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HOMOLOGY GROUPS OF RELATIONS

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Any relation between the elements of a set X and the elements of a set Y is associated with two simplicial complexes K and L. A simplex of K is a finite set of elements of X related to a common element of Y; a simplex of L is a finite set of elements of Y related to a common element of X. In particular, the relation of being an element of a set of a covering is a relation between the points of a space and the sets of the covering the space. One of the complexes associated with this relation is the nerve of the covering; its homology and cohomology groups are used in defining the Čech homology and cohomology groups of the space. The other associated complex, which has points of the space as its vertices, is used in defining the Vietoris homology groups and the Alexander cohomology groups.

The two complexes associated with a relation will be shown (Theorem 1) to have isomorphic homology and cohomology groups; if the complexes are geometrically realized, they even have the same homotopy type. In particular, the nerve and the Vietoris complex of any covering have isomorphic homology and cohomology groups. It follows that, when the Čech [5] and Vietoris [12] homology groups are based on the same family of coverings, these groups are isomorphic for arbitrary spaces.[1] It also follows that, when the Alexander [2] and Čech cohomology groups are based on the same family of coverings, they are isomorphic for arbitrary spaces.[2]

The Alexander cohomology theory based on all open coverings is found to satisfy the seven Eilenberg-Steenrod axioms.[3] The proof consists of showing that this cohomology theory is isomorphic with the Čech cohomology theory which is known [6] to satisfy the axioms.

1. Complexes associated with relations

Let X and Y be two sets and let R be a relation between X and Y, that is, R is a subset of the product set $X \times Y$. If $(x, y) \in R$ we write $x\, R\, y$. Let the simplicial complexes K and L be defined as follows. A finite subset s of X is a simplex of K whenever the elements of s are all related to a common element of Y, that is, whenever there is an element y_s of Y such that, for each $x \in s$, $x\, R\, y_s$. Similarly a finite subset t of Y is defined to be a simplex of L whenever some element

[1] The isomorphism of the Čech and Vietoris homology groups for compact metric spaces is well known; see [10], page 273.

[2] In the particular case when the family is the family of all open coverings, this isomorphism has been proved for paracompact spaces by Hurewicz, Dugundji and Dowker [9, page 405], for compact spaces by Spanier [11] and for arbitrary spaces by Alexander (unpublished).

[3] This was proved for compact spaces by Spanier [11].

of X is related to all the elements of t. It is easily verified that K and L are indeed simplicial complexes.

Let R_1 be a relation between X_1 and Y_1 and let R_2 be a relation between X_2 and Y_2. Then a map

$$f:(X_2, Y_2, R_2) \to (X_1, Y_1, R_1)$$

of X_2 into X_1 and Y_2 into Y_1, such that the subset R_2 of $X_2 \times Y_2$ is mapped into the subset R_1 of $X_1 \times Y_1$, is called a map of the relation R_2 into the relation R_1. Let K_1 and L_1 be the complexes associated with the relation R_1 and let K_2 and L_2 be the complexes associated with R_2. If s is a simplex of K_2, let y_s be an element of Y_2 such that $x\ R_2\ y_s$ for each $x \in s$. Then, for each $x \in s$, $(x, y_s) \in R_2$ and hence $(f(x), f(y_s)) \in R_1$, that is, $f(x)\ R_1\ f(y_s)$. Thus the images of the elements of s form a simplex of K_1. Thus f induces a simplicial map $f_{21}:K_2 \to K_1$. Similarly f induces a simplicial map $\bar{f}_{21}:L_2 \to L_1$.

Clearly, if $g:(X_3, Y_3, R_3) \to (X_2, Y_2, R_2)$ is a map of the relation R_3 into the relation R_2, then

$$(fg)_{31} = f_{21}g_{32}:K_3 \to K_1$$

and

$$(\overline{fg})_{31} = \bar{f}_{21}\bar{g}_{32}:L_3 \to L_1 .$$

If X_2 is a subset of X_1, if Y_2 is a subset of Y_1 and if R_2 is a subset of $(X_2 \times Y_2) \cap R_1$, we say that the relation R_2 is a subrelation of the relation R_1, and we write $(X_2, Y_2, R_2) \subset (X_1, Y_1, R_1)$. In this case the map

$$f:(X_2, Y_2, R_2) \to (X_1, Y_1, R_1)$$

which maps each element of X_2 on itself and maps each element of Y_2 on itself is called the inclusion map. The complex K_2 is now a subcomplex of K_1 and L_2 is a subcomplex of L_1. Thus with the pair (R_1, R_2) of relations are associated the pairs (K_1, K_2) and (L_1, L_2) of complexes.

If $(R_{\alpha 1}, R_{\alpha 2})$ is a pair of relations and $(R_{\beta 1}, R_{\beta 2})$ is another pair, a map

$$f:(X_{\beta 1}, X_{\beta 2}, Y_{\beta 1}, Y_{\beta 2}, R_{\beta 1}, R_{\beta 2}) \to (X_{\alpha 1}, X_{\alpha 2}, Y_{\alpha 1}, Y_{\alpha 2}, R_{\alpha 1}, R_{\alpha 2})$$

of $X_{\beta 1}$ into $X_{\alpha 1}$ and $Y_{\beta 1}$ into $Y_{\alpha 1}$ which maps $X_{\beta 2}$, $Y_{\beta 2}$, $R_{\beta 1}$, $R_{\beta 2}$ respectively into $X_{\alpha 2}$, $Y_{\alpha 2}$, $R_{\alpha 1}$, $R_{\alpha 2}$ is called a map of the pair $(R_{\beta 1}, R_{\beta 2})$ into the pair $(R_{\alpha 1}, R_{\alpha 2})$. One sees that the submap $f_1:(X_{\beta 1}, Y_{\beta 1}, R_{\beta 1}) \to (X_{\alpha 1}, Y_{\alpha 1}, R_{\alpha 1})$ induces a simplicial map $f_{\beta\alpha 1}:K_{\beta 1} \to K_{\alpha 1}$ and the submap $f_2:(X_{\beta 2}, Y_{\beta 2}, R_{\beta 2}) \to (X_{\alpha 2}, Y_{\alpha 2}, R_{\alpha 2})$ induces a simplicial map $f_{\beta\alpha 2}:K_{\beta 2} \to K_{\alpha 2}$ such that, if $i_\alpha:K_{\alpha 2} \to K_{\alpha 1}$ and $i_\beta:K_{\beta 2} \to K_{\beta 1}$ are inclusion maps, then $i_\alpha f_{\beta\alpha 2} = f_{\beta\alpha 1} i_\beta:K_{\beta 2} \to K_{\alpha 1}$. That is, f induces a simplicial map $f_{\beta\alpha}:(K_{\beta 1}, K_{\beta 2}) \to (K_{\alpha 1}, K_{\alpha 2})$ of a pair of complexes into a pair of complexes. Similarly f induces a simplicial map $\bar{f}_{\beta\alpha}:(L_{\beta 1}, L_{\beta 2}) \to (L_{\alpha 1}, L_{\alpha 2})$.

2. Barycentric subdivisions

The barycentric subdivision K' of a simplicial complex K is defined as follows. The vertices of K' are the simplexes of K, and a finite set of vertices of K' form a simplex of K' if they can be simply ordered by inclusion. If a complex K_2

is a subcomplex of a simplicial complex K_1, then the barycentric subdivision K_2' is a subcomplex of K_1'.

Assume that the vertices of K_1 are ordered so that the vertices of any simplex of K_1 have a simple order. Then we define a simplicial map

$$\phi:(K_1', K_2') \rightarrow (K_1, K_2)$$

as follows. If $x' = x_0x_1 \cdots x_p$ is a vertex of K_1', that is, a simplex of K_1, let $\phi x'$ be the least vertex of the simplex x' in the given fixed order, i.e., $\phi x' \in x'$ and, for each $x_i \in x'$, $\phi x' \leqq x_i$. It is easily verified that ϕ is a simplicial map and that reordering the vertices of K_1 will replace ϕ by a contiguous[4] map. Clearly if $x_1' \subset x_2'$ then $\phi x_1' \geqq \phi x_2'$; thus ϕ is order reversing.

The simplicial map $\phi:(K_1', K_2') \rightarrow (K_1, K_2)$ induces a homomorphism $\phi^*:H^p(K_1, K_2) \rightarrow H^p(K_1', K_2')$ of the cohomology groups, and, since contiguous simplicial maps induce the same homomorphism, ϕ^* does not depend on the order given to the vertices. It is known [10, pages 166–167] that ϕ^* is an isomorphism of $H^p(K_1, K_2)$ onto $H^p(K_1', K_2')$. Similarly ϕ induces an isomorphism $\phi_*:H_p(K_1', K_2') \rightarrow H_p(K_1, K_2)$ of the homology groups.

The complexes K_1', K_2' have barycentric subdivisions K_1'', K_2'', and the ordering of the vertices of K_1' by inclusion determines a simplicial map $\phi':(K_1'', K_2'') \rightarrow (K_1', K_2')$, and so on. The map ϕ' induces an isomorphism ϕ'^* of $H^p(K_1', K_2')$ onto $H^p(K_1'', K_2'')$ and an isomorphism ϕ_*' of $H_p(K_1'', K_2'')$ onto $H_p(K_1', K_2')$.

Let (L_1, L_2) be a second pair of complexes and let $\psi:(L_1, L_2) \rightarrow (K_1, K_2)$ be a simplicial map. Then ψ induces a simplicial map $\psi':(L_1', L_2') \rightarrow (K_1', K_2')$ of the barycentric subdivisions. For, if $y' = y_0 \cdots y_p$ is a vertex of L_1', [respectively L_2'], i.e., a simplex of $L_1[L_2]$, then, since ψ is simplicial, the vertices $\psi y_0, \cdots, \psi y_p$ form a simplex of $K_1[K_2]$ or a vertex of $K_1'[K_2']$. Let $\psi' y'$ be this vertex. Clearly ψ' is order preserving, i.e., if $y_1' \subset y_2'$ then $\psi' y_1' \subset \psi' y_2'$; therefore a simplex $y_0' \cdots y_q'$ is mapped into a simplex. Thus ψ' is a simplicial map.

Let $\bar{\phi}:(L_1', L_2') \rightarrow (L_1, L_2)$ be the simplicial map which maps each vertex of L_1' on its first vertex in L_1.

LEMMA 1. *The maps* $\phi\psi'$ and $\psi\bar{\phi}:(L_1', L_2') \rightarrow (K_1, K_2)$ *are contiguous.*

PROOF. If $y'' = y_0' \cdots y_q'$ is a simplex of L_1', let $\hat{y}'$ be its largest vertex. Then, for each $y_i' \in y''$, $y_i' \subset \hat{y}'$ and hence $\psi' y_i' \subset \psi' \hat{y}'$. Hence, since $\phi\psi' y_i' \in \psi' y_i'$ $\phi\psi' y_i' \in \psi' \hat{y}'$. Also $\bar{\phi} y_i' \in y_i' \subset \hat{y}'$ and hence $\psi\bar{\phi} y_i' \in \psi' \hat{y}'$. Thus the images of the vertices of the simplex y'' both by $\phi\psi'$ and by $\psi\bar{\phi}$ are contained in the simplex $\psi' \hat{y}'$. Moreover, if y'' is a simplex of L_2', $\psi' \hat{y}'$ will be a simplex of K_2. Thus $\phi\psi'$ and $\psi\bar{\phi}$ are contiguous.

3. The homomorphisms η and ω

Let (R_1, R_2) be a pair of relations and let (K_1, K_2) and (L_1, L_2) be the pairs of complexes associated with $(X_1, X_2, Y_1, Y_2, R_1, R_2)$. We introduce a map $\psi:(L_1', L_2') \rightarrow (K_1, K_2)$ of the barycentric subdivision of (L_1, L_2) into (K_1, K_2).

[4] Two simplicial maps ϕ and ϕ_1 are called contiguous if for each simplex x' of K_1' [respectively K_2'] the images of the vertices of x' by ϕ and by ϕ_1 all lie in a common simplex of K_1 [respectively K_2].

Each vertex y' of L_1' is a simplex of L_1; we choose $\psi y' \in X_1$ so that $\psi y' R_1 y$ for each $y \in y'$ and so that, if y' is a vertex of L_2', $\psi y' \in X_2$ and $\psi y' R_2 y$ for each $y \in y'$. By the definition of (L_1, L_2) the choice is possible.

If $y'' = y_0' \cdots y_q'$ is a simplex of L_1', let $\tilde{y}'$ be its least vertex. Then, for each $y_i' \in y''$, $\bar{\phi}\tilde{y}' \in \tilde{y}' \subset y_i'$ and hence $\psi y_i' R_1 \bar{\phi}\tilde{y}'$. Thus the elements $\psi y_0', \cdots, \psi y_q'$ of X_1 form a simplex of K_1. If the simplex y'' is contained in L_2', it is seen that, for each $y_i' \in y''$, $\psi y_i' R_2 \bar{\phi}\tilde{y}'$, and hence the simplex with vertices $\psi y_0', \cdots, \psi y_q'$ is contained in K_2. Thus $\psi:(L_1', L_2') \rightarrow (K_1, K_2)$ is a simplicial map.

The definition of ψ depended on a choice. If $\hat{\psi}$ is the result of any other choice, we similarly have $\hat{\psi} y_i' R_1 \bar{\phi}\tilde{y}'$ or, if y'' is in L_2', $\hat{\psi} y_i' R_2 \bar{\phi}\tilde{y}'$. Thus $\psi y_0', \cdots, \psi y_q'$, $\hat{\psi} y_0', \cdots, \hat{\psi} y_q'$ are the vertices of a simplex of K_1, respectively K_2. Hence ψ and $\hat{\psi}$ are contiguous. Hence the homomorphisms $\psi^*:H^p(K_1, K_2) \rightarrow H^p(L_1', L_2')$ and $\psi_*:H_p(L_1', L_2') \rightarrow H_p(K_1, K_2)$ are uniquely determined.

Since $\bar{\phi}^*$ is an isomorphism onto it has an inverse $\bar{\phi}^{*-1}:H^p(L_1', L_2') \rightarrow H^p(L_1, L_2)$. We define

$$\eta = \bar{\phi}^{*-1}\psi^*:H^p(K_1, K_2) \rightarrow H^p(L_1, L_2).$$

Let $\eta_2 = (\bar{\phi} \mid L_2')^{*-1}(\psi \mid L_2')^*$ be the corresponding homomorphism of $H^p(K_2)$ into $H^p(L_2)$. Let $\delta:H^p(K_2) \rightarrow H^{p+1}(K_1, K_2)$ be the coboundary homomorphism in (K_1, K_2) and let $\bar{\delta}$ and $\bar{\delta}'$ be the coboundary homomorphisms in (L_1, L_2) and (L_1', L_2') respectively.

Lemma 2. *The homomorphism η commutes with the coboundary homomorphism, that is*

$$\eta\delta = \bar{\delta}\eta_2:H^p(K_2) \rightarrow H^{p+1}(L_1, L_2).$$

Proof. By the third axiom of Eilenberg and Steenrod [8; also 6, page 278],

$$\begin{array}{ccccc}
H^p(K_2) & \xrightarrow{(\psi/L_2')^*} & H^p(L_2') & \xleftarrow{(\bar{\phi}/L_2')^*} & H^p(L_2) \\
\downarrow \delta & & \downarrow \bar{\delta}' & & \downarrow \bar{\delta} \\
H^{p+1}(K_1, K_2) & \xrightarrow{\psi^*} & H^{p+1}(L_1', L_2') & \xleftarrow{\bar{\phi}^*} & H^{p+1}(L_1, L_2)
\end{array}$$

Figure 1.

commutativity holds in each rectangle of Figure 1. Hence $\bar{\phi}^{*-1}\psi^*\delta = \bar{\delta}(\bar{\phi} \mid L_2')^{*-1}(\psi \mid L_2')^*$, that is, $\eta\delta = \bar{\delta}\eta_2$.

Let

$$\omega = \psi_*\bar{\phi}_*^{-1}:H_p(L_1, L_2) \rightarrow H_p(K_1, K_2)$$

and let $\omega_2 = (\psi \mid L_2')_*(\bar{\phi} \mid L_2')_*^{-1}$ be the corresponding homomorphism of $H_p(L_2)$ into $H_p(K_2)$. Let ∂ and $\bar{\partial}$ be the boundary homomorphisms in (K_1, K_2) and (L_1, L_2) respectively.

Lemma 2a. *The homomorphism ω commutes with the boundary homomorphism, that is,*

$$\partial\omega = \omega_2\bar{\partial}:H_p(L_1, L_2) \rightarrow H_{p-1}(K_2).$$

Proof. By the third Eilenberg-Steenrod axiom for homology, ∂ commutes with ψ_* and with $\bar{\phi}_*$. Hence ∂ commutes with $\omega = \psi_*\bar{\phi}_*^{-1}$.

LEMMA 3. *If f is a map of a pair of relations $(R_{\beta 1}, R_{\beta 2})$ into a pair $(R_{\alpha 1}, R_{\alpha 2})$, then $f_{\beta\alpha}\psi_\beta$ and $\psi_\alpha \bar{f}'_{\beta\alpha} : (L'_{\beta 1}, L'_{\beta 2}) \to (K_{\alpha 1}, K_{\alpha 2})$ are contiguous.*

PROOF. Let $y'' = y'_0 \cdots y'_q$ be a simplex of $L'_{\beta 1}$, and let $\tilde{y}'$ be its least vertex. Let y_0 be any fixed element of $\tilde{y}'$. Then, for each $y'_i \in y''$, $y_0 \in \tilde{y}' \subset y'_i$ and hence $\psi_\beta y'_i R_{\beta 1} y_0$. Therefore $f(\psi_\beta y'_i) R_{\alpha 1} f(y_0)$, that is, $f_{\beta\alpha}\psi_\beta y'_i R_{\alpha 1} f(y_0)$. Also, since $y_0 \in y'_i$, $f(y_0) = \bar{f}_{\beta\alpha} y_0 \in \bar{f}'_{\beta\alpha} y'_i$ and hence $\psi_\alpha \bar{f}'_{\beta\alpha} y'_i R_{\alpha 1} f(y_0)$. Hence all the vertices $f_{\beta\alpha}\psi_\beta y'_i$ and $\psi_\alpha \bar{f}'_{\beta\alpha} y'_i$ belong to a common simplex of $K_{\alpha 1}$. Hence $f_{\beta\alpha 1}\psi_{\beta 1}$ is contiguous with $\psi_{\alpha 1}\bar{f}'_{\beta\alpha 1}$. Similarly $f_{\beta\alpha 2}\psi_{\beta 2}$ and $\psi_{\alpha 2}\bar{f}'_{\beta\alpha 2}$ are contiguous. Therefore $f_{\beta\alpha}\psi_\beta$ and $\psi_\alpha \bar{f}'_{\beta\alpha}$ are contiguous.

LEMMA 4. *If f is a map of $(R_{\beta 1}, R_{\beta 2})$ into $(R_{\alpha 1}, R_{\alpha 2})$ then f^* commutes with η, that is*

$$\eta_\beta f^*_{\beta\alpha} = \bar{f}^*_{\beta\alpha}\eta_\alpha : H^p(K_{\alpha 1}, K_{\alpha 2}) \to H^p(L_{\beta 1}, L_{\beta 2}).$$

PROOF. It follows from Lemma 3 that commutativity holds in the left rec-

$$\begin{array}{ccccc}
H^p(K_\alpha) & \xrightarrow{\psi^*_\alpha} & H^p(L'_\alpha) & \xleftarrow{\bar{\phi}^*_\alpha} & H^p(L_\alpha) \\
\downarrow f^*_{\beta\alpha} & & \downarrow \bar{f}'^*_{\beta\alpha} & & \downarrow \bar{f}^*_{\beta\alpha} \\
H^p(K_\beta) & \xrightarrow{\psi^*_\beta} & H^p(L'_\beta) & \xleftarrow{\bar{\phi}^*_\beta} & H^p(L_\beta)
\end{array}$$

Figure 2.

tangle of Figure 2, and from Lemma 1 that commutativity holds in the right rectangle. Hence $\bar{\phi}^{*-1}_\beta \psi^*_\beta f^*_{\beta\alpha} = \bar{f}^*_{\beta\alpha}\bar{\phi}^{*-1}_\alpha \psi^*_\alpha$, that is, $\eta_\beta f^*_{\beta\alpha} = \bar{f}^*_{\beta\alpha}\eta_\alpha$.

LEMMA 4a. *If f is a map of $(R_{\beta 1}, R_{\beta 2})$ into $(R_{\alpha 1}, R_{\alpha 2})$ then f_* commutes with ω, that is,*

$$f_{\beta\alpha *}\omega_\beta = \omega_\alpha \bar{f}_{\beta\alpha *} : H_p(L_{\beta 1}, L_{\beta 2}) \to H_p(K_{\alpha 1}, K_{\alpha 2}).$$

PROOF. It follows from Lemma 3 that f_* commutes with ψ_* and from Lemma 1 that f_* commutes with $\bar{\phi}_*$. Hence f_* commutes with $\omega = \psi_* \bar{\phi}_*^{-1}$.

4. The isomorphisms η and ω

It is to be shown that $\eta : H^p(K_1, K_2) \to H^p(L_1, L_2)$ and $\omega : H_p(L_1, L_2) \to H_p(K_1, K_2)$ are isomorphisms onto. We first prove the following lemmas.

LEMMA 5. *Let (R_1, R_2) be a pair of relations and (K_1, K_2) and (L_1, L_2) the associated pairs of complexes. Then the maps $\psi\bar{\psi}'$ and $\phi\phi' : (K''_1, K''_2) \to (K_1, K_2)$ are contiguous.*

PROOF. Let $x''' = x''_0 \cdots x''_q$ be a simplex of K''_1 and let $\tilde{x}''$ be its least vertex.

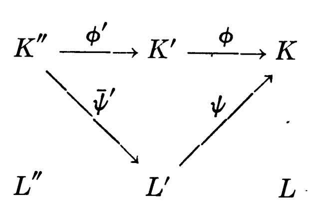

Figure 3.

Thus, for each x''_i, $\tilde{x}'' \subset x''_i$, and, since ϕ' is order reversing, $\phi' x''_i \subset \phi' \tilde{x}''$. Let $y = \bar{\psi}\phi'\tilde{x}''$; then xR_1y for each $x \in \phi'\tilde{x}''$. Hence, since $\phi\phi' x''_i \in \phi' x''_i \subset \phi'\tilde{x}''$, $\phi\phi' x''_i R_1 y$.

For each vertex x' of x''_i, $\bar{\psi}x' \in \bar{\psi}' x''_i$. Hence, since $\phi'\tilde{x}'' \in \tilde{x}'' \subset x''_i$, $\bar{\psi}\phi'\tilde{x}'' \in \bar{\psi}' x''_i$. Thus $y \in \bar{\psi}' x''_i$ for each x''_i. Hence, for each x''_i, $\psi\bar{\psi}' x''_i R_1 y$.

Thus all the vertices $\phi\phi' x''_i$ and $\psi\bar{\psi}' x''_i$ are vertices of a simplex of K_1. Hence $\phi_1\phi_1'$ and $\psi_1\bar{\psi}_1'$ are contiguous. Similarly $\phi_2\phi_2'$ and $\psi_2\bar{\psi}_2'$ are contiguous. Hence $\phi\phi'$ and $\psi\bar{\psi}'$ are contiguous, as was to be shown.

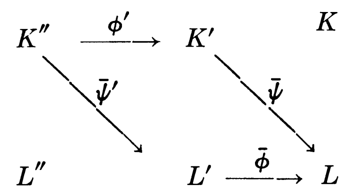

Figure 4.

LEMMA 6. *The maps $\bar{\phi}\bar{\psi}'$ and $\bar{\psi}\phi' : (K''_1, K''_2) \to (L_1, L_2)$ are contiguous.*

PROOF. This follows immediately from Lemma 1.

THEOREM 1. *If (R_1, R_2) is a pair of relations and (K_1, K_2) and (L_1, L_2) are the associated pairs of complexes, then*

$$\eta : H^p(K_1, K_2) \to H^p(L_1, L_2)$$

is an isomorphism onto.

PROOF. From Lemma 5 (Figure 3) we see that

$$\phi^{*-1}\phi'^{*-1}\bar{\psi}'^{*}\psi^{*} = 1 : H^p(K_1, K_2) \to H^p(K_1, K_2).$$

Also from Lemma 6 (Figure 4) we see that $\phi'^{*-1}\bar{\psi}'^{*} = \bar{\psi}^{*}\bar{\phi}^{*-1}$. Hence, substituting, we have

$$\phi^{*-1}\bar{\psi}^{*}\bar{\phi}^{*-1}\psi^{*} = 1,$$

that is,

$$\bar{\eta}\eta = 1 : H^p(K_1, K_2) \to H^p(K_1, K_2).$$

Similarly

$$\eta\bar{\eta} = 1 : H^p(L_1, L_2) \to H^p(L_1, L_2).$$

Hence

$$\eta : H^p(K_1, K_2) \to H^p(L_1, L_2)$$

is an isomorphism onto.

THEOREM 1a. *If (R_1, R_2) is a pair of relations and if (K_1, K_2) and (L_1, L_2) are the associated pairs of complexes, then*

$$\omega : H_p(L_1, L_2) \to H_p(K_1, K_2)$$

is an isomorphism onto.

The proof is similar to that of Theorem 1 and is omitted.

5. Isomorphism of limit groups

Let X be a space[5] and let A be a subset of X. A covering α of the pair (X, A) is a pair (α_1, α_2) consisting of a collection α_1 of subsets of X whose union is X and a subcollection α_2 whose union contains A. The nerve K_α of α is a pair $(K_{\alpha 1}, K_{\alpha 2})$ of simplicial complexes; a simplex of $K_{\alpha 1}$ is a finite set of elements of α_1 whose intersection is not empty, and a simplex of $K_{\alpha 2}$ is a finite set of elements of α_2 whose intersection meets A. The complex $K_{\alpha 2}$ is a subcomplex of $K_{\alpha 1}$. The Vietoris pair $L_\alpha = (L_{\alpha 1}, L_{\alpha 2})$ of simplicial complexes is defined thus: A simplex of $L_{\alpha 1}$ is a finite set of points of X contained in a common element of α_1 and a simplex of $L_{\alpha 2}$ is a finite set of points of A contained in a common element of α_2. The complex $L_{\alpha 2}$ is a subcomplex of $L_{\alpha 1}$.

Clearly K and L are the pairs of complexes associated with the relation pair $(X, A, \alpha_1, \alpha_2, \epsilon, \epsilon)$ where each ϵ means "element of".

If $\beta = (\beta_1, \beta_2)$ is also a covering of (X, A) we say that β is a refinement of α, or that $\alpha < \beta$, if each element of β_1 is contained in some element of α_1 and if each element of β_2 is contained in some element of α_2. A relation map π of $(X, A, \beta_1, \beta_2, \epsilon, \epsilon)$ into $(X, A, \alpha_1, \alpha_2, \epsilon, \epsilon)$ is called a projection if π maps each point of X on itself and maps each element U of β_1 on an element V of α_1 such that $U \subset V$ and such that, whenever $U \in \beta_2$, $V \in \alpha_2$. If $\alpha < \beta$ such a projection π exists. Let the induced simplicial maps of the associated complexes be $\pi_{\beta\alpha}$: $(K_{\beta 1}, K_{\beta 2}) \rightarrow (K_{\alpha 1}, K_{\alpha 2})$ and $\bar{\pi}_{\beta\alpha} : (L_{\beta 1}, L_{\beta 2}) \rightarrow (L_{\alpha 1}, L_{\alpha 2})$. Given the coverings β and α with $\alpha < \beta$, the relation map π is in general not uniquely determined, but a second choice is associated with a simplicial map contiguous with $\pi_{\beta\alpha}$. Hence $\pi^*_{\alpha\beta}$ and $\pi_{\beta\alpha *}$ are uniquely determined. The map $\bar{\pi}_{\beta\alpha}$ is a uniquely determined inclusion map.

Let Ω be a family of coverings of (X, A) such that, if α and β are in Ω, there exists in Ω a common refinement γ of α and β; $\alpha < \gamma$, $\beta < \gamma$ and $\gamma \in \Omega$. Then Ω, with $<$, forms a directed set. Let G be a discrete abelian group. The groups $H^p(K_\alpha) = H^p(K_{\alpha 1}, K_{\alpha 2}; G)$, together with the homomorphisms $\pi^*_{\beta\alpha}: H^p(K_\alpha) \rightarrow H^p(K_\beta)$, form a direct spectrum $S^p(X, A; G, \Omega)$ whose limit group $H^p(X, A; G, \Omega)$ is the p-dimensional Čech cohomology group of (X, A) with coefficient group G and fundamental family Ω. The groups $H^p(L_\alpha) = H^p(L_{\alpha 1}, L_{\alpha 2}; G)$, together with the homomorphisms $\bar{\pi}^*_{\beta\alpha}: H^p(L_\alpha) \rightarrow H^p(L_\beta)$ form a direct spectrum $\bar{S}^p(X, A; G, \Omega)$ whose limit group is the Alexander[6] cohomology group $\bar{H}^p(X, A; G, \Omega)$.

Similarly the homology groups $H_p(K_\alpha) = H_p(K_{\alpha 1}, K_{\alpha 2}; G)$ and homomorphisms $\pi_{\beta\alpha *}: H_p(K_\beta) \rightarrow H_p(K_\alpha)$ form an inverse spectrum $S_p(X, A; G, \Omega)$

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[5] Until Section 9 no topology will be assumed in the space X, thus X may be merely a set.

[6] Alexander [2] defined only the absolute cohomology groups. For the special case when the family Ω is the family of all open coverings, Spanier [11] gives two definitions of the relative cohomology groups (i. e., cohomology groups of a pair (X, A)) one, attributed to A. D. Wallace, in terms of neighborhoods of the diagonal in a product space and the other in terms of coverings. Because of an error in the second definition, the results in his Appendix A are incorrect as may be seen from the following counter example. Let X be the subset of the plane for which $x^2 + y^2 \leqq 1$, and let A be the non-closed subset defined by $x^2 + y^2 = 1$, $x < 1$. Then, in Spanier's notation, $H^2(X, A) = 0$, $\bar{H}^2(X, A) = G$ and so $H \neq \bar{H}$.

whose limit group is the Čech homology group $H_p(X, A; G, \Omega)$. The homology groups $H_p(L_\alpha) = H_p(L_{\alpha 1}, L_{\alpha 2}; G)$ and homomorphisms $\bar{\pi}_{\beta\alpha *}: H_p(L_\beta) \to H_p(L_\alpha)$ form an inverse spectrum $\bar{S}_p(X, A; G, \Omega)$ whose limit group is the Vietoris homology group $\bar{H}_p(X, A; G, \Omega)$.

THEOREM 2. *The Čech and Alexander cohomology groups are isomorphic;*

$$H^p(X, A; G, \Omega) \approx \bar{H}^p(X, A; G, \Omega).$$

PROOF. By Theorem 1, $\eta_\alpha: H^p(K_\alpha) \to H^p(L_\alpha)$ is an isomorphism onto. By Lemma 4, π^* commutes with η, that is,

$$\eta_\alpha \pi^*_{\alpha\beta} = \bar{\pi}^*_{\alpha\beta}\eta_\beta : H^p(K_\beta) \to H^p(L_\alpha).$$

Thus, if we identify each $H^p(K_\alpha)$ with $H^p(L_\alpha)$ by the isomorphism η_α the two spectra are identified, and hence also their limit groups.

THEOREM 2a. *The Čech and Vietoris homology groups are isomorphic;*

$$H_p(X, A; G, \Omega) \approx \bar{H}_p(X, A; G, \Omega).$$

The proof is similar to that of Theorem 2 and is omitted.

6. The homomorphism f^*

Let f be a map[7] of (X, A) into a pair (Y, B), that is, f maps X into Y so that $f(A) \subset B$. Let α be a covering of (X, A) and let σ be a covering of (Y, B). We say that f maps α into σ, or that $\sigma < \alpha$, if the image $f(U)$ of each element U of α_1 is contained in some element of σ_1 and the image of each element of α_2 is contained in some element of σ_2. Corresponding to f there is at least one relation map of $(X, A, \alpha_1, \alpha_2, \epsilon, \epsilon)$ into $(Y, B, \sigma_1, \sigma_2, \epsilon, \epsilon)$ which maps each $x \in X$ onto $f(x)$ and each $U \in \alpha_1$ onto some $V \in \sigma_1$ such that $f(U) \subset V$ and such that, if $U \in \alpha_2$, $V \in \sigma_2$.

Let the associated simplicial maps be $f_{\alpha\sigma}: (K_{\alpha 1}, K_{\alpha 2}) \to (K_{\sigma 1}, K_{\sigma 2})$ and $\bar{f}_{\alpha\sigma}: (L_{\alpha 1}, L_{\alpha 2}) \to (L_{\sigma 1}, L_{\sigma 2})$. The relation map may not be uniquely determined by f but a second choice is associated with a simplicial map contiguous with $f_{\alpha\sigma}$, while the simplicial map $\bar{f}_{\alpha\sigma}$ is unchanged. Thus the homomorphisms $f^*_{\alpha\sigma}$, $\bar{f}^*_{\alpha\sigma}$, $f_{\alpha\sigma *}$ and $\bar{f}_{\alpha\sigma *}$ are uniquely determined.

Let Ω be the fundamental family of coverings of (X, A), let Ω_1 be the fundamental family of coverings of (Y, B) and let f be a map of (X, A) into (Y, B) such that, for each $\sigma \in \Omega_1$, there is some $\alpha \in \Omega$ which is mapped into σ by f. Then the homomorphisms $f^*_{\alpha\sigma}$, defined whenever $\sigma < \alpha$, constitute a map[8] of the direct spectrum $S^p(Y, B; G, \Omega_1)$ into the spectrum $S^p(X, A; G, \Omega)$ and this map of the spectrum induces a homomorphism $f^*: H^p(Y, B; G, \Omega_1) \to H^p(X, A; G, \Omega)$. Similarly the homomorphisms $\bar{f}^*_{\alpha\sigma}$ determine a homomorphism $\bar{f}^*: \bar{H}^p(Y, B; G, \Omega_1) \to \bar{H}^p(X, A; G, \Omega)$.

LEMMA 7. *When the Čech and Alexander cohomology groups are identified the homomorphism f^* coincides with $\bar{f}^*$.*

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[7] No continuity is assumed here, in fact X may not be a topological space.

[8] See [6], page 279.

PROOF. It is sufficient to show that, under the identification, the homomorphisms $f^*_{\alpha\sigma}$ and $\bar{f}^*_{\alpha\sigma}$ coincide, that is, that

$$\eta_\alpha f^*_{\alpha\sigma} = \bar{f}^*_{\alpha\sigma}\eta_\sigma : H^p(K_\sigma) \to H^p(L_\alpha).$$

But this follows from lemma 4.

7. The homomorphism δ

Let α be a covering of (X, A) and σ a covering of A. We say that $\sigma < \alpha$ if, for each element U of α_2, $U \cap A$ is contained in some element of σ. There is then at least one relation map θ of (A, α_2, ϵ) into (A, σ, ϵ) which maps each $x \in A$ on itself and maps each element U of α_2 on an element V of σ such that $U \cap A \subset V$. Let the associated simplicial maps be $\theta_{\alpha\sigma} : K_{\alpha 2} \to K_\sigma$ and $\bar{\theta}_{\alpha\sigma} : L_{\alpha 2} \to L_\sigma$. The relation map θ may not be uniquely determined, but a second choice is associated with a simplicial map contiguous with $\theta_{\alpha\sigma}$. The map $\bar{\theta}_{\alpha\sigma}$ is a uniquely determined inclusion map. Thus the homomorphisms $\theta^*_{\alpha\sigma}$, $\bar{\theta}^*_{\alpha\sigma}$, $\theta_{\alpha\sigma *}$ and $\bar{\theta}_{\alpha\sigma *}$ are uniquely determined.

Let $\delta_\alpha : H^p(K_{\alpha 2}) \to H^{p+1}(K_{\alpha 1}, K_{\alpha 2})$ be the coboundary homomorphism for the pair $(K_{\alpha 1}, K_{\alpha 2})$ and let $\bar{\delta}_\alpha : H^p(L_{\alpha 2}) \to H^{p+1}(L_{\alpha 1}, L_{\alpha 2})$ be that for $(L_{\alpha 1}, L_{\alpha 2})$. If $\sigma < \alpha$ we define

$$\delta_{\alpha\sigma} = \delta_\alpha \theta^*_{\alpha\sigma} : H^p(K_\sigma) \to H^{p+1}(K_{\alpha 1}, K_{\alpha 2})$$

and

$$\bar{\delta}_{\alpha\sigma} = \bar{\delta}_\alpha \bar{\theta}^*_{\alpha\sigma} : H^p(L_\sigma) \to H^{p+1}(L_{\alpha 1}, L_{\alpha 2}).$$

Let Ω be the fundamental family of coverings of (X, A), let Ω_1 be the fundamental family of coverings of A and, for each $\sigma \in \Omega_1$, let there exist some $\alpha \in \Omega$ with $\sigma < \alpha$. Then the homomorphisms $\delta_{\alpha\sigma}$, defined whenever $\sigma < \alpha$, constitute a map of the direct spectrum $S^p(A; G, \Omega_1)$ into the spectrum $S^{p+1}(X, A; G, \Omega)$, and this map of the spectrum induces a homomorphism

$$\delta : H^p(A; G, \Omega_1) \to H^{p+1}(X, A; G, \Omega).$$

Similarly the homomorphisms $\bar{\delta}_{\alpha\sigma}$ determine a homorphism

$$\bar{\delta} : \bar{H}^p(A; G, \Omega_1) \to \bar{H}^{p+1}(X, A; G, \Omega).$$

LEMMA 8. *When the Čech and Alexander cohomology groups are identified the homomorphism δ coincides with $\bar{\delta}$.*

PROOF. It is sufficient to show that, under the identification, the homomorphisms $\delta_{\alpha\sigma}$ and $\bar{\delta}_{\alpha\sigma}$ coincide, that is, that

$$\eta_\alpha \delta_{\alpha\sigma} = \bar{\delta}_{\alpha\sigma}\eta_\sigma : H^p(K_\sigma) \to H^{p+1}(L_{\alpha 1}, L_{\alpha 2}).$$

By Lemma 4,

$$\eta_{\alpha 2}\theta^*_{\alpha\sigma} = \bar{\theta}^*_{\alpha\sigma}\eta_\sigma : H^p(K_\sigma) \to H^p(L_{\alpha 2}),$$

and by Lemma 2,

$$\eta_\alpha \delta_\alpha = \bar{\delta}_\alpha \eta_{\alpha 2} : H^p(K_{\alpha 2}) \to H^{p+1}(L_{\alpha 1}, L_{\alpha 2}).$$

Therefore $\eta_\alpha \delta_\alpha \theta^*_{\alpha\sigma} = \bar{\delta}_\alpha \eta_{\alpha 2} \theta^*_{\alpha\sigma} = \bar{\delta}_\alpha \bar{\theta}^*_{\alpha\sigma} \eta_\sigma$; thus we have $\eta_\alpha \delta_{\alpha\sigma} = \bar{\delta}_{\alpha\sigma} \eta_\sigma$ as was to be shown.

8. The homomorphisms f^* and ∂

As in Section 6, let f be a map of (X, A) into (Y, B), let Ω and Ω_1 be fundamental families of coverings of (X, A) and (Y, B) respectively and, for each $\sigma \in \Omega_1$, let there exist some $\alpha \in \Omega$ which is mapped into σ by f. An element of $H_p(X, A; G, \Omega)$ is a thread of the spectrum $S_p(X, A; G, \Omega)$, that is, a set $\{a_\alpha\}$ of elements $a_\alpha \in H_p(K_\alpha)$, one for each $\alpha \in \Omega$, such that, whenever $\alpha < \beta$, $\pi_{\beta\alpha *} a_\beta = a_\alpha$. It is easily verified that the set of all images $f_{\alpha\sigma *} a_\alpha$, for a_α in a given thread, is a thread of the spectrum $S_p(Y, A; G, \Omega_1)$. Thus there is defined a map $f_*: H_p(X, A; G, \Omega) \to H_p(Y, B; G, \Omega_1)$ which is easily seen to be a homomorphism. Similarly the set of homomorphisms $\bar{f}_{\alpha\sigma *}$ induces a homomorphism $\bar{f}_*: \bar{H}_p(X, A; G, \Omega) \to \bar{H}_p(Y, B; G, \Omega_1)$.

LEMMA 7a. *When the Čech and Vietoris homology groups are identified the homomorphism f_* coincides with $\bar{f}_*$.*

PROOF. It is sufficient to show that

$$f_{\alpha\sigma *} \omega_\alpha = \omega_\sigma \bar{f}_{\alpha\sigma *}: H_p(L_\alpha) \to H_p(K_\sigma).$$

This follows from Lemma 4a.

Again let Ω and Ω_1 be fundamental families of coverings of (X, A) and A respectively such that for each $\sigma \in \Omega_1$ there is some $\alpha \in \Omega$ with $\sigma \prec \alpha$. Whenever $\sigma \prec \alpha$ there are simplicial maps $\theta_{\alpha\sigma}: K_{\alpha 2} \to K_\sigma$ and $\bar{\theta}_{\alpha\sigma}: L_{\alpha 2} \to L_\sigma$ whose induced homomorphisms $\theta_{\alpha\sigma *}$ and $\bar{\theta}_{\alpha\sigma *}$ are uniquely determined. Let $\partial_{\alpha\sigma} = \theta_{\alpha\sigma *} \partial_\alpha$ and $\bar{\partial}_{\alpha\sigma} = \bar{\theta}_{\alpha\sigma *} \bar{\partial}_\alpha$ where ∂_α and $\bar{\partial}_\alpha$ are the boundary homomorphisms for the pairs $(K_{\alpha 1}, K_{\alpha 2})$ and $(L_{\alpha 1}, L_{\alpha 2})$ respectively. Then the set of all images $\partial_{\alpha\sigma} a_\alpha$, for a_α in a given thread of $S_p(X, A; G, \Omega)$, form a thread of $S_{p-1}(A; G, \Omega_1)$ and the resulting map $\partial: H_p(X, A; G, \Omega) \to H_{p-1}(A; G, \Omega_1)$ is a homomorphism. Similarly the homomorphisms $\bar{\partial}_{\alpha\sigma}$ determine a homomorphism $\bar{\partial}: \bar{H}_p(X, A; G, \Omega) \to \bar{H}_{p-1}(A; G, \Omega_1)$.

LEMMA 8a. *When the Čech and Vietoris homology groups are identified the homomorphism ∂ coincides with $\bar{\partial}$.*

The proof is similar to that of Lemma 8 and is omitted.

9. Cohomology theory of topological spaces

If X is a topological space, if A is a subset of X and if Ω is the family of all coverings of (X, A) by open sets of X then we write $H^p(X, A; G)$ and $\bar{H}^p(X, A; G)$ respectively, omitting explicit mention of Ω, for the Čech and Alexander cohomology groups of (X, A) with coefficient group G.

If $f: (X, A) \to (Y, B)$ is continuous and if σ is a covering of (Y, B) by open sets then there is a covering α of (X, A) by open sets which is mapped into σ by f. For example, let α_1 be the set of all $f^{-1}(V)$ with $V \in \sigma_1$, and let α_2 be the set of all $f^{-1}(V)$ with $V \in \sigma_2$. Hence for each continuous map $f: (X, A) \to (Y, B)$ the homomorphisms $f^*: H^p(Y, B; G) \to H^p(X, A; G)$ and $\bar{f}^*: \bar{H}^p(Y, B; G) \to \bar{H}^p(X, A; G)$ exist.

If σ is any covering of A by open sets there is a covering α of (X, A) by open sets such that $\sigma < \alpha$. For example, let α_1 be the set of all open sets of X and let α_2 be the set of all open sets U of X such that $U \cap A$ is an element of σ. Hence, if X is a topological space and if $A \subset X$, the homomorphisms $\delta: H^p(A; G) \to H^{p+1}(X, A; G)$ and $\bar{\delta}: \bar{H}^p(A; G) \to \bar{H}^{p+1}(X, A; G)$ exist.

The category of groups $H^p(X, A; G)$ for fixed G, with $p = 0, 1, \cdots$ and arbitrary topological (X, A), together with the homomorphisms f^* for arbitrary continuous maps $f:(X, A) \to (Y, B)$ and the coboundary homomorphisms $\delta: H^p(A; G) \to H^{p+1}(X, A; G)$, forms a cohomology theory, the Čech cohomology theory. Similarly the groups $\bar{H}^p(X, A; G)$ and homomorphisms $\bar{f}^*$ and $\bar{\delta}$ form the Alexander cohomology theory.

THEOREM 3. *When the Čech and Alexander cohomology groups $H^p(X, A; G)$ and $\bar{H}^p(X, A; G)$ are identified for each pair (X, A) then the cohomology theories are identified.*

PROOF. It is sufficient to show that under the identification f^* coincides with $\bar{f}^*$ and δ with $\bar{\delta}$. But this was proved in Lemmas 7 and 8.

THEOREM 4. *The Alexander cohomology theory satisfies the seven Eilenberg-Steenrod axioms.*

PROOF. This is shown by identifying the Alexander cohomology theory with the Čech cohomology theory which is known [6] to satisfy the axioms. (A direct proof that the Alexander cohomology theory satisfies six of the seven axioms is given by Spanier [11]; he also proves that the remaining axiom, the homotopy axiom, is satisfied when the pairs (X, A) are required to be compact.)

10. Remarks

(i) Since (Section 5 above) the projection maps of the Vietoris complexes are uniquely determined the induced homomorphisms of the cochains are also uniquely determined. The cochain groups $C^p(L_\alpha) = C^p(L_{\alpha 1}, L_{\alpha 2}; G)$ with these homomorphisms form a direct spectrum whose limit group $\bar{C}^p(X, A; G, \Omega)$ is a group of Alexander cochains of the pair (X, A). The coboundary operator for $\bar{C}^p(X, A; G, \Omega)$ is induced by that for the groups $C^p(L_\alpha)$, and the cohomology group found from these limit cochains is isomorphic [3, page 306] with the limit cohomology group $\bar{H}^p(X, A; G, \Omega)$. There is a homomorphism

$$\bar{\pi}_\alpha: C^p(L_{\alpha 1}, L_{\alpha 2}; G) \to \bar{C}^p(X, A; G, \Omega)$$

which maps each cochain of $C^p(L_\alpha)$ on the bundle containing it; it is easily seen that this homomorphism is onto. Let the kernel of $\bar{\pi}_\alpha$ be $C_0^p(L_\alpha)$; then $\bar{C}^p \approx C^p(L_\alpha)/C_0^p(L_\alpha)$. A cochain of $C^p(L_\alpha)$ is in $C_0^p(L_\alpha)$ if and only if it belongs to the zero bundle; such cochains are called "locally zero" [1, page 511].

(ii) In the case of the Čech groups, the projection maps $\pi_{\beta\alpha}$ of the nerves are not in general uniquely determined and in general the cochain groups $C^p(K_{\alpha 1}, K_{\alpha 2}; G)$ do not form a spectrum. If X is a topological space let a covering (α_1, α_2) of (X, A) by open sets be called fine if each open subset of an element of α_1 is in α_1 and each open subset of an element of α_2 is in α_2. If Ω is the family of all fine coverings of (X, A), Ω is cofinal in the directed set of all coverings of (X, A) by open sets and hence $H^p(X, A; G, \Omega) \approx H^p(X, A; G)$. If α and β are fine coverings and if $\alpha < \beta$ then $K_{\beta 1}$ is a subcomplex of $K_{\alpha 1}$, $K_{\beta 2}$ is a subcomplex of $K_{\alpha 2}$ and $\pi_{\beta\alpha}: K_\beta \to K_\alpha$ can be chosen to be the inclusion map. The cochain groups $C^p(K_\alpha) = C^p(K_{\alpha 1}, K_{\alpha 2}; G)$ for fine α together with the cochain homomorphisms induced by the inclusion maps $\pi_{\beta\alpha}$ form a direct spectrum whose limit group $C^p(X, A; G)$ is the group of Čech cochains of (X, A). As in (i), the cohomology group $H^p(X, A; G)$ can be found from this cochain group $C^p(X, A; G)$, and $C^p(X, A; G)$ is isomorphic with $C^p(K_\alpha)/C_0^p(K_\alpha)$ where $C_0^p(K_\alpha)$ is the group of "locally zero" cochains of K_α.

(iii) The Čech and Vietoris homology groups, being inverse limit groups, are topological groups; their topology is however usually not of interest. But, if the coverings α of Ω are all finite, the nerves K_α are finite complexes and, if G now is taken to be a division-closure topological group, the chain group $C_p(K_\alpha) = C_p(K_{\alpha 1}, K_{\alpha 2}; G)$ becomes also a topological group, the cycle group $Z_p(K_\alpha)$ is a topological group and the boundary group $B_p(B_\alpha)$ is a closed subgroup of $Z_p(K_\alpha)$. Hence $H_p(K_\alpha)$ is a topological group and the inverse limit group $H_p(X, A; G, \Omega)$ has in general a non-trivial topology. However, if the number of points is infinite, the Vietoris complexes L_α are not finite and the groups $C_p(L_\alpha)$ must be taken as discrete groups. One can however in this case use the isomorphism $\omega: H_p(L_\alpha) \to H_p(K_\alpha)$ to introduce a topology in $H_p(L_\alpha)$. In this way the limit group $H_p(X, A; G, \Omega)$ will acquire a possibly non-trivial topology; in fact the Vietoris and Čech homology groups will be topologically isomorphic.

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