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CLASSIFICATION OF ORIENTED SPHERE BUNDLES OVER A 4-COMPLEX

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1. Introduction

The classification of $(n-1)$-sphere bundles with structure group $SO(n)$ (special orthogonal group) over a complex K of dimension at most 4 has been carried out in several special cases. If $n = 2$ or if the dimension of K does not exceed 3 then the characteristic class W_2 is a complete invariant (see [17]; note that W_2 is an integer class for $n = 2$, and is a class mod 2 if $n \geqq 3$). 2-sphere bundles with vanishing class W_2 were classified by the second author in an unpublished manuscript* (1938; announced in [19], 4); these bundles are not determined by their characteristic classes (see our example in Section 3). Pontrjagin in 1945 (see [9]) gave a solution for arbitrary n provided that $H^4(K; Z)$ has no 2-torsion; in this case $(n-1)$-sphere bundles are characterized by W_2, W_4 (for $n \geqq 4$) and P_4 (see [9] or the Corollary in Section 3). In this paper we give the classification for the general case. Throughout the paper we assume $n \geqq 3$.

The $(n-1)$-sphere bundles over a complex K are generated by mappings of K into some Grassmann manifold G_n (see [17]); in fact, they are in natural one-to-one correspondence with the homotopy classes (see [14], [8], and [15], 19). The manifold G_n has vanishing 1^{st} and 3^{rd} homotopy groups (see (3)); since we assume $\dim(K) \leqq 4$, our problem reduces to the standard problem of classifying mappings of a complex into a space (G_n) with only two non-vanishing homotopy groups. This problem has been studied in many papers, beginning with special cases in [7], [11], and in general in [13], [4]. In particular there are explicit solutions if the homotopy groups of the image space are in dimensions m and $m+2$ (see [13]), but unfortunately not for $m = 2$, the case in which we are interested. A partial solution in this case was given in [2].

* This classification involves the cohomology operation Ψ: $H^1(K; Z_2) \to H^4(K; Z)$ which is obtained by applying the Bockstein homomorphism first and then taking the cup square; see (23). In the manuscript Ψ was defined by a cochain formula (in a simplicial complex with ordered vertices) as follows. For any cochain y mod 2, let ωy be the corresponding integral cochain (compare [18, 11]) whose coefficients are 0 or 1. Then $\omega(y \smile y') = \omega y \smile \omega y'$, and for 1-cocycles c mod 2, we see that $\frac{1}{2}\delta\omega c = \omega' c \smile c)$. Therefore the function $\phi(c) = \omega(c \smile c \smile c \smile c)$ coincides with $\frac{1}{2}\delta\omega c \smile \frac{1}{2}\delta\omega c$ and induces the operation Ψ. Unfortunately the operation ω was omitted in [19], giving a wrong statement, as noted by Pontrjagin [9].

The general result in [4, Theorem 14.2], leaves one with the task of expressing the k-invariant $k(G_n)$ of G_n by familiar cohomology operations. To be more precise, it is only the suspension of $k(G_n)$ and the deviation of $k(G_n)$ from additivity which is needed in [4, Theorem 14.2]. In the following Section 2, we make these computations and formulate the corresponding classification theorem. In Section 3 we relate the resulting invariants to the characteristic classes of the bundles involved. As an application we give a short proof of a recent theorem of Massey (see [6, Theorem V]). The last Section contains a correction of Pontrjagin's Theorem 2 in [9].

2. The classification theorem ($n \geqq 3$)

Let G_n be the Grassmann manifold of oriented n-planes through the origin in a euclidean space of dimension at least $n+5$. Then G_n is the base of a 5-universal $SO(n)$-bundle (see [15, 19.6–19.7]), and we have natural isomorphisms (see [15, 19.9])

$$\pi_i(G_n) \approx \pi_{i-1}(SO(n)) , \qquad i \leqq 4 . \tag{1}$$

According to [15, 24.11],

$$\begin{cases} \pi_0(SO(n)) \approx \pi_2(SO(n)) = 0 , \\ \pi_1(SO(n)) \approx Z_2 = \text{cyclic of order 2} , \\ \pi_3(SO(n)) \approx \begin{cases} Z = \text{free cyclic for } n \neq 4 , \\ Z + Z \text{ for } n = 4 ; \end{cases} \end{cases} \tag{2}$$

hence

$$\begin{cases} \pi_1(G_n) \approx \pi_3(G_n) = 0 , \\ \pi_2(G_n) \approx Z_2 , \\ \pi_4(G_n) \approx \begin{cases} Z \text{ for } n \neq 4 , \\ Z + Z \text{ for } n = 4 . \end{cases} \end{cases} \tag{3}$$

We choose generators for $\pi_3(SO(n))$ as in [15, 22.3, 22.7 and 23.6] and thereby identify this group with Z (for $n \neq 4$) or $Z+Z$ (for $n = 4$); similarly for the groups $\pi_4(G_n)$, using the isomorphisms (1).

The k-invariant of G_n is an element $k(G_n) \in H^5(Z_2, 2; \pi_4(G_n))$ (see [4, 11]). We want to compute the *suspension* and the *deviation from additivity* of $k(G_n)$.

The suspension of $k(G_n)$ is an element of $H^4(Z_2, 1; \pi_4(G_n)) = H^4(Z_2, 1; \pi_3(SO(n)))$. Up to sign it coincides with $k(SO(n))$, the k-invariant of $SO(n)$ (see [16]).

Consider the case $n = 3$ first. Now $SO(3)$ is homeomorphic with real projective 3-space P_3. A space $K(Z_2, 1)$ is given by real projective space P_∞ of infinite dimension. The k-invariant of P_3 is the first obstruction for retracting P_∞ onto P_3, i.e., the obstruction for retracting P_4 onto P_3. Since such a retraction is not possible the invariant $k(SO(3)) = k(P_3)$ is not zero; hence it is the only non-vanishing element in $H^4(Z_2, 1; Z) = H^4(P_\infty; Z)$. If v is the non-trivial element of $H^1(Z_2, 1; Z_2)$, then since $\beta v \smile \beta v \neq 0$, we have

$$k(SO(3)) = \beta v \smile \beta v \; ; \tag{4}$$

β is the Bockstein homomorphism associated with the coefficient sequence

$$0 \longrightarrow Z \xrightarrow{\;2\;} Z \longrightarrow Z_2 \longrightarrow 0 \, . \tag{5}$$

Since v is the basic cohomology class of $P_\infty = K(Z_2, 1)$, the formula (4) also describes the cohomology operation defined by $k(SO(3))$ (see [4, 7] or [12, 4]): If X is any space and $x \in H^1(X, Z_2)$, then

$$k(SO(3)) \vdash x = \beta x \smile \beta x \, . \tag{6}$$

For $n = 4$ the situation is similar since $SO(4)$ is homeomorphic with $SO(3) \times S^3 = P_3 \times S^3$ (see [15, 22.6]). We have

$$\begin{aligned} k(SO(4)) \in H^4(Z_2, 1; \pi_3(SO(4))) &= H^4(Z_2, 1; Z + Z) \\ &= H^4(Z_2, 1; Z) + H^4(Z_2, 1; Z) \, . \end{aligned}$$

If we write $k(SO(4)) = k_1(SO(4)) + k_2(SO(4))$ according to this decomposition, then

$$k_1(SO(4)) \vdash x = \beta x \smile \beta x \, , \qquad k_2(SO(4)) \vdash x = 0 \, . \tag{7}$$

For $n \geqq 5$ the invariant $k(SO(n)) \in H^4(Z_2, 1; Z)$ is zero since there exists a mapping $r : P_{n-1} \rightarrow SO(n)$ with $r_* : \pi_1(P_{n-1}) \approx \pi_1(SO(n))$ (see [15, 23.3–23.4]);

$$k(SO(n)) = 0 \qquad \text{for } n \geqq 5 \, . \tag{8}$$

We have now to compute the deviation of $k(G_n)$ from additivity. We claim that

$$k(G_3) \vdash (y, z) = \beta(y \smile z) \, , \tag{9}$$

$$k_1(G_4) \vdash (y, z) = \beta(y \smile z) \, , \qquad k_2(G_4) \vdash (y, z) = 0 \, , \tag{10}$$

$$k(G_n) \vdash (y, z) = 0 \qquad \text{for } n \geqq 5 \, . \tag{11}$$

Here y and z denote 2-dimensional cohomology classes mod 2 of an arbitrary space Y, and $k_1(G_4)$, $k_2(G_4)$ are the components of $k(G_4)$ with respect to the decomposition

$$H^5(Z_2, 2; \pi_4(G_4)) = H^5(Z_2, 2; Z + Z) = H^5(Z_2, 2; Z) + H^5(Z_2, 2; Z) \, .$$

The formulas (9)–(11) follow from our computation (6)–(8) of the suspension of $k(G_n)$, together with the

LEMMA. *Take* $a \in H^5(Z_2, 2; Z)$ *and let* $\sigma a \in H^4(Z_2, 1; Z)$ *be the suspension of* a. *Then*

$$a \vdash (y, z) = \beta(y \smile z) \text{ or } a \vdash (y, z) = 0 \qquad (\textit{for all } y,\ z \textit{ as above})$$

according to whether

$$\sigma a \vdash x = \beta x \smile \beta x \text{ or } \sigma a \vdash x = 0 \qquad (\textit{for all } x \textit{ as in } (6)).$$

We remark first that $\beta(y \smile z)$ is a non-trivial cohomology operation, even when reduced mod 2: If we denote reduction mod 2 by a horizontal bar, then

(12) $$\overline{\beta(y \smile z)} = Sq^1(y \smile z) = Sq^1 y \smile z + y \smile Sq^1 z\ ,$$

and this is different from zero for instance in $S^2 \times X$, where S^2 is a 2-sphere and X is a space consisting of a 2-sphere with a 3-cell attached by a mapping of its boundary of degree 2 (i.e., X is the suspension of the real projective plane).

To prove the lemma we look at the universal example

$$Y = K(Z_2, 2) \times K(Z_2, 2)$$

(see [4]). Let u be the basic class of $H^2(Z_2, 2; Z_2)$. Then in Y we have, by definition of $a \vdash (y, z)$,

(13) $$a \vdash (u \otimes 1 + 1 \otimes u) = a \otimes 1 + 1 \otimes a + a \vdash (u \otimes 1, 1 \otimes u)\ .$$

If we compute $H^5(Y; Z)$ by applying the Künneth formula, using $H^2(Z_2, 2; Z) = 0$, $H^3(Z_2, 2; Z) = Z_2$ (see [3, Ch. IV]), we find

(14) $$H^5(Y; Z) = H^5(Z_2, 2; Z) \otimes H^0(Z_2, 2; Z) + H^0(Z_2, 2; Z) \otimes H^5(Z_2, 2; Z) + Z_2\ ;$$

the cross term Z_2 has only one non-zero element. This element must be

(15) $$\beta(u \otimes 1 \smile 1 \otimes u) = \beta(u \otimes u)\ ,$$

since (15) is a cross term element of $H^5(Y; Z)$ which is different from zero, as we have seen. (Cross term elements can be characterized as lying in the intersection of the kernels of the two homomorphisms $H^*(Y) \to H^*(Z_2, 2)$ which are induced by the two natural inclusions $K(Z_2, 2) \to Y$.) Since $a \vdash (u \otimes 1, 1 \otimes u)$ is also a cross term element, we have

(16) $$a \vdash (u \otimes 1, 1 \otimes u) = \lambda\beta(u \otimes u) \qquad \text{with } \lambda = 0 \text{ or } 1\ .$$

Reducing this equation modulo 2 gives

(17) $$\bar{a} \vdash (u \otimes 1, 1 \otimes u) = \lambda Sq^1(u \otimes u)\ .$$

The element $\bar{a}$ lies in $H^5(Z_2, 2; Z_2)$. This module (over Z_2) has a base consisting of Sq^2Sq^1u and $u \smile Sq^1u$ (see [12, 9]). Therefore

(18) $$\bar{a} = \mu Sq^2Sq^1u + \nu(u \smile Sq^1u) \qquad \text{with } \mu, \nu = 0 \text{ or } 1 .$$

If we apply Sq^1 to (18) the left side gives zero because $\bar{a}$ comes from an integer class. Sq^1 applied to either of Sq^2Sq^1u or $u \smile Sq^1u$ gives $Sq^3Sq^1u = Sq^1u \smile Sq^1u \neq 0$. Therefore the coefficients μ and ν must be equal:

(19) $$\bar{a} = \mu(Sq^2Sq^1u + u \smile Sq^1u) .$$

If we compute $\bar{a} \vdash (u \otimes 1, 1 \otimes u)$ from (19) we find $\mu Sq^1(u \otimes u)$, hence (by (17)) $\mu = \lambda$, and

(20) $$\bar{a} = \lambda(Sq^2Sq^1u + u \smile Sq^1u) .$$

Now apply the suspension homomorphism σ to (20). It commutes with Steenrod squares and kills cup products; therefore

(21) $$\sigma\bar{a} = \lambda Sq^2Sq^1v = \lambda(Sq^1v \smile Sq^1v) ,$$

where v is the basic class in $H^1(Z_2, 1; Z_2)$. Comparing (16) and (21) now proves the lemma; we have only to remark that $\sigma\bar{a} = \overline{\sigma a}$ and $\overline{\beta x \smile \beta x} = Sq^1x \smile Sq^1x$.

REMARK. With some more effort the invariant $k(G_n)$ itself can be computed. It turns out that

(22) $$\begin{aligned} k(G_3) &= \pm\beta_4\mathfrak{p}(u) , \\ k_1(G_4) = \pm\beta_4\mathfrak{p}(u) , &\quad k_2(G_4) = 0 , \\ k(G_n) &= 2\beta_4\mathfrak{p}(u) \qquad \text{for } n \geqq 5 , \end{aligned}$$

where $u \in H^2(Z_2, 2; Z_2)$ is the basic class, $\mathfrak{p} : H^2(Z_2, 2; Z_2) \to H^4(Z_2, 2; Z_4)$ is the Pontrjagin square, and β_4 is the Bockstein homomorphism associated with the coefficient sequence $0 \longrightarrow Z \xrightarrow{4} Z \longrightarrow Z_4 \longrightarrow 0$. The formula for $k(G_4)$ was given to the authors by F. Peterson.

Inserting our computations into [4, 14.2] gives the

THEOREM 1 (CLASSIFICATION THEOREM). *Let* $\mathfrak{B}_1, \mathfrak{B}_2$ *be principal* $SO(n)$-*bundles,* $n \geqq 3$ (*or equivalently* $(n-1)$-*sphere bundles with structure group* $SO(n)$; see [15, 8.2]) *over a complex* K *whose dimension is at most* 4, *and let* $h_i : K \to G_n$ *be a classifying mapping for* $\mathfrak{B}_i$ (see [15, 19]). *Assume* $W_2(\mathfrak{B}_1) = W_2(\mathfrak{B}_2) = w_2$. *This implies that the parts of the* $\mathfrak{B}_i$ *over the* 3-*skeleton* K^3 *of* K *are equivalent* (see [17]); *we can therefore assume* (see [15, 19]) *that* h_1 *and* h_2 *agree on* K^3. *Then the difference cocycle of* (h_1, h_2) *is defined; its cohomology class* $d(h_1, h_2)$ *is an element of* $H^4(K; \pi_4(G_n))$.

The bundles $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are equivalent if and only if*:

(a) *for* $n = 3$ *there exists a cohomology class* $x \in H^1(K; Z_2)$ *such that*

$$d(h_1, h_2) = \beta x \smile \beta x + \beta(x \smile w_2) \tag{23}$$

where β *is the Bockstein homomorphism associated with the coefficient sequence* (5), *and* $\pi_4(G_3)$ *has been identified with* Z *by* (3);

(b) *for* $n = 4$ *there is an* $x \in H^1(K; Z_2)$ *such that*

$$d_1(h_1, h_2) = \beta x \smile \beta x + \beta(x \smile w_2)\ , \qquad d_2(h_1, h_2) = 0\ , \tag{24}$$

where $d(h_1, h_2) = d_1(h_1, h_2) + d_2(h_1, h_2)$ *is the decomposition corresponding to* $\pi_4(G_4) = \pi_3(SO(4)) = Z + Z$, *using the chosen generators*;

(c) *for* $n \geqq 5$ *we have*

$$d(h_1, h_2) = 0\ . \tag{25}$$

Since for given $w_2 \in H^2(K; Z_2)$ and $d \in H^4(K; \pi_4(G_n))$ one can always find bundles $\mathfrak{B}_1$, $\mathfrak{B}_2$ and classifying mappings h_1, h_2 with $W_2(\mathfrak{B}_i) = w_2$ and $d(h_1, h_2) = d$, this theorem gives a complete classification.

REMARK. Since $\pi_5(G_n) = \pi_4(SO(n)) = 0$ for $n \geqq 6$, $\pi_6(G_n) = 0$ for $n \geqq 7$ (see [15, 24.11]), and $\pi_7(G_n) = 0$ for $n \geqq 6$ (see [1, Proposition 19.3]), we see that the classification theorem extends to 5-complexes K for $n \geqq 6$, and to 7-complexes for $n \geqq 7$ (for G_n we have now to take oriented n-planes in a euclidean space of dimension at least $n+8$).

3. Characteristic classes

The following theorem is essentially contained in Pontrjagin [9].

THEOREM 2. *Let* $\mathfrak{B}_1$, $\mathfrak{B}_2$ *be bundles with classifying mappings* h_1, h_2 *as in Theorem* 1. *Let* $P(\mathfrak{B}_i) \in H^4(K; Z)$ *denote the Pontrjagin class of* $\mathfrak{B}_i$, *and* (*for* $n = 4$) $W_4(\mathfrak{B}_i) \in H^4(K; Z)$ *respectively* (*for* $n > 4$) $W_4(\mathfrak{B}_i) \in H^4(K; Z_2)$, *its* 4th *Stiefel-Whitney class.*

(a) *If* $n = 3$ *then*

$$P(\mathfrak{B}_1) - P(\mathfrak{B}_2) = -4d(h_1, h_2).$$

(b) *If* $n = 4$ *then*

$$P(\mathfrak{B}_1) - P(\mathfrak{B}_2) = -4d_1(h_1, h_2) - 2d_2(h_1, h_2),$$
$$W_4(\mathfrak{B}_1) - W_4(\mathfrak{B}_2) = d_2(h_1, h_2).$$

(c) *If* $n \geqq 5$ *then*

$$P(\mathfrak{B}_1) - P(\mathfrak{B}_2) = -2d(h_1, h_2),$$
$$W_4(\mathfrak{B}_1) - W_4(\mathfrak{B}_2) \equiv d(h_1, h_2) \bmod 2.$$

PROOF. Let $\rho_n \in \pi_4(G_n)$ for $n \neq 4$ respectively ρ_4^1, $\rho_4^2 \in \pi_4(G_4)$ for $n = 4$

be the generators as in Section 2 (see (3)). We show that

(26) $\langle P, \rho_3\rangle = -4$; $\langle P, \rho_4^1\rangle = -4$, $\langle P, \rho_4^2\rangle = -2$; $\langle P, \rho_n\rangle = -2$ for $n \geqq 5$;

(27) $\langle W_4, \rho_4^1\rangle = 0$, $\langle W_4, \rho_4^2\rangle = 1$; $\langle W_4, \rho_n\rangle \equiv 1 \bmod 2$ for $n \geqq 5$,

where P, W_4 are the characteristic classes of the universal $SO(n)$-bundle over G_n, and $\langle\ \ \rangle$ denotes the evaluation of a cohomology class on a homotopy class (considered as a homology class).

Consider W_4 first. The generators ρ_4^1, ρ_4^2 correspond to the elements α_3, $\beta_3 \in \pi_3(SO(4))$ (see [15, 22.7]) under the isomorphisms (1); therefore the equations (27) follow from the definition of W_4 as an obstruction (see [15, 38.2]).

The class P is invariant under the inclusion $SO(n) \subset SO(n+1)$, i.e., the class $P(\mathfrak{B})$ of an $SO(n)$-bundle $\mathfrak{B}$ remains the same if we consider $\mathfrak{B}$ as an $SO(n+1)$-bundle (see [5, § 4]). Since the element $\alpha_3 - 2\beta_3$ goes into zero under the inclusion $SO(4) \rightarrow SO(5)$ (see [15, 23.5 – 23.6]), this shows that $\langle P, \rho_4^1 - 2\rho_4^2\rangle = 0$, or $\langle P, \rho_4^1\rangle = 2\langle P, \rho_4^2\rangle$. Next we show that $\langle P, \rho_4^2\rangle = -2$. Then the remaining equations (26) follow since ρ_3 goes into ρ_4^1 and ρ_4^2 goes into ρ_n $(n \geqq 5)$ under the appropriate inclusions $SO(m) \rightarrow SO(m+k)$ (see [15, 22.7 and 23.6]).

To prove $\langle P, \rho_4^2\rangle = -2$ we consider the inclusion $U(2) \rightarrow SO(4)$, where $U(2)$ is the unitary group in 2 variables. Thereby every $U(2)$-bundle defines an $SO(4)$-bundle; if $\mathfrak{B}$ is an $SO(4)$-bundle over the 4-sphere S^4 which is obtained in this way then $P(\mathfrak{B}) = -2W_4(\mathfrak{B})$ (see [20, Theorem 9]). In particular this holds for the bundle which is defined by ρ_4^2 (see [15, 25.1]), hence $\langle P, \rho_4^2\rangle = -2\langle W_4, \rho_4^2\rangle = -2$.

Using (26) we now prove the case (a) of the theorem; (b) and (c) follow similarly. Assume that h_1, h_2 are cellular mappings (i.e., map the i-skeleton of K into the i-skeleton of G_3 for some cellular decomposition of K and G_3), which agree on the 3-skeleton of K. Let $\Delta(\mathrm{h}_1, h_2)$ be their difference cocycle (its cohomology class is $d(h_1, h_2)$), and let p be a cellular cocycle in the cohomology class P. We show that

(28) $$h_1^*(p) - h_2^*(p) = -4\Delta(h_1, h_2)\ ,$$

where h_i^* is the homomorphism on cellular cochains which is induced by h_i. This equation implies (a) since $h_i^*(P) = P(\mathfrak{B}_i)$.

It is sufficient to verify (28) on every 4-cell of K; we can therefore assume that K consists of a single (closed) 4-cell e, and h_1, h_2 are mappings of e into the 4-skeleton $G_3^{(4)}$ of G_3 which agree on the boundary ∂e of e. Then h_1, h_2 determine an element $\alpha \in \pi_4(G_3^{(4)})$, the "difference" of h_1 and

h_2, which under the injection $G_3^{(4)} \to G_3$ goes into $\Delta(h_1, h_2)\cdot e$.

Let $e \vee S^4$ be the wedge (= union with a single common point) of e with a 4-sphere. Let $f : e \to e \vee S^4$ be a mapping which is the identity on ∂e, and which covers both e and S^4 with degree one. Finally let $h : e \vee S^4 \to G_3^{(4)}$ be a mapping whose restriction to e is h_2 and whose restriction to S^4 represents the class α. Then the composite mapping

$$e \xrightarrow{f} e \vee S^4 \xrightarrow{h} G_3^{(4)}$$

is homotopic rel ∂e to h_1 (i.e., there is a homotopy which does not move the image of ∂e); hence

(29) $$f^*h^*(p) = (hf)^*(p) = h_1^*(p) .$$

But $h^*(p)$ restricted to e is $h_2^*(p)$, and the value of $h^*(p)$ on S^4 is $-4\Delta(h_1, h_2)\cdot e$, by (26). Since f covers both e and S^4 with degree one this shows that $f^*h^*(p) = h_2^*(p) - 4\Delta(h_1, h_2)$, as asserted in (28).

COROLLARY (Pontrjagin [9]). *If $H^4(K; Z)$ has no 2-torsion, then oriented $(n-1)$-sphere bundles $\mathfrak{B}$ over a 4-complex K are determined by their characteristic classes $W_2(\mathfrak{B})$, $W_4(\mathfrak{B})$ (if $n \geqq 4$) and $P(\mathfrak{B})$.*

Indeed, if $\mathfrak{B}_1$, $\mathfrak{B}_2$ have the same characteristic classes then the left sides of the equations in Theorem 2 are zero, and hence $4d(h_1, h_2) = 0$, where h_1, h_2 are classifying mappings. If there is no 2-torsion in $H^4(K; Z)$ this implies that $d(h_1, h_2) = 0$, and hence $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent.

APPLICATION. We re-prove the following theorem ([6, Theorem V]) of Massey.

THEOREM 3 (Massey). *Let $\mathfrak{B}$ be a 2-sphere bundle with structure group $SO(3)$ over a 4-complex K, and assume $H^4(K; Z)$ has no 2-torsion. Then $\mathfrak{B}$ admits a cross-section if and only if there exists an integer class $\gamma \in H^2(K; Z)$ such that*

(a) $\gamma \equiv W_2(\mathfrak{B}) \bmod 2$,

(b) $\gamma \smile \gamma = P(\mathfrak{B})$.

PROOF. $\mathfrak{B}$ admits a cross-section if and only if its structure group can be reduced to $SO(2)$ (see [15, 9.5]); For an $SO(2)$-bundle $\mathfrak{B}$ the class $W_2(\mathfrak{B})$ is an integer class whose cup-square $W_2(\mathfrak{B}) \smile W_2(\mathfrak{B})$ is the Pontrjagin class $P(\mathfrak{B})$ (see [20, Theorem 9] or [5, Satz 4.5.1 with $q = 1$]). This proves the "only if" part of the theorem. Assume now that a class γ with the properties (a) and (b) exists, and let $\mathfrak{B}'$ be a $SO(2)$-bundle over K with $W_2(\mathfrak{B}') = \gamma$. Then, as $SO(3)$-bundles, $\mathfrak{B}$ and $\mathfrak{B}'$ have the same characteristic classes, and hence are equivalent by the Corollary above.

EXAMPLE of a non-trivial $SO(3)$-bundle over a 4-complex with vanishing characteristic classes: Let K be a complex consisting of a 3-sphere S^3 with a 4-cell e^4 attached by a mapping $\varphi : \partial e^4 \to S^3$ of its boundary of degree 2. Then $H^i(K; Z) = 0$ for $i \neq 0, 4$, and $H^4(K; Z) = Z_2$. Let $h : K \to G_3$ be a mapping which sends S^3 into a point and is such that $d(h, c)$, the difference cohomology class with a constant mapping $c : K \to G_3$, is the non-zero element of $H^4(K; Z)$. The $SO(3)$-bundle $\mathfrak{B}$ over K which is induced by h is non-trivial, by Theorem 1, but its characteristic classes are zero (see Theorem 2). A similar example can be constructed with the product $P_2 \times P_2$ of two real projective planes as base.

4. Remark on Pontrjagin's Theorem 2 in [9]

This theorem can be formulated as follows. Let $\mathfrak{B} = \{E, p, B, \Gamma, \Gamma\}$ be a principal bundle, with B a complex, Γ a connected topological group. Assume the part of $\mathfrak{B}$ over the m-skeleton B^m of B is trivial, i.e., there exists a cross-section f over B^m (see [15, 9]). The obstruction for extending f over the $(m+1)$-skeleton B^{m+1} is an $(m+1)$-cocycle $z(f)$ of B with coefficients in $\pi_m(\Gamma)$. Let Σ_m be the image of the Hurewicz homomorphism $\Phi : \pi_m(\Gamma) \to H_m(\Gamma)$; we can factor Φ as follows:

$$\Phi : \pi_m(\Gamma) \xrightarrow{\varphi} \Sigma_m \subset H_m(\Gamma) .$$

Now $\varphi \circ z(f)$ is a cocycle with coefficients in Σ_m, and the theorem states that the cohomology class w_f^{m+1} of $\varphi \circ z(f)$ is an invariant of $\mathfrak{B}$, i.e., does not depend on f. We show by a counter-example that this is not correct. However, following the indication for a proof as given by Pontrjagin one finds that *the cohomology class of* $\Phi \circ z(f)$ *is an invariant* (see below).

For an example, let Γ be the group $SO(3)$, take $m = 3$, and let B be a 4-complex. Under the Hurewicz homomorphism the group $\pi_3(SO(3))$ maps isomorphically onto the subgroup $\Sigma_3 \subset H_3(\Gamma)$ of spherical homology classes. Therefore we may consider w_f^4 as the cohomology class of $z(f)$ which in turn can be identified with the class $d(h, c)$ of Theorem 1, where $h : B \to G_3$ is a classifying mapping for the given bundle $\mathfrak{B}$, and $c : B \to G_3$ is a constant mapping. But if B is real projective 4-space P_4, and if $d(h, c)$ is the non-zero element of $H^4(P_4; Z)$, then $\mathfrak{B}$ is trivial by Theorem 1 (a); hence $d(h, c)$ is not an invariant.

We now show that the cohomology class of $\Phi \circ z(f)$ (in the notation above) does not depend on the choice of the cross-section f. Let $g : B^m \to E$ be a second cross-section. Then for every $b \in B^m$ there is a unique $\gamma(b) \in \Gamma$ which (under right translation) transforms $g(b)$ into $f(b)$: $g(b)\gamma(b) = f(b)$. This defines a continuous mapping $\gamma : B^m \to \Gamma$; let $z(\gamma)$ be its obstruction cocycle. We claim that

(30) $$z(g) + z(\gamma) = z(f) .$$

It is sufficient to verify (30) on every $(m + 1)$-cell σ of B. But over σ the bundle $\mathfrak{B}$ is trivial and (30) expresses the fact that the addition in $\pi_m(\Gamma)$ is induced by the multiplication in Γ (see [15, 16.7]).

Next we show that $\Phi \circ z(\gamma)$ is a coboundary; then the invariance of the class of $\Phi \circ z(f)$ follows if we apply Φ to (30). The group $C_m(B)$ of cellular m-chains of B can be written as a direct sum $C_m(B) = Z_m(B) + D_m$, where $Z_m(B)$ denotes the cycles and D_m is a complementary summand. Now define an m-cochain y of B with coefficients in $H_m(\Gamma)$ as follows: y maps each cycle $\mathfrak{Z}_m \in Z_m$ into the homology class of $\gamma(\mathfrak{Z}_m)$, and y maps D_m into zero. Then for every $(m + 1)$-cell σ, $(\delta y)\cdot\sigma = y\cdot(\partial\sigma)$ is the homology class of Γ which is represented by $\gamma|\partial\sigma$, and this is nothing but $(\Phi \circ z(\gamma))\cdot\sigma$; hence $\delta y = \Phi \circ z(\gamma)$, q.e.d. (This is virtually the same proof as in [10, proof of Satz 3, § 3]).

In our example ($\Gamma = SO(3)$, $m = 3$) the inclusion $Z = \Sigma_3 \to H_3(SO(3)) = Z$ is multiplication by 2; therefore $2d(h, c)$ (notation from above) is an invariant of $\mathfrak{B}$. This can also be deduced from Theorem 1.

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