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# A DUALITY THEOREM FOR REIDEMEISTER TORSION

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This paper will show that the torsion invariant of Reidemeister, Franz, and de Rham for a manifold satisfies a duality relation, analogous to Poincaré duality. As an application one obtains a new proof that the Alexander polynomial of a knot is symmetric (a result first proved by Seifert [11]).

## 1. The duality theorem

First some algebraic preliminaries. Let $P$ be a ring with an anti-automorphism $\rho \to \bar{\rho}$ of period two. Given any left $P$-module $A$ define the *dual module* $A^*$ to be $\mathrm{Hom}_P(A, P)$, considered as a left $P$-module in the following way. For each $\rho \in P$ and $f: A \to P$ define $\rho f: A \to P$ by the formula

$$[a, \rho f] = [a, f]\bar{\rho} ,$$

where $[a, f]$ denotes the value of the function $f$ at $a$. If $A$ is free and finitely generated, then clearly $A^*$ is free and finitely generated, and $A^{**}$ can be identified with $A$. Note that any homomorphism $h: A_1 \to A_2$ gives rise to a dual homomorphism $h^*: A_2^* \to A_1^*$.

As an example consider the following geometrical situation. Let $M$ be a simplical complex whose underlying space is an oriented $n$-manifold without boundary. Let $\Pi$ be a group of fixed point free simplicial automorphisms of $M$. Then the chain group $C_q(M; Z)$ can be considered as a free left module over the integral group ring $Z[\Pi]$.

Now suppose that $M$ has a dual cell subdivision $M'$. Then the chain group $C_{n-q}(M'; Z)$ is also a free left $Z[\Pi]$-module. We will assume that the quotient space $M/\Pi$ is compact, so that these modules are finitely generated.

There is a canonical anti-automorphism $\rho \to \bar{\rho}$ of $Z[\Pi]$ which takes each group element $\pi$ into $\pi^{-1}$.

LEMMA 1 (Reidemeister). *If the elements of $\Pi$ are orientation preserving automorphisms of $M$, then the $Z[\Pi]$-module $C_{n-q}(M'; Z)$ is canonically isomorphic to the dual of $C_q(M; Z)$. Furthermore the boundary homomorphism*

$$\partial: C_{n-q}(M'; Z) \longrightarrow C_{n-q-1}(M'; Z)$$

*is (up to sign) dual to the corresponding homomorphism*

$$\partial\colon C_{q+1}(M; Z) \longrightarrow C_q(M; Z) .$$

PROOF. For each chain $c' \in C_{n-q}(M'; Z)$ define a homomorphism

$$[\ \ , c']\colon C_q(M; Z) \longrightarrow Z[\Pi]$$

by the formula

$$[c, c'] = \textstyle\sum_{\pi \in \Pi} \langle c, \pi c' \rangle \pi ;$$

where $\langle c, \pi c' \rangle$ denotes the (integer) intersection number of $c$ and $\pi c'$. The required identities

$$[\pi c, c'] = \pi[c, c'] ,$$
$$[c, \pi c'] = [c, c']\bar{\pi} ,$$

and

$$[\partial c, c'] = \pm[c, \partial c']$$

are easily verified. (Compare Reidemeister [9], Burger [2], Blanchfield [1].) This proves Lemma 1.

Now let us apply this duality to the torsion invariant. Given a commutative field $F$ and a homomorphism

$$h\colon Z[\Pi] \longrightarrow F ,$$

we can consider $F$ as a right $Z[\Pi]$-module and hence form the vector spaces

$$C_q = F \otimes_\Pi C_q(M; Z) , \qquad C'_{n-q} = F \otimes_\Pi C_{n-q}(M'; Z)$$

over $F$.

Assume that we are given an involution $a \to \bar{a}$ of $F$ which satisfies the identity $h(\bar{a}) = \bar{h}(a)$. Then the vector space $C'_{n-q}$ is canonically isomorphic to the dual of $C_q$.

If the chain complex $C_*$ is acyclic—i.e., if the sequence

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

is exact—then the torsion invariant

$$\Delta_h M \in F_0/\pm h(\Pi)$$

is defined. (See Franz [6] or de Rham [10]. The definition as given in Milnor [7] will be repeated below.) Here $F_0$ denotes the multiplicative group $F - 0$ and $\pm h(\Pi)$ denotes the subgroup consisting of all elements $\pm h(\pi)$, $\pi \in \Pi$.

THEOREM 1. *This torsion invariant $\Delta = \Delta_h M$ satisfies the identity*

$$\Delta\bar{\Delta}^{\varepsilon(n)} = \pm h(\Pi)$$

*where* $\varepsilon(n) = (-1)^n$.

Thus if $n$ is odd the torsion invariant is self-conjugate:

$$\bar{\Delta}_h M = \Delta_h M ,$$

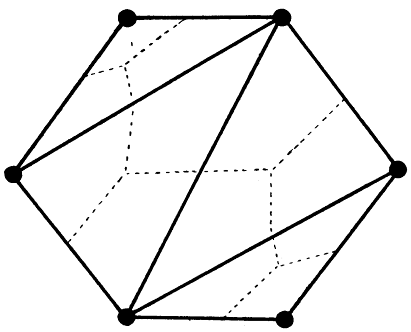
and if $n$ is even the torsion is unitary:

$$\bar{\Delta}_h M = (\Delta_h M)^{-1} .$$

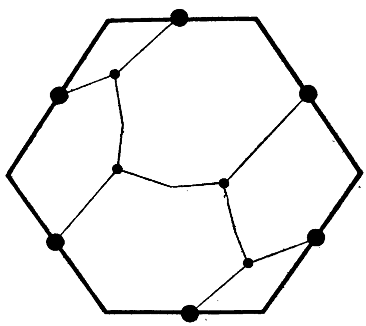
The classical illustration of torsion is provided by the (odd dimensional) lens manifolds. (See [6], [10].) One uses the field of complex numbers, and the usual complex conjugation operation. Hence Theorem 1 asserts that the resulting torsion element is essentially a real number.

For an even dimensional manifold, with $F$ the field of complex numbers, the torsion must be a number on the unit circle defined up to multiplication by certain roots of unity. I do not know any non-trivial examples.

Before proving Theorem 1, it is convenient to consider the more general case of a triangulated $n$-manifold $M$ with boundary $bM$. Again we assume that each simplex of $M$ has a dual cell. For a $q$-simplex of $bM$ one can define not only the dual $(n - q)$-cell in $M$, but also the dual $(n - q - 1)$-cell in $bM$. Taking the cells of both types we obtain a dual complex $M'$ with subcomplex $bM'$.



$M$



$M'$
with $bM'$ emphasized

Again assume that $\Pi$ acts freely, preserving orientation, and that $M/\Pi$ is compact.

LEMMA 2. *The left* $Z[\Pi]$*-module* $C_{n-q}(M', bM'; Z)$ *is dual to* $C_q(M; Z)$*; and the boundary operator in* $C_*(M', bM'; Z)$ *is (up to sign) dual to the boundary operator in* $C_*(M; Z)$.

The proof is straightforward.

Again let $h: Z[\Pi] \to F$ be a homomorphism compatible with the conjugation operations.

THEOREM 1′. *If the torsion* $\Delta_h M$ *is defined* (i.e., *if the chain complex* $F \otimes_\Pi C_*(M; Z)$ *is acyclic*) *then* $\Delta_h(M, bM)$ *is defined, and conversely. Furthermore*

$$\Delta_h(M, bM) = (\bar{\Delta}_h M)^{-\varepsilon(n)}$$

*where* $\varepsilon(n) = (-1)^n$.

Clearly this result contains Theorem 1 as a special case. The following reformulation is usually more convenient to use. According to [7, Lemma 4], if both $\Delta_h M$ and $\Delta_h(M, bM)$ are defined, then $\Delta_h(bM)$ is also defined, and

$$\Delta_h(M, bM) = \Delta_h M / \Delta_h(bM) .$$

Combining this information with Theorem 1′, one obtains:

THEOREM 2. *If* $\Delta_h M$ *is defined then* $\Delta_h(bM)$ *is also defined, and*

$$\Delta_h(bM) = (\Delta_h M)(\bar{\Delta}_h M)^{\varepsilon(n)} .$$

Examples will be given in § 2.

In order to prove Theorems 1 and 1′ it is first necessary to review the definition of torsion, as given in [7]. For any vector space $C$ of dimension $d$ over $F$, let $\Lambda C$ denote the $d^{\text{th}}$ exterior power $\Lambda^d C$. This is a one-dimensional vector space.

Given a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 ,$$

and given generators $a \in \Lambda A, b \in \Lambda B$, construct a generator $b/a \in \Lambda C$ as follows. Choose a basis $x_1, \cdots, x_n$ for $B$ so that the last $n - m$ vectors $x_{m+1}, \cdots, x_n$ form a basis for the subspace $A \subset B$. Thus $a$ and $b$ can be written in the form

$$a = f x_{m+1} \wedge \cdots \wedge x_n , \qquad b = g x_1 \wedge \cdots \wedge x_n$$

for appropriate field elements $f, g \neq 0$. Define $b/a$ to be the image in $\Lambda C$ of

$$g f^{-1} x_1 \wedge \cdots \wedge x_m .$$

This construction does not depend on the choice of basis.

Now suppose that one is given a long exact sequence

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

and a preferred generator $v_q \in \Lambda C_q$ for each $q$. Using the short exact sequence

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \partial C_{n-1} \longrightarrow 0 ,$$

one constructs a generator

$$v_{n-1}/v_n \in \Lambda(\partial C_{n-1}) .$$

Proceeding inductively, using the sequences

$$0 \longrightarrow \partial C_{q+1} \longrightarrow C_q \longrightarrow \partial C_q \longrightarrow 0 ,$$

one constructs generators

$$v_q/\big(v_{q+1}/\big(\cdots/(v_{n-1}/v_n)\cdots\big)\big) \in \Lambda(\partial C_q)$$

for each $q$. These will be written briefly as $(v_q/v_{q+1}/\cdots/v_{n-1}/v_n)$.

In particular one obtains a generator

$$(v_1/v_2/\cdots/v_{n-1}/v_n) \in \Lambda(\partial C_1) = \Lambda C_0 .$$

Taking the ratio of this with the given generator $v_0 \in \Lambda C_0$ we obtain a field element

$$D = v_0/(v_1/v_2/\cdots/v_n) .$$

This ratio $D \in F_0$ is called the *torsion*[1] associated with $\{C_q, v_q\}$.

Note the identity

$$(1) \qquad (v_1/v_2/\cdots/v_n) = D^{-1}v_0 .$$

Now let us apply duality to this situation. For any vector space $C$, note that $\Lambda(C^*)$ is canonically isomorphic to the dual $(\Lambda C)^*$ of $\Lambda C$. In fact given elements

$$v = x_1 \wedge \cdots \wedge x_n \in \Lambda C , \qquad w = y_1 \wedge \cdots \wedge y_n \in \Lambda(C^*) ,$$

define $[v, w]$ to be the determinant of the matrix $\| [x_i, y_j] \|$. As an example, if $x_1, \cdots, x_n$ is a basis for $C$ and $x_1^*, \cdots, x_n^*$ is the dual basis for $C^*$ then $[x_1 \wedge \cdots \wedge x_n, x_1^* \wedge \cdots \wedge x_n^*] = 1$.

Consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

together with the dual sequence

$$0 \longleftarrow A^* \longleftarrow B^* \longleftarrow C^* \longleftarrow 0 .$$

Let $a \in \Lambda A$, $b \in \Lambda B$, $b^* \in \Lambda B^*$, and $c^* \in \Lambda C^*$ be generators; with $[b, b^*] = 1$.

LEMMA 3. *Then* $[b/a, c^*]$ *is equal to* $\pm[a, b^*/c^*]^{-1}$.

PROOF. Choose a basis $x_1, \cdots, x_n$ for $B$ so that $x_{m+1}, \cdots, x_n$ form a basis for $A$. Let $x_1^*, \cdots, x_n^*$ denote the dual basis for $B^*$, so that $x_1^*, \cdots, x_m^*$ form a basis for $C^* \subset B^*$. Define field elements $f, g, h$ by

[1] Actually this element $D$ is the reciprocal of the torsion as defined by Franz.

$$a = fx_{m+1} \wedge \cdots \wedge x_n , \qquad b = gx_1 \wedge \cdots \wedge x_n ,$$
$$b^* = \bar{g}^{-1}x_1^* \wedge \cdots \wedge x_n^* , \qquad c^* = \bar{h}x_1^* \wedge \cdots \wedge x_m^* .$$

Then $[b/a, c^*] = gf^{-1}h$ and $[a, b^*/c^*] = \pm fg^{-1}h^{-1}$. This completes the proof.

Now given an acyclic chain complex $0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0$ with generators $v_q \in \Lambda C_q$, form the dual complex

$$0 \longrightarrow C_0^* \longrightarrow C_1^* \longrightarrow \cdots \longrightarrow C_n^* \longrightarrow 0$$

and the dual generators $v_q^* \in \Lambda C_q^*$. Just as before one constructs generators

$$(v_q^*/v_{q-1}^*/ \cdots /v_1^*/v_0^*) \in \Lambda(\partial^* C_q^*)$$

by induction on $q$. The torsion $D_1 \in F$ of the dual complex is then defined by the identity

$$D_1 = v_n^*/(v_{n-1}^*/\cdots/v_1^*/v_0^*)$$

or

$$(v_{n-1}^*/v_{n-2}^*/\cdots/v_1^*/v_0^*) = D_1^{-1}v_n^* . \tag{2}$$

Applying Lemma 3 to the dual exact sequences

$$0 \longrightarrow \partial C_{q+1} \longrightarrow C_q \longrightarrow \partial C_q \longrightarrow 0 ,$$

and

$$0 \longleftarrow \partial^* C_q^* \longleftarrow C_q^* \longleftarrow \partial^* C_{q-1}^* \longleftarrow 0 ,$$

one obtains the identity

$$[v_q/(v_{q+1}/\cdots/v_{n-1}/v_n), (v_{q-1}^*/\cdots/v_1^*/v_0^*)]$$
$$= \pm[(v_{q+1}/\cdots/v_{n-1}/v_n), v_q^*/(v_{q-1}^*/\cdots/v_1^*/v_0^*)]^{-1} .$$

In other words the field element

$$\pm f = [(v_{q+1}/\cdots/v_{n-1}/v_n), (v_q^*/v_{q-1}^*/\cdots/v_1^*/v_0^*)]^{\varepsilon(q)}$$

is independent of $q$.

Taking $q = 0$ and making use of formula (1) this gives

$$\pm f = [(v_1/\cdots/v_{n-1}/v_n), v_0^*] = [D^{-1}v_0, v_0^*] = D^{-1} .$$

On the other hand, taking $q = n - 1$ and using the formula (2) it gives

$$\pm f = [v_n, D_1^{-1}v_n^*]^{\varepsilon(n-1)} = \bar{D}_1^{\varepsilon(n)} .$$

Therefore the torsion $D_1$ of the dual complex satisfies the identity

$$D\bar{D}_1^{\varepsilon(n)} = \pm 1 . \tag{3}$$

This completes the purely algebraic part of the proof.

The remainder of the proof is straightforward. The $q$-cells of $M$ determine a preferred basis for $C_q(M; Z)$ and hence a preferred generator

$$v_q \in \Lambda C_q' = \Lambda(F \otimes_\Pi C_q(M; Z)) .$$

This generator is well defined up to multiplication by elements of the form $\pm h(\pi)$. If $D \in F_0$ denotes the torsion associated with $\{C_q, v_q\}$ then $\Delta_h M$ is defined to be the coset $\Delta_h M = \pm h(\Pi) D \in F_0/\pm h(\Pi)$.

The $(n-q)$-cells of $M' - bM'$ determine a dual basis for $C_{n-q}(M', bM'; Z)$ and hence determine the dual generator

$$v_q^* \in \Lambda C_q^* = \Lambda(F \otimes_\Pi C_{n-q}(M', bM'; Z)) .$$

Thus if $D_1$ denotes the torsion associated with $\{C_q^*, v_q^*\}$ then

$$\Delta_h(M', bM') = \pm h(\Pi) D_1 \in F_0/\pm h(\Pi) .$$

According to formula (3):

$$(\Delta_h M)(\bar{\Delta}_h(M', bM'))^{\varepsilon(n)} = \pm h(\Pi) .$$

But since $M$ and $M'$ have a common subdivision, we have

$$\Delta_h(M, bM) = \Delta_h(M', bM') .$$

This completes the proof of Theorems 1, 1′ and 2.

## 2. Applications to knot theory

In this section, $\Pi$ will always be an infinite cyclic multiplicative group with generator $t$. As field $F$ take the quotient field of the integral group ring $Z[\Pi]$. Thus $F$ can be described as the field $Q(t)$ of rational functions in one variable $t$ over the rational numbers. Let $I: Z[\Pi] \to Q(t)$ be the imbedding, and define the conjugation operation in $Q(t)$ by $\overline{f(t)} = f(t^{-1})$.

LEMMA 4. *Let $K$ be a finite polyhedron having the homology of the circle, and let $L = \tilde{K}$ be its infinite cyclic covering complex. Then the torsion*

$$\Delta_I L \in Q(t)_0/\pm\Pi$$

*is defined.*

The proof will be given later.

As an application consider any differentiable imbedding of the $(n-2)$-sphere in the $n$-sphere. Removing a tubular neighborhood of the $(n-2)$-sphere from $S^n$ we obtain a compact manifold $K$ with boundary. Choose some $C^1$-triangulation of $K$ (see Whitehead [13]) and let $M = \tilde{K}$ denote the infinite cyclic covering complex.

THEOREM 3. *If $M$ is obtained from a differentiable imbedding $S^{n-2} \to S^n$ as above, then the torsion $\Delta_I M$ is defined and satisfies the symmetry relation*

$$\Delta_I M = \bar{\Delta}_I M \qquad \textit{for } n \textit{ odd},$$
$$(\Delta_I M)(\bar{\Delta}_I M) = \pm\Pi/(t-1)^2 \qquad \textit{for } n \textit{ even}.$$

PROOF. It follows from Lemma 4 that $\Delta_I M$ is defined. According to Theorem 2 we have

$$(*) \qquad (\Delta_I M)(\bar{\Delta}_I M)^{\varepsilon(n)} = \Delta_I(bM) .$$

But the boundary $bM$ is clearly equivariantly diffeomorphic to the product $\tilde{S}^1 \times S^{n-2}$; where $\tilde{S}^1$ denotes the universal covering space of the circle. Thus $bM$ can be given an equivariant cell structure consisting of four cells

$$e^1 \times e^{n-2}, \qquad e^0 \times e^{n-2}, \qquad e^1 \times e^0, \qquad e^0 \times e^0$$

together with their translates under the action of $\Pi$. The boundary relations are

$$\partial(e^1 \times e^{n-2}) = (t-1)(e^0 \times e^{n-2}), \qquad \partial(e^1 \times e^0) = (t-1)(e^0 \times e^0).$$

The torsion invariant

$$D = e^0 \times e^0/(e^1 \times e^0/1/\cdots/1/e^0 \times e^{n-2}/e^1 \times e^{n-2})$$

for this chain complex can easily be computed. In fact $D$ is equal to $(t-1)^{-1-\varepsilon(n)}$. The torsion computed from any $C^1$-triangulation of $bM$ will be the same, since this cell complex has a $C^1$-triangulation as an equivariant subdivision. Therefore

$$\Delta_I(bM) = \pm\Pi/(t-1)^{1+\varepsilon(n)} .$$

Combining this information with the formula $(*)$ we obtain Theorem 3.

As an example, consider the standard imbedding of $S^{n-2}$ in $S^n$. Then

$$\Delta_I M = \pm\Pi/(t-1) .$$

Therefore $(\Delta_I M)(\bar{\Delta}_I M)^{\varepsilon(n)} = \pm\Pi/(t-1)(t^{-1}-1)^{\varepsilon(n)}$ is equal to $\pm\Pi/(t-1)^{1+\varepsilon(n)}$, as required by Theorem 3.

Now let us specialize to the case $n = 3$.

THEOREM 4. *If $K$ is a complex of dimension $\leqq 2$, or a manifold of dimension $\leqq 3$, having the homology of a circle, then the torsion invariant $\Delta_I \tilde{K}$ is equal to $\pm\Pi A(t)/(t-1)$, where $A(t)$ denotes the Alexander polynomial of the fundamental group.*

Combining this with Theorem 3 we see that the Alexander polynomial $A(t)$ for the complement of a knot in $S^3$ satisfies the relation

$$A(t)/(t-1) = \pm t^i A(t^{-1})/(t^{-1}-1)$$

hence

$$A(t^{-1}) = \mp t^{-i-1} A(t) .$$

Combining this with the fact that $A(1) = \pm 1$ (see for example Fox [4]) one easily obtains the following:

COROLLARY (Seifert). *The Alexander polynomial $A(t)$ of a knot has the form $\pm t^j(a_0 + a_1 t + \cdots + a_{2r}t^{2r})$ with $a_0 = a_{2r}$, $a_1 = a_{2r-1}, \cdots, a_{r-1} = a_{r+1}$.*

*Note.* The possibility that this corollary could be proved in this way was conjectured by Reidemeister [8, § 5]. The above proof is close to that given by Blanchfield [1].

The rest of this paper will be devoted to the proofs of Lemma 4 and Theorem 4.

In order to prove Lemma 4 one must show that the chain complex $Q(t) \otimes_\Pi C_*(L; Z)$ is acyclic. Let $e_1^q, \cdots, e_{\alpha_q}^q$ denote the standard basis for $C_q(L; Z)$, with one basis element for each $q$-cell of $K = L/\Pi$. The boundary homomorphism on $C_q(L; Z)$ can be described by a matrix $\| a_{ij}^q(t) \|$ over the integral domain $Z[\Pi]$, which is defined by:

$$\partial e_i^q = \sum_j a_{ij}^q(t) e_j^{q-1} .$$

Let $r_q$ denote the rank of this matrix. Note that the difference $\alpha_q - r_q - r_{q+1}$ is equal to the $q^{\text{th}}$ Betti number of the chain complex

$$C_* = Q(t) \otimes_\Pi C_*(L; Z) .$$

Now consider the chain complex $C_*(K; Z)$. Note that the boundary homomorphism

$$\partial\colon C_q(K; Z) \longrightarrow C_{q-1}(K; Z)$$

can be described by a matrix of integers $\| a_{ij}^q(1) \|$ which is a homomorphic image of the matrix above. Therefore the rank $r'_q$ of this boundary homomorphism satisfies $r'_q \leqq r_q$. This implies that the Betti numbers $\alpha_q - r'_q - r'_{q+1}$ of $C_*(K; Z)$ are greater than or equal to the corresponding Betti numbers of $C_*$. But $K$ has the homology of a circle. Therefore the chain complex $C_*$ can have non-trivial homology at most in the dimensions 0 and 1.

On the other hand $H_0(C_*)$ is zero. For if $e_i^0$ denotes any vertex, then there certainly exists a 1-chain $c$ with $\partial c = te_i^0 - e_i^0$. Since $t - 1$ is a unit in $Q(t)$, this implies that $e_i^0$ itself is a boundary.

Finally, since the Euler characteristic $\alpha_0 - \alpha_1 + \alpha_2 - + \cdots \pm \alpha_n$ is zero, the group $H_1(C_*)$ must also be zero. This completes the proof of Lemma 4.

PROOF OF THEOREM 4. If $K$ is a 3-manifold, then by pushing in one free face at a time, we can collapse $K$ down to a 2-dimensional subcomplex. These collapsings do not affect the torsion. (Compare Whitehead [14].)

Furthermore the simplicial complex $K$ can be replaced by a CW-complex $K_1$ with a single vertex $e_1^0$. For let $e^1$ be any edge of $K$, and let $K'$ be the CW-complex obtained from $K$ by collapsing $e^1$ to a point. It is not difficult to show that $\Delta_I \widetilde{K}' = \Delta_I \widetilde{K}$. Now iterate this procedure, always selecting an edge $e^1$ which has distinct end points. After a finite number of stages we obtain a complex $K_1$ with a single vertex.

Let $L = \widetilde{K}_1$. The boundary of any edge $e_j^1$ of $L$ clearly has the form

$$\partial e_j^1 = (t^r - t^s)e_1^0 .$$

Without loss of generality we may assume that the first edge $e_1^1$ has boundary equal to $(t-1)e_1^0$. For otherwise it is only necessary to adjoin a suitable 2-cell to $K$, having such an edge as free face. (Again compare Whitehead [14].)

Now consider the matrices $\| a_{ij}^2(t) \|$ and $\| a_{j1}^1(t) \|$ as defined earlier. If $m = \alpha_2$ denotes the number of 2-cells, then the relation $\alpha_2 - \alpha_1 + \alpha_0 = 0$ implies that $\alpha_1 = m + 1$. Thus $\| a_{ij}^2(t) \|$ is an $m \times (m+1)$-matrix and $\| a_{j1}^1 \|$ is an $(m+1) \times 1$-matrix. It is well known that $\| a_{ij}^2(t) \|$ is precisely the Alexander matrix of that presentation of the fundamental group of $K$ which is associated with the given cell structure. (Compare Fox [3, p. 547] and [4].) Hence the Alexander polynomial $A(t)$ is defined to be a generator of the principal ideal in $Z[\Pi]$ generated by the $m \times m$ minor determinants of $\| a_{ij}^2(t) \|$.

The first column of $\| a_{ij}^2(t) \|$ can be expressed as a linear combination of the remaining columns. This follows from the relation

$$\textstyle\sum_j a_{ij}^2 a_{j1}^1 = 0 ,$$

together with the fact that $a_{11}^1 = t - 1$, and that each $a_{j1}^1$ has the form $t^r - t^s$; and hence is divisible by $t - 1$. Therefore $A(t)$ can be defined as the determinant of the matrix obtained by deleting the first column of $\| a_{ij}^2 \|$.

Now let us compute the torsion invariant. Let

$$v_2 = e_1^2 \wedge \cdots \wedge e_m^2 , \qquad v_1 = e_1^1 \wedge \cdots \wedge e_{m+1}^1 , \qquad v_0 = e_1^0 ,$$

and note that

$$e_1^1 \wedge (\partial e_1^2 \wedge \partial e_2^2 \wedge \cdots \wedge \partial e_m^2) = A(t) v_1 .$$

Therefore, using the exact sequence

$$0 \longrightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \longrightarrow 0 ,$$

we have $A(t)v_1/v_2 = \partial e_1^1 = (t-1)v_0$, hence $v_0/(v_1/v_2) = A(t)/(t-1)$. This completes the proof of Theorem 4.

REMARK. A similar computation can be carried out for a knot with $n$ components in $S^3$, $n \geqq 2$. In this case the torsion

$$\Delta_I M \in Q(t_1, \cdots, t_n)/\pm \Pi$$

associated with the maximal abelian covering is precisely equal to the Alexander polynomial $A(t_1, \cdots, t_n)$. Applying the duality theorem, one obtains the symmetry relation

$$A(t_1^{-1}, \cdots, t_n^{-1}) = \pm t_1^{i_1} \cdots t_n^{i_n} A(t_1, \cdots, t_n)$$

which is due to Torres [12]. (See also [1], [5].)

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