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Spherical Universe topology and the Casimir effect

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Abstract

Recent interest in the possible non-trivial topology of the Universe, and the resulting analysis of the Laplacian eigenproblem, has prompted a reprise of calculations done by ourselves some time ago. The mode problem on the fixed-point-free factored 3-sphere, S^3/Γ , is re-addressed and applied to some field theory calculations for massless fields of spin 0, 1/2 and 1. In particular the degeneracies on the factors, including lens spaces, are rederived more neatly in a geometric fashion. Likewise, the vacuum energies are re-evaluated by an improved technique and expressed in terms of the polyhedrally invariant polynomial degrees, being thus valid for all cases without angle substitution. An alternative, but equivalent expression is given employing the cyclic decomposition of Γ . The scalar functional determinants are also determined. As a bonus, the spectral asymmetry function, $\eta(s)$ is treated by the same approach and explicit forms are given for $\eta(-2n)$ on one-sided lens spaces.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The current interest in the topology of the Universe has led to calculations involving the modes on discrete factors of the sphere, S^d / Γ , with Γ freely acting. These are required both for the spectral analysis of the appearance of the Universe and for quantum field theory calculations.

Weyl raised the question of the topology ('inter-connection') of the Universe in his classic book 'Space–Time–Matter' in 1922 and later cosmic speculations were made by Ellis [1] in connection with the Friedman–Robertson–Walker metric. Milnor [2] has also considered the observational consequences of a non-trivial topology. Some other references can be found in [3, 4], for example. Starkman [5] includes a translation of the pre-GR work of Schwarzschild [6].

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The enumeration of manifolds locally isometric to the sphere, in particular to the 3-sphere, is a textbook matter and factored spheres occur frequently in various contexts seeing that they provide examples of multiply connected spaces that are relatively easy to control. I might mention the topic of analytic torsion. Spheres occur as hypersurfaces and boundaries and these can be replaced by factored spheres as in the analysis of boundary terms in the index theorem, e.g., Gibbons *et al* [7], and in the generalized cone [8, 9].

Our interest in such spaces was originally as examples in connection with quantum mechanics on multiply connected spaces. It was suggested in [10], for example, that the target space in the σ -model could just as well be S^3/Γ as S^3 . Pion perturbation theory would not distinguish between these. The quantum mechanical and field theoretic propagators on S^3/Γ or $T \times S^3/\Gamma$ are given as pre-image sums of those for the full S^3 . The nice review by Camporesi [11] contains extensive information on these sphere, and other homogeneous space, quantities.

In [12], we presented some field theory calculations on multiply connected Clifford– Klein spaces, including the flat (Hantsche and Wendt), $T \times \mathbb{R}^3/\Gamma$, ones and the curved (Seifert and Threlfall) ones, $T \times S^3/\Gamma$. For simplicity, we chose those Γ that produce homogeneous manifolds and the techniques used involved ζ -functions and images. In later calculations, concerned with symmetry breaking by 'Wilson lines' [13, 14], similar ingredients were employed.

In the course of the evaluations we naturally encountered mode properties and expressions for degeneracies. These have occurred in some recent works, e.g., [15], dealing with cosmic topology. In the present work, I wish to re-examine these technical questions while filling in some gaps and extending the earlier discussions. In [12], the Casimir energies on lens spaces S^3/Z_q were given generally in terms of polynomials in q and I wish here to extend these to prism spaces, S^3/D'_q . Some results along these lines have already been given in [13] taken from [16]. One might also wish to consider the case of non-homogeneous manifolds, which were only mentioned in [12]. Of course, any 'realistic' cosmology must be time dependent but since my aim here is simply to exhibit some mathematical details, I consider only the static Einstein Universe, $T \times M$.

In addition to lens and prism spaces, in [12] we also computed the Casimir energies for the other binary polyhedral groups. An objective is to rederive these, the point being that they turned out to be rational quantities arising from combinations of terms containing irrational quantities. The geometric reason is clear. Roughly, each group can be expressed in terms of cyclic groups and we then only have to combine appropriately the above-mentioned polynomials. An elaboration of this might be a good starting point, however some necessary preliminaries have to be recounted. I refer to the mode problem.

The expressions for scalar modes on the full *d*-dimensional sphere go back as far as Green in 1837 and were developed by Hill in 1883 [17]. Later discussions naturally abound and have entered the standard reference works so it is unnecessary to give any sort of comprehensive history here. In the following section I present some basic facts.

2. Modes and degeneracies on the 3-sphere and factored 3-sphere

The 3-sphere case is special because of the isomorphism $SO(4) \sim SU(2) \times SU(2)/Z_2$, essentially a consequence of the isomorphism $S^3 \sim SU(2)$, the two factors corresponding to left and right group actions. The fact that, for a free, discrete action, the factors must be binary polyhedral groups was derived by Seifert and Threlfall [18], although known to Hopf [19]. The classic discussions of these groups are due to Klein [20] and Cayley [21]. There are, of course, many later treatments. Wolf [22] is one standard reference and he also treats the *d*-sphere, see also Milnor [23]. Handy information is available in Coxeter and Moser [24] and in Coxeter [25].

The binary groups also occur in the quantum mechanics of electrons in crystals, the original examination being by Bethe [26]. He calls them *double groups* and his technique has passed into physics textbooks, e.g., Landau and Lifshitz [27]. A more rigorous analysis is provided by Opechowski [28]. The standard mathematical reference [25] does not mention Bethe's work.

The spatial manifold I am concerned with here is, therefore, $\mathcal{M} = S^3/(\Gamma_L \times \Gamma_R)$ with, to repeat, both Γ_L and Γ_R binary polyhedral groups. For a homogeneous space, one of the factors will be trivial, equal to 1, but for a while I keep to the general situation.

One final scene-setting point has to be raised before the calculation is begun. The quantum mechanics, and therefore scalar quantum field theory, on spaces with a non-trivial first homotopy group, $\pi_1(\mathcal{M})$ (which is isomorphic to Γ for free actions) has a freedom coded by the homomorphism, $\pi_1(\mathcal{M}) \rightarrow U(1)$. I do not wish to invoke this freedom in the following. It can easily be incorporated but to do so would extend the algebra, and this paper, unnecessarily.

In order to evaluate the Casimir energy, for example, one needs the equations of motion on $T \times M$. This amounts to a choice of scalar Laplacian on M. One choice is the bare Laplacian, Δ , and another is the 'conformal' Laplacian $\Delta + R/6$, on S^3 . (We define Δ with the sign such that its spectrum is non-negative.) This choice will affect the eigenvalues but not the degeneracies nor the eigenfunctions, and, since it is these I wish to spotlight, I work, for preference, with the conformal Laplacian which makes the eigenvalues on the full sphere squares of integers, say l^2/a^2 , $l = 1, 2, \ldots$, up to a scaling, *a* being the radius. The Laplacian on S^3 coincides with the Casimir operator on SU(2), up to a scale, and the eigenfunctions can be taken as proportional to the complete set of representation matrices, $\mathcal{D}_{mn}^j(g)$, $g \in SU(2)$ with dimensions, l = 2j + 1. This is true for any Lie group. Square integrable completeness is the content of the Peter–Weyl theorem. The classic book by Vilenkin [29] provides all the details one could require. Talman [30] and Miller [31] are also very useful. In this paper, I restrict attention to the 3-sphere where one has the full array of angular momentum techniques to play with.

As is well known, going back at least to Rayleigh, the effect of the factoring, $\mathcal{M} \to \mathcal{M}/\Gamma$, amounts to a cull of the modes on \mathcal{M} . In solid state physics this process is referred to as symmetry adaptation and functions on \mathcal{M}/Γ can be obtained by projection from those on \mathcal{M} , which amounts to averaging over Γ . This process can be traced back, in its general form, to Cartan and Weyl. Making this projection does not always immediately yield quantities of practical value.

Stiefel [32] makes some useful remarks on the application of group theory to the solution of boundary value problems.

One must begin therefore, again, with the scalar modes on the full sphere, S^3 , for which it is sufficient to take the hyperspherical harmonics, $\mathcal{D}_{mn}^j(g)$.

If one requires the explicit form of the eigenmodes, then the traditional method is to select an appropriate coordinate system, separate variables and solve some ordinary differential equations by various means. In this way, the modes were known to Green for arbitrary dimensions, were developed by Hill and related to ambient harmonic polynomials. This is the way the $\mathcal{D}_{mn}^{j}(g)$ are usually evaluated in standard angular momentum references using, for example, Euler angles and involving Jacobi polynomials. Vilenkin [29] has the details, and much else.

As a rule, it is more elegant to use as much group theory (here angular momentum theory) as possible.

Instead of the $\mathcal{D}_{mn}^{j}(g)$ an equivalent set of (scalar) harmonics may be defined by some left–right recoupling,

$$Y_{n:LM}(g) = \left[\frac{(2J+1)(2L+1)}{|\mathcal{M}|}\right]^{1/2} \begin{pmatrix} J & L & m \\ m' & M & J \end{pmatrix} \mathcal{D}_m^{Jm'}(g)$$

where n = 2J + 1 and $|\mathcal{M}|$ is the volume of $SU(2), 2\pi^2 a^3$.

For some purposes these functions are more convenient than the $\mathcal{D}s$. They are associated with the polar coordinate system, (χ, ξ, η) on $S^3 \sim SU(2)$. A group element, $g \in SU(2)$, is parametrized by an angle of rotation, 2χ , and the S^2 angles, ξ, η , specify an axis of rotation, using the language of rotation in the light of the isomorphism, $SO(3) \sim SU(2)/Z_2$.

Explicit formulae for the $Y_{n:LM}$ are derived in the literature (e.g., [30, 33]). There is a neater method than the one used in these references but, since the eigenfunctions are not needed in this paper, I leave it until a later time. Formally, I just use the Ds.

The projected eigenfunctions on S^3/Γ are periodized sums on S^3 in the standard way,

$$\phi_{mn}^{j}(g) = \left[\frac{2j+1}{2\pi^{2}a^{3}|\Gamma_{L}||\Gamma_{R}|}\right]^{1/2} \sum_{\gamma=(\gamma_{L},\gamma_{R})} \mathcal{D}_{mn}^{j}(\gamma_{L}g\gamma_{R}).$$
(1)

This is an example of the more formal, and general, statement that if $\tilde{\phi}_{\lambda_n}(q)$ are the eigenfunctions on the covering manifold, $\tilde{\mathcal{M}}$, then [12, 34]

$$\phi_{\lambda_n}(q) = \frac{1}{\sqrt{|\Gamma|}} \sum_{\gamma} \widetilde{\phi}_{\lambda_n}(\gamma q), \qquad q \in \mathcal{M},$$
⁽²⁾

are periodic eigenfunctions on $\mathcal{M} = \widetilde{\mathcal{M}} / \Gamma$. For convenience, I make no notational distinction between points, q, of $\widetilde{\mathcal{M}}$ and \mathcal{M} .

The standard difficulty is that these projected eigenfunctions are not independent, as constructed, and a certain amount of diagonalization is required. I summarize this well-known state of affairs in the present notation.

From general self-adjointness arguments, both $\phi_{\lambda_n}(q)$ and $\widetilde{\phi}_{\lambda_n}(q)$ must be orthogonal, on \mathcal{M} and $\widetilde{\mathcal{M}}$, respectively, for distinct eigenvalues. They will also form complete sets. Orthogonality means that one can work eigenspace by eigenspace.

Label, in the usual way, the covering eigenfunctions by the eigenvalue λ and an index *i* to take care of any degeneracy. Instead of (2), then define

$$\phi_{\lambda,i}(q) = \frac{1}{\sqrt{|\Gamma|}} \sum_{\gamma} \widetilde{\phi}_{\lambda,i}(\gamma q), \qquad q \in \mathcal{M},$$
(3)

and construct the scalar product

$$P_{ij} \equiv \int_{\mathcal{M}} \mathrm{d}q \,\phi_{\lambda,i}^*(q)\phi_{\lambda,j}(q). \tag{4}$$

Using completeness and eigenspace orthogonality, it is easy to show that P is a projection operator, $P^2 = P$. For the proof start with

$$P_{ij}P_{jk} = \sum_{j} \int_{\mathcal{M}} \mathrm{d}q \int_{\mathcal{M}} \mathrm{d}q' \,\phi_{\lambda,i}^{*}(q)\phi_{\lambda,j}(q)\phi_{\lambda,j}^{*}(q')\phi_{\lambda,k}(q'),\tag{5}$$

and consider the quantity

$$\sum_{j} \phi_{\lambda,j}(q) \phi_{\lambda,j}^{*}(q').$$
(6)

One has completeness on \mathcal{M} ,

$$\sum_{\lambda} \sum_{j} \phi_{\lambda,j}(q) \phi_{\lambda,j}^{*}(q') = \delta(q,q') = \sum_{\gamma} \widetilde{\delta}(\gamma q,q').$$
⁽⁷⁾

Incidentally, this is consistent with the factors in (3) after a group translation. The left-hand side of (7) is

$$\frac{1}{|\Gamma|} \sum_{\gamma,\gamma'} \sum_{\lambda,j} \widetilde{\phi}_{\lambda,j}(\gamma q) \widetilde{\phi}_{\lambda,j}^*(\gamma' q') = \frac{1}{|\Gamma|} \sum_{\gamma,\gamma'} \widetilde{\delta}(\gamma q, \gamma' q') \\
= \frac{1}{|\Gamma|} \sum_{\gamma,\gamma'} \widetilde{\delta}(\gamma'^{-1} \gamma q, q') \\
= \sum_{\gamma} \widetilde{\delta}(\gamma q, q').$$
(8)

Using eigenvalue λ orthogonality on \mathcal{M} , the quantity (6) occurring in (5) can be replaced by the full quantity (7) and the double integral reduced to a single one recognized as P_{ik} as required.

Orthogonality on \mathcal{M} implies the following identity on the covering space:

$$\frac{1}{|\Gamma|} \sum_{\gamma} \int_{\widetilde{\mathcal{M}}} \widetilde{\phi}^*_{\lambda,i}(\gamma q) \widetilde{\phi}_{\lambda',j}(q) \, \mathrm{d}q = \delta_{\lambda\lambda'} P_{ij}, \tag{9}$$

and completeness on $\widetilde{\mathcal{M}}$, used in (8), is

$$\sum_{\lambda,i} \widetilde{\phi}_{\lambda,i}(q) \widetilde{\phi}^*_{\lambda,i}(q') = \widetilde{\delta}(q,q').$$
⁽¹⁰⁾

Now consider the combination

$$\sum_{i} \widetilde{\phi}_{\lambda,i}(q) P_{ij} = \frac{1}{|\Gamma|} \sum_{\gamma} \int_{\widetilde{\mathcal{M}}} \mathrm{d}q' \sum_{i} \widetilde{\phi}_{\lambda,i}(q) \widetilde{\phi}_{\lambda,i}^{*}(\gamma q') \widetilde{\phi}_{\lambda,j}(q') \tag{11}$$

using either (4) + (3) or (9). Replace the sum over *i* in (11) by the complete sum (10) and use (9) to show that the sum over λ is restricted to the single term $\lambda = \lambda$ and so makes no change, but the integral can now be performed and I regain the sum (projection) in (3), so that

$$\phi_{\lambda,j} = \sqrt{|\Gamma|} \sum_{i} \widetilde{\phi}_{\lambda,i}(q) P_{ij}$$

which is the algebraic expression of the projection $\widetilde{\mathcal{M}} \to \mathcal{M}$.

The diagonalization referred to earlier is more precisely that of *P*, which has eigenvalues 1 and 0, the number of 1s, i.e., Tr *P*, being just the degeneracy of the λ level on \mathcal{M} . There is no need to perform the diagonalization to determine this.

Therefore, the degeneracy of the λ eigenvalue is, from (9),

$$d_{\lambda} = \frac{1}{|\Gamma|} \sum_{\gamma} \int_{\widetilde{\mathcal{M}}} \sum_{i} \widetilde{\phi}_{\lambda,i}^{*}(\gamma^{-1}q) \widetilde{\phi}_{\lambda,i}(q) \,\mathrm{d}q.$$
(12)

Diagonalization would be required to determine the independent modes in this direct approach which is not necessarily a practical one.

Equation (12) is a standard result in the theory of symmetry adaptation, familiar in quantum mechanics, and applying it to (1) yields, after some mild group theory [12],

$$d_l = \frac{1}{|\Gamma_L||\Gamma_R|} \sum_{\gamma = (\gamma_L, \gamma_R)} \chi_l(\gamma_L) \chi_l(\gamma_R),$$
(13)

where $\chi_l(g)$ is the character of the spin *j* representation, with l = 2j + 1,

$$\chi_l(g) = \frac{\sin l\theta}{\sin \theta}.$$

 $a\theta$ is the radial distance on S^3 between the origin, corresponding to the unit element of SU(2), and the point q, corresponding to the group element, g. The character is a class function. θ is the colatitude in the polar coordinate system on SU(2). It was denoted by χ earlier and $2\theta = \omega$ equals, as mentioned, the SO(3) rotation angle.

One thus encounters in (13) the quantities

$$d_{l}(\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_{l}(\theta_{\gamma}) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\sin l\theta_{\gamma}}{\sin \theta_{\gamma}},$$
(14)

which can be evaluated for each binary polyhedral group, if desired, since the angles, θ_{γ} , are known and the conjugacy class decompositions can be used to ease the arithmetic. An example for the ordinary cubic group, **O**, is given by Stiefel [32].

Of course, the degeneracy is often combined with other quantities in an eigenmode sum over *l* and then it may be advantageous to leave (14) alone. For example the conformal ζ -function on S^3/Γ is

$$\zeta_{\Gamma}(s) = a^{2s} \sum_{l=1}^{\infty} \frac{d_l(\Gamma_L) d_l(\Gamma_R)}{l^{2s}},$$
(15)

and the sum over l produces two Epstein ζ -functions, in this case. An expression is given later.

Another example is the generating function for χ_l ,

$$\sum_{l=1}^{\infty} \chi_l(\theta) \,\mathrm{e}^{-2\gamma l} = \frac{1}{2} \frac{1}{\cosh(2\gamma) - \cos\theta},\tag{16}$$

obtained by trivial geometric summation. This can often be used for *ad hoc* evaluations. For example, it directly yields the standard generating function for lens space degeneracies. For $\Gamma = Z_q$ the angles θ_{γ} are $\theta_p = p(2\pi/q)$ for p = 0, 1, ..., q - 1 and so one is led to the (binary) cyclic generating function (heat-kernel) setting $t = e^{-2\gamma}$,

$$G(t,q) = \sum_{l=1}^{\infty} d_l(q) t^l = \frac{1}{2q} \sum_{p=0}^{q-1} \frac{1}{\cosh(2\gamma) - \cos(2p\pi/q)}$$
$$= \frac{t(1+t^q)}{(1-t^2)(1-t^q)}.$$
(17)

Expansion of the right-hand side is sufficient to yield expressions for the cyclic degeneracies. If q is even, it follows that $d_l(q)$ is zero for l even, and for l odd we can use the SO(3) result

$$g(\sigma, q) = \sum_{l=0}^{\infty} (2[l/q] + 1)\sigma^{l} = \frac{1}{1 - \sigma} \frac{1 + \sigma^{q}}{1 - \sigma^{q}},$$
(18)

to read off the degeneracy, $d_{2l+1}(q)$, having relabelled $l \rightarrow 2l + 1$ and set $\sigma = t^2$.

It is, nevertheless, still of interest to look at expression (14) directly. Similar finite trigonometric sums have been considered for many years. Most involve sums related to cyclic groups, Z_q . The basic sum is classic and given in Bromwich [35] p 272, example 18,

$$\sum_{p=1}^{q-1} \frac{\sin(kl\pi p/q)}{\sin(l\pi p/q)} = q - k, \quad \text{for} \quad (l,q) = 1,$$

and *k* odd, k < 2q - 1.

Using this formula, one can show that

$$\sum_{p=1}^{q-1} \frac{\sin(2\pi t p/q)}{\sin(2\pi p/q)} = \begin{cases} -t & t \text{ even} \\ q-t & t \text{ odd,} \end{cases}$$

for all integer t and q, 0 < t < q. This allows one to find the Z_q group-averaged SU(2) character,

$$d_{l}(q) = \langle \chi_{l} \rangle_{q} = \frac{1}{q} \sum_{p=0}^{q-1} \chi_{l}(2\pi p/q)$$

$$= \begin{cases} r & t \text{ even} \\ r+1 & t \text{ odd} \end{cases} q \text{ odd}$$

$$= \begin{cases} 0 & l \text{ even} \\ 2r+1 & l \text{ odd} \end{cases} q \text{ even}$$
(19)

where I have made the mod q residue class decomposition, l = rq + t, i.e., $r = \lfloor l/q \rfloor$.

These results are of course in agreement with the preceding calculations. The last result in (19), with $q \rightarrow 2q$ and $l \rightarrow 2l + 1$, is equivalent to the SO(3) character sum,

$$\frac{1}{q} \sum_{p=0}^{q-1} \frac{\sin((2l+1)\pi p/q)}{\sin(\pi p/q)} = 2[l/q] + 1, \qquad l = 0, 1, \dots.$$
(20)

These are all standard results and, in particular, (19) gives the Laplacian degeneracies on simple lens spaces, when multiplied by the left degeneracy, $d_l(1) = l$, according to (13).

The analysis can be extended to general lens spaces by using linked two-sided actions so that γ is labelled by θ_L and θ_R as follows. Going over to the combinations,

$$\alpha = \theta_R + \theta_L, \qquad \beta = \theta_R - \theta_L, \tag{21}$$

the lens space, $L(q; l_1, l_2)$, is defined by setting

$$\alpha = \frac{2\pi p \nu_1}{q}, \qquad \beta = \frac{2\pi p \nu_2}{q}, \tag{22}$$

where p = 0, 1, ..., q - 1, labels γ . ν_1 and ν_2 are integers coprime to q, with l_1 and l_2 their mod q inverses. The simple, 'one-sided' lens space, L(q; 1, 1), corresponds to setting $\nu_2 = \nu_1 = \nu = 1$, say, so that $\theta_L = 0$ and $\theta_R = 2\pi p/q$.

The degeneracy is

$$d_l(q; l_1, l_2) = \frac{1}{q} \sum_{p=0}^{q-1} \frac{\sin(l(\alpha - \beta)/2)}{\sin((\alpha - \beta)/2)} \frac{\sin(l(\alpha + \beta)/2)}{\sin((\alpha + \beta)/2)}$$
$$= \frac{1}{q} \sum_{p=0}^{q-1} \frac{\cos l\alpha - \cos l\beta}{\cos \alpha - \cos \beta}.$$

It is convenient to leave off the group average and consider the (partial) generating function

$$\sum_{l=1}^{\infty} d_l(\alpha, \beta) t^l \equiv \sum_{l=0}^{\infty} \frac{\cos l\alpha - \cos l\beta}{\cos \alpha - \cos \beta} t^l$$
$$= t(1 - t^2) \left(\frac{1}{1 + t^2 - 2t \cos \alpha}\right) \left(\frac{1}{1 + t^2 - 2t \cos \beta}\right), \quad (23)$$

using the elementary summation, cf(16),

$$2\sum_{l=0}^{\infty}\cos(l\alpha)t^{l} = 1 + \frac{1-t^{2}}{1+t^{2}-2t\cos\alpha}.$$
(24)

Equation (23) is the same generating function derived by Ray [36]. (Actually he does p-forms and d-spheres.)

The full degeneracy follows upon averaging over the group elements, i.e., the angles α and β given, for a lens space, by (22). Except for the one-sided case, $\alpha = \pm \beta$, it does not seem possible to complete the sum over p. In this particular case we obtained (17) for the one-sided degeneracy and this can also be found, as a check, from (23) setting $\alpha = \beta$, say, and using the integrated form of (23),

$$\sum_{l=1}^{\infty} \frac{d_l(\alpha, \alpha)}{l} t^l = \frac{t}{1 + t^2 - 2t \cos \alpha}.$$
(25)

The division by *l* on the left corresponds to the removal of the left degeneracy.

Turning to the other binary groups, we need their structure, which is, of course well documented.

The ordinary polyhedral groups, considered as subgroups of SO(3), have a natural action on the 2-sphere. They are generated by rotations through $2\pi/\lambda$, $2\pi/\mu$, $2\pi/\nu$ about the vertices of a spherical triangle of angles π/λ , π/μ , π/ν on S^2 . A fundamental domain is comprised of such a triangle together with its reflection. For the dihedral group, D_q , the fundamental domain can be taken to be the lune, or digon, of apex angle, π/q .

The binary groups are obtained by lifting the action of the ordinary ones using the isomorphism, $SO(3) = SU(2)/Z_2$. Opechowski [28], for example, spells this out.

Coxeter and Moser denote the ordinary group by (λ, μ, ν) and its double by $\langle \lambda, \mu, \nu \rangle$. The double of an ordinary group *G* is variously denoted by *G'* [27], *G[†]* [28], 2*G* [37] and *G*^{*} [22].

The lifting can be accommodated geometrically by replacing the 2-sphere by a two-sheeted Riemann surface with branch points at the vertices of the above spherical triangulation [25], which is, of course, the same triangulation that results from the application of the complete symmetry group of a regular solid.

Algebraically, this doubling is mirrored by the formal introduction, following Bethe, into the presentation of the group of an element, denoted by Q, that commutes with the other generators and satisfies $Q^2 = E$ ($E \equiv id$). Q corresponds to a rotation through 2π . The double group $\langle \lambda, \mu, \nu \rangle$ is generated by L, M and N with relations

$$L^{\lambda} = M^{\mu} = N^{\nu} = LMN = Q$$

$$Q^{2} = E, \qquad [L, Q] = [M, Q] = [N, Q] = 0.$$

I first look at the group with an infinite number of members. This is the binary dihedral group, D'_q . Because, for two-sided actions, one has to substitute in the angles, θ_{γ} , by hand I consider only right actions. I choose to write the generator-relation structure as

$$A^q = B^2 = (AB)^2 = Q, \qquad Q^2 = E,$$

and can thus formally write D'_{q} as the direct sum

$$D'_a = Z_{2a} \oplus Z_{2a}B,$$

where Z_{2q} is generated by A.

The angles θ_{γ} are

$$\theta_{\gamma} = \pi p/q, \qquad \pi \mp \pi p/q, \quad p = 0, \dots, q-1$$

for A^p . The minus sign adjusts the range of θ to be between 0 and π , as is appropriate for the colatitude in polar coordinates on S^3 . Equivalently, in order to be more in tune with the action on the doubly covered 2-sphere, θ can be 'unrolled' to run from 0 to 2π , as on a circle, a great circle in fact. Doing this corresponds to taking the plus sign. Remember, the angle θ is half the rotation angle. (Actually θ can be completely unrolled to be a coordinate on the real line, but this is not relevant here.)

For $\gamma = A^p B$, i.e., those 2q elements containing a (binary) dihedral rotation, $\theta_{\gamma} = \pi/2$ for all γ .

Hence, from (14), the right action degeneracy is

$$d_l(D'_q) = \frac{1 - (-1)^l}{4q} \sum_{p=0}^{q-1} \frac{\sin(l\pi p/q)}{\sin(\pi p/q)} + \frac{1}{2}\sin(l\pi/2),$$

so that *l* is restricted to odd values, when, with $l \rightarrow 2l + 1$,

$$d_{2l+1}(D'_q) = [l/q] + \frac{1}{2}(1 + (-1)^l), \qquad l = 0, 1, \dots,$$
(26)

using the SO(3) formula (20).

A generating function can also be found. Having obtained (26), a simple way is to use (18) which yields

$$\sum_{l=0}^{\infty} d_{2l+1}(D'_q)\sigma^l = \frac{1}{2} \left(\frac{1}{1-\sigma} \frac{1+\sigma^q}{1-\sigma^q} + \frac{1}{1+\sigma} \right) = \frac{1+\sigma^{q+1}}{(1-\sigma^2)(1-\sigma^q)}.$$
 (27)

Recall that the *full* degeneracy on S^3 is obtained by multiplying by the left action degeneracy, 2l + 1, to give

$$(2l+1)d_{2l+1}(D'_q).$$

Note that the S^3 formula for the essential part (27) of the right generating function for D'_q actually coincides with the S^2 formula for D_q . Similar considerations hold for the other double groups as I now discuss.

Referring to the formula for the right degeneracy on $S^3/\Gamma'(14)$, the doubling means that for every θ_{γ} between 0 and π there is another, $\theta_{\gamma} + \pi$, in the range π to 2π . Hence, one can write

$$d_{l}(\Gamma') = \frac{1}{|\Gamma'|} \sum_{\substack{\gamma \in \Gamma' \\ 0 \leqslant \theta_{\gamma} < \pi}} \frac{\sin l\theta_{\gamma} - \sin l(\pi + \theta_{\gamma})}{\sin \theta_{\gamma}}$$
$$= \frac{1 - (-1)^{l}}{2|\Gamma|} \sum_{\substack{\gamma \in \Gamma' \\ 0 \leqslant \theta_{\gamma} < \pi}} \frac{\sin l\theta_{\gamma}}{\sin \theta_{\gamma}},$$
(28)

whence *l* is odd = 2l + 1 so

$$d_{2l+1}(\Gamma') = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\sin(2l+1)\theta_{\gamma}}{\sin \theta_{\gamma}} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\sin(2l+1)\omega_{\gamma}/2}{\sin \omega_{\gamma}/2}$$
(29)

which is the scalar Laplacian degeneracy on the rotational orbifold, S^2/Γ , denoted by $d(l; \Gamma)$. This is best discussed as follows.

For the purely rotational polyhedral groups, let n_q be the number of conjugate q-fold axes. Then the S^2/Γ scalar Laplacian degeneracy is (cf [32, 38]),

$$d(l; \Gamma) = \left(1 - \sum_{q} n_{q}\right) \frac{2l+1}{|\Gamma|} + \frac{1}{|\Gamma|} \sum_{q} q n_{q} d_{q}(l)$$

= $\frac{1}{|\Gamma|} \sum_{q} q n_{q} d_{q}(l) - \frac{2l+1}{2}, \qquad l = 0, 1, \dots,$ (30)

where $d_q(l)$ is the Z_q cyclic degeneracy on S^2 given above as $d_q(l) = 2[l/q] + 1$. The final equality does not hold for Γ itself a cyclic group.

Thus, on S^2 , all that is necessary is to combine the cyclic degeneracies [39, 40]. Expressing things rather in terms of generating functions, for the 2-sphere we have

$$g(\sigma; \Gamma) \equiv \sum_{l=0}^{\infty} d(l; \Gamma) \sigma^{l}$$
$$= \frac{1}{|\Gamma|} \sum_{q} q n_{q} g(\sigma, q) - \frac{1}{2} g(\sigma, 1), \qquad (31)$$

where $g(\sigma, q) = g(\sigma; Z_q)$ is given by (18) and $g(\sigma, 1) = g(\sigma; 1)$.

For example, for the dihedral group, D_q , $n_q = 1$, $n_2 = q$ and simple arithmetic gives

$$g(\sigma; D_q) = \frac{1 + \sigma^{1+q}}{(1 - \sigma^2)(1 - \sigma^q)},$$
(32)

agreeing with (27), and is the standard formula for the dihedral Poincaré series, e.g., [39, 41]. The powers of σ on the denominator are the degrees associated with the dihedrally invariant polynomial basis, e.g., [42].

For the regular solids (not the dihedron), (30) and (31) simplify an application of the orbit-stabilizer relation, $|\Gamma| = 2qn_q$, $\forall q$,

$$d(l;\Gamma) = \frac{1}{2} \left(\sum_{q} d_{q}(l) - 2l - 1 \right), \qquad l = 0, 1, \dots,$$
(33)

and

$$g(\sigma; \Gamma) = \frac{1}{2} \left(\sum_{q} g(\sigma, q) - g(\sigma, 1) \right), \tag{34}$$

which is a rather neat result.

As an example take the octahedral group **O**, for which $n_2 = 6$, $n_3 = 4$ and $n_4 = 3$. Simple arithmetic yields

$$g(\sigma; \mathbf{0}) = \frac{1 + \sigma^9}{(1 - \sigma^4)(1 - \sigma^6)},$$
(35)

for the generating function, obtainable in other ways.

We can use the identity

$$d_{2l+1}(\Gamma') = d(l; \Gamma), \tag{36}$$

together with (33) to get the right degeneracies on S^3/Γ' , most easily,

$$d_{2l+1}(\mathbf{O}') = d(l; \mathbf{O})$$

$$= [l/2] + [l/3] + [l/4] + 1 + l$$

$$d_{2l+1}(\mathbf{T}') = [l/2] + 2[l/3] + 1 + l$$

$$d_{2l+1}(\mathbf{Y}') = [l/2] + [l/3] + [l/5] + 1 + l.$$
(37)

These results are therefore better appreciated as having an \mathbb{R}^3 geometric origin.

The derivation of these somewhat standard formulae given by Ikeda [43] is more involved although he does treat higher spheres. His technique is algebraic and involves resolving the groups into subgroups. His expressions for the lens space degeneracies differ in form from mine.

As I have remarked, the corresponding evaluations in the case of double-sided actions are much harder. Ikeda and Yamamoto [44] examine two-sided lens spaces.

3. Heat-kernels and partition functions

On the unit 3-sphere the eigenvalues of the conformal Laplacian equal l^2 , l = 1, ... and so the integrated heat-kernel associated with the *square root* of this Laplacian (the so-called *cylinder* kernel) on S^3/Γ' equals

$$K^{1/2}(\tau) = \sum_{l=1}^{\infty} l d_l(\Gamma') e^{-l\tau},$$
(38)

which on setting $t = e^{-\tau}$ is recognized as a generating function. This can be related to the polyhedral generating functions $g(\sigma; \Gamma)$ as follows. Define

$$G_{\text{tot}}(t; \Gamma') = K^{1/2}(\tau) = \sum_{l=1}^{\infty} l d_l(\Gamma') t^l.$$
(39)

The filtering process giving the eigenproblem on S^3/Γ' restricts *l* to odd values, as has been shown, and so

$$G_{\text{tot}}(t; \Gamma') = \sum_{l=0}^{\infty} (2l+1) d_{2l+1}(\Gamma') e^{-(2l+1)\tau}$$

= $-\frac{d}{d\tau} \sum_{l=0}^{\infty} d_{2l+1}(\Gamma') e^{-(2l+1)\tau}$
= $-\frac{d}{d\tau} e^{-\tau} \sum_{l=0}^{\infty} d(l; \Gamma) e^{-2l\tau}$
= $-\frac{d}{d\tau} e^{-\tau} g(\sigma; \Gamma),$ (40)

with $\sigma = t^2 = e^{-2\tau}$.

As an organizational point I note that these results do not apply, directly, to odd lens spaces, in particular to Z_1 , i.e., to the *full* 3-sphere. This remark applies to later results too.

We can write the general rotation generating function

$$g(\sigma; \Gamma) = \frac{1 + \sigma^{\delta_0}}{(1 - \sigma^{\delta_1})(1 - \sigma^{\delta_1})},\tag{41}$$

in terms of the degrees δ_0 , δ_1 , δ_2 and can take things further as in our developments in [38]. Simple algebra gives, from (40),

$$K^{1/2}(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\cosh(\delta_0 \tau)}{2\sinh(\delta_2 \tau)\sinh(\delta_1 \tau)}.$$
(42)

I just mention that a possible direct physical interpretation of this quantity occurs in thermal field theory on the spacetime, $T \times S^3 / \Gamma'$, because the free energy is given by

$$F(\beta) = E - \frac{1}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} K^{1/2}(m\beta),$$
(43)

where $\beta = 1/kT$. Kennedy [45] gives some discussion of thermal quantities on this factored Einstein Universe.

In (43), E is the vacuum, zero temperature energy and can be called the Casimir energy and I now turn to its evaluation. The numbers were derived in [12] essentially by direct substitution of group properties. I now present a more systematic method.

4. Casimir energies on spherical factors

Since I am concerned, at least initially, with exposing general techniques, I restrict to a conformally invariant scalar field theory. In this case, for a freely acting Γ' on $T \times S^3 / \Gamma'$, there are no divergences to bother us. As a consequence, the only other basic result one needs is that the Casimir energy is given by the value of the ζ -function on S^3 / Γ' , $\zeta(s)$, at s = -1/2,

$$E = \frac{1}{2}\zeta\left(-\frac{1}{2}\right).\tag{44}$$

The essential calculational point is that the ζ -function, $\zeta(s)$, for the Laplacian on S^3/Γ' is related to the ζ -function for the square root of the Laplacian by simply

$$\zeta(s) = \zeta^{1/2}(2s), \tag{45}$$

and the latter quantity is given by the standard Mellin transform

$$\zeta^{1/2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}\tau \ \tau^{s-1} K^{1/2}(\tau), \tag{46}$$

with $K^{1/2}$ as in (42). One then has the continuation

$$\zeta(s) = \frac{i\Gamma(1-2s)}{2\pi} \int_C d\tau (-\tau)^{2s-1} K^{1/2}(\tau),$$
(47)

where *C* is the Hankel contour.

Looking at (42) and integrating by parts gives

$$\zeta(s) = \frac{i\Gamma(2-2s)}{2\pi} \int_C d\tau(-\tau)^{2s-2} H(\tau),$$
(48)

where

$$H(\tau) = \frac{\cosh(\delta_0 \tau)}{2\sinh(\delta_2 \tau)\sinh(\delta_1 \tau)}.$$
(49)

Note that $H(\tau/2)$ is a square-root heat-kernel on the unit orbifold S^2/Γ [38], the operator being the conformal one, $\mathbf{L}^2 + 1/4$, with eigenvalues $(2l + 1)^2/4$, l = 0, 1, ..., and degeneracy 2l + 1.

This means that it is possible to relate the ζ -functions on S^3/Γ' and S^2/Γ . Each is given by the general formulae (45) and (46) where, for S^3/Γ' , $K^{1/2}$ is given by (42) while, for the unit S^2/Γ , it equals $H(\tau/2)$ (49), so one has the relation

$$K_{S^3/\Gamma'}^{1/2}(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau} K_{S^2/\Gamma}^{1/2}(2\tau).$$

Substitution of this into the previous equations easily yields the relation

$$\zeta_{S^3/\Gamma'}(s) = 2^{1-2s} \zeta_{S^2/\Gamma}(s-1/2), \tag{50}$$

the simplest example of which is for $\Gamma = 1$, when the ζ -functions are Riemann, or better, Hurwitz ones. The details are mildly instructive. For $\Gamma = 1$, the doubled group is $\Gamma' = Z_2$, giving the *projective* 3-sphere. It is well known (e.g., [46, 10]) that *l* is then restricted to odd values so

$$\zeta_{S^3/Z_2}(s) = \sum_{\text{odd}} \frac{n^2}{n^{2s}},$$
(51)

and we know that

$$\zeta_{S^2}(s) = 2^{2s} \sum_{\text{odd}} \frac{n}{n^{2s}},$$

which confirms (50). Only even lens spaces are accessible via (50). To avoid misunderstandings, it should be emphasized that S^2/Γ refers to an *orbifold* quotient. Γ has fixed points on the 2-sphere.

Equation (50) just reflects the relation (36) between the degeneracies $(2l + 1)d_{2l+1}(\Gamma') = (2l + 1)d(l; \Gamma)$, and can, of course, be deduced immediately from this.

Equation (50) at the point s = 0 relates the conformal anomaly in three dimensions to the Casimir energy in two. Both are zero.

It is now a simple matter to set s = -1/2 and evaluate the integral (48) by residues. One finds

$$E_{\Gamma'} = \frac{15\delta_0^4 - 30\delta_0(\delta_1^2 + \delta_2^2) + 7\delta_1^4 + 10\delta_1^2\delta_2^2 + 7\delta_2^4}{720\delta_1\delta_2},\tag{52}$$

the actual numbers being

$$E_{\mathbf{T}'} = -\frac{3761}{8640}, \qquad E_{\mathbf{O}'} = -\frac{11\,321}{17\,280}, \qquad E_{\mathbf{Y}'} = -\frac{43\,553}{43\,200},$$
 (53)

in agreement with our earlier evaluations [12], but without the rather *ad hoc* computations employed there and outlined in section 6. It is obvious from the start that the values are rational. Although $\delta_0 = \delta_1 + \delta_2 - 1$, the expressions are neater if δ_0 is retained.

The expression for the dihedral D'_q case is easily obtained from (52) as

$$E_{D'_q} = -\frac{8q^4 + 20q^2 + 180q - 7}{1440q},$$
(54)

and the cyclic Z_q values are

$$E_{Z_q} = -\frac{q^4 + 10q^2 - 14}{720q}.$$
(55)

5. Cyclic decompositions

Instead of treating each group one by one, labelled by the corresponding degrees, it is possible, perhaps more economically, to use the cyclic decompositions (31) or (34) [39], p 139, which clearly translate into cyclic decompositions of the ζ -functions [38], and thence of the Casimir energies. For example, from (34) and (50)

$$\zeta_{\Gamma'}(s) = \frac{1}{2} \left(\sum_{q} \zeta_{Z_{2q}}(s) - \zeta_{Z_2}(s) \right),$$
(56)

and so, in particular,

$$E_{\Gamma'} = \frac{1}{2} \left(\sum_{q} E_{Z_{2q}} - E_{Z_2} \right), \tag{57}$$

which works for \mathbf{T}' , \mathbf{O}' and \mathbf{Y}' using (55).

The ζ -function (48) is related to the Barnes ζ -function employed in earlier works. In view of (50) we can equivalently repeat the formula for the 2-sphere case [38]

$$\zeta_{S^2/\Gamma}(s) = \zeta_2(2s, 1/2 \mid \delta_1, \delta_2) + \zeta_2(2s, \delta_1 + \delta_2 - 1/2 \mid \delta_1, \delta_2),$$
(58)

where, generally,

$$\zeta_d(s, a \mid \omega) = \frac{\mathrm{i}\Gamma(1-s)}{2\pi} \int_C \mathrm{d}\tau \, \frac{\exp(-a\tau)(-\tau)^{s-1}}{\prod_{i=1}^d (1-\exp(-\omega_i\tau))}$$
$$= \sum_{\mathbf{m}=0}^\infty \frac{1}{(a+\omega\cdot\mathbf{m})^s}.$$
(59)

The residues and values of the Barnes function are given in terms of generalized Bernoulli functions, of which (52) is an example and it is clear that this whole process can easily be automated and extended to the higher spheres.

6. Cosecant sums

From the basic definition (15) the one-sided 3-sphere ζ -function emerges directly as a sum of derivatives of Epstein ζ -functions [12]

$$\zeta(s) = -\frac{1}{2|\Gamma'|} \sum_{\gamma} \frac{1}{\sin \theta_{\gamma}} \frac{\partial}{\partial \theta_{\gamma}} Z \begin{vmatrix} 0\\ \theta_{\gamma}/2\pi \end{vmatrix} (2s), \tag{60}$$

where Z is the simplest Epstein function (it has other names),

$$Z \begin{vmatrix} 0\\ \theta/2\pi \end{vmatrix} (2s) = \sum_{-\infty}^{\infty} \frac{e^{in\theta}}{n^{2s}}.$$
(61)

I will denote it by $Z_E(\theta, s)$, for short.

From this expression an alternative form of the Casimir energy was derived in [12]. From the standard formula

$$2\sum_{l=1}^{\infty} \sin l\theta = \cot(\theta/2), \tag{62}$$

one finds

$$E = \frac{1}{|\Gamma'|} \left[\frac{1}{240} - \frac{1}{16} \sum_{\gamma \neq \mathbf{1}} \operatorname{cosec}^4 \left(\frac{\theta_{\gamma}}{2} \right) \right].$$
(63)

More generally, in the two-sided case

$$E = \frac{1}{|\Gamma'|} \left[\frac{1}{240} - \frac{1}{16} \sum_{\gamma \neq \mathbf{1}} \operatorname{cosec}^2\left(\frac{\alpha}{2}\right) \operatorname{cosec}^2\left(\frac{\beta}{2}\right) \right],\tag{64}$$

in terms of the angles (21). For right actions only, $\alpha = \beta = \theta_R = \theta_{\gamma}$.

For lens and prism spaces one can use standard, and often very old (some dating to Euler) finite sums of powers of cosecants to give polynomials in q, in agreement with the results stated earlier. For the other groups, direct substitution of the angles yielded the values in (53) after cancellations. See also the computations in Gibbons *et al* [7].

Of course, some derivations of these cosecant sums boil down to residue evaluations and so these sums in themselves are somewhat of a detour. A brief history was attempted in [47]

and more references can be found in Berndt and Yeap [48]. Some explicit expressions are given later in connection with the corresponding spinor calculation.

From the purely numerical aspect, an Epstein approach is made more attractive by the existence of an exponentially convergent series involving the incomplete Γ -function, $\Gamma(s, a)$, for which there is a rapid, continued fraction algorithm. Against this must be set the fact that the angles θ_{γ} have to be individually put in.

We have used this method before. Here, I consider its use for the evaluation of $\zeta'(0)$. The relevant expression is [13]

$$\pi^{-s}\Gamma(s)\frac{\partial}{\partial\theta}Z_E(\theta,s) = -2\sum_{n=1}^{\infty}n\sin(n\theta)\Gamma(s,\pi n^2) - \sum_{n=-\infty}^{\infty}(n+h)\Gamma((3-2s)/2,\pi(n+h)^2)$$
(65)

with $h = \theta/2\pi$. The transformations leading to this expression are already in Epstein [49].

An important analytical fact about this formula is that it has exactly the combination needed to compute $\zeta'(0)$. To see this we need only note that

$$\lim_{s \to 0} \Gamma(s) f(s) \sim f'(0) + f(0) \left(\frac{1}{s} + \gamma\right)$$

and that $Z_E(\theta, 0) = 0$. (There is no conformal anomaly on S³/ Γ' .) Therefore, one has quite simply

$$\frac{\partial}{\partial \theta} Z'_E(\theta, 0) = -2\sum_{n=1}^{\infty} n \sin(n\theta) \Gamma(0, \pi n^2) - \sum_{n=-\infty}^{\infty} (n+h) \Gamma(3/2, \pi (n+h)^2), \tag{66}$$

which can be substituted into

$$\begin{aligned} \zeta'(0) &= -\frac{1}{2|\Gamma'|} \sum_{\gamma} \frac{1}{\sin \theta_{\gamma}} \frac{\partial}{\partial \theta_{\gamma}} Z'_{E}(\theta_{\gamma}, 0) \\ &= \frac{2}{|\Gamma'|} \left(\zeta'_{R}(-2) - \frac{1}{4} \sum_{\gamma \neq 1} \frac{1}{\sin \theta_{\gamma}} \frac{\partial}{\partial \theta_{\gamma}} Z'_{E}(\theta_{\gamma}, 0) \right), \end{aligned}$$
(67)

and the sum over $\gamma \neq 1$ for T', O' and Y' performed angle by angle, as mentioned before.

This expression is not pursued here because another route to this quantity is given in the next section.

7. Functional determinants

Formula (67) allows one to compute the Laplacian determinant, $\exp(-\zeta'(0))$. Alternatively, equations (50) and (58) mean that it is possible to find expressions for the functional determinants on the factored 3-sphere in terms of the Barnes function, and thence, if desired, of the Hurwitz ζ -function, which is often how these answers are left. In [50, 51] we have discussed such questions and again can make use of this work here. Of course, there are many other relevant references, but this is not a historical work.

From relation (50) one gets

$$\zeta_{S^3/\Gamma'}'(0) = 2\zeta_{S^2/\Gamma}'(-1/2),\tag{68}$$

where I have used the vanishing of $\zeta_{S^3/\Gamma'}(0)$, at least for conformal scalars and spinors. From (58) it is seen that one is required to evaluate

$$\zeta'_2(-1, a \mid \delta_1, \delta_2)$$
 $(a = 1/2, \ \delta_1 + \delta_2 - 1/2)$

and the problem devolves upon computing the derivative of the Barnes function at negative integers. This has been treated in [50, 51]. The analysis in [50] allows one to obtain 'exact' expressions in terms of derivatives of the 'lower' Hurwitz ζ -function. The procedure involves breaking up the summation over **m** in the Barnes function (59) using residue classes.

Rather than treating general degrees, it is somewhat easier to calculate $\zeta'(0)$ on lens spaces and then use the cyclic decomposition (56). Relation (68) specializes to

$$\zeta_{S^3/Z_{2a}}'(0) = 2\zeta_{S^2/Z_a}'(-1/2).$$
(69)

The degrees for the lens case are $\delta_1 = q$, $\delta_2 = 1$. Using residue classes mod q, manipulation of the sum definition of the Barnes function yields the expression

$$\zeta_{S^2/Z_q}(s) = \frac{2}{q} \zeta_R\left(2s-1, \frac{1}{2}\right) + \zeta_R\left(2s, \frac{1}{2}\right) - \frac{1}{q^{2s+1}} \sum_{p=0}^{q-1} (2p+1)\zeta_R\left(2s, \frac{2p+1}{2q}\right)$$
(70)

which was referred to as the orbifolded S^2 rotational ζ -function in [38] and was used in [10] to compute 2-sphere determinants. Here one requires the value of the derivative at s = -1/2,

$$\begin{aligned} \zeta_{S^2/Z_q}'\left(-\frac{1}{2}\right) &= 2\zeta_R'\left(-1,\frac{1}{2}\right) + \frac{4}{q}\zeta_R'\left(-2,\frac{1}{2}\right) + 2\log q \sum_{p=0}^{q-1} (2p+1) \\ &\times \zeta_R\left(-1,\frac{2p+1}{2q}\right) - 2\sum_{p=0}^{q-1} (2p+1)\zeta_R'\left(-1,\frac{2p+1}{2q}\right) \\ &= \frac{1}{12}\log(q/2) - \zeta_R'(-1) - \frac{3}{q}\zeta_R'(-2) - 2\sum_{p=0}^{q-1} (2p+1)\zeta_R'\left(-1,\frac{2p+1}{2q}\right) \end{aligned}$$
(71)

which could, possibly, be thought of as 'exact' but is, at least, in a form suitable for numerical treatment, cf Nash and O'Connor [52]. In the derivation of this formula further use has been made of the fact that $\zeta_{S^2/Z_q}(-1/2)$ is zero.

It is possible to find an alternative expression for the ζ -function that displays this vanishing and allows the derivatives to be avoided, in analogy to the result $\zeta'_R(-2) = -\zeta_R(3)/4\pi^2$. The details are given in [38] that give rise to the alternative form

$$\zeta_{S^2/Z_q}(s) = \frac{2}{q} \zeta_R \left(2s - 1, \frac{1}{2} \right) + \frac{2^{2s} \Gamma(1 - 2s) \cos \pi s}{q \pi^{1 - 2s}} \\ \times \sum_{p=1}^{q-1} \frac{1}{\sin(\pi p/q)} \left(\zeta_R \left(1 - 2s, \frac{p}{q} \right) - 2^{2s} \zeta_R \left(1 - 2s, \frac{p + q}{2q} \right) \right), \tag{72}$$

showing the zeros at $s = -(2k+1)\pi/2$ with k = 0, 1, ... Despite appearances, the only pole is at s = 1, correctly.

The required derivative follows as the numerically easier formula

$$\zeta_{S^2/Z_q}'\left(-\frac{1}{2}\right) = \frac{3}{4\pi^2 q} \zeta_R(3) + \frac{1}{2q\pi} \sum_{p=1}^{q-1} \frac{1}{\sin(\pi p/q)} \left(\zeta_R\left(2, \frac{p}{q}\right) - \frac{1}{2} \zeta_R\left(2, \frac{p+q}{2q}\right)\right).$$

In order to cover \mathbf{T}' , \mathbf{O}' and \mathbf{Y}' the values q = 1, 2, 3, 4 and 5 are needed, the simple q = 1 case being already given in (51).



Figure 1. $W = -\log \det$ for conformal scalars on lens spaces of order 2q.

I present the numbers for the scalar (conformal) determinant, det = $e^{-\zeta'(0)}$, on S^3/Γ' , det(**T**') = 0.202 0887, det(**O**') = 0.128 7757, det(**Y**') = 0.073 0560, and display a graph of $W = -\log$ det for the even, one-sided lens spaces (see figure 1). The determinant tends to zero as $q \to \infty$.

8. Spinors

To analyse the astrophysical data one needs only the scalar harmonics. However, it is within our scope to consider other fields and I now lay out some comments on the Dirac field without going into too many details as most of these are available elsewhere.

The eigenproblem for spin-half on spheres, and therefore on the Einstein Universe, is well known, going back at least to Schrödinger. Basic facts are that the eigenvalues of the squared, massless Dirac operator are given by

$$\lambda_n = \frac{1}{a^2} (n+1/2)^2, \qquad n = 1, 2, \dots$$
 (73)

with degeneracies 2n(n + 1) for a two-component field on the full sphere.

On S^3/Γ' the total degeneracies (left times right) are

$$D_n(\Gamma') = \frac{1}{|\Gamma'|} \sum_{\gamma} ((n+1)\chi_n(\theta_{\gamma}) + n\chi_{n+1}(\theta_{\gamma})).$$
(74)

The two parts to D_n correspond to the fact that the positive and negative eigenvalues of the Dirac operator have been combined into (73). Look at the two parts in turn using the previous analysis of the right degeneracy (14) on S^3/Γ' . The first part is zero unless *n* is odd and the second is zero unless *n* is even, see (29). (I am again excluding odd lens spaces.) Thus in the first part, set n = 2l + 1, and in the second n = 2l + 2 with l = 0, 1, ... in both cases. Hence, from (36)

$$D_{2l+1}(\Gamma') = (2l+2)d(l;\Gamma), \qquad D_{2l+2}(\Gamma') = (2l+2)d(l+1;\Gamma), \tag{75}$$

in terms of the S^2/Γ degeneracies. Our previous formulae, e.g., (37), can be used to make (75) more explicit.

The heat-kernel for the (positive) square root of the squared massless Dirac operator on S^3/Γ' is then, cf (38),

$$K_{S}^{1/2}(\tau) = \sum_{n=1}^{\infty} D_{n}(\Gamma') e^{-(n+1/2)\tau}$$

$$= \sum_{l=0}^{\infty} (2l+2)d(l;\Gamma) e^{-(2l+3/2)\tau} + \sum_{l=0}^{\infty} (2l+2)d(l+1;\Gamma) e^{-(2l+5/2)\tau}$$

$$= -e^{\tau/2} \frac{d}{d\tau} \sum_{l=0}^{\infty} d(l;\Gamma) e^{-(2l+2)\tau} - e^{-\tau/2} \frac{d}{d\tau} \sum_{l=0}^{\infty} d(l+1;\Gamma) e^{-(2l+2)\tau}$$

$$= -e^{\tau/2} \frac{d}{d\tau} e^{-2\tau} \sum_{l=0}^{\infty} d(l;\Gamma) e^{-2l\tau} - e^{-\tau/2} \frac{d}{d\tau} \sum_{l=0}^{\infty} d(l+1;\Gamma) e^{-(2l+2)\tau}$$

$$= -e^{\tau/2} \frac{d}{d\tau} e^{-2\tau} \sum_{l=0}^{\infty} d(l;\Gamma) e^{-2l\tau} - e^{-\tau/2} \frac{d}{d\tau} \sum_{l=0}^{\infty} d(l;\Gamma) e^{-2l\tau}, \quad (76)$$

where a zero term has been added to the second sum. One can set $\sigma = e^{-2\tau}$ in order to make contact with the *SO*(3) generating functions (31) which one notes from [38] are related to the S^2/Γ Laplacian square-root heat-kernels, $H(\tau/2)$ (49) by

$$g(\sigma; \Gamma) = e^{\tau} H(\tau) \tag{77}$$

and so

$$K_{S}^{1/2}(\tau) = -e^{\tau/2} \frac{d}{d\tau} e^{-\tau} H - e^{-\tau/2} \frac{d}{d\tau} e^{\tau} H$$

= $-2 \sinh(\tau/2) H - 2 \cosh(\tau/2) \frac{d}{d\tau} H.$ (78)

The spinor ζ -function is given by the general formula (47) with (78) and the derivative can again be removed by an integration by parts yielding two integrals

$$\zeta_{S}(s) = \frac{i\Gamma(2-2s)}{\pi} \int_{C} d\tau (-\tau)^{2s-2} \cosh(\tau/2) H(\tau) - \frac{i\Gamma(1-2s)}{2\pi} \int_{C} d\tau (-\tau)^{2s-1} \sinh(\tau/2) H(\tau).$$
(79)

We can confirm from this that $\zeta_S(0) = 0$.

For the two-component spinor Casimir energy

$$E_{\Gamma'} = -\frac{1}{2}\zeta_S\left(-\frac{1}{2}\right),\tag{80}$$

a standard residue evaluation gives

$$E_{\Gamma'} = \frac{1}{5760\delta_1\delta_2} \left(128\delta_1^4 + 128\delta_2^4 - 640\delta_1^2\delta_2^2 + 1920\delta_1^2\delta_2 + 1920\delta_1\delta_2^2 - 4320\delta_1\delta_2 - 1440\delta_1^2 - 1440\delta_2^2 + 2400\delta_1 + 2400\delta_2 - 1005 \right).$$
(81)

In particular,

$$E_{Z_{2q}} = \frac{128q^4 - 160q^2 + 83}{5760q}$$

$$E_{D'_q} = \frac{128q^4 - 160q^2 + 1440q + 83}{11520q}$$

$$E_{\mathbf{T}'} = \frac{40\,211}{69\,120}, \qquad E_{\mathbf{O}'} = \frac{135\,251}{138\,240}, \qquad E_{\mathbf{Y}'} = \frac{567\,443}{345\,600}.$$
(82)

The last values can also be obtained from the cyclic decomposition (57), which is true generally.

9. The angle form

As a check, I derive the spinor equivalent of (63), which can be called the image form of the vacuum energy. This follows on a direct evaluation of the original summation expression for the ζ -function

$$\zeta(s) = \sum_{n} \frac{d_{n}}{\lambda_{n}^{s}} = \frac{1}{|\Gamma'|} \sum_{\gamma} \sum_{n=1}^{\infty} \frac{1}{(n+1/2)^{2s}} ((n+1)\chi_{n}(\theta_{\gamma}) + n\chi_{n+1}(\theta_{\gamma})).$$
(83)

The only divergent term is for the identity, $\gamma = E = id$, $\theta_{\gamma} = 0$, which is easily treated by continuing to a Hurwitz ζ -function:

$$\zeta_E(s) = \frac{2}{|\Gamma'|} \sum_{n=1}^{\infty} \frac{n(n+1)}{(n+1/2)^{2s}}$$
$$= \frac{2}{|\Gamma'|} \left(\zeta_R(2s-2, 1/2) - \frac{1}{4} \zeta_R(2s, 1/2) \right),$$
(84)

a very old expression as, apart from the volume factor, $|\Gamma'|$, this is just the full sphere result. It could be rearranged in several inessential ways. For example, at the sum level, one can introduce $\overline{n} = 2n + 1$ and rewrite the sum over odds as all minus evens.

The other terms, $\gamma \neq E$, do not diverge at s = -1/2, and I can proceed directly with the sum as it stands. The Casimir energy (80) is

$$E_{\Gamma'} = -\frac{1}{|\Gamma'|} \left(\zeta_R(-3, 1/2) - \frac{1}{4} \zeta_R(-1, 1/2) \right) - \frac{1}{|\Gamma'|} \sum_{\gamma \neq E} \sum_{n=1}^{\infty} \left(n^2 + \frac{1}{2} \right) \chi_n(\theta_{\gamma})$$

$$= \frac{1}{8|\Gamma'|} \left(\frac{17}{120} - \sum_{\gamma \neq E} (\operatorname{cosec}^2 \theta_{\gamma}/2 - \operatorname{cosec}^4 \theta_{\gamma}/2) \right),$$
(85)

using (62).

The $cosec^4$ sum is that occurring in the scalar vacuum energy and the $cosec^2$ part is a novelty occasioned by the spectral asymmetry of the Dirac operator on a factored space.

Defining the cosecant sums

$$C(r; \Gamma') = \frac{1}{|\Gamma'|} \sum_{\gamma \neq E} \operatorname{cosec}^{2r} \theta_{\gamma}/2,$$
(86)

brute force angle substitution gives

$$C(1; \mathbf{T}') = \frac{167}{72}, \qquad C(2; \mathbf{T}') = \frac{1505}{216}$$

$$C(1; \mathbf{O}') = \frac{383}{144}, \qquad C(2; \mathbf{O}') = \frac{4529}{432}$$

$$C(1; \mathbf{Y}') = \frac{1079}{360}, \qquad C(2; \mathbf{Y}') = \frac{87\,109}{5400},$$
(87)

and the old summations mentioned earlier are

$$C(1; Z_q) = \frac{q^2 - 1}{3q}, \qquad C(2; Z_q) = \frac{(q^2 + 11)(q^2 - 1)}{45q}$$

$$C(1; D'_q) = \frac{4q^2 + 12q - 1}{12q}, \qquad C(2; D'_q) = \frac{16q^4 + 40q^2 + 360q - 11}{180q},$$
(88)

the last sum by Jadhav, in this way. Combining these values yields the spinor Casimir energies (82) previously obtained by the alternative method involving the degrees.

10. The Maxwell field

To complete the set of standard fields I now consider massless spin-one. Actually, it is possible to treat all three spins, 0, 1/2 and 1, together [53], but, for transparency, it has been decided to keep them apart.

The solution of Maxwell's equations on the Einstein Universe is well known and again goes back to Schrödinger. I deal with transverse fields. The eigenvalues of the square of the first-order curl operator are

$$\lambda_n = n^2, \qquad n = 2, 3, \dots \tag{89}$$

with degeneracies $d_n = 2(n^2 - 1)$ on the full sphere. On the factored sphere, the (total) degeneracies are

$$d_{n}(\Gamma') = \frac{1}{|\Gamma'|} \sum_{\gamma} ((n+1)\chi_{n-1}(\theta_{\gamma}) + (n-1)\chi_{n+1}(\theta_{\gamma}))$$
(90)

and again the existence of two parts can be ascribed to a spectral asymmetry.

For Maxwell theory, there is a gauge question. In addition to the transverse field (i.e., coexact 1-form), one must subtract a harmonic zero form. One way of doing this, formally, on the full sphere is to extend and double up the summation range. For the ζ -function,

$$\zeta(s) = \sum_{-\infty}^{\infty} \frac{n^2 - 1}{n^{2s}} = 2(\zeta_R(2s - 2) - \zeta_R(2s)).$$
(91)

Although it seems nothing has been done, the value $\zeta(0) = 1$ can now be interpreted as a consequence of the ghost zero mode and *not* as an indication of a constant term in the expansion of the heat-kernel [54]. These considerations can be dispensed with if one is concerned just with the vacuum, zero-point energy. They would come into play for functional determinants but these are left for another time.

Returning to (90), the previous analysis shows that *n* must be even, n = 2l + 2, and so for the Maxwell cylinder heat-kernel

$$K_{M}^{1/2}(\tau) = \sum_{l=0}^{\infty} ((2l+3)d_{l}(\Gamma) + (2l+1)d_{l+1}(\Gamma)) e^{-(2l+2)\tau}$$
$$= e^{\tau} \frac{d}{d\tau} e^{-2\tau} H(\tau) + e^{-\tau} \frac{d}{d\tau} e^{2\tau} (H(\tau) - e^{-\tau})$$
(92)

using (31) and (77). The corresponding ζ -function is

$$\zeta_M(s) = \frac{i\Gamma(2-2s)}{2\pi} \int_C d\tau (-\tau)^{2s-2} (2\cosh\tau H(\tau) - 1) - \frac{i\Gamma(1-2s)}{2\pi} \int_C d\tau (-\tau)^{2s-1} (2\sinh\tau H(\tau) - 1).$$
(93)

A simple check is the value $\zeta_M(0) = 1$ which arises from the '1' term in the second integrand and which has a zero mode connotation. One confirms that it does not contribute to the residue when evaluating $\zeta_M(-1/2)$ and finds for the Casimir energy

$$E_{\Gamma'} = -\frac{1}{90\delta_1\delta_2} \left(2\delta_1^4 + 2\delta_2^4 - 10\delta_1^2\delta_2^2 + 30\delta_1^2\delta_2 + 30\delta_1\delta_2^2 - 135\delta_1\delta_2 - 45\delta_1^2 - 45\delta_2^2 + 105\delta_1 + 105\delta_2 - 60 \right).$$
(94)

Explicit values are

$$E_{Z_{2q}} = -\frac{2q^4 - 25q^2 + 2}{90q}$$

$$E_{D'_q} = -\frac{2q^4 - 25q^2 - 45q + 2}{180q}$$

$$E_{\mathbf{T}'} = \frac{79}{270}, \qquad E_{\mathbf{O}'} = \frac{23}{1080}, \qquad E_{\mathbf{Y}'} = -\frac{698}{1350}.$$
(95)

The Maxwell vacuum energy is negative on dodecahedron space.

The angle form can again be produced as a check, and for interest. We have [13]

$$E_{\Gamma'} = \frac{1}{|\Gamma'|} (\zeta_R(-3, 1) - \zeta_R(-1, 1)) + \frac{1}{|\Gamma'|} \sum_{\gamma \neq E} \sum_{n=1}^{\infty} (n^2 + 2) \chi(\theta_{\gamma})$$
$$= \frac{1}{2|\Gamma'|} \left(\frac{11}{60} + \sum_{\gamma \neq E} \left(\operatorname{cosec}^2 \theta_{\gamma} / 2 - \frac{1}{4} \operatorname{cosec}^4 \theta_{\gamma} / 2 \right) \right), \tag{96}$$

and, of course, calculation gives agreement with (95).

11. Spectral asymmetry

As another example of the use of the eigenvalue expressions I will derive expressions for the spectral asymmetry quantity that occurs as a boundary correction to the index theorem, applied to four dimensions. Textbook discussions concern the Dirac equation, the signature and the de Rham complex, e.g., [55], and the corresponding literature is extensive. My treatment of context and content will be brief.

The Atiyah–Patodi–Singer spectral asymmetry function $\eta(s)$ is

$$\eta(s) = \sum_{\lambda}^{\prime} \frac{(\operatorname{sign} \lambda)}{|\lambda|^s}.$$

Restricting to one-sided quotients, the construction of η corresponds, effectively, to changing the sign of the second, negative spectrum part of (74), (90) or of (76), (92) and also

setting $2s \rightarrow s$. Following this through gives

$$\eta_{S}(s) = -\frac{i\Gamma(2-s)}{\pi} \int_{C} d\tau (-\tau)^{s-2} \sinh(\tau/2) H(\tau) + \frac{i\Gamma(1-s)}{2\pi} \int_{C} d\tau (-\tau)^{s-1} \cosh(\tau/2) H(\tau),$$
(97)

for spin-half and

$$\eta_M(s) = -\frac{i\Gamma(2-s)}{2\pi} \int_C d\tau (-\tau)^{s-2} (2\sinh\tau H(\tau) - 1) + \frac{i\Gamma(1-s)}{2\pi} \int_C d\tau (-\tau)^{s-1} (2\cosh\tau H(\tau) - 1),$$
(98)

in the Maxwell case. It can again be seen that $\eta(-(2n + 1)) = 0$ from the vanishing of any residues. Furthermore, other values can be readily found. Consider $\eta(-2n)$ and start with $\eta(0)$. Straightforward computation of residues yields

$$\eta_{S}(0) = \frac{4\delta_{1}^{2} + 4\delta_{2}^{2} + 12\delta_{1}\delta_{2} - 12\delta_{2} - 12\delta_{1} + 7}{12\delta_{1}\delta_{2}}$$

$$\eta_{M}(0) = \frac{2\delta_{1}^{2} + 2\delta_{2}^{2} + 3\delta_{1}\delta_{2} - 6\delta_{2} - 6\delta_{1} + 5}{3\delta_{1}\delta_{2}}$$
(99)

and one can, once more, avoid the angle substitution employed, in this context, by Gibbons *et al* [7]. The results agree in detail with the values in this reference some of which we repeat

$$\eta_{\mathbf{T}'} = \frac{167}{144}, \qquad \eta_{\mathbf{O}'} = \frac{383}{288}, \qquad \eta_{\mathbf{Y}'} = \frac{1079}{720}.$$

The cyclic decomposition could also have been employed, and I will now do so for the other values of η by specializing to even lens spaces, L(2q; 1, 1). By putting in the particular expression for H, (49), one has

$$\eta_{S}(s) = -\frac{i\Gamma(2-s)}{4\pi} \int_{C} d\tau (-\tau)^{s-2} \frac{\coth q\tau}{\cosh(\tau/2)} + \frac{i\Gamma(1-s)}{8\pi} \int_{C} d\tau (-\tau)^{s-1} \frac{\coth q\tau}{\sinh(\tau/2)}, \quad (100)$$

for spin-half and

$$\eta_M(s) = -\frac{i\Gamma(2-s)}{2\pi} \int_C d\tau (-\tau)^{s-2} (\coth q\tau - 1) + \frac{i\Gamma(1-s)}{2\pi} \int_C d\tau (-\tau)^{s-1} (\coth q\tau \coth \tau - 1),$$
(101)

in the Maxwell case. Standard expansions allow one to write, n > 0,

$$\eta_{S}(-2n) = \frac{2^{-2n-4}}{n+1} \sum_{m=0}^{n+1} \binom{2n+2}{2m} 2^{4m} B_{2m} \left(E_{2n-2m+2} + \frac{D_{2n-2m+2}}{2n+1} \right) q^{2m-1}$$

$$\eta_{M}(-2n) = \frac{2^{2n+2}}{2n+1} \left(B_{2n+2}q^{2n+1} + \frac{1}{2n+2} \sum_{m=0}^{n} \binom{2n+2}{2m} B_{2m} B_{2n-2m+2}q^{2m-1} \right).$$
(102)

 E_n are Euler numbers and the D_n are related to the Bernoulli numbers by $D_n = 2(1-2^{n-1})B_n$. The expressions are related to the expansion coefficients of the relevant heat-kernel [38]. These and earlier results are derived on the assumption that q is even. They can be extended to odd lens spaces by setting $2q = \overline{q}$ when they will apply to $S^3/Z_{\overline{q}}$ for all \overline{q} .

Some particular values are

$$\eta_{S}(0) = \frac{1}{6\overline{q}}(\overline{q}^{2} - 1)$$

$$\eta_{S}(-2) = \frac{1}{360\overline{q}}(\overline{q}^{2} - 1)(4\overline{q}^{2} + 29)$$

$$\eta_{S}(-4) = \frac{1}{10080\overline{q}}(\overline{q}^{2} - 1)(48\overline{q}^{4} + 272\overline{q}^{2} + 1609)$$

and

$$\begin{split} \eta_M(0) &= \frac{1}{3\overline{q}}(\overline{q} - 1)(\overline{q} - 2) \\ \eta_M(-2) &= \frac{1}{45\overline{q}}(\overline{q}^2 - 1)(\overline{q}^2 - 4) \\ \eta_M(-4) &= \frac{1}{45\overline{q}}(\overline{q}^2 - 1)(\overline{q}^2 - 4)(3\overline{q}^2 + 8) \\ \eta_M(-6) &= \frac{1}{315\overline{q}}(\overline{q}^2 - 1)(\overline{q}^2 - 4)(3\overline{q}^4 + 10\overline{q}^2 + 24) \\ \eta_M(-8) &= \frac{1}{1260\overline{q}}(\overline{q}^2 - 1)(\overline{q}^2 - 4)(25\overline{q}^6 + 92\overline{q}^4 + 272\overline{q}^2 + 640). \end{split}$$

These results exhibit the fact that $\eta_M(s)$ vanishes on the full sphere ($\overline{q} = 1$) and on the projective sphere, L(2; 1, 1). The latter fact follows immediately from the angle sum form of Atiyah, Patodi and Singer, since the only $\theta_{\gamma} = \pi$. It also can be seen in the contour integral forms, (101), (100). Note that only this lens space retains the full, global symmetry of S^3 . The spin-one η is really the *signature*, which vanishes when there is an orientation preserving isometry, as on the projective sphere, see e.g., [56].

The Maxwell $\eta_M(-2n)$ (102) was derived by ourselves some time ago using the more involved techniques in (13) and (16). It can be rearranged using an identity of Apostol [57] in terms of a generalized Dedekind sum [38] and in other ways.

The lens space values given above can be combined to give those on the other quotients by using the cyclic decomposition which reads here

$$\eta_{\Gamma'}(s) = \frac{1}{2} \left(\sum_{q} \eta_{Z_{2q}}(s) - \eta_{Z_2}(s) \right).$$
(103)

The numbers evaluated using this relation provide a useful check.

It should be mentioned that Seade [58] has looked at the Dirac η invariant on the factored 3-sphere and, more recently, Cisneros-Molina [59] has extended the discussion to the twisted case. General calculations can be found in Goette [60].

12. Discussion and conclusion

A number of points arising can be mentioned. Concerning the cosecant sums (86), the fact that they are rational numbers for the cyclic and dihedral cases follows from a residue

evaluation. For the other groups it is not so evident directly, but follows from the cyclic decomposition.

Our discussion of spinors was restricted to the natural, trivial spin structure on the factored 3-sphere. The dependence of the spectrum on the spin structures is discussed in general by Bär [61, 62] who also considers the squashed (Berger) sphere.

It is also possible to calculate the functional determinants for spinors on S^3/Γ' and this will be given at another time. The full sphere results exist already. The evaluation for the Maxwell field is complicated by the nonzero value of $\zeta_M(0)$.

On homogeneous quotients of the Einstein Universe, the vacuum energy density, $\langle T_0^0 \rangle$, is obtained simply by dividing *E* by the volume. It is also possible to obtain the spatial densities $\langle T_i^j \rangle$ [12]. Because the symmetry group is generally reduced, these contain geometric structure over and above that arising from the metric.

The case of non-homogeneous quotients is much harder but in certain circumstances an exact $\langle T_0^0 \rangle$ can be found with some work [16].

As mentioned, it is possible to introduce an equivariant twisting according to $Hom(\Gamma, U(N))$, say. The analysis is one in character theory. The scalar summations can still be performed in the case of one-sided lens spaces and result in generalized Bernoulli polynomials.

The construction of the eigenfunctions is left aside as a chapter in the theory of symmetry adaptation most familiar, perhaps, in solid state physics.

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