F

AN ETALE APPROACH TO THE NOVIKOV CONJECTURE

A.N. Dranishnikov, S. Ferry, and S, Weinberger

Abstract. We show that the rational Novikov conjecture for a group Γ of finite homological type follows from the mod 2 acyclicity of the Higson compactifcation of an $E\Gamma$. We then show that for groups of finite asymptotic dimension the Higson compactification is mod p acyclic for all p, and deduce the integral Novikov conjecture for these groups.

§1 Introduction

Ten years ago, the most popular approach to the Novikov conjecture went via compactifications. If a compact aspherical manifold, say, has a universal cover which suitably equivariantly compactifies, already Farrell and Hsiang [FH] proved that the Novikov conjecture follows. Subsequent work by many authors weakened their hypotheses and extended the idea to other settings. (See e.g. [CP, FW, Ro].)

In recent years other coarse methods have supplanted the compactification method (most notably the embedding method of [STY] in the C* algebra setting). The reason for this is that it seemed to have a better chance of applying generally, while compactifications effective for proving the Novikov conjecture seemed to require some special geometry for their construction. For a brief time, it seemed that this could conceivably not be the case. Higson introduced a general compactification of metric spaces (somewhat reminiscent of the Stone-Cech compactification) that automatically has half of the properties necessary for application to the Novikov conjecture. The missing property was acyclicity, which holds for the Stone Čech compactification in dimensions > 1, so it was natural to hope that the Higson compactification of a universal cover is automatically acyclic. (See [Ro1].)

Unfortunately, it was soon realized by and others (see [Ke, DF]) that the Higson compactification, even for manifolds as small as $\mathbf{R}$, has nontrivial rational cohomology. Thus, it was felt that general compactifications were not suitable for the problem – one has to use geometric compactifications. However, Gromov has recently harpooned the embedding approach by constructing finitely generated groups which do not uniformly embed in Hilbert space [Gr]. Moreover, a number of authors (e.g. [HLS] [O]) have shown that Gromovs groups can be used to construct counterexamples to general forms of the Baum-Connes conjecture.

1991 *Mathematics Subject Classification*. Primary 20H15.

Key words and phrases. dimension, asymptotic dimension, absolute extensor, Higson corona, Novikov conjecture.

This work was partially supported by supported by NSF.

Moreover, for reasons that are not entirely clear, the embedding method has never been translated into pure topology; the results on the integral Novikov conjecture so obtained, from the L-theoretic viewpoint never account for the prime 2, and there do not seem to be many results in pure algebraic K-theory (or A-theory) provable by this method.

This paper seeks to rehabilitate the Higson compactification approach. We will show that

Theorem 1. *If the Higson compactification of $E\Gamma$ is mod 2 acyclic, and $B\Gamma$ is finite type, then the Novikov conjecture for Γ holds at the prime 2.*

In other words, the assembly map localized at the prime two is an injection on homotopy groups. At odd primes we do not know how to prove any corresponding statement, in general, for reasons related to the examples in [DFW]: The L-spectrum away from 2 is periodic K-theory, and cohomologically acyclic spaces may not be acyclic for periodic K-theory.

This difficulty is an infinite dimensional phenomenon, so for groups of finite asymptotic dimension this issue does not arise, and it is possible to prove integral results.

Theorem 2. *If $B\Gamma$ is a finite complex and Γ has finite asymptotic dimension and R is a ring with involution that has finitely generated L-groups, then the $\mathbf{L}(R)$ assembly map for Γ is injective.*

In this paper, by $\mathbf{L}(R)$, we will always mean $\mathbf{L}^{-\infty}(R)$.

Theorem 2 is not a new theorem. For $K(C^*\Gamma)$, this is due to [Yu]. In algebraic K-theory there are papers [Ba, CG], which apply to L-theory as well (without the restrictions on R); moreover [CFY] has yet another approach to the stronger L-theory result. The novelty is the method of using finite primes to repair the acyclicity of the Higson compactification and then using this "etale" idea to get a Novikov result.[1]

Theorem 2 follows from the general discussion above, together with our third theorem.

[1]Note that the celebrated paper of [BHM] on the algebraic K-theory analogue of the Novikov conjecture is a very nice illustration of a completely different etale idea. We thank Alain Connes for suggesting this evocative description of our method.

Theorem 3. *If* Γ *is as in Theorem 2, then the Higson compactification of* $B\Gamma$ *is mod p acyclic for every p.*

To motivate and clarify our results, we close with a brief discussion of the Higson compactification and of the Keesling elements. If X is a metric space, and $f : X \to \mathbf{R}$ is a bounded continuous function, we say that f has *decaying variation* if

$$\lim \operatorname{diam}(f(B_r(x))) = 0$$

for any fixed r and any sequence of points x going to infinity in X. Higson's compactification is the smallest one containing X densely so that all functions with decaying variation extend. Let C_h be the aggregate of such functions. We obtain the Higson compactification of X by embedding X into $Maps[C_h : R]$ and taking the closure of the image.

Let us now consider the first cohomology of the compactification of a uniformly simply connected X. We represent $\check{H}^1(\bar{X})$ by maps $f : \bar{X} \to S^1$. On X, we can lift $f|$ and view our class as being given by $\exp(2\pi i\tilde{f})$, where $\tilde{f}$ has decaying variation but is not necessarily bounded. Maps $f, g : \bar{X} \to S^1$ represent the same class iff $\tilde{f}$ and $\tilde{g}$ differ by a bounded function.

Note therefore, that $H^1(\bar{X}; Z)$ is now an $\mathbf{R}$-vector space – one can multiply the function $\tilde{f}$ by a real number. In particular, this cohomology vanishes with mod p coefficients. In this regard, the cohomology of the Higson compactification resembles the uniformly finite homology of a uniformly contractible n manifold, [BW], which is known to be an $\mathbf{R}$ vector space, aside from the class coming from the fundamental class of X.

Our paper is organized as follows. Section 2 is devoted to descent, i.e. deducing Novikov conjectures from metric results. The last section is devoted to verifying the mod p acyclicity result for the finite asymptotic dimension case. This is done using ideas of quantitative algebraic topology. The key idea is that when homotopy groups are finite, it is sometimes possible to get extra Lipschitz conditions on maps, a priori.

§2 L-theory and assembly map with $\mathbf{Z}_p$ coefficients

For every ring with involution R (even for any additive category with involution) Ranicki defines a 4-periodic spectrum $\mathbf{L}_*(R)$ with $\pi_i(\mathbf{L}_*(R)) = L_i(R)$ [Ran1,2]. We use the notation $\mathbf{L} = \mathbf{L}_*(\mathbf{Z})$. Strictly speaking, $\mathbf{L}$-spectra are indexed by K-groups.

Our $\mathbf{L}(\mathbf{R})$ equals $\mathbf{L}^{-\infty}(\mathbf{R})$, the limit of the $\mathbf{L}$-spectra associated to the negative K-groups. This applies to all $\mathbf{L}$-spectra occurring in this paper.

For a metric space (X,d) we will denote by $C_X(R)$ the boundedly controlled Pedersen-Weibel category whose objects are locally finite direct sums $A = \oplus_{x\in X} A(x)$ of finite dimensional free R-modules and whose morphisms are given by matrices with bounded propagations (see [FRR]). For a subset $V \subset X$ we denote by $A(V)$ the sum $\oplus_{x\in V} A(x)$. Ranicki [Ran2] defined *X-bounded quadratic L-groups* $L_*(C_X(R))$ and the corresponding spectrum $\mathbf{L}_*(C_X(R))$. We will use the notation $\mathbf{L}^{bdd}(X) = \mathbf{L}_*(C_X(\mathbf{Z}))$.

Suppose that $\bar{X}$ is a compactification of X with corona $Y = \bar{X} \setminus X$. Then one can define the *continuously controlled category* $B_{X,Y}(R)$ by taking the same objects as above with morphisms $f : A \to B$ in $B_{X,Y}(R)$ taken to be homomorphisms such that for every $y \in Y$ and every neighborhood $y \in U \subset \bar{X}$ there is a smaller neighborhood $y \in V \subset U$ such that $f(A(V)) \subset A(U)$. This category is additive and hence also admits an L-theory. The corresponding spectrum for $R = \mathbf{Z}$ we denote by $\mathbf{L}^{cc}(X) = \mathbf{L}_*(B_{X,Y}(\mathbf{Z}))$.

If $\mathbf{E} = \{E_k \mid k \in \mathbf{Z}\}$ is a spectrum, we will denote the $\mathbf{E}$-homology of X by $H_i(X;\mathbf{E})$. If X is a complex, then $H_i(X;\mathbf{E}) = \pi_i(X_+ \wedge \mathbf{E}) = \lim_{k\to\infty} \pi_{i+k}(X_+ \wedge E_k)$. If X is a compact metric space, we denote the Steenrod $\mathbf{E}$-homology of X by $H_i(X;\mathbf{E})$, i.e. $H_i(X;\mathbf{E}) = \pi_i(\mathrm{holim}\{N^i_+ \wedge \mathbf{E}\})$ where $X = \varprojlim N^i$ is the inverse limit of polyhedra [CP2], [EH].

The following theorem was proven in [P], [CP1], [CP2].

Theorem 2.1. *$L_i(B_{X,Y}(R)) = \bar{H}_{i-1}(Y;\mathbf{L}(R))$ for all i where the homology on the right is reduced Steenrod $\mathbf{L}_*(R)$-homology.*

A metric space X is called *proper* if every ball $B_r(x)$ in X is compact. Subsets $A, B \subset X$ of a proper metric space (X,d) are called *diverging* if

$$\lim_{r\to\infty} d(A \setminus B_r(x_0), B \setminus B_r(x_0)) = \infty$$

where $x_0 \in X$ is any basepoint, $B_r(x_0)$ is the r-ball centered at x_0, and the distance $d(A', B')$ between sets is the infimum of distances between $d(a,b)$, $a \in A'$, $b \in B'$. We note that the minimal compactification of a proper metric space X with respect to the property that the closures of every pair of diverging subsets in X have empty intersection in the corona is the Higson compactification.

A compactification $\tilde{X}$ of X is called *Higson dominated* if the identity map $1_X : X \to X$ admits a continuous extension $\bar{X} \to \tilde{X}$, where $\bar{X}$ is the Higson compactification. We note that for every Higson dominated compactification $\tilde{X}$ of X with corona $Y = \tilde{X} - X$, there is a forgetful functor $C_X(R) \to B_{X,Y}(R)$. This functor defines a map of spectra $\phi : \mathbf{L}^{bdd}(X) \to \mathbf{L}^{cc}(X)$.

If X is the universal covering of a finite complex with fundamental group Γ, then Γ acts on $C_X(R)$ and hence on $\mathbf{L}_*(C_X(R))$ with fixed set $\mathbf{L}_*(C_X(R))^\Gamma = \mathbf{L}_*(\mathrm{R}\Gamma)$ [CP]. If a compactification of X with corona Y is equivariant, then Γ acts on $B_{X,Y}(R)$ and hence on $\mathbf{L}_*(B_{X,Y}(R))$.

We recall that the homotopy fixed set $X^{h\Gamma}$ of a pointed space X with a Γ action on it is defined as the space of equivariant maps $\mathrm{Map}_\Gamma(\mathrm{E}\Gamma_+, X)$.

For a general spectrum $\mathbf{E}$ and a torsion free group Γ it was proven [C], [CP1] that

$$\mathbf{H}_*(\mathrm{B}\Gamma; \mathbf{E}) \cong \mathbf{H}^{lf}_*(\mathrm{E}\Gamma; \mathbf{E})^\Gamma \cong \mathbf{H}^{lf}_*(\mathrm{E}\Gamma; \mathbf{E})^{h\Gamma}.$$

For a locally compact space X, $H^{lf}_i(X; \mathbf{E})$ is defined to be the Steenrod $\mathbf{E}$-homology rel infinity of the one point compactification αX. We note that for a complex N, $\mathbf{H}_*(N; \mathbf{E}) = N_+ \wedge \mathbf{E}$ and for a compact space Y, $\mathbf{H}_*(Y; \mathbf{E}) = \mathrm{holim}\{N^\alpha_+ \wedge \mathbf{E}\}$ where N^α runs over nerves of all finite open covers of Y. If Γ acts on a compact space Y, then it acts on the set of all finite open covers of Y and hence on the spectrum $\mathbf{H}_*(Y; \mathbf{E})$.

The following theorem is due to Carlsson-Pedersen [CP1],[CP2]. See also Ranicki [Ran3]. The first part of it is discussed in a more general setting in [Ros], Theorem 3.3.

Theorem 2.2. *For every group Γ with finite classifying complex $\mathrm{B}\Gamma$ there is a morphism of spectra called the bounded control assembly map*

$$A^{bdd} : \mathbf{H}^{lf}_*(\mathrm{E}\Gamma; \mathbf{L}) \to \mathbf{L}^{bdd}(\mathrm{E}\Gamma)$$

which fits into a homotopy commutative diagram

$$\begin{array}{ccc}
\mathbf{H}_*(\mathrm{B}\Gamma; \mathbf{L}) & \xrightarrow{A} & \mathbf{L}_*(\mathbf{Z}\Gamma) \\
\simeq \big\downarrow & & trf \big\downarrow \\
\mathbf{H}^{lf}_*(\mathrm{E}\Gamma; \mathbf{L})^{h\Gamma} & \xrightarrow{A^{bdd,h\Gamma}} & \mathbf{L}^{bdd}(\mathrm{E}\Gamma)^{h\Gamma}
\end{array}$$

where A is the standard assembly map, and the vertical arrows are the natural maps from fixed sets to homotopy fixed sets.

If the universal covering space $\mathrm{E}\Gamma$ admits a Higson dominated equivariant compactification X with corona Y then the diagram can be extended

$$\begin{array}{ccc}
\mathbf{H}_*(\mathrm{B}\Gamma;\mathbf{L}) & \xrightarrow{A} & \mathbf{L}_*(\mathbf{Z}\Gamma) \\
\simeq\Big\downarrow & & trf\Big\downarrow \\
\mathbf{H}^{lf}_*(\mathrm{E}\Gamma;\mathbf{L})^{h\Gamma} & \xrightarrow{A^{bdd,h\Gamma}} & \mathbf{L}^{bdd}(\mathrm{E}\Gamma)^{h\Gamma} \\
=\Big\downarrow & & \phi^{h\Gamma}\Big\downarrow \\
\mathbf{H}^{lf}_*(\mathrm{E}\Gamma;\mathbf{L})^{h\Gamma} & \xrightarrow{A^{cc,h\Gamma}} & \mathbf{L}^{cc}(\mathrm{E}\Gamma)^{h\Gamma}
\end{array}$$

where $A^{cc,h\Gamma}=\phi^{h\Gamma}\circ A^{bdd,h\Gamma}$. Moreover, A^{cc} coincides with the boundary map in the Steenrod $\mathbf{L}$-homology exact sequence of pair (X,Y):

$$A^{cc}_p=\partial:\mathbf{H}^{lf}_*(\mathrm{E}\Gamma;\mathbf{L})\to\mathbf{H}_{*-1}(Y;\mathbf{L}).$$

Let $M(p)$ denote the Moore spectrum for the group $\mathbf{Z}_p=\mathbf{Z}/p\mathbf{Z}$.

Lemma 2.1. *Let X be a finite-dimensional compact metric space. Suppose that X is $\mathbf{Z}_p$-acyclic, i.e., $\tilde{H}^*(X;\mathbf{Z}_p)=0$ for reduced Čech cohomology. Then X is acyclic for the reduced Steenrod $\mathbf{L}\wedge M(p^k)$-homology for all k.*

Proof. In view of the universal coefficient formula [M], we note that $\tilde{H}_*(X;\mathbf{Z}_p)=0$ for reduced Steenrod homology. The coefficient exact sequence shows that $\tilde{H}_*(X;\mathbf{Z}_{p^k})=0$ for all k. For every spectrum $\mathbf{S}$ there is a Steenrod homology Atiyah-Hirzebruch spectral sequence with

$$E^2_{i,j}=\tilde{H}_i(X;H_j(S^0,\mathbf{S}))$$

which converges to $\tilde{H}_*(X;\mathbf{S})$ provided that X is a finite dimensional compact metric space [EH],[KS]. Since all of the groups $H_q(S^0,\mathbf{L}\wedge M(p^k))$ are p-primary, we have $E^2_{i,j}=0$ for all i and j. □

REMARK 1. The finite dimensionality condition is essential here. There are acyclic compacta that have nontrivial mod p complex K-theory [T]. Namely, by results of Adams and Toda there is a map $f:\Sigma^kM(\mathbf{Z}_p,m)\to M(\mathbf{Z}_p,m)$ which induces an isomorphism in K-theory. The inverse limit X of suspensions of f is a an acyclic compactum, since all bonding maps are trivial in cohomology, yet it has nontrivial K-theory and nontrivial mod p K-theory. Hence, for p odd, X has nontrivial mod p L-theory.

REMARK 2. The spectrum $\mathbf{L}$ localized at 2 is equivalent (see [MM]) to the Eilenberg-MacLane 4-periodic spectrum generated by the space

$$\prod_{i=1}^{\infty} K(\mathbf{Z}_{(2)}, 4i) \times K(\mathbf{Z}_2, 4i-2).$$

Thus, $\mathbf{L} \wedge M(2) = \mathbf{L}_{(2)} \wedge M(2)$ is the Eilenberg-MacLane spectrum generated by the space

$$\prod_{i=1}^{\infty} K(\mathbf{Z}_2, 4i) \times K(\mathbf{Z}_2, 4i-1) \times K(\mathbf{Z}_2, 4i-2).$$

Indeed, for any R, $\mathbf{L}(R)$ is Eilenberg-MacLane at 2 [TW]. Thus, if a compactum X is $\mathbf{Z}_2$-acyclic, then it is $\mathbf{L} \wedge M(2)$-acyclic without any finite dimensionality assumption on X.

DEFINITION. The mod p assembly map A_p is

$$A \wedge 1_{M(p)} : \mathbf{H}_*(\mathrm{B}\Gamma; \mathbf{L}) \wedge M(p) \to \mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(p).$$

Theorem 2.3. *Suppose that the universal cover X of a finite aspherical complex B has finite dimensional mod p acyclic Higson compactification. Then the mod p^k assembly map for $\Gamma = \pi_1(B)$ is a split monomorphism for every k.*

Proof. Let let m be the dimension of the Higson corona $\dim \nu X$. Using Schepin's spectral theorem [Dr1] one can obtain a metrizable $\mathbf{Z}_p$-acyclic Γ-equivariant compactification $\bar{X}$ of X with corona Y such that $\dim Y = m$ (see the proof of Lemma 8.3. in [Dr1]). We introduce coefficients to the second diagram of Theorem 2.2 by forming the smash product with $M(p^k)$.

$$\begin{array}{ccc}
\mathbf{H}_*(B; \mathbf{L}) \wedge M(p^k) & \xrightarrow{A_p} & \mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(p^k) \\
\simeq \big\downarrow & & \big\downarrow \phi^{h\Gamma} trf \wedge 1 \\
\mathbf{H}^{lf}_*(X; \mathbf{L})^{h\Gamma} \wedge M(p^k) & \xrightarrow{A^{cc,h\Gamma} \wedge 1} & \mathbf{L}^{cc}(X)^{h\Gamma} \wedge M(p^k)
\end{array}$$

Note also that $\mathbf{L} \wedge p^k = \mathrm{cof}(\times p^k : \mathbf{L} \to \mathbf{L})$ and $\times p^k$ commutes with all of the structure mapsdefining the spectral structure and the Γ-action. We therefore have the commutative diagram

$$\begin{array}{ccc} \mathbf{H}_*(B;\mathbf{L}) \wedge M(p^k) & \xrightarrow{A_p} & \mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(p^k) \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \phi^{h\Gamma} trf \wedge 1} \\ (\mathbf{H}^{lf}_*(X;\mathbf{L}) \wedge M(p^k))^{h\Gamma} & \xrightarrow{(A^{cc}\wedge 1)^{h\Gamma}} & (\mathbf{L}^{cc}(X) \wedge M(p^k))^{h\Gamma}. \end{array}$$

It suffices to show that

$$A^{cc} \wedge 1 : \mathbf{H}^{lf}_*(X;\mathbf{L}) \wedge M(p^k) \to \mathbf{L}^{cc}(X) \wedge M(p^k)$$

is an isomorphism because this implies that $(A^{cc} \wedge 1)^{h\Gamma}$ is an isomorphism and hence that A_{p^k} is a split monomorphism.

In view of Theorems 2.1 and 2.2 we need to show that

$$\partial \wedge 1 : \mathbf{H}^{lf}_*(X;\mathbf{L}) \wedge M(p^k) \to \mathbf{H}_{*-1}(Y;\mathbf{L}) \wedge M(p^k)$$

is isomorphism. Note that $\partial \wedge 1_{M(p^k)}$ is equivalent to the boundary homomorphism for the pair $(\bar{X}, Y)$ in $\mathbf{L} \wedge M(p^k)$-homology. By Lemma 2.1, $\mathbf{H}_*(\bar{X};\mathbf{L} \wedge M(p^k)) = \mathbf{H}_*(pt;\mathbf{L} \wedge M(p^k))$ and hence $\partial \wedge 1_{M(p^k)}$ is indeed an isomorphism. □

It is known that the universal coefficient formula with $\mathbf{Z}_p$ coefficients (UCF) holds true for every generalized homology theory and is natural with respect to morphisms of spectra:

Proposition 2.1. *For every morphism of spectra $A : \mathbf{E}_1 \to \mathbf{E}_2$ and every p and i there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_i(\mathbf{E}_1) \otimes \mathbf{Z}_p & \longrightarrow & \pi_i(\mathbf{E}_1 \wedge M(p)) & \longrightarrow & \pi_{i-1}(\mathbf{E}_1) * \mathbf{Z}_p & \longrightarrow & 0 \\ & & {\scriptstyle A_*\otimes 1}\downarrow & & {\scriptstyle (A\wedge 1)_*}\downarrow & & {\scriptstyle (A_{*-1})*1}\downarrow & & \\ 0 & \longrightarrow & \pi_i(\mathbf{E}_2) \otimes \mathbf{Z}_p & \longrightarrow & \pi_i(\mathbf{E}_2 \wedge M(p)) & \longrightarrow & \pi_{i-1}(\mathbf{E}_2) * \mathbf{Z}_p & \longrightarrow & 0. \end{array}$$

Proof. We apply the smash product with $\mathbf{E}_i$, $i = 1, 2$ to the cofibration of spectra $\mathbf{S} \to \mathbf{S} \to M(p)$, where $\mathbf{S}$ is the sphere spectrum. Then the result follows from the homotopy exact sequence of the resulting cofibrations of spectra and the induced morphism between them

$$\begin{array}{ccccc} E_1 & \longrightarrow & E_1 & \longrightarrow & E_1 \wedge M(p) \\ \downarrow & & \downarrow & & \downarrow \\ E_2 & \longrightarrow & E_2 & \longrightarrow & E_2 \wedge M(p). \end{array}$$

□

Theorem 2.4. *Suppose that the the universal cover of a finite complex* $\mathrm{B}\Gamma$ *has finite dimensional mod* p *acyclic Higson compactification. Then the integral assembly map* A *is a monomorphism.*

Proof. In view of compactness of $\mathrm{B}\Gamma$ we have $\nu\Gamma = \nu\mathrm{E}\Gamma$. By Theorem 2.3 $A \otimes 1_G$ is a split monomorphism for every finite abelian group G. Since $\mathrm{B}\Gamma$ is a finite complex, the standard induction argument on the number of cells show that the group $H_i(\mathrm{B}\Gamma; \mathbf{L})$ is finitely generated for every i. Hence for every $\alpha \in H_i(\mathrm{B}\Gamma; \mathbf{L})$ there is p such that $\alpha \otimes 1 \in H_i(\mathrm{B}\Gamma; \mathbf{L}) \otimes \mathbf{Z}_{p^k}$ is not zero. By the UCF there is a monomorphism $H_i(\mathrm{B}\Gamma; \mathbf{L}) \otimes \mathbf{Z}_{p^k} \to H_i(\mathrm{B}\Gamma; \mathbf{L}(\mathbf{Z}) \wedge M(p^k))$ which, together with the assembly map, produces a commutative diagram

$$\begin{array}{ccc} H_i(\mathrm{B}\Gamma; \mathbf{L}) \otimes \mathbf{Z}_{p^k} & \longrightarrow & H_i(\mathrm{B}\Gamma; \mathbf{L} \wedge M(p^k)) \\ {\scriptstyle A\otimes 1}\downarrow & & {\scriptstyle A_p}\downarrow \\ L_i(\mathbf{Z}\Gamma) \otimes \mathbf{Z}_{p^k} & \longrightarrow & \pi_i(\mathbf{L}(\mathbf{Z}\Gamma) \wedge M(p^k)). \end{array}$$

This diagram implies that $A(\alpha) \neq 0$. □

Theorem 2.5. *Suppose that* Γ *is a group with* $\mathrm{B}\Gamma$ *a finite complex. If* $\mathrm{E}\Gamma$ *has mod 2 acyclic Higson compactification, then the rational Novikov conjecture holds for* Γ.

Proof. In view of Remark 2 the argument of Theorem 2.3 for $p = 2$ works without the assumption $\dim \mathrm{E}\Gamma < \infty$. Hence the mod 2^k assembly map

$$A \wedge 1_{M(2^k)} : \mathbf{H}_*(\mathrm{B}\Gamma; \mathbf{L}) \wedge M(2^k) \to \mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(2^k)$$

is a split monomorphism. On the group level by the Universal Coefficient Formula we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_i(\mathrm{B}\Gamma; \mathbf{L}) \otimes \mathbf{Z}_{2^k} & \longrightarrow & H_i(\mathrm{B}\Gamma; \mathbf{L} \wedge M(2^k)) & \longrightarrow & H_{i-1}(\mathrm{B}\Gamma; \mathbf{L}) * \mathbf{Z}_{2^k} & \longrightarrow & 0 \\ & & {\scriptstyle A_*\otimes 1}\downarrow & & {\scriptstyle (A\wedge 1)_*}\downarrow & & {\scriptstyle (A_{*-1})*1}\downarrow & & \\ 0 & \longrightarrow & \pi_i(\mathbf{L}_*(\mathbf{Z}\Gamma)) \otimes \mathbf{Z}_{2^k} & \longrightarrow & \pi_i(\mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(2^k)) & \longrightarrow & \pi_{i-1}(\mathbf{L}_*(\mathbf{Z}\Gamma) * \mathbf{Z}_{2^k} & \longrightarrow & 0. \end{array}$$

Since the group $H_i(\mathrm{B}\Gamma; \mathbf{L})$ is finitely generated, it can be presented as $\oplus_{F_i}\mathbf{Z} \oplus \mathrm{Tor}_i$. Since $A_* \otimes 1_{\mathbf{Z}_{2^k}}$ is a monomorphism, the kernel $\ker(A_*)$ consists of 2^k divisible elements. Since k is arbitrary, $\ker(A_*|_{\oplus \mathbf{Z}}) = 0$. Therefore, $A_* \otimes 1_{\mathbf{Q}}$ is a monomorphism. $\square$

This proves Theorem 1 of the introduction.

§3 Mod p acyclicity of Higson compactifications of asymptotically finite dimensional spaces

We recall that a map $f : X \to Y$ between metric spaces is λ-Lipschitz if $d_Y(f(x), f(x')) \leq \lambda d_X(x, x')$ for all $x, x' \in X$. Denote by

$$\mathrm{Lip}(f) = \sup_{x \neq x'} \{ \frac{d_Y(f(x), f(x'))}{d_X(x, x')} \}$$

the minimal Lipschitz constant of f.

Every simplicial complex K carries a metric where all simplexes are isometric to the standard euclidean simplex. We will call the maximal such metric on K *uniform* and usually we will denote the corresponding metric space by K_U.[2] Note that the metric space K_U is geodesic. If no metric is specified, we will assume that a finite complex is supplied with this uniform metric. In particular, the complexes in the two lemmas below are assumed to have the uniform metric.

The following lemma is a special case of Theorem A from [SW].

Lemma 3.1. *Let Y be a finite simplicial complex with $\pi_n(Y)$ finite. Then for every $\lambda > 0$ there is a $\mu > 0$ such that every map $f : B^n \to Y$ with $\mathrm{Lip}(f|_{S^{n-1}}) \leq \lambda$ can be deformed to a μ-Lipschitz map $g : B^n \to Y$ by means of a homotopy $h_t : B^n \to Y$ with $h_t|_{S^{n-1}} = f|_{S^{n-1}}$.*

Lemma 3.2. *Let L be a finite dimensional complex and let K be a finite complex with finite homotopy groups $\pi_i(K)$ for $i \leq \dim L + 1$. Let $f, g : L \to K$ be homotopic Lipschitz maps. Then every homotopy between f and g can be deformed to a Lipschitz homotopy $H : L \times [0,1] \to K$.*

Proof. Let $F : L \times I \to K$ be a homotopy between f and g. By induction on n using Lemma 3.1, we construct a μ_n-Lipschitz map $H_n : L^{(n)} \times I \cup L \times \{0,1\} \to K$

[2] An important exception to this convention occurs with *asymptotic polyhedra*, which are defined below.

which is a deformation of F restricted to the n-skeleton $L^{(n)}$ such that H_n extends H_{n-1}. Here the fact that L (and hence $L \times I$) is geodesic is essential for the argument because this condition guarantees that the union of λ-Lipschitz maps on $\Delta^n \times I$ is λ-Lipschitz on $L^{(n)} \times I$ (see [Dr2] or [Dr3] for details). Then $H = H_m$ for $m = \dim L$. $\square$

Let $x_0 \in X$ be a basepoint in a metric space X. For $x \in X$ and $A \subset X$ we denote by $\|x\| = \mathrm{dist}(x, x_0)$ and $\|A\| = \max\{\|z\| \mid z \in A\}$. By $B_r(x)$ we denote the closed r-ball in X centered at x. We use notation $B_r = B_r(x_0)$.

Let σ be an n-dimensional simplex spanned in a Euclidean space. By $s_\sigma : \sigma \to \Delta^n$ we denote a simplicial homeomorphism onto the standard n-simplex $\Delta^n = \{(x_i) \in \mathbf{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0\}$. A locally finite simplicial complex L with a geodesic metric on it is called an *asymptotic polyhedron* if every simplex in L is isometric to a simplex σ spanned in a Euclidean space and $\lim_{\|\sigma\| \to \infty} \mathrm{Lip}(s_\sigma) = 0$. Asymptotic polyhedra usually do not have the uniform metric.

Proposition 3.3. *Let (K, d) be an asymptotic polyhedron and let d_U denote the uniform geodesic metric on K. Then the identity map $u : (K, d) \to (K, d_U)$ satisfies the condition $\lim_{x \to \infty} \mathrm{diam}(u(B_r(x))) = 0$.*

Proof. Let r and $\epsilon > 0$ be given. There is a t so that $\mathrm{Lip}(s_\sigma) < \epsilon/(2r)$ for every σ with $\|\sigma\| \geq t$. Choose x so that $\|x\| \geq t + 2r$. For every two points $z, z' \in B_r(x)$, a geodesic segment $J = [z, z']$ joining them does not intersect B_t. There is a partition of J: $z = z_0 < z_1 \cdots < z_m = z'$ such that every segment $J_i = [z_i, z_{i+1}]$ lies in a simplex σ_i. By our choice of t, $\mathrm{Lip}(s_{\sigma_i}) < \epsilon/(2r)$ for each i. Therefore $d_U(z_i, z_{i+1}) \leq \epsilon/(2r)d(z_i, z_{i+1})$ and hence $d_U(z, z') \leq \sum \epsilon/(2r)d(z_i, z_{i+1}) = \epsilon/(2r)d(z, z') \leq \epsilon$. $\square$

A family of subsets $\mathcal{A}$ is said to be d-disjoint if for every pair of sets A and A' in $\mathcal{A}$ we have $d(x, x') > d$ for all $x \in A$ and $x' \in A'$. A family of subsets $\mathcal{U}$ of a metric space X is *uniformly bounded* if there is a $B > 0$ so that the diameter of each element of $\mathcal{U}$ is less than B. Gromov, [G1], defined the *asymptotic dimension* $\mathrm{asdim}\, X$ of a metric space X as follows: $\mathrm{asdim}\, X \leq n$ if for every d there are $n+1$ d-disjoint uniformly bounded families $\mathcal{U}_i$, $i = 0, \dots, n$ of subsets of X such that the union $\mathcal{U} = \cup \mathcal{U}_i$ is a cover of X.

Gromov's definition can be equivalently reformulated as follows: $\mathrm{asdim}\, X \leq n$ if for any arbitrary large number d there is a uniformly bounded open cover $\mathcal{U}$ of X with multiplicity $\leq n+1$ and with Lebesgue number $\geq d$ (see Assertion 1 in [BD2] for a proof).

We note that for every n-dimensional asymptotic polyhedron L, $\operatorname{asdim} L \le n$.

Let $\mathcal{U}$ be a locally finite open cover of a metric space X. The *canonical projection* to the nerve $p : X \to \mathrm{Nerve}(\mathcal{U})$ is defined using the partition of unity $\{\phi_U : X \to \mathbf{R}\}_{U\in\mathcal{U}}$ defined by $\phi_U(x) = d(x, X \setminus U)/\sum_{V\in\mathcal{U}} d(x, X \setminus V)$. The family $\{\phi_U : X \to \mathbf{R}\}_{U\in\mathcal{U}}$ defines a map p to the Hilbert space $l_2(\mathcal{U})$ of square summable functions on $\mathcal{U}$ with the Dirac functions δ_U, $U \in \mathcal{U}$ as the basis. The nerve $N(\mathcal{U})$ of the cover $\mathcal{U}$ is realized in $l_2(\mathcal{U})$ by taking every vertex U to δ_U. Clearly, the image of p lies in the nerve. Given a family of positive numbers $\bar\lambda = \{\lambda_U\}_{U\in\mathcal{U}}$ we can change the above imbedding of the nerve $N(\mathcal{U})$ into $l_2(\mathcal{U})$ by taking each vertex U to $\lambda_U\delta_U$. Then the projection $p^{\bar\lambda}_{\mathcal{U}} : X \to N(\mathcal{U})$ to modified realization is given by the formula $p^{\bar\lambda}_{\mathcal{U}}(x) = \sum_{U\in\mathcal{U}} \lambda_U\phi_U(x)\delta_U$.

For a subset $A \subset X$ we denote by $L(\mathcal{U})|_A$ the Lebesgue number of $\mathcal{U}$ restricted to A. More precisely, $L(\mathcal{U})|_A = \inf_{y\in A} \max_{V\in\mathcal{U}} d(y, X \setminus V)$.

Proposition 3.4. *Let $\mathcal{U}$ be a locally finite cover of a geodesic metric space X with the multiplicity $\le m$. Then the above projection to the nerve $p^{\bar\lambda}_{\mathcal{U}} : X \to N = N(\mathcal{U})$ for $\bar\lambda = \{\lambda_U\}$ with $\lambda_U = L(\mathcal{U})|_U/(2m + 1)^2$ is 1-Lipschitz where the nerve is taken with the intrinsic metric induced from $l_2(\mathcal{U})$.*

Proof. We will show that the map $\bar p = p^{\bar\lambda}_{\mathcal{U}}$ is 1-Lipschitz as a map to $l_2(\mathcal{U})$. Then the partition of N into simplices defines a locally finite partition on X such that $\bar p$ is 1-Lipschitz on every piece, considered as the map to N with the induced path metric. Since X is a geodesic metric space, this will imply that $\bar p$ is 1-Lipschitz.

Let $x, y \in X$ and $U \in \mathcal{U}$. The triangle inequality implies

$$|d(x, X \setminus U) - d(y, X \setminus U)| \le d(x, y).$$

It is easy to see that $\sum_{V\in\mathcal{U}} d(x, X \setminus V) \ge L(\mathcal{U})|_U = (2m + 1)^2\lambda_U$ for $x \in U$. Then $|\phi_U(x)-\phi_U(y)| = |\frac{d(x,X\setminus U)}{\sum_{V\in\mathcal{U}} d(x,X\setminus V)} - \frac{d(y,X\setminus U)}{\sum_{V\in\mathcal{U}} d(x,X\setminus V)} + \frac{d(y,X\setminus U)}{\sum_{V\in\mathcal{U}} d(x,X\setminus V)} - \frac{d(y,X\setminus U)}{\sum_{V\in\mathcal{U}} d(y,X\setminus V)}| \le$

$$\frac{1}{\sum_{V\in\mathcal{U}} d(x, X \setminus V)} d(x, y) + d(y, X \setminus U)|\frac{1}{\sum_{V\in\mathcal{U}} d(x, X \setminus V)} - \frac{1}{\sum_{V\in\mathcal{U}} d(y, X \setminus V)}|$$

$$\le \frac{1}{(2m + 1)^2\lambda_U} d(x, y) + \frac{1}{(2m + 1)^2\lambda_U}(\sum_{V\in\mathcal{U}} |d(x, X\setminus V)-d(y, X\setminus V)|) \le \frac{1}{\sqrt{2m + 1}\lambda_U} d(x, y).$$

Then $\|p(x) - p(y)\| =$

$$= (\sum_{U\in\mathcal{U}} \lambda_U^2(\phi_U(x) - \phi_U(y))^2)^{\frac12} \le ((2m)\frac{1}{2m + 1} d(x, y)^2)^{\frac12} \le d(x, y).$$

□

Lemma 3.5. *Suppose that X is a geodesic metric space with* $\operatorname{asdim} X \leq n$ *and let $f : X \to \mathbf{R}_+$ be a proper function. Then there exist a compact set $C \subset X$ and a map $\phi : X \to N$ to a $2n+1$-dimensional asymptotic polyhedron with* $\operatorname{diam}(\phi^{-1}(\Delta)) \leq f(z)$ *for all $z \in \phi^{-1}(\Delta) \setminus C$ and for all simplices Δ of N. Moreover, every vertex of N lies in the image of ϕ.*

Proof. We construct ϕ as the projection $p_{\mathcal{U}}^{\bar{\lambda}}$ from Proposition 3.4 to the nerve of a cover $\mathcal{U}$ of X.

Fix a monotone sequence $\{l_i\}$ tending to infinity. Since $\operatorname{asdim} X \leq n$, for every i there is a uniformly bounded cover $\mathcal{U}_i$ of multiplicity $n+1$ with the Lebesgue number $L(\mathcal{U}_i) > l_i$. Let m_i be an upper bound for the diameter of elements of $\mathcal{U}_i$. Choose a sequence $\{r_i\}$ such that $f(X \setminus B_{r_i - m_i}) \geq 2m_{i+1}$ and take $\mathcal{U} = \cup_i \mathcal{U}_i'$ where

$$\mathcal{U}_i' = \{U \setminus B_{r_{i-1}} \mid U \in \mathcal{U}_i, U \cap B_{r_i} \neq \emptyset\}.$$

We take $C = B_{r_1}$ and check that the conclusion of our Lemma holds. By definition, the preimage $\phi^{-1}(\Delta)$ lies in a union $\cup_{j \in J} U_j'$ of sets from $\mathcal{U}$ with nonempty intersection. Let $x \in \cap_{j \in J} U_j'$. Each U_j' belongs to the cover $\mathcal{U}_i'$ for some i. Let k be maximal among those i's for all $j \in J$. Then $x \notin B_{r_{k-1}}$. Therefore $U_j' \subset X \setminus B_{r_{k-1} - m_{k-1}}$ for all j. Hence $f(z) \geq 2m_k \geq \operatorname{diam} \cup_j U_j'$ for $z \in X \setminus B_{r_{k-1} - m_{k-1}}$ and $k > 1$.

Since $l_i \to \infty$ the nerve N realized in $l_2(\mathcal{U})$ as above with the intrinsic metric is an asymptotic polyhedron. Since the covers $\mathcal{U}_i'$ and $\mathcal{U}_j'$ do not intersect for $|i-j| > 1$, the multiplicity of $\mathcal{U}$ is $\leq 2n+2$.

Finally, if we replace $\mathcal{U}$ by an irreducible subcover, ϕ is onto the vertices of N. □

REMARK. By a standard dimension theoretic trick the polyhedron N can be chosen to be n-dimensional.

A metric space X is called *uniformly contractible* if there is a function $S : \mathbf{R}_+ \to \mathbf{R}_+$ such that every ball $B_r(x)$ is contractible to a point in the ball $B_{S(r)}(x)$. The function S is called a *contractibility function* for X. We will always take our contractibility functions to be strictly monotone.

Lemma 3.6. *Let X be a uniformly contractible proper metric space with* $\operatorname{asdim} X \leq n$*. Then given a proper function $g : X \to \mathbf{R}_+$ there exist a $2n+1$-dimensional asymptotic polyhedron N, a proper 1-Lipschitz map $\phi : X \to N$, and a proper homotopy inverse map $\gamma : N \to X$ with $d(x, \gamma\phi(x)) < g(x)$ for all $x \in X$. Moreover, there is a compact set $C \subset X$ such that* $\operatorname{diam}(\phi^{-1}(\Delta)) \leq g(x)$ *for all $x \in \Delta \setminus C$.*

Proof. Let S be a contractibility function. We define $\rho(t) = S^{-1}(t/2)$ where S^{-1} is the inverse function for S. Then we take $f = \rho^{2n+1} \circ g$, the composition of g and the $2n+1$-fold iteration of ρ. Clearly $g \leq f$. We assume here that $g(x) \leq \|x\|/2$. We define by induction on i a lift γ on the i-skeleton $N^{(i)}$ of the nerve of a cover of X given by Lemma 3.5 for this choice of f. We take $\gamma(v) \in \phi^{-1}(v)$ for every vertex v. Then using the uniform contractibility of X we can extend γ with control over the 1-skeleton $N^{(1)}$ and so on. Without loss of generality (and for simplicity of exposition) we may assume that X is a polyhedron of the dimension n supplied with a triangulation of mesh ≤ 1. By induction on i we define a homotopy $H : X^{(i)} \times I \to X$ joining the identity map with $\gamma \circ \phi$. We consider a function $\psi(x) = \|x\| - \max\{d(x,y) \mid y \in H(x \times I)\}$. If ψ tends to infinity, then the map H is proper, but it is easy to verify that $\psi(x) \geq \|x\|/2$. □

We recall that the Higson compactification $\bar{X}$ of a proper metric space X can be defined as the maximal ideal space of the completion of the ring of bounded functions with the gradient tending to zero at infinity [Ro1]. The defining property of the Higson corona is the following:

(*)*A continuous map $f : X \to Z$ to a compact metric space is extendable to the Higson corona νX if and only if it satisfies the condition: For arbitrary large R*

$$\lim_{\|x\| \to \infty} \operatorname{diam}(f(B_R(x)) = 0.$$

Note that a proper Lipschitz map $f : X \to Y$ induces a continuous mapping between the Higson coronas $\bar{f} : \nu X \to \nu Y$.

Theorem 3.7. *Let X be a uniformly contractible geodesic proper metric space with finite asymptotic dimension and let $\bar{X}$ be the Higson compactification. Then $\check{H}^n(\bar{X}; \mathbf{Z}_p) = 0$ for all $n > 0$ and all p.*

Proof. We show that every map $\alpha : \bar{X} \to K(\mathbf{Z}_p, n)$ is null homotopic. Since $\bar{X}$ is compact, the image $\alpha(\bar{X})$ is contained in the k-skeleton $K = K(\mathbf{Z}_p, n)^{(k)}$ for some k. We will assume that K is a finite complex of dimension at least $2n+2$. For convenience, we replace K by a Riemannian manifold with Riemannianly collared boundary. Let $4\epsilon_K$ be a convexity radius for K, i.e. a positive constant such that every two points in an $4\epsilon_K$-ball can be joined by a unique geodesic in that $4\epsilon_K$-ball. Since the map $\alpha|_X \to K$ is extendable over the Higson corona, the function $R_\alpha(t) = \operatorname{Lip}(\alpha|_{X \setminus B_t(x_0)})$ tends to zero at infinity. We apply Lemma 3.6 with $g(x) \leq \min\{\epsilon_K / R_\alpha(\|x\|/2), \|x\|/4\}$ to obtain an asymptotic polyhedron N and maps $\phi : X \to N$ and $\gamma : N \to X$. We may assume that ϕ is surjective on vertices. Let $[u,v]$ be an edge in N, then $d_K(\alpha\gamma(u),\alpha\gamma(v)) \le R_\alpha(t_0)d_X(\gamma(u),\gamma(v))$ where $t_0=\min\{\|\gamma(u)\|,\|\gamma(v)\|\}$. We may assume that there are $x,y\in X$ such that $\phi(x)=u$ and $\phi(y)=v$. Then

$$d_X(\gamma(u),\gamma(v)) \le d_X(x,y)+\epsilon_K/R_\alpha(\frac{1}{2}\|x\|)+\epsilon_K/R_\alpha(\frac{1}{2}\|y\|) \le \operatorname{diam}\phi^{-1}[u,v]+2\epsilon_K/R_\alpha(\frac{1}{2}\|x\|)$$

provided that $\|x\| \le \|y\|$. Because of the inequality $g(x) \le \|x\|/4$ we have that $R_\alpha(t_0) \le R_\alpha(\|x\|/2)$. By Lemma 3.6, $\operatorname{diam}(\phi^{-1}([u,v])) < g(x)$. As a result, we obtain the inequality $d_K(\alpha\gamma(u),\alpha\gamma(v)) \le 3\epsilon_K$. This means that there is a map $\beta : N \to K$ which coincides on vertices with the map $\alpha\circ\gamma$ and which is $3\epsilon_K$-Lipschitz for $N_U = N$ taken with the uniform metric: β is obtained by linear extension of the restriction of $\alpha\circ\gamma$ to vertices. Note that β is ϵ_K-close to $\alpha\circ\gamma$ and that these maps are therefore homotopic.

Since X is contractible, the map $\alpha\circ\gamma$ is null homotopic. Therefore, so is β. Note that the homotopy groups $\pi_i(K)$ are finite for $i \le \dim N+1$. We apply Lemma 3.2 to obtain a λ-Lipschitz homotopy $H : N_U\times I \to K$ of β to a constant map. This homotopy defines a Lipschitz map $\tilde{H} : N_U \to K^I_\lambda$ to the space of λ-Lipschitz mappings of the unit interval I to K. We note that the space K^I_λ is compact. Then, by Proposition 3.3, $\tilde{H}\circ u : N \to K^I_\lambda$ satisfies the Higson extendibility condition (*). Let $\bar{h} : \bar{N} \to K^I_\lambda$ be the extension over the Higson corona. This extension defines a map $\bar{H} : \bar{N}\times I \to K$. The map $\bar{H}$ is a homotopy between the extension $\overline{\beta}$ and a constant map. To complete the proof, we show that α is homotopic to $\bar{\beta}\circ\bar{\phi}$ where $\bar{\phi}$ is the extension of the Lipschitz map ϕ to the Higson compactifications. Note that

$$d_K(\alpha(x),\alpha\gamma\phi(x)) \le R_\alpha(t_0)d(x,\gamma\phi(x)) \le R_\alpha(t_0)\epsilon_K/R_\alpha(\|x\|/2) \le \epsilon_K$$

where $t_0=\min\{\|x\|,\|\gamma\phi(x)\|\} \ge \|x\|/2$. Therefore, $d_K(\alpha(x),\,\beta\circ\phi(x)) \le 4\epsilon_K$. Then for every $x\in X$ we join the points $\alpha(x)$ and $\beta\phi(x)$ by the unique geodesic $\psi_x : I \to K$. This defines a map $\tilde{\psi} : X \to K^I_\mu$. Since both α and $\beta\circ\phi$ satisfy the condition (*), the map $\tilde{\psi}$ has the property (*). Let $\bar{\psi} : \bar{X} \to K^I_\mu$ be the extension of $\tilde{\psi}$ to the Higson corona. The map $\bar{\psi}$ defines a homotopy $\Psi : \bar{X}\times I \to K$ between α and $\bar{\beta}\circ\bar{\phi}$. □

We have now proven Theorem 3. The following asserts Theorem 2:

Corollary 3.8. *Suppose that a group Γ has finite asymptotic dimension and that $\mathrm{B}\Gamma$ is a finite complex. Then the integral assembly map $A : H_*(\mathrm{B}\Gamma;\mathbf{L}) \to L_*(\mathbf{Z}\Gamma)$ is a monomorphism.*

Proof. In view of the inequality $\dim\nu\Gamma \le \operatorname{asdim}\Gamma$ [DKU] we can apply Theorem 2.4. □

Corollary 3.9. *If* Γ *and* R *satisfy the finiteness conditions of Theorem 2, then* $H_i(B\Gamma; \mathbf{L}(R)) \to L_i(R\Gamma)$ *is injective for all* i*.*

References

[Ba] A. C. Bartels, *Squeezing and higher algebraic K-theory*, K-theory **28** (2003), 19–37.

[BD1] G. Bell and A. Dranishnikov, *On asymptotic dimension of groups*, Algebr. Geom. Topol. **1** (2001), 57-71.

[BD2] G. Bell and A. Dranishnikov, *On asymptotic dimension of groups acting on trees*, Geom. Dedicata **103** (2004), 89-101.

[BW] J. Block and S. Weinberger, *Aperiodic tilings, positive scalar curvature and amenability of spaces*, J. Amer. Math. Soc. **5** (1992), 907–918.

[BHM] M. Bökstedt,W.-C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Inventiones Math. **111** (1993), 465–539.

[C] G. Carlsson, *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), 5-127, London Math. Soc. Lecture Note Ser., 226,*, Cambridge Univ. Press, Cambridge, 1995.

[CG1] G. Carlsson and B. Goldfarb, *On homological coherence of discrete groups*, Journal of Algebra **276 No 2** (2004), 77-97.

[CG2] G. Carlsson and B. Goldfarb, *The integral K-theoretic Novikov conjecture for groups with finite asymptotic dimension*, Inventiones Math. (2004).

[CP1] G. Carlsson and E. K. Pedersen, *Controlled algebra and the Novikov conjectures for K- and L-theory*, Topology **34** (1993), 731-758.

[CP2] G. Carlsson and E. K. Pedersen, *Čech homology and the Novikov conjectures for K- and L-theory*, Math Scand (1997).

[CFY] S. Chang, S. Ferry, and G. Yu, *Bounded rigidity of manifolds and asymptotic dimension growth*, to appear in K-Theory (2005).

[Dr1] A. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys **55:6** (2000), 71-116.

[Dr2] A. Dranishnikov, *On hypersphericity of manifolds with finite asymptotic dimension*, Trans. Amer. Math. Soc. **355 no 1** (2003), 155-167.

[Dr3] A. Dranishnikov, *On Hypereuclidean manifolds*, Preprint (2000).

[DF] A. Dranishnikov and S. Ferry, *On the Higson-Roe corona*, Russian Math. Surveys **52 no 5** (1997), 1017-1028.

[DJ] A. Dranishnikov and T. Januszkiewicz, *Every Coxeter group acts amenably on a compact space*, Topology Proceedings **24** (1999), 135-141.

[DKU] A. Dranishnikov, J. Keesling, V. V. Uspenskij, *On the Higson corona of uniformly contractible spaces*, Topology **37** (1999), n0. 4, 791-803.

[EH] D. A. Edwards and H. M. Hastings, *Čech and Steenrod Homology Theories with Applications to Geometric Topology*, Lecture Notes in Mathematics Proceedings **542** (1976), Springer.

[FRR] S. Ferry, A. Ranicki and J. Rosenberg; Editors, *Novikov conjectures, index theorems and rigidity, Vol. 1, 2. London Math. Soc. Lecture Note Ser., 226,*, Cambridge Univ. Press, Cambridge, 1995.

[FW] S. Ferry and S. Weinberger, *A coarse approach to the Novikov conjecture*, Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), 147–163, London Math. Soc. Lecture Note Ser., 226, (1995), Cambridge Univ. Press, Cambridge.

[G1] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, vol 2, Cambridge University Press, 1993.

[G2] M. Gromov, *Spaces and questions*, GAFA **Special Volume, Part I** (2000), 118-161.

[HLS] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geom. Funct. Anal. **12** (2002), 330–354.

[HR] N. Higson and J. Roe, *Analytic K-homology*, Oxford University Press, Oxford, 2000.

[Ji] L. Ji, *Asymptotic dimension of arithmetic groups*, preprint (2003).

[KS] J. Kaminker and C. Schochet, *K-theory and Steenrod homology: applications to the Brown-Douglas-Fillmore theory of operator algebras*, Trans. Amer. Math. Soc. **227** (1977), 63-107.

[Kee] J. Keesling, *The one-dimensional Čech cohomology of the Higson compactification and its corona*, Topology Proc. **19** (1994), 129–148.

[M] W. Massey, *Homology and cohomology theory*, Dekker, New York, 1978.

[MM] I. Madsen and R. J. Milgram, *The universal smooth surgery class*, Comm. Math. Helv. **50** (1975), 281-310.

[O] N. Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 691–695.

[P] E. K. Pedersen, *Bounded and continuous control*, Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), 277-284, London Math. Soc. Lecture Note Ser., 226, (1995), Cambridge Univ. Press, Cambridge.

[Ran1] A.A. Ranicki, *Algebraic L-theory and topological manifolds*, Cambridge Univ. Press, Cambridge, 1992.

[Ran2] A.A. Ranicki, *Additive L-theory*, K-theory **3** (1989), 163-195.

[Ran3] A.A. Ranicki, *On the Novikov conjecture*, in the book "Novikov conjectures, index theorems and rigidity", Vol. 1 (Oberwolfach, 1993), 272-337, London Math. Soc. Lecture Note Ser., 226, (1995), Cambridge Univ. Press, Cambridge.

[Ro1] J. Roe, *Coarse cohomology and index theory for complete Riemannian manifolds*, Memoirs Amer. Math. Soc. No. 497, 1993.

[Ro2] J. Roe, *Lectures on coarse geometry*, University Lecture series, Volume 31 (2003), AMS.

[Ros] J. Rosenberg, *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), 338-372, London Math. Soc. Lecture Note Ser., 226,*, Cambridge Univ. Press, Cambridge, 1995.

[SW] J. Siegel and F. Williams, *Uniform bounds for isoperimetric problems*, Proc. Amer. Math. Soc. **107** (1989), no 2, 459–464.

[STY] G., Skandalis, Tu, J. L., and Yu, G., *The coarse Baum-Connes conjecture and groupoids*, Topology **41** (2002), 807–834.

[T] J. L. Taylor, *A counter-Example in Shape Theory*, Bull. Amer. Math. Soc. **81** (1975), 629-632.

[TW] L. Taylor, B. Williams, *Surgery spaces: structure and formulae*, Lecture Notes in Mathematics **741** (1979), 170–195.

[Yu1] G. Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Ann. of Math **147** (1998), no. 2, 325-355.

University of Florida, Department of Mathematics, P.O. Box 118105, 358 Little Hall, Gainesville, FL 32611-8105, USA

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019

Department of Mathematics, The University of Chicago, Chicago, IL 60637
E-mail address: dranish@math.ufl.edu