbf{R}^n$, V a neighborhood of K in $\mathbf{R}^n$ and $r\colon V\to K$ a retraction. Let $f\colon K\to K$ be a continuous map with fixed point set F. Let F^+ be the corresponding map of $K^+=(K,\varnothing)$.

obtains the commutative diagram in Stab

$$\begin{array}{ccccc}
\Sigma^n S^0 & & \xrightarrow{\Sigma^n \sigma f^+} & & \Sigma^n S^0 \\
\wr\Vert & & & & \wr\Vert \\
(\mathbf{R}^n, \mathbf{R}^n \setminus 0) & & & & (\mathbf{R}^n, \mathbf{R}^n \setminus 0) \\
\wr\Vert \uparrow \cup & & & & \uparrow \\
(\mathbf{R}^n, \mathbf{R}^n \setminus B) & \subset & (\mathbf{R}^n, \mathbf{R}^n \setminus K) & & \Big| \; \begin{matrix} v - frv \\ \uparrow \\ v \end{matrix} \\
 & & \wr\Vert \uparrow \cup & & \Big| \; v \\
 & & (V, V \setminus K) & \subset & (V, V \setminus F)
\end{array}$$

Going around the lower part of this diagram and applying $H_n(\,;\mathbf{Z})$, one gets a homomorphism which is multiplication by the fixed point index of f. This is the definition of the index in Dold [1a], p. 202. Hence

$$\operatorname{index} f = H_n(\Sigma^n \sigma f^+, \mathbf{Z}) = \tilde{H}_0(\sigma f^+; \mathbf{Z}) = \tilde{H}(\sigma f^+; \mathbf{Q}) = \sigma \tilde{H} f^+ = \sigma H f.$$

4.6. Corollary (Lefschetz-Hopf). *If f is a map of a compact ENR into itself then the index of f equals the Lefschetz number of Hf: $HK \to HK$.*

We conclude this section by listing some formal properties of the trace.

4.7. Proposition. *Let $\mathscr{C}$ be any monoidal category.*
(a) $\sigma f = f$ for all f: $I \to I$.
(b) $\sigma Df = \sigma f$ if f: $A \to A$ and A is strongly dualizable.
(c) $\sigma(f_1 \otimes f_2) = \sigma f_1 \otimes \sigma f_2 = (\sigma f_1) \circ (\sigma f_2)$, if f_i: $A_i \to A_i$ and A_i is strongly dualizable, $i = 1, 2$.
(d) $\sigma(fg) = \sigma(gf) = \sigma(\gamma \circ (f \otimes g))$ if f: $A \to B$, g: $B \to A$ and A and B are strongly dualizable.

Proof. (a) is trivial. (b) and (c) follow from Proposition 4.4 and the fact that the monoidal functors D: $\check{\mathscr{C}} \to \mathscr{C}$ and $\otimes$: $\mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$ preserve tensor products (cf. the discussion preceding Proposition 2.5). The proof of (d) is left to the reader; using (b) it becomes simple diagram chasing.

§5. Transfer

Let $\mathscr{C}$ be a monoidal category. An object A together with a "diagonal" morphism d: $A \to A \otimes A$ is called a *coalgebra*: c: $A \to I$ is called a *counit* of (A, d) if the diagram

$$\begin{array}{ccccc}
I \otimes A & \xleftarrow{=} & A & \xrightarrow{=} & A \otimes I \\
 & {\scriptstyle c \otimes \mathrm{id}} \nwarrow & \downarrow d & \nearrow {\scriptstyle \mathrm{id} \otimes c} & \\
 & & A \otimes A & &
\end{array}$$

commutes. There is an obvious duality to algebras and units, for which a theory analogous to the following can also be formulated. We are giving preference to the coalgebra case, because we want to apply it in the next section to $\mathscr{C} = \mathrm{Stab}_B$. For the rest of this section let (A, d) always be a coalgebra, A a strongly dualizable object of $\mathscr{C}$ and f an endomorphism of A.

$$(5.2) \qquad \tau f: I \xrightarrow{\eta} A \otimes DA \underset{\cong}{\xrightarrow{\gamma}} DA \otimes A \xrightarrow{Df \otimes d} DA \otimes A \otimes A \xrightarrow{c \otimes \mathrm{id}} I \otimes A = A.$$

5.3. Proposition. *Without changing the composite morphism τf one may replace $DF \otimes d$ in (5.2) by*
(a) $DA \otimes A \xrightarrow{\mathrm{id} \otimes d} DA \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} DA \otimes A \otimes A$,
(b) $DA \otimes A \xrightarrow{\mathrm{id} \otimes f} DA \otimes A \xrightarrow{\mathrm{id} \otimes d} DA \otimes A \otimes A$, *or*
(c) $DA \otimes A \xrightarrow{\mathrm{id} \otimes d} DA \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes f \otimes f} DA \otimes A \otimes A$,
provided that f is a morphism of coalgebras in case (c).

The easy proof is omitted. That f is a morphism of coalgebras means of course that the diagram

$$\begin{array}{ccc}
A & \xrightarrow{d} & A \otimes A \\
f \downarrow & & \downarrow f \otimes f \\
A & \xrightarrow{d} & A \otimes A
\end{array}$$

commutes.

5.4. Proposition. *If f: $A \to A$ is a morphism of coalgebras then the diagram*

$$\begin{array}{ccccc}
 & & I & & \\
 & {\scriptstyle \tau f} \swarrow & & \searrow {\scriptstyle \tau f} & \\
A & & \xrightarrow{f} & & A
\end{array}$$

commutes.

Loosely speaking this means that the image of τf lies in the "fixed point set" of f. The proof follows easily from Proposition 5.3(c).

A morphism x: $A \to I$ may be called a *coelement* of A. *Right translation* by x is defined to be the morphism

$$r_x: A \xrightarrow{d} A \otimes A \xrightarrow{\mathrm{id} \otimes x} A \otimes I = A.$$

5.5. Proposition. $x \circ (\tau f) = \sigma(f r_x) = \sigma(r_x f)$.

The proof is strightforward.

5.6. Corollary. *If (A, d) has a counit c then $c \circ (\tau f) = \sigma f$.*

§6. Parametrized stable homotopy

As in §3 we consider compactly generated (not necessarily Hausdorff) spaces. For a fixed space B we consider "spaces over B", i.e., continuous maps p: $E \to B$. One should think of them mainly as families $(E_b | b \in B)$ of the "fibers" $E_b = p^{-1} b$, parametrized by B. The topology of E has just the purpose to make it meaningful to say that something happening in the family (E_b) depends continuously on b. Guided by this principle one can translate the whole content of §3 to the parametrized case.

whose objects are (up to a formal suspension and desuspensions Σ^n, $n \in \mathbf{Z}$) "well-sectioned" spaces over B, i.e., commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{s} & E \\ & \mathrm{id} \searrow \quad \swarrow p & \\ & B & \end{array}$$

such that s is a cofibration in the category of spaces over B. If $E' \to E$ is a mapping of spaces over B, it makes sense to consider (E, E') as an object of Stab_B by forming the fiberwise mapping cone. By a literal translation of Theorem 3.1 and its proof we get

6.1. THEOREM. *Let B be metric and K a subset of $B \times \mathbf{R}^n$ such that the projection map $p\colon K \to B$ is proper. Let K be an ENR_B, i.e. there exist a neighborhood V of K in $B \times \mathbf{R}^n$ and a retraction $r\colon V \to K$ which is fiber-wise, i.e. $r(V \cap (b \times \mathbf{R}^n)) \subset b \times \mathbf{R}^n$ for each $b \in B$. Then $(K, \varnothing)$ and $\Sigma^{-n}(B \times \mathbf{R}^n, (B \times \mathbf{R}^n)\setminus K)$ are strongly dual in* Stab_B.

Applying our general theory of trace (§4) and transfer (§5) to this dual pair, one recovers a large part of the results of Dold [4].

Similar results can be obtained for all strongly dualizable objects of Stab_B. In Becker-Gottlieb [1] it was proved that a well-sectioned $p\colon E \to B$ is strongly dualizable in Stab_B if

(a) *p is a Hurewicz fibration,*

(b) *E_b has the stable homotopy type of a finite complex for each $b \in B$, and*

(c) *B has the homotopy type of a finite-dimensional CW-complex.*

One can avoid the hypothesis "finite-dimensional" in (c). For this one has first to extend the category Stab_B to a larger category, namely to a category of spectra over B. This can be done by translating the construction of the category Sch in Puppe [14] to the parametrized case obtaining Sch_B. Using a representation theorem of Schön [15], VIII. 23 and [16](4), Mónica Prieto has proved(5) that a well-sectioned p: $E \to B$ is strongly dualizable in Sch_B if (a), (b) and

(c′) *B has the homotopy type of a CW-complex.*

This allows us to strengthen Theorem 1.1 of Becker-Gottlieb [1] as follows

6.2. THEOREM. *Let $p\colon E \to B$ be a space over B (without section) which satisfies* (a), (b) *and* (c′). *Let the diagram*

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & p \searrow \quad \swarrow p & \\ & B & \end{array}$$

be commutative. Choose a based point $b_0 \in B$ and assume that $F = p^{-1}b_0$ is connected. Then the map $\Omega B \to F$ in the fiber sequence of p, considered as an element of $\mathrm{Stab}(\Omega B, F)$ *and multiplied by the Lefschetz number of $f \setminus F\colon F \to F$, gives the zero element of* $\mathrm{Stab}(\Omega B, F)$.

(4) *Russian translator's note.* See also [28].

(5) Cf. the remarks added in proof at the end of the paper.

about the existence of duals in Sch_B can be replaced by

(a′) *There is a numerable covering (U_λ) of B such that p is fiber-homotopy trivial over each U_λ.*

§7. Equivariant stable homotopy

Let G be a compact topological group. In analogy to Stab (§3) one can construct a monoidal category Stab^G whose objects are pairs (X, α), where X is a G-well-pointed compactly generated G-space and α is an element of the real representation ring of G. The set of morphisms from (X, α) to (Y, β) is

$$\mathrm{Stab}^G((X, \alpha), (Y, \beta)) = \operatorname*{colim}_W [S^{\alpha \oplus W} \wedge X, S^{\beta \oplus W} \wedge Y]^G,$$

where W runs through a cofinal set of orthogonal representations of W, directed by inclusion. If W is large enough, $\alpha \oplus W$ can be considered as a representation and $S^\alpha \mathbf{U}^W$ denotes its one-point-compactification. (Compare G. Segal [17], Kosniowski [8], Hauschild [6] and Waner [21].)

As in §3 a pair of G-spaces defines an object of Stab^G, and we have a system of suspension functors Σ^α which are automorphisms of Stab^G.

7.1. THEOREM. *Let $W \cong \mathbf{R}^n$ be an orthogonal representation of the compact group G, and let K be a compact G-invariant subset of W which is also an equivariant neighborhood retract (G-ENR). Then $(K, \varnothing)$ and $\Sigma^{-W}(W, W \setminus K)$ are strongly dual in* Stab^G.

The proof is completely analogous to the proof of Theorem 3.1.

Strong duality in Stab^G coincides with equivariant S-duality in the sense of Wirthmüller [22]. If G is a compact Lie group, Jaworowski [7] gives a characterization of G-ENR-spaces which implies in particular that a finite G-equivariant CW-complex is a G-ENR.

It is clear that our methods of §3 also apply to the case of parametrized equivariant stable homotopy, i.e. G-spaces over B. Thus we get a common generalization of Theorems 6.1 and 7.1 to $G\text{-ENR}_B$-space. In particular we obtain a transfer in this case, which includes the case of equivariant bundles studied by Nishida [12].

Along the lines of Becker-Gottlieb [1], duals and transfer maps for G-fibrations have been constructed by Waner [12]. This also fits into our general framework. It is very likely that one can show the existence of strong duals for G-fibrations in the same way as indicated in the second part of §6 for trivial G.

§8. Stable shape theory

Let Shape be the category of pointed shapes of compact spaces (Dydak-Segal [5]). The ordinary smash product induces a bifunctor of Shape into itself. (This is not at all clear if we do not restrict to compact spaces.) Just as Stab has been constructed in §3 from the homotopy category of well-pointed spaces, one constructs from Shape a stable shape category Stab-Shape. The smash product induces a monoidal structure.

8.1. THEOREM. *Let K be a compact subset of $\mathbf{R}^n$ such that the Čech cohomology groups of K (with integer coefficients) are finitely generated. Then $(K, \varnothing)$ and $\Sigma^{-n}(\mathbf{R}^n, \mathbf{R}^n \setminus K)$ are strongly dual in* Stab-Shape.

First we have to explain what $(\mathbf{R}^n, \mathbf{R}^n \setminus K)$ means in this context. The mapping cone C of the inclusion $\mathbf{R}^n \setminus K \subset \mathbf{R}^n$ is not compact. But C has the homotopy type of a CW-complex, and by Alexander duality its integral homology groups are finitely generated. This implies that it has the stable homotopy type of a finite CW-complex. Hence it defines an object of Stab-Shape up to equivalence.

The proof of 8.1 is again analogous to the proof of 3.1. The only difference is that we do not have the retraction $r\colon V \to K$ in (3.7). But we do get a compatible system of homotopy classes

$$\Sigma^n S^0 \to (V, \varnothing) \otimes (\mathbf{R}^n, \mathbf{R}^n \setminus K)$$

where V runs through all neighborhoods of K in $\mathbf{R}^n$ and this is enough to define the shape morphism

$$\Sigma^n \eta\colon \Sigma^n S^0 \to (K, \varnothing) \otimes (\mathbf{R}^n, \mathbf{R}^n \setminus K).$$

Theorem 8.1 can be applied to the theory of fixed points, but we shall not do it here.

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Added in proof (July 1980): In the meantime the results have been extended and some of the proofs which are only indicated here can be found in the literature.

Theorem 3.1 has been generalized to the relative case by Henn [23], [24]. A proof is also contained in Hommel [25] where all details are given. In addition, all the other results of our §§1 and 3 are proved there.

The results of Mônica Prieto quoted in §6 are contained in [26] and [27].

In a slightly different way as indicated above a relative version of Theorem 8.1 was proved by Henn [23], [24].

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Translated by A. DOLD