$\angle CBN < \angle LCB < 90°$. Since smaller chords of a circle subtend smaller acute angles, and $BL < CN$,

$$\angle LCB < \angle CBN.$$

We thus have a contradiction.

Editorial note. Martin Gardner, in his review of Coxeter's *Introduction to geometry* (Scientific American, 204 (1961) 166–168) described this famous theorem in such an interesting manner that hundreds of readers sent him their own proofs. He took the trouble to refine this massive lump of material until only the above gem remained. This theorem was proposed in 1840 by C. L. Lehmus, and proved by Jacob Steiner. For its history until 1940 see J. A. McBride, Edinburgh Math. Notes, 33 (1943) 1–13.

THE CANTOR EXPANSION OF REAL NUMBERS

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The Cantor expansion of a real number α in a given base-sequence $\{b_n\}$ of natural numbers $b_n \geqq 2$ is

$$\alpha = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \cdots b_n} \tag{1}$$

with a_0 an integer and nonnegative integers (digits) $a_n \leqq b_n - 1$, $n \geqq 1$.

The following formula proves to be useful in establishing irrationality of some numbers:

$$a_n = [b_n b_{n-1} \cdots b_2 b_1 \alpha] - b_n [b_{n-1} \cdots b_2 b_1 \alpha] \tag{2}$$

in which $[\xi]$ denotes the greatest integer not exceeding ξ. Here are some examples.

1. If $b_n = n+1$, $a_0 = 2$, $a_n = 1$, the Cantor expansion (1) represents the number e, the irrationality of which follows by (2) immediately: if $e = r/q$ take $n = q$ to get the contradiction $1 = 0$.

2. In an analogous way one can prove the irrationality of the numbers: sinh 1, cosh 1, and $I_k(1)$ and $I_k(2)$ ($k = 0, 1, 2, \cdots$) for the Bessel functions

$$I_k(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{2^n n!(n+k)!}.$$

3. If b_n is the nth prime p_n in the natural sequence and $a_n = 1$, the irrationality of the number

$$\epsilon = \sum_{n=1}^{\infty} \frac{1}{p_1 p_2 \cdots p_n}$$

can be proved from (2). If $\epsilon = r/q$ choose n in (2) so that $q \leqq p_{n-1}$. If all the prime factors of q occur with exponent 1 only, formula (2) gives the contradiction $1=0$. If some prime factors occur with exponent higher than 1, write

$$p_n p_{n-1} \cdots p_2 p_1 \frac{r}{q} = A + \frac{a}{s}, \qquad p_{n-1} \cdots p_2 p_1 \frac{r}{q} = B + \frac{b}{s}$$

with natural numbers A, B, $1 \leqq a < s$, $1 \leqq b < s$. It follows that

$$\text{(i)} \qquad 2s < q$$

because not all primes would cancel with the prime factors of q. Thus, formula (2) would give $1 = A - p_n B$ whence, in view of $A + (a/s) = p_n(B + (b/s))$ it would follow that $p_n = (a+s)/b < 2s$, in contradiction to (i).

It is well known (see, e.g. [1]) that a sufficient condition for an infinite Cantor expansion to represent an irrational number is that each prime divides infinitely many of the b_n's. The number e gives an extreme example showing that the condition is not necessary. If a Cantor expansion of π were known it would yield an elementary proof (without using integrals) that π is irrational.

Reference

1. I. Niven, Irrational Numbers, Carus Mathematical Monograph no. 11, 1956, pg. 10–11.

A NOTE ON THE DERIVATION OF RODRIGUES' FORMULAE

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In the study of special functions, solutions to differential equations in the form of Rodrigues' Formulae are of considerable interest. The following elementary method provides the derivation of these formulae for a particular class of second order differential equations.

Suppose we have the differential equation

$$(1) \qquad (Ax^2 + Bx + C)y'' + (Dx + E)y' + Fy = 0,$$

where A, 5, C, D, E, and F are independent of x, and the associated equation

$$(2) \qquad As^2 + (A - D)s + F = 0$$

has a positive integral root, say $s=j$.

Construct a second differential equation

$$(3) \qquad (Ax^2 + Bx + C)z'' + [(D - 2Aj)x + (E - Bj)]z' = 0,$$

whose solution is

$$(4) \qquad z' = K(Ax^2 + Bx + C)^j \exp\left\{-\int \frac{Dx + E}{Ax^2 + Bx + C}\, dx\right\},$$

where K is an arbitrary constant.

Now differentiate (3) j times. It is easily verified that this result is