K-THEORY AND DERIVED EQUIVALENCES

DANIEL DUGGER AND BROOKE SHIPLEY

Abstract. We show that if two rings have equivalent derived categories then they have the same algebraic K-theory. Similar results are given for G-theory, and for a large class of abelian categories.

Contents

1. Introduction

Algebraic K-theory began as a collection of elaborate invariants for a ring R. Quillen [Q2] constructed these by feeding the category of finitely-generated projective R-modules into the so-called Q-construction. In fact, the Q-construction can take as input any category with a sensible notion of exact sequence. Waldhausen later realized in [Wa] that the same kind of invariants can be defined for a very broad class of *homotopical* situations (Waldhausen used 'categories with cofibrations and weak equivalences'). To define the algebraic K-theory of a ring using the Waldhausen approach, one takes as input the category of *bounded chain complexes* of finitely-generated projective modules.

As soon as one understands this perspective it becomes natural to ask whether the Waldhausen K-theory construction really depends on the whole input category or just on the associated *homotopy category* (where the weak equivalences have been inverted). Or, in the algebraic case, one asks whether the K-theory of a ring depends only on the the associated derived category. In this paper we answer the latter question in the affirmative; if one is given the derived category of a ring, together with its triangulation—but without knowing which ring it is—then it is theoretically possible to recover the algebraic K-theory of the ring.

Date: September 6, 2002; 1991 AMS Math. Subj. Class.: 19D99, 18E30, 55U35.
Second author supported in part by an NSF grant.

We now give a more detailed description of the results. If R is a ring, let $\mathcal{D}_R$ denote the derived category of unbounded chain complexes of R-modules. Recall that $\mathcal{D}_R$ is a triangulated category in a standard way [We, 10.4]. Also, let $K_*(R)$ denote the algebraic K-groups of R. Our first theorem is the following:

Theorem A. *If R and S are two rings for which $\mathcal{D}_R$ and $\mathcal{D}_S$ are equivalent as triangulated categories, then their algebraic K-groups are isomorphic: $K_*(R) \cong K_*(S)$.*

When the hypothesis of the theorem holds we say that R and S are **derived equivalent**, and so the result says that derived equivalent rings have isomorphic K-theories. (This definition of 'derived equivalent' is not manifestly the same as that of [R1, Def 6.5], but they do in fact agree—see Theorem 4.2).

We can actually state somewhat stronger results. Recall that $K_n(R)$ is the nth homotopy group of a certain space $K(R)$ produced by ones favorite K-theory machine. Let $\mathcal{D}_c(R)$ denote the full subcategory of $\mathcal{D}_R$ consisting of the perfect complexes—that is, those complexes which are isomorphic in $\mathcal{D}_R$ to a bounded complex of finitely-generated projectives. (The 'c' is for 'compact', a term which is defined in Example 3.4).

Theorem B. *If R and S are rings such that $\mathcal{D}_c(R)$ and $\mathcal{D}_c(S)$ are equivalent as triangulated categories, then $K_n(R) \cong K_n(S)$ for all $n \geq 0$. Even more, one has a weak equivalence of K-theory spaces $K(R) \simeq K(S)$.*

The K_0 part of this result is very simple (see [R1, 9.3]), and so our contribution is the extension to higher K-theory. We should mention that one can even weaken the hypotheses somewhat, to require only an equivalence between $\mathcal{D}_c(R)$ and $\mathcal{D}_c(S)$ which commutes with the shift or suspension functor; see Remark 4.4.

There are similar results for the G-theory of a ring. Recall that when R is Noetherian $G(R)$ is the Quillen K-theory of the category of finitely-generated R-modules (as opposed to finitely-generated projectives); see [Sr, Chapter 5]. In terms of the Waldhausen machinery, it is the algebraic K-theory of the category of bounded chain complexes of finitely-generated R-modules—we denote the associated homotopy category by $\mathcal{D}_b(\text{mod-}R)$.

Theorem C. *Suppose that R and S are Noetherian rings.*

(a) If R and S are derived equivalent, then $G(R) \simeq G(S)$; in particular, $G_n(R) \cong G_n(S)$ for all $n \geq 0$.
(b) If $\mathcal{D}_b(\text{mod-}R)$ is triangulated-equivalent to $\mathcal{D}_b(\text{mod-}S)$, then $G(R) \simeq G(S)$.

Results along these lines first appeared in the work of Neeman [N1]. Neeman has had the much more ambitious goal of actually constructing the algebraic K-theory space directly from the derived category. It seems he has accomplished this in the case of abelian categories (cf. [N1, Thm. 7.1, p336]), and so for instance can construct $G(R)$ from $\mathcal{D}_b(\text{mod-}R)$ when R is Noetherian. Using this result Neeman is able to prove Theorem C(b), and from this he is able to deduce Theorem B in the case of regular rings (because for regular rings one has $G_*(R) \cong K_*(R)$). Theorem B in the above generality is new, however, as are the other results above.

Neeman's work is quite long and intricate, and it has sometimes been met with a certain amount of suspicion—mostly because experts just did not believe that K-theory could depend only on the derived category. The point we would like to accentuate is that our proofs of the above theorems are all quite simple. The only 'new' tool which enters the mix is the use of model categories. Although model categories are not often used in these contexts, their use effectively streamlines our work. There are two main points underlying the above theorems:

(1) Any equivalence of model categories yields a weak equivalence of K-theory spaces (see Proposition 3.6), and

(2) If two rings R and S are derived equivalent then tilting theory shows that their model categories of chain complexes Ch_R and Ch_S are in fact equivalent as model categories (see Theorem 4.2).

The first observation can be seen as an improvement of [TT, 1.9.8], see Remark 3.11. The second is a more structured version of [R1, 6.4] and [R2, 3.3, 5.1]; note that unlike [R2], we do not require any flatness hypotheses. See also [SS2, 5.1.1, B.1].

The observation in (2) is definitely surprising, although it turns out that it is not hard to prove (in fact, considering the extra structure in the model category seems to simplify the classical tilting theory proofs). The reason it is surprising is that the derived category of R is the 'homotopy category' of Ch_R, and this usually represents only first-order information in the model category. Equivalent model categories have equivalent homotopy categories, but it almost never works the other way around. So something special happens when dealing with chain complexes over a ring; the first order information here determines all of the higher order information. Note that this does not happen in arbitrary 'abelian' model categories. See also Remarks 2.5 and 6.8.

We state one last theorem along these lines, where we replace the category of R-modules by any rich enough abelian category. Of course any abelian category $\mathcal{A}$ has an unbounded derived category $\mathcal{D}_{\mathcal{A}}$, and we'll say that $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent if $\mathcal{D}_{\mathcal{A}}$ is triangulated-equivalent to $\mathcal{D}_{\mathcal{B}}$. Let $K_c(\mathcal{A})$ denote the Waldhausen K-theory of the compact objects in $\mathcal{C}h_{\mathcal{A}}$. It turns out that the space $K_c(\text{Mod-}R)$ is just $K(R)$.

Recall that if $\mathcal{A}$ is an abelian category, we say that an object P is a *strong generator* if $X = 0$ whenever $\hom_{\mathcal{A}}(P, X) = 0$; when $\mathcal{A}$ has arbitrary coproducts, the object P is called *small* if $\oplus_\alpha \hom_{\mathcal{A}}(P, X_\alpha) \longrightarrow \hom_{\mathcal{A}}(P, \oplus_\alpha X_\alpha)$ is a bijection for every set of objects $\{X_\alpha\}_\alpha$. Gabriel [G, V.1] has classified the abelian categories which are equivalent to categories of modules over a ring: these are the co-complete abelian categories with a single strong generator. Freyd [F, 5.3H] generalized this to include the case of many generators; see Theorems 6.1 and 7.1. Using these basic tools, we can extend our above statements to prove the following:

Theorem D. *Let $\mathcal{A}$ and $\mathcal{B}$ be co-complete abelian categories which have sets of small, projective, strong generators. Then*

(a) $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent if and only if $\mathcal{C}h_{\mathcal{A}}$ and $\mathcal{C}h_{\mathcal{B}}$ are equivalent as model categories.

(b) If $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent, then $K_c(\mathcal{A}) \simeq K_c(\mathcal{B})$.

Neeman [N1, 7.1] has proven that if $\mathcal{A}$ and $\mathcal{B}$ are small abelian categories for which $\mathcal{D}_b(\mathcal{A})$ is triangulated-equivalent to $\mathcal{D}_b(\mathcal{B})$, then $K(\mathcal{A}) \simeq K(\mathcal{B})$ where $K(\mathcal{A})$ denotes the Quillen K-theory of the exact category $\mathcal{A}$. There is little overlap between this result and the above one: the abelian categories in Theorem D have infinite direct sums, so it follows from the Eilenberg-Swindle that $K(\mathcal{A})$ and $K(\mathcal{B})$ are both trivial. We do not know how to apply our methods to the kinds of abelian categories Neeman deals with.

One final note: The reader may have noticed that we have always talked about K-theory *spaces*, rather than K-theory *spectra*. In fact, all of the results in this paper hold when restated in terms of spectra, and there is no difference in the proofs. We have chosen to avoid the added complications in an attempt to streamline the presentation.

1.1. **Organization.** The proofs of Theorems A–C are given in Section 5, and the paper has been structured so that the reader can get to them as soon as possible. The sections previous to that build up the necessary machinery, but with most of the technical proofs postponed until later. Section 3 recasts Waldhausen K-theory as an invariant for model categories, and proves that it is preserved by Quillen equivalences. Section 4 explains what tilting theory has to say about Quillen equivalences between model categories of chain complexes. Finally, in Section 7 we develop the many-generators version of tilting theory, and prove Theorem D.

1.2. **Notation and terminology.** Being topologists, our convention is to always work with *chain* complexes C_* rather than cochain complexes. So the differentials have the form $d\colon C_n \to C_{n-1}$, and the shift operator is denoted as ΣC: it is the chain complex with $(\Sigma C)_n = C_{n-1}$.

Throughout this paper we deal with right modules (and our rings are not necessarily commutative). Everything could be translated to left modules as well, but because of the usual conventions for composing maps, right modules are what naturally arise in some of our results; see Theorems 6.1 and 6.4 for example. Mod-R denotes the category of all R-modules, whereas mod-R denotes the category of finitely-generated R-modules (we only use this when R is right-Noetherian). Likewise, Proj-R is the category of all projective R-modules and proj-R is the subcategory of finitely-generated projectives.

Finally, if $\mathcal{C}$ is a category then we write $\mathcal{C}(X,Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

1.3. **Acknowledgments.** We are grateful to Mike Mandell and Stefan Schwede for several helpful conversations related to this paper.

2. Model category preliminaries

A model category is a category equipped with certain extra structures which allow one to 'do homotopy theory'. The theory is based on three standard examples: the category of topological spaces, the category of simplicial sets, and the category of chain complexes over a given ring. In this section we recall the basic axioms of model categories, and state the main facts we need in the body of the paper. [DwSp], [Hi] and [Ho] are good references for this material.

Definition 2.1. A **model category** is a category $\mathcal{M}$ equipped with three distinguished classes of maps: the *weak equivalences*, the *cofibrations*, and the *fibrations*. Cofibrations are depicted as $\rightarrowtail$, fibrations as $\twoheadrightarrow$, and weak equivalences as $\xrightarrow{\sim}$. Maps which are both cofibrations and weak equivalences are called *trivial cofibrations*, and denoted by $\overset{\sim}{\rightarrowtail}$; *trivial fibrations* are defined similarly. The following axioms are required:

Axiom 1: $\mathcal{M}$ is complete and co-complete.

Axiom 2: (Two-out-of-three axiom) If $f\colon A \to B$ and $g\colon B \to C$ are maps in $\mathcal{M}$ and any two of f, g, and gf are weak equivalences, then so is the third.

Axiom 3: (Retract axiom) A retract of a weak equivalence (respectively cofibration, fibration) is again a weak equivalence (respectively cofibration, fibration).

Axiom 4: (Lifting axiom) Suppose

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads] \\ B \arrow[r] & Y \end{tikzcd}$$

is a square (in which $A \to B$ is a cofibration and $X \to Y$ is a fibration). Then if either of the two vertical maps is a weak equivalence, there is a lifting $B \to X$ making the diagram commute.

Axiom 5: (Factorization axiom) Any map $A \to X$ may be functorially factored in two ways, as $A \overset{\sim}{\rightarrowtail} B \twoheadrightarrow X$ and as $A \rightarrowtail Y \overset{\sim}{\twoheadrightarrow} X$.

Suppose maps $A \to B$ and $X \to Y$ are given. When any square as in Axiom 4 has a lifting $B \to X$, we say that $A \to B$ has the *left-lifting-property* with respect to $X \to Y$.

Example 2.2. In this paper we only deal explicitly with model categories on categories of chain complexes.

(a) The category $\mathcal{C}h_R^+$ of non-negatively graded chain complexes over a ring R has a model structure where the weak equivalences are the maps inducing homology isomorphisms (the **quasi-isomorphisms**), the fibrations are the maps which are surjective in positive degrees, and the cofibrations are the monomorphisms with degreewise projective cokernels; see [Q1, II p. 4.11, Remark 5], [DwSp, Sec. 7]. This model structure on $\mathcal{C}h_R^+$ is referred to as the *projective* model structure since there are other model structures on $\mathcal{C}h_R^+$.

(b) The category $\mathcal{C}h_R$ of unbounded chain complexes over a ring R also has a **(projective) model structure** with weak equivalences the homology isomorphisms, and fibrations the epimorphisms; see [Ho, 2.3.11], [SS1, 5]. Every cofibration is still a degreewise split injection and the cokernel is levelwise projective, but not all such degreewise split injections are cofibrations.

(c) $\mathcal{C}h_R$ has another model structure with the same weak equivalences, but where the cofibrations are the monomorphisms. The fibrations are harder to describe, but any fibration is a degreewise surjection with levelwise-injective kernel. This is the *injective model structure* on $\mathcal{C}h_R$. In this paper we only need to use the projective model structure on $\mathcal{C}h_R$, however.

When $\mathcal{M}$ is a model category, one may formally invert the weak equivalences $\mathcal{W}$ to obtain the category-theoretic localization $\mathcal{W}^{-1}\mathcal{M}$. This is the **homotopy category** of $\mathcal{M}$, written $\mathrm{Ho}\,\mathcal{M}$; see [Q1, I.1], [DwSp, 6.2]. Since the weak equivalences in $\mathcal{C}h_R$ are the quasi-isomorphisms, the homotopy category $\mathrm{Ho}\,\mathcal{C}h_R$ is equivalent to the (unbounded) derived category $\mathcal{D}_R$ (cf. [We, Example 10.3.2]).

A model category is called **pointed** if the initial object and terminal object are the same. The homotopy category of any pointed model category turns out to have a *suspension functor* Σ. For topological spaces this is ordinary suspension, whereas for $\mathcal{C}h_R^+$ and $\mathcal{C}h_R$ it is the functor sending a chain complex C to the shift ΣC with $(\Sigma C)_n = C_{n-1}$. As the example of $\mathcal{C}h_R^+$ shows, this functor need not be an equivalence. When it *is* an equivalence we say that $\mathcal{M}$ is a **stable** model category, and in this case $\mathrm{Ho}\,\mathcal{M}$ becomes a triangulated category in a natural way [Ho, 7.1]. (When $\mathcal{M}$ is not stable, $\mathrm{Ho}\,\mathcal{M}$ only has a 'partial' triangulation; see [Q1, I.2, I.3], [Ho, 6.5] for details). For $\mathcal{C}h_R$ this of course specializes to the usual triangulation on $\mathcal{D}_R$.

Definition 2.3. A **Quillen map** of model categories $\mathcal{M} \to \mathcal{N}$ consists of a pair of adjoint functors $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ such that L preserves cofibrations and trivial cofibrations (it is equivalent to require that R preserves fibrations and trivial fibrations). In this case the pair (L, R) is also called a *Quillen pair*.

Example 2.4. Let $R \to S$ be a map of rings. The adjoint pair of functors $\mathcal{L}\colon \text{Mod-}R \rightleftarrows \text{Mod-}S\colon \mathcal{R}$ defined by $\mathcal{L}(M) = M \otimes_R S$ and $\mathcal{R}(N) = \mathrm{Hom}_R(S, N)$ prolongs to an adjoint pair between categories of chain complexes. One readily checks that these prolongations are Quillen maps $\mathcal{C}h_R^+ \to \mathcal{C}h_S^+$ and $\mathcal{C}h_R \to \mathcal{C}h_S$.

A Quillen map induces adjoint total derived functors between the homotopy categories [Q1, I.4]. The map is a **Quillen equivalence** if the total derived functors are adjoint equivalences of the homotopy categories. This is equivalent to Quillen's original definition by [Ho, 1.3.13]. More generally we say that $\mathcal{M}$ and $\mathcal{N}$ are **Quillen equivalent** if they are connected by a zig-zag of Quillen equivalences, and we write $\mathcal{M} \simeq_Q \mathcal{N}$. As one simple example, the identity functors give a Quillen equivalence $\mathcal{C}h_R^{\mathrm{proj}} \to \mathcal{C}h_R^{\mathrm{inj}}$ between the projective and injective model structures on $\mathcal{C}h_R$.

Remark 2.5. In general, having a Quillen equivalence of model categories is much stronger than just having an equivalence between the associated homotopy categories. This is because of the added structure required for a Quillen map; functors on the homotopy categories may not lift to the model category level, and even if they do they may not be compatible with the model category structures. For example, it follows from [Q1, I.4 Thm. 3] that Quillen maps between stable model categories induce triangulated functors between the homotopy categories. Quillen maps preserve even more structure, for example the simplicial mapping space structures [DK80, 5.4], [Ho, 5.6.2]. There are simple topological examples—see [SS2, 3.2.1], for instance—of stable model categories which have the same triangulated homotopy category, but which are nevertheless not Quillen equivalent. In Remark 6.8 we discuss another example (based on [Sc]) which is entirely algebraic.

The following theorem shows that for the special case of the model categories $\mathcal{C}h_R$, Quillen equivalence is not a stronger notion than triangulated equivalence of homotopy categories. In some sense this happens because rings are determined by 'first order' information—compared, for example, to differential graded rings which are not. This is proved in Section 6 as Theorem 4.2 (Parts 1 and 2).

Theorem 2.6. *Two rings R and S are derived equivalent if and only if their associated model categories of chain complexes $\mathcal{C}h_R$ and $\mathcal{C}h_S$ are Quillen equivalent.*

This theorem cannot be extended to cover the case where R or S is a differential graded algebra; we give an example in [DS] which is discussed a little in Remark 6.8. In Corollary 7.7 we do give a certain extension of this theorem to abelian categories, however. The situation is a little confusing, because these two sentences may seem contradictory. They are not though; see Remark 7.8.

3. K-THEORY AND MODEL CATEGORIES

In [Wa, Section 1] Waldhausen defined a notion of *category with cofibrations and weak equivalences* and showed how to construct a K-theory space from such data. The purpose of this section is to adapt Waldhausen's machinery to the context of model categories. This is almost entirely straightforward, but it has the advantage of streamlining the theory somewhat.

Let $\mathcal{M}$ be a pointed model category with initial object $*$. An object A is called **cofibrant** if $* \rightarrowtail A$ is a cofibration. By a **Waldhausen subcategory** of $\mathcal{M}$ we mean a full subcategory $\mathcal{U}$ with the properties that

(i) $\mathcal{U}$ contains the initial object $*$;
(ii) Every object of $\mathcal{U}$ is cofibrant;
(iii) If $A \rightarrowtail B$ and $A \to X$ are maps in $\mathcal{U}$, then the pushout $B \amalg_A X$ (computed in $\mathcal{M}$) belongs to $\mathcal{U}$.

The proof of the following is just a matter of chasing through the definitions:

Lemma 3.1. *Any Waldhausen subcategory of $\mathcal{M}$, equipped with the notions of cofibrations and weak equivalences from $\mathcal{M}$, is a* 'category with cofibrations and weak equivalences' *in the sense of* [Wa, 1.2]*; also, it satisfies the saturation axiom* [Wa, p. 327].

The lemma says that we may apply Waldhausen's $S_\bullet$-construction [Wa, 1.3] to obtain a simplicial category $wS_\bullet(\mathcal{U})$. Taking the nerve in every dimension gives a simplicial space $[n] \mapsto N(wS_n(\mathcal{U}))$, and $K(\mathcal{U})$ is defined to be loops on the realization of this simplicial space: $K(\mathcal{U}) = \Omega|NwS_\bullet(\mathcal{U})|$. One defines the algebraic K-groups of $\mathcal{U}$ by $K_n(\mathcal{U}) = \pi_n(K(\mathcal{U}))$.

We give a partial description of $wS_\bullet(\mathcal{U})$ here, because we need it later in Appendix A. Let $wF_n(\mathcal{U})$ denote the category whose objects are sequences $\{A\}$ of cofibrations $A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n$ in $\mathcal{U}$, and whose morphisms are commutative diagrams $\{A\} \to \{A'\}$ in which every map $A_n \to A'_n$ is a weak equivalence. One can almost make $[n] \mapsto wF_{n-1}(\mathcal{U})$ into a simplicial category (where $wF_{-1}(\mathcal{U})$ is interpreted as the trivial category with one object and an identity map) by defining

$$d_i([A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n]) = \begin{cases} [A_0 \rightarrowtail \cdots \rightarrowtail \hat{A}_i \rightarrowtail \cdots \rightarrowtail A_n] & \text{if } i \neq 0, \\ [A_1/A_0 \rightarrowtail A_2/A_0 \rightarrowtail \cdots \rightarrowtail A_n/A_0] & \text{if } i = 0. \end{cases}$$

The difficulty is that with this definition the simplicial identities do not hold on the nose, in the end because there are different possible choices for the quotients A_i/A_0 (they are canonically isomorphic, but still different). The category $wS_n(\mathcal{U})$ is equivalent to $wF_{n-1}(\mathcal{U})$, but is slightly 'fatter' in a way that allows one to make the face and degeneracy maps commute on the nose. The reader is referred to [Wa, p. 328] for the precise definition—it should be noted, though, that the basic ideas in the present paper can all be understood by pretending that $wS_n(\mathcal{U})$ is just $wF_{n-1}(\mathcal{U})$. The only time the details of $wS_n(\mathcal{U})$ are needed is in the Appendix.

Example 3.2. Let R be a ring. The following are Waldhausen subcategories of $\mathcal{C}h_R$ (as is easily verified).

(1) $\mathcal{U}_K = \{$all bounded complexes of finitely-generated projectives$\}$.
(2) $\mathcal{U}_G = \{$all bounded below complexes C of finitely-generated projectives such that $H_k(C) \neq 0$ for only finitely-many values of $k\}$.

Let $K(R)$ and $G(R)$ denote the Quillen K-theory spaces for the exact categories of finitely-generated projectives and finitely-generated modules, respectively. Then we have:

Lemma 3.3. *$K(\mathcal{U}_K) \simeq K(R)$, and if R is Noetherian then $K(\mathcal{U}_G) \simeq G(R)$.*

Proof. A reference for $K(\mathcal{U}_K) \simeq K(R)$ is [TT, 1.11.7]. For the G-theory, the reference is [TT, 3.11.10, 3.12, 3.13]—however, since the terminology of that paper is fairly cumbersome, we repeat the proof for the reader's convenience.

Let $\mathcal{V}$ denote the subcategory of $\mathcal{C}h_R$ consisting of all bounded complexes of finitely-generated modules; [TT, 1.11.7] shows that $K(\mathcal{V})$ is the same as $G(R)$. Let $\mathcal{W}$ denote the subcategory of $\mathcal{C}h_R$ consisting of all chain complexes quasi-isomorphic to an element of $\mathcal{V}$. One can check that if R is Noetherian then $\mathcal{U}_G$ consists precisely of the cofibrant objects in $\mathcal{W}$. Then [TT, 1.9.8] shows that $\mathcal{U}_G \hookrightarrow \mathcal{W}$ and $\mathcal{V} \hookrightarrow \mathcal{W}$ induce equivalences of K-theory spaces. □

Example 3.4. If $\mathcal{T}$ is a triangulated category with infinite sums, an object $X \in \mathcal{T}$ is called **compact** if the natural map $\oplus_\alpha \mathcal{T}(X, Z_\alpha) \to \mathcal{T}(X, \oplus_\alpha Z_\alpha)$ is a bijection for every collection $\{Z_\alpha \in \mathcal{T}\}$. If $\mathcal{M}$ is a stable model category, it is easy to check that the homotopy category $\operatorname{Ho}\mathcal{M}$ has all infinite sums. We'll say that an object in $\mathcal{M}$ is compact if its image in $\operatorname{Ho}\mathcal{M}$ is compact. The subcategory $\mathcal{M}_c \subseteq \mathcal{M}$ consisting of all compact, cofibrant objects is a complete Waldhausen category.

We are especially interested in this for the case $\mathcal{M} = \mathcal{C}h_R$, where a theorem of Bökstedt-Neeman [BN, 6.4] identifies the compact objects as the perfect complexes, i.e. the complexes which are quasi-isomorphic to a bounded complex of finitely-generated projectives.

Example 3.5. Waldhausen never explicitly used model categories, but he could have been working in this context all along. Waldhausen developed his machinery to apply to the following case. Let X be a simplicial set, and let $(X \downarrow s\mathcal{S}et \downarrow X)$ denote the category of retractive spaces over X. This has a natural model structure inherited from the category of simplicial sets [Q1, II.3] by forgetting the retraction over X (cf. [Hi, 7.6.5]). Take $\mathcal{U}$ to be the subcategory consisting of those retractive spaces $X \hookrightarrow Z \to X$ for which the map $X \hookrightarrow Z$ is obtained by attaching finitely many simplices. This is a Waldhausen subcategory, and the associated K-theory space is denoted $A(X)$; see [Wa, 2.1].

If $\mathcal{U}$ is a subcategory of $\mathcal{M}$, write $\overline{\mathcal{U}}$ for the full subcategory of $\mathcal{M}$ consisting of all cofibrant objects which are weakly equivalent to an object in $\mathcal{U}$. From Example 3.4 above it follows that $(\mathcal{C}h_R)_c = \overline{\mathcal{U}_K}$ (where $\mathcal{U}_K$ is from Example 3.2). Call the Waldhausen category $\mathcal{U}$ **complete** if $\mathcal{U} = \overline{\mathcal{U}}$.

Suppose that $(L, R)\colon \mathcal{M} \to \mathcal{N}$ is a Quillen map of pointed model categories. Let $\mathcal{U}$ and $\mathcal{V}$ be Waldhausen subcategories of $\mathcal{M}$ and $\mathcal{N}$ such that L maps $\mathcal{U}$ into $\mathcal{V}$. Since L preserves cofibrations, one checks easily that it induces a well-defined map $K(\mathcal{U}) \to K(\mathcal{V})$.

Proposition 3.6. *Suppose that (L, R) is a Quillen equivalence, and that $\mathcal{U}$ is a complete Waldhausen subcategory of $\mathcal{M}$. Let $\mathcal{V} = \overline{L\mathcal{U}}$—i.e., $\mathcal{V}$ consists of all cofibrant objects which are weakly equivalent to an object in $L(\mathcal{U})$. Then $\mathcal{V}$ is a complete Waldhausen subcategory of $\mathcal{N}$, and $L\colon K(\mathcal{U}) \to K(\mathcal{V})$ is a weak equivalence.*

The proof is simple but long winded, so we defer it to an appendix.

Remark 3.7. The proposition also works in the following way. Let Q be a cofibrant replacement functor for $\mathcal{M}$; for example one can take the map $* \to X$ and apply the functorial factorization $* \rightarrowtail QX \xrightarrow{\sim} X$ in $\mathcal{M}$ to define Q. Similarly, let F be a fibrant-replacement functor for $\mathcal{N}$ with $Y \overset{\sim}{\rightarrowtail} FY \twoheadrightarrow *$ for Y in $\mathcal{N}$. Suppose that $\mathcal{V}$ is a complete Waldhausen subcategory of $\mathcal{N}$. Define $R\mathcal{V}$ to be the set of all objects of the form $QRFX$ where $X \in \mathcal{V}$, and let $\mathcal{U} = \overline{R\mathcal{V}}$. Then $\mathcal{U}$ is a complete Waldhausen subcategory of $\mathcal{M}$, and $\overline{L\mathcal{U}} = \mathcal{V}$. The functor L induces a map $K(\mathcal{U}) \to K(\mathcal{V})$, and the proposition says this is an equivalence. So we have actually proven:

Corollary 3.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories connected by a zig-zag of Quillen equivalences. Let $\mathcal{U}$ be a complete Waldhausen subcategory of $\mathcal{M}$, and let $\mathcal{V}$ consist of all cofibrant objects in $\mathcal{N}$ which are carried into $\mathcal{U}$ by the composite of the derived functors of the Quillen equivalences. Then $\mathcal{V}$ is a complete Waldhausen subcategory of $\mathcal{N}$, and there is an induced zig-zag of weak equivalences between $K(\mathcal{U})$ and $K(\mathcal{V})$.*

Corollary 3.9. *A Quillen equivalence $\mathcal{M} \to \mathcal{N}$ between stable model categories induces a weak equivalence of K-theory spaces $K(\mathcal{M}_c) \xrightarrow{\sim} K(\mathcal{N}_c)$, where $\mathcal{M}_c$ and $\mathcal{N}_c$ denote the subcategories of cofibrant, compact objects.*

Proof. Write the functors of the Quillen equivalence as (L, R). The derived functors of L and R induce an equivalence between the homotopy categories, and so in particular they take compact objects to compact objects. This clearly implies $\mathcal{N}_c \supseteq \overline{L\mathcal{M}_c}$; it basically gives the opposite inclusion as well, but we now explain this in more detail.

If X is in $\mathcal{N}_c$, let FX be a fibrant-replacement $X \overset{\sim}{\rightarrowtail} FX \twoheadrightarrow *$ in $\mathcal{N}$ and let $Q(RFX)$ be a cofibrant-replacement $* \rightarrowtail Q(RFX) \xrightarrow{\sim} RFX$ of RFX in $\mathcal{M}$. Because the derived functors of L and R take compact objects to compact objects, $QRFX$ must still be compact—i.e., $QRFX \in \mathcal{M}_c$. Yet $LQRFX$ is weakly equivalent to X, and so $X \in \overline{L\mathcal{M}_c}$. At this point we have shown $\mathcal{N}_c = \overline{L\mathcal{M}_c}$, and so we can just apply Proposition 3.6. □

Corollary 3.10. *If $\mathcal{C}h_R$ and $\mathcal{C}h_S$ are Quillen equivalent (perhaps through a zig-zag of Quillen equivalences), then $K(R) \simeq K(S)$.*

Proof. We have already remarked that $K(R) \simeq K(\mathcal{U}_K)$, and $\mathcal{U}_K = (\mathcal{C}h_R)_c$. All the intermediate model categories in the zig-zag must be stable because 'stability' is preserved under Quillen equivalence. Therefore Corollary 3.9 applies. □

Remark 3.11. The above corollary has two improvements over similar results in the literature. The first is that we are allowing a zig-zag of Quillen equivalences, rather than just an equivalence $\mathcal{C}h_R \to \mathcal{C}h_S$; in particular, note that our zig-zag could conceivably pass through very non-algebraic model categories. For just a single Quillen equivalence $\mathcal{C}h_R \to \mathcal{C}h_S$, the closest result in the literature seems to be [TT, 1.9.8]. In that result, however, the functor $L\colon \mathcal{C}h_R \to \mathcal{C}h_S$ is required to be *complicial*, meaning in part that it is induced via prolongation from a functor Mod-$R \to$ Mod-S. In some of our applications L is the functor which tensors with a chain complex of projectives (rather than just a single projective), and so the [TT] result is not applicable.

4. Tilting Theory

In this section we determine the *algebraic content* of having a Quillen equivalence between $\mathcal{C}h_R$ and $\mathcal{C}h_S$ for rings R and S. A nice, complete answer can be given in terms of *tilting theory*. Originally tilting theory only dealt with derived equivalences, but it turns out that for rings derived equivalence and Quillen equivalence coincide.

We begin with a classical analogue of tilting theory, namely Morita theory. Morita theory describes necessary and sufficient conditions for when two categories of modules are equivalent. Call a (right) R-module P a **strong generator** if $\hom_R(P, X) = 0$ implies $X = 0$ for any (right) R-module X.

Theorem 4.1. **(Morita Theory)** *Given rings R and S, the following conditions are equivalent:*

1. *The categories of (right) modules over R and S are equivalent.*
2. *There is an R-S bimodule M and an S-R bimodule N such that $M \otimes_S N \cong R$ as R-bimodules and $N \otimes_R M \cong S$ as S-bimodules.*
3. *There is a (right) R-module P which is finitely-generated, projective and a strong generator such that $\hom_R(P, P) \cong S$.*

Proof. We only give a brief sketch because this is classical, see [We, 9.5]. For (2) implies (1), the functors $- \otimes_R M\colon$ Mod-$R \longrightarrow$ Mod-S and $- \otimes_S N\colon$ Mod-$S \longrightarrow$ Mod-R give the inverse equivalences. For (1) implies (3), given an equivalence $F\colon$ Mod-$S \longrightarrow$ Mod-R one may take $P = F(S)$. For (3) implies (2), take $N = P$ since P is a $\hom_R(P, P)$-R bimodule and take $M = \hom_R(P, R)$ which is an R-$\hom_R(P, P)$ bimodule. ☐

Now we turn to the analogue of Morita theory for categories of chain complexes, called 'tilting theory'. This analogue was developed by Rickard in [R1, 6.4] to classify derived equivalences of rings. Later, Keller [Kr, 8.2] broadened tilting theory to apply to more general derived equivalences of abelian categories. We extend both sets of results to give Quillen equivalences underlying the derived equivalences. Theorem 4.2 below extends Rickard's work, whereas the generalization to abelian categories is considered in Section 7. These results can also be used to remove certain flatness assumptions in [R2, 3.3, 5.1].

Let $\mathcal{T}$ be a triangulated category. Recall that a **full subtriangulated category** $\mathcal{S}$ is a full subcategory which is (i) closed under isomorphisms, (ii) closed under the suspension functor, and (iii) has the property that if two objects of a distinguished triangle in $\mathcal{T}$ lie in $\mathcal{S}$ then so does the third object. When $\mathcal{T}$ has infinite sums, a full subtriangulated category is called **localizing** if it is closed under coproducts of sets of objects [N2, 1.5.1, 3.2.6]. A complex P in $\mathcal{T}$ is a (weak) **generator** if the only localizing subcategory of $\mathcal{T}$ which contains P is $\mathcal{T}$. Although this definition looks much different than the definition of a strong generator, it is not. If P is compact (see Example 3.4 for a definition), then P is a (weak) generator if and only if $\mathcal{T}(P, X)_* = 0$ implies X is trivial (see [SS2, 2.2.1] for a proof that these are equivalent). Here $\mathcal{T}(-, -)_*$ denotes the graded maps with $\mathcal{T}(X, Y)_n = \mathcal{T}(\Sigma^n X, Y)$.

An object $P \in \mathcal{C}h_R$ is called a **tilting complex** if it is a bounded complex of finitely-generated projectives, a generator of $\mathcal{D}_R$, and $\mathcal{D}_R(P, P)_*$ is concentrated in degree zero [R1, Def. 6.5]. Here is our generalization of Rickard's result [R1, Thm 6.4]:

Theorem 4.2. (Tilting theorem) *The following conditions are equivalent for rings R and S:*

1. *There is a zig-zag of Quillen equivalences between the model categories of chain complexes of R- and S-modules:*
$$\mathcal{C}h_R \simeq_Q \mathcal{C}h_S.$$
2. *The unbounded derived categories are triangulated equivalent:*
$$\mathcal{D}_R \simeq_\Delta \mathcal{D}_S.$$
3. *The naive homotopy categories of bounded chain complexes of finitely generated projective R and S-modules are triangulated equivalent:*
$$K_b(\text{proj-}R) \simeq_\Delta K_b(\text{proj-}S).$$
4. *The model category $\mathcal{C}h_R$ has a tilting complex P whose endomorphism ring in $\mathcal{D}_R$ is isomorphic to S: $\mathcal{D}_R(P,P) \cong S$.*

Remark 4.3. Rickard [R1, 6.4] showed that (3) and (4) are equivalent and that both these are equivalent to having a triangulated equivalence $\mathcal{D}_b(\text{Mod-}R) \simeq \mathcal{D}_b(\text{Mod-}S)$. He defined two rings to be 'derived equivalent' if any of these conditions holds. We defined 'derived equivalent' to mean (2), and so the result shows that our use agrees with Rickard's. Note that [R1, 6.4] gives several other equivalent conditions involving variations of the derived category; see Proposition 5.1 as well.

Proof of $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. Every Quillen equivalence of stable model categories induces an equivalence of triangulated homotopy categories [Q1, I.4 Theorem 3], so (1) implies (2). Any triangulated equivalence restricts to an equivalence between the respective subcategories of compact objects. Since $K_b(\text{proj-}R)$ is equivalent to the full subcategory of compact objects in $\mathcal{D}_R$ by [BN, Prop. 6.4], (2) implies (3).

Now we assume condition (3) and choose a triangulated equivalence between $K_b(\text{proj-}R)$ and $K_b(\text{proj-}S)$. Let $S[0]$ be the free S-module on one generator, viewed as a complex in $\mathcal{C}h_S$ concentrated in dimension zero; let T be its image in $K_b(\text{proj-}R)$. We have $\mathcal{D}_R(T,T) \cong \mathcal{D}_S(S[0], S[0]) \cong S$. Since $S[0]$ generates $K_b(\text{proj-}S)$, T generates $K_b(\text{proj-}R)$. Since $R[0]$ is a generator of $\mathcal{D}_R$ and $R[0] \in K_b(\text{proj-}R)$, the only localizing subcategory of $\mathcal{D}_R$ containing $K_b(\text{proj-}R)$ is $\mathcal{D}_R$; so T generates $\mathcal{D}_R$. Hence T is a tilting complex and condition (4) holds. $\square$

The real content of the theorem, of course, is the proof that $(4) \Rightarrow (1)$. This is given in Section 6, after we have developed a little more machinery.

Remark 4.4. We could have put one more intermediary condition in Theorem 4.2. Instead of a triangulated equivalence (in either (2) or (3)) we could have required only an equivalence of categories which commutes with the shift or suspension functor. Such an equivalence would preserve compact objects and preserve the graded maps $\mathcal{D}(-,-)_*$. It would also preserve the property of being a compact generator, since an object is a compact generator if and only if it detects trivial objects by [SS2, 2.2.1]. Thus, such equivalences preserve tilting complexes. We do not have very interesting examples of such equivalences, though (other than triangulated equivalences).

Remark 4.5. The two tilting theory results in this paper, Theorem 4.2 and its analogue Theorem 7.5, also appear in disguised form in [SS2]. Chain complexes do not satisfy the stated hypotheses of the tilting theorem in [SS2, 5.1.1], but in [SS2, Appendix B.1] chain complexes are shown to be Quillen equivalent to a model category which does satisfy the stated hypotheses. So Theorems 4.2 and 7.5 can be considered as special cases of [SS2, 5.1.1]. Here, though, we have removed all hypotheses and the proofs are much simplified—they only use categories of chain complexes, whereas the proofs in [SS2] require the use of the new symmetric monoidal category of symmetric spectra [HSS].

5. Proofs of the main results

If you accept the basic results stated so far, it becomes easy to prove the first three theorems cited in the introduction.

Proof of Theorem B. This follows from Corollary 3.10 together with the equivalence of Parts 1 and 3 in Theorem 4.2. Note that $\mathcal{D}_c(R)$ and $K_b(\text{proj-}R)$ are two names for the same thing, by [BN, 6.4]. □

Proof of Theorem A. If $\mathcal{D}_R$ and $\mathcal{D}_S$ are equivalent as triangulated categories, then so are their full subcategories of compact objects. So Theorem B applies. This also follows from Corollary 3.10 and the equivalence of Parts 1 and 2 in Theorem 4.2. □

We now turn our attention to the proof of Theorem C, which is the G-theory result. We begin with a proposition which is fairly interesting in its own right. Consider a function $\mathcal{C}$ which assigns to each ring R a subcategory of $\mathcal{D}_R$. We say that the assignment **preserves equivalences** if every triangulated equivalence $F\colon \mathcal{D}_R \to \mathcal{D}_S$ restricts to an equivalence between $\mathcal{C}(R)$ and $\mathcal{C}(S)$.

Here is some new notation: $\mathcal{D}_{h+}(\text{Mod-}R)$ denotes the full subcategory of $\mathcal{D}_R$ consisting of chain complexes with bounded below homology, and $\mathcal{D}_{hb}(\text{Mod-}R)$ denotes the full subcategory of complexes with bounded homology. One can similarly define $K_{hb}(\text{proj-}R)$, etc. The notation $K_{+,hb}(\text{proj-}R)$ means the intersection of $K_+(\text{proj-}R)$ and $K_{hb}(\text{proj-}R)$. It is an easy exercise to check that $\mathcal{D}_{h+}(\text{Mod-}R) = K_+(\text{Proj-}R)$ and $\mathcal{D}_{hb}(\text{Mod-}R) = \mathcal{D}_b(\text{Mod-}R)$.

Proposition 5.1. *The assignments $R \mapsto \mathcal{C}(R)$ preserve equivalences, where $\mathcal{C}(R)$ is any of the following:*

$$K_b(\text{proj-}R), \quad K_+(\text{Proj-}R) = \mathcal{D}_{h+}(\text{Mod-}R), \quad \mathcal{D}_{h-}(\text{Mod-}R),$$

$$\mathcal{D}_{hb}(\text{Mod-}R) = \mathcal{D}_b(\text{Mod-}R), \quad K_+(\text{proj-}R), \quad K_{+,hb}(\text{proj-}R).$$

Proof. The result [BN, 6.4] identifies $K_b(\text{proj-}R)$ with the subcategory of compact objects in $\mathcal{D}_R$. Any equivalence $\mathcal{D}_R \to \mathcal{D}_S$ must preserve direct sums, and so it takes compact objects to compact objects.

A complex X lies in $\mathcal{D}_{h+}(\text{Mod-}R)$ if and only if it satisfies the following property: for any compact object A, there exists an N such that $\mathcal{D}_R(\Sigma^{-k}A, X) = 0$ for $k > N$. Since triangulated equivalences preserve compact objects and the suspension, they preserve these objects as well.

Similarly, a complex X lies in $\mathcal{D}_{h-}(\text{Mod-}R)$ if and only if for any compact object A, there exists an N such that $\mathcal{D}_R(\Sigma^k A, X) = 0$ for all $k > N$. The same argument as above applies. For $\mathcal{D}_{hb}(\text{Mod-}R)$, note that this is just the intersection of $\mathcal{D}_{h+}(\text{Mod-}R)$ and $\mathcal{D}_{h-}(\text{Mod-}R)$.

The case of $K_+(\text{proj-}R)$ is harder, but was proven by Rickard—see the first paragraph in the proof of [R1, 8.1]. Finally, $K_{+,hb}(\text{proj-}R)$ is just the intersection of $K_+(\text{proj-}R)$ and $\mathcal{D}_{hb}(\text{Mod-}R)$. $\square$

$K_{+,hb}(\text{proj-}R)$ is the full subcategory of $\mathcal{D}_R$ consisting of complexes which are quasi-isomorphic to a bounded-below complex of finitely-generated projectives, and which also have bounded homology. So one has the inclusions $\mathcal{D}_c(R) \subseteq K_{+,hb}(\text{proj-}R) \subseteq \mathcal{D}_R$. Note that $K_{+,hb}(\text{proj-}R)$ is the image in $\mathcal{D}_R$ of the Waldhausen subcategory $\mathcal{U}_G(R) \subseteq \mathcal{C}h_R$. It is an easy exercise to check that when R is right-Noetherian one has $K_{+,hb}(\text{proj-}R) = \mathcal{D}_b(\text{mod-}R)$, where the latter denotes the full subcategory of $\mathcal{D}_R$ consisting of the bounded complexes of finitely-generated modules.

Theorem C follows immediately from the following more comprehensive statement:

Theorem 5.2. *Let R and S be right-Noetherian.*

(a) If R and S are derived equivalent, then $G(R) \simeq G(S)$.
(b) R and S are derived equivalent if and only if $K_{+,hb}(\text{proj-}R)$ and $K_{+,hb}(\text{proj-}S)$ are equivalent as triangulated categories.
(c) If $\mathcal{D}_b(\text{mod-}R) \simeq_\Delta \mathcal{D}_b(\text{mod-}S)$, then $G(R) \simeq G(S)$ and $K(R) \simeq K(S)$.

Proof. Part (b) is entirely due to Rickard [R1, 8.1,8.2]. (Note that Rickard uses cochain complexes whereas we use chain complexes, and writes $K^{-,b}(\text{proj-}R)$ for what we call $K_{+,hb}(\text{proj-}R)$, etc.)

For (a), suppose that R and S are derived equivalent. Then Theorem 4.2 says that there is a chain of Quillen equivalences between $\mathcal{C}h_R$ and $\mathcal{C}h_S$. On the homotopy categories, this gives us a chain of triangulated equivalences between $\mathcal{D}_R$ and $\mathcal{D}_S$. Proposition 5.1 says that this triangulated equivalence between $\mathcal{D}_R$ and $\mathcal{D}_S$ restricts to an equivalence between $K_{+,hb}(\text{proj-}R)$ and $K_{+,hb}(\text{proj-}S)$. So the complete Waldhausen subcategory $\mathcal{U}_G(R)$ is carried to $\mathcal{U}_G(S)$ via the various adjoint functors in the chain of Quillen equivalences. One can now use Corollary 3.8 to deduce that $K(\mathcal{U}_G(R)) \simeq K(\mathcal{U}_G(S))$. That is, $G(R) \simeq G(S)$.

For (c), recall that when R is Noetherian $\mathcal{D}_b(\text{mod-}R)$ is just another name for $K_{+,hb}(\text{proj-}R)$, and the same for S. So if $\mathcal{D}_b(\text{mod-}R) \simeq \mathcal{D}_b(\text{mod-}S)$ then by (b) R and S are derived equivalent; so we can apply (a) and Theorem B. $\square$

6. Derived equivalence implies Quillen equivalence

In this section we prove the Tilting Theorem 4.2. The only difficult part of this theorem follows from a differential graded analogue of the following result from [G, V.1]. This can also be viewed as another perspective on Morita theory.

Theorem 6.1. (Gabriel) *Let $\mathcal{A}$ be a co-complete abelian category with a small, projective, strong generator P. Then the functor*

$$\hom_{\mathcal{A}}(P, -)\colon\ \mathcal{A} \longrightarrow \text{Mod-}\hom_{\mathcal{A}}(P, P)$$

is the right adjoint of an equivalence of categories.

There is also a version of this theorem for a set of small generators, due to Freyd; see Section 7.

We begin by defining a chain complex of morphisms between any two chain complexes. For M, N in $\mathcal{C}h_R$ define $\mathrm{Hom_R}(M,N)$ in $\mathcal{C}h_{\mathbb{Z}}$ by

$$\mathrm{Hom_R}(M,N)_n = \prod_k \hom_R(M_k, N_{n+k}).$$

The differential for $\mathrm{Hom_R}(M,N)$ is given by $df_n = d_N f_n + (-1)^{n+1} f_n d_M$. This structure gives an enrichment of $\mathcal{C}h_R$ over $\mathcal{C}h_{\mathbb{Z}}$. So instead of an endomorphism *ring*, an object in $\mathcal{C}h_R$ has a differential graded ring of endomorphisms.

Definition 6.2. The tensor product of X and Y in $\mathcal{C}h_{\mathbb{Z}}$ is defined by

$$(X \otimes Y)_n = \bigoplus_k X_k \otimes Y_{n-k}$$

where $d(x_p \otimes y_q) = dx_p \otimes y_q + (-1)^p x_p \otimes dy_q$. A **differential graded algebra** is a chain complex A in $\mathcal{C}h_{\mathbb{Z}}$ with an associative and unital multiplication $\mu\colon A \otimes A \longrightarrow A$ [We, 4.5.2]. A (right) **differential graded module** M over a differential graded algebra A is a chain complex M with an associative and unital action $\alpha\colon M \otimes A \longrightarrow A$. Denote the category of such modules by Mod-A.

For any P in $\mathcal{C}h_R$ let $\mathrm{End}_R(P) = \mathrm{Hom_R}(P,P)$. Notice that $\mathrm{End}_R(P)$ is a differential graded ring with the product structure coming from composition. Also, for any $X \in \mathcal{C}h_R$ the complex $\mathrm{Hom_R}(P,X)$ is a right differential graded $\mathrm{End}_R(P)$-module with the action given by precomposition. So $\mathrm{Hom_R}(P,-)$ induces a functor from $\mathcal{C}h_R$ to Mod-$\mathrm{End}_R(P)$. Its left adjoint is denoted $- \otimes_{\mathrm{End}_R(P)} P$. This left adjoint can be defined as the coequalizer that the notation suggests using the evaluation map $\mathrm{Hom_R}(P,P) \otimes P \longrightarrow P$.

Our differential graded analogue of Gabriel's theorem produces a Quillen equivalence of model categories instead of an equivalence of categories. So before stating it we need to establish the model category structure on a category of differential graded modules. The following proposition is proved in [Hi, 2.2.1, 3.1] and in [SS1, 4.1.1].

Proposition 6.3. *Let A be a DGA. The category* Mod-A *has a model category structure where the weak equivalences are the maps inducing an isomorphism in homology and the fibrations are the surjections. The cofibrations are then determined to be the maps with the left-lifting-property with respect to the trivial fibrations.*

We can now state the following differential graded version of Gabriel's theorem.

Theorem 6.4. *Let P in $\mathcal{C}h_R$ be a bounded complex of finitely generated projectives. If P is a (weak) generator for $\mathcal{C}h_R$, then there is a Quillen equivalence*

$$\text{Mod-}\mathrm{End}_R(P) \longrightarrow \mathcal{C}h_R$$

in which the right-adjoint is the functor $\mathrm{Hom_R}(P,-)$.

Before proving this theorem we need the following lemma.

Lemma 6.5. *Let $M, N \in \mathcal{C}h_R$. Then $H_* \mathrm{Hom_R}(M,N) \cong \mathcal{D}_R(M,N)_*$ when M is cofibrant.*

Proof. It is easy to see in general that $H_n \mathrm{Hom}_R(M,N) \cong H_n \mathrm{Hom}_R(\Sigma^n M, N) \cong [\Sigma^n M, N]$ where $[-,-]$ denotes chain-homotopy-classes of maps. When A is cofibrant one has that $\mathcal{D}_R(A,B) \cong [A,B]$ (since all objects are fibrant in $\mathcal{C}h_R$), and so we can write

$$H_n \operatorname{Hom}_R(M,N) \cong [\Sigma^n M, N] \cong \mathcal{D}_R(\Sigma^n M, N) = \mathcal{D}_R(M,N)_n.$$

□

Proof of Theorem 6.4. For any complex of projectives P, $\mathrm{Hom}_{\mathrm{R}}(P,-)$ preserves surjections (fibrations) and hence is exact. We next show that $\mathrm{Hom}_{\mathrm{R}}(P,-)$ preserves trivial fibrations; since $\mathrm{Hom}_{\mathrm{R}}(P,-)$ is exact, we only need to show that $H_* \mathrm{Hom}_{\mathrm{R}}(P,K) = 0$ when $H_*K = 0$ and apply this to the kernel K of the trivial fibration. P is cofibrant by [Ho, 2.3.6] because P is a bounded complex of projectives. Thus, by Lemma 6.5, if K is acyclic then $H_* \mathrm{Hom}_{\mathrm{R}}(P,K) \cong \mathcal{D}_R(P,K)_* \cong 0$. Hence, the functor $\mathrm{Hom}_{\mathrm{R}}(P,-)$ preserves fibrations and trivial fibrations; see also [Ho, 4.2.13]. So its left adjoint is a Quillen map, and therefore the adjoint pair induces total derived functors on the level of homotopy categories [Q1, I.4]. Denote these derived functors by $\mathrm{RHom}_{\mathrm{R}}(P,-)$ and $-\otimes^L_{\mathrm{End}_R(P)} P$ respectively.

Since Ch_R and $\text{Mod-}\mathrm{End}_R(P)$ are stable model categories, both total derived functors preserve shifts and triangles in the homotopy categories, i.e., they are exact functors of triangulated categories by [Q1, I.4 Prop. 2]. Since $-\otimes^L_{\mathrm{End}_R(P)} P$ is a left adjoint it commutes with coproducts. To see that $\mathrm{RHom}_{\mathrm{R}}(P,-)$ commutes with coproducts it is enough to show that $\mathcal{D}_{\mathrm{End}_R(P)}(\mathrm{End}_R(P), \mathrm{RHom}_{\mathrm{R}}(P,-))$ commutes with coproducts since $\mathrm{End}_R(P)$ is a compact generator of $\text{Mod-}\mathrm{End}_R(P)$. By adjointness, this functor is isomorphic to $\mathcal{D}_R(\mathrm{End}_R(P)\otimes^L_{\mathrm{End}_R(P)} P, -)$ which in turn is isomorphic to $\mathcal{D}_R(P,-)$ since $\mathrm{End}_R(P)$ is cofibrant. Since P is compact [BN, 6.4] this functor commutes with coproducts.

Now consider the full subcategories of those M in $\mathrm{Ho}(\text{Mod-}\mathrm{End}_R(P))$ and X in $\mathcal{D}_R$ respectively for which the unit of the adjunction

$$\eta \ : \ M \longrightarrow \mathrm{RHom}_{\mathrm{R}}(P, M \otimes^L_{\mathrm{End}_R(P)} P)$$

or the counit of the adjunction

$$\nu \ : \mathrm{RHom}_{\mathrm{R}}(P,X) \otimes^L_{\mathrm{End}_R(P)} X \longrightarrow X$$

are isomorphisms. Since both derived functors are exact and preserve coproducts, these are localizing subcategories. The map η is an isomorphism on the free module $\mathrm{End}_R(P)$ and the map ν is an isomorphism on P. Since the free module $\mathrm{End}_R(P)$ generates the homotopy category of $\mathrm{End}_R(P)$-modules and P generates Ch_R, the derived functors are inverse equivalences of the homotopy categories. □

Before completing the proof of the Tilting Theorem, here are two important statements.

Lemma 6.6. *Suppose that A is a DGA and R is a ring (considered as a DGA concentrated in degree zero). Then A and R are quasi-isomorphic if and only if $H_k(A) \cong H_k(R)$ for all k. (That is, if and only if $H_k(A) = 0$ for $k \neq 0$ and $H_0(A) \cong R$.)*

Proof. Given $H_k(A) \cong H_k(R)$ for all k, then there are quasi-isomorphisms of DGAs $A \leftarrow A\langle 0\rangle \rightarrow H_0(A) \cong R$. Here $A\langle 0\rangle$ is the (-1)-connected cover of A with $A\langle 0\rangle_k = 0$ for $k<0$, $A\langle 0\rangle_k = A_k$ for $k>0$ and $A\langle 0\rangle_0 = Z_0 A$ the zero cycles. □

Proposition 6.7. *Any quasi-isomorphism $A \longrightarrow B$ of differential graded algebras induces a Quillen equivalence Mod-$A \to$* Mod-B.

Proof. Any map $f\colon A \longrightarrow B$ induces a Quillen adjoint pair between Mod-A and Mod-B, just as in Example 2.4. The right adjoint is given by restriction of scalars and the left adjoint is $-\otimes_A B$. [SS1, 4.3] shows that this adjoint pair is a Quillen equivalence. □

Completion of the proof of Theorem 4.2. We must show that (4) ⇒ (1), so suppose that $\mathcal{C}h_R$ has a tilting complex T. Then T satisfies the hypotheses of Theorem 6.4, hence $\mathcal{C}h_R$ is Quillen equivalent to the category of modules over the differential graded algebra $\mathrm{End_R}(T)$. Since T is a bounded complex of projectives, it is cofibrant by [Ho, 2.3.6]; hence from Lemma 6.5 we have $H_* \,\mathrm{End_R}(T) \cong \mathcal{D}_R(T,T)_* \cong S$ concentrated in dimension zero. By Lemma 6.6 this implies that $\mathrm{End_R}(T)$ is quasi-isomorphic to S. Thus the categories of $\mathrm{End_R}(T)$-modules and right differential graded S-modules ($\mathcal{C}h_S$) are Quillen equivalent by Proposition 6.7:

$$\mathcal{C}h_R \quad \simeq_Q \quad \text{Mod-}\,\mathrm{End}(T) \quad \simeq_Q \quad \mathcal{C}h_S.$$

□

Remark 6.8. We have now shown that when R and S are rings, their model categories of dg-modules are Quillen equivalent if and only if the associated homotopy categories are triangulated equivalent. This is false if R and S are allowed to be DGAs rather than rings, essentially because the analog of Lemma 6.6 fails: the quasi-isomorphism type of an arbitrary DGA is not determined by its homology (not even if you include all its Massey products, see [S, A.3]).

In [DS] we give an explicit example of two DGAs which are derived equivalent, but where the model categories of dg-modules are not Quillen equivalent. The example is based on [Sc] which considers model categories underlying the stable category of modules over the Frobenius rings $R = \mathbb{Z}/p^2$ and $R' = \mathbb{Z}/p[\epsilon]/\epsilon^2$. The homotopy categories are triangulated equivalent but the corresponding K-theory groups are non-isomorphic at K_4. So by Corollary 3.9 these model categories cannot be Quillen equivalent. In [DS] we give a simpler proof of this by studying certain endomorphism DGAs, where we can detect the difference in the second Postnikov sections instead of in K_4.

7. Many generators version of proofs

In this section we generalize the work in Section 6 to the case where we have a set of generators instead of just one. Here the analogue for abelian categories is in [F, 5.3H]. For derived equivalences, Keller [Kr, 8.2] gave the corresponding extension of Rickard's work [R1, 6.4]. As always, our purpose is just to upgrade the derived equivalences to Quillen equivalences.

As in Section 6, before moving to a differential graded setting we first recall the classical setting. Define a *ring with many objects* to be a small $\mathcal{A}b$-category (a category enriched over abelian groups); this terminology makes sense because an $\mathcal{A}b$-category with one object corresponds to a ring, with composition corresponding to the ring multiplication. Given a ring with many objects $\underline{R}$, a (right) $\underline{R}$-module M is a contravariant additive functor from $\underline{R}$ to $\mathcal{A}b$. This means that for any two objects P, P' in $\underline{R}$ there are maps $M(P') \otimes \underline{R}(P, P') \longrightarrow M(P)$. The category of right $\underline{R}$-modules is an abelian category.

If $\mathcal{A}$ is an abelian category and $\mathcal{P}$ is a set of objects, say that $\mathcal{P}$ is a **set of strong generators** if $X = 0$ whenever $\hom_{\mathcal{A}}(P, X) = 0$ for every P in $\mathcal{P}$. Define $\underline{\mathrm{End}}_{\mathcal{A}}(\mathcal{P})$ to be the full subcategory of $\mathcal{A}$ (enriched over $\mathcal{A}b$) with object set $\mathcal{P}$. The following theorem from [F, 5.3H] classifies abelian categories with a set of strong generators:

Theorem 7.1. (Freyd) *Let $\mathcal{A}$ be a co-complete abelian category with a set of small, projective, strong generators $\mathcal{P}$. Then the functor*

$$\mathrm{hom}_{\mathcal{A}}(P,-)\colon\ \mathcal{A} \longrightarrow \text{Mod-}\,\underline{\mathrm{End}}_{\mathcal{A}}(\mathcal{P})$$

is the right adjoint of an equivalence of categories.

In order to generalize this result to a more homotopical setting, we need to replace $\mathcal{A}b$-categories with $\mathcal{C}h$-categories (categories enriched over $\mathcal{C}h = \mathcal{C}h_{\mathbb{Z}}$.) Since a $\mathcal{C}h$-category with one object is a differential graded algebra, one may think of a small $\mathcal{C}h$-category as a **DGA with many objects**. Given a small $\mathcal{C}h$-category $\mathcal{R}$, a (right) $\mathcal{R}$-module $\mathcal{M}$ is a contravariant $\mathcal{C}h$-functor from $\mathcal{R}$ to $\mathcal{C}h$. This means that for any two objects P, P' of $\mathcal{R}$ there is a structure map of chain complexes $\mathcal{M}(P') \otimes \mathcal{R}(P,P') \longrightarrow \mathcal{M}(P)$. See [Ky, 1.2] or [B, 6.2] for more details.

Notice that $\mathcal{C}h_R$ and $\mathcal{C}h_{\underline{R}}$ are both $\mathcal{C}h$-categories, where R is a ring and $\underline{R}$ is a ring with many objects. The enrichment of $\mathcal{C}h_R$ over $\mathcal{C}h$ was discussed in the previous section. Since any two $\underline{R}$-modules have an abelian group of morphisms $\mathrm{hom}_{\underline{R}}(M,N)$, the enrichment for $\mathcal{C}h_{\underline{R}}$ follows similarly.

Definition 7.2. Let $\mathcal{P}$ be a set of objects in a $\mathcal{C}h$-category $\mathcal{C}$. We denote by $\mathcal{E}(\mathcal{P})$ the full subcategory of $\mathcal{C}$ (enriched over $\mathcal{C}h$) with objects $\mathcal{P}$, i.e., $\mathcal{E}(\mathcal{P})(P,P') = \mathrm{Hom}_{\mathcal{C}}(P,P')$. We let

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{P},-)\,:\,\mathcal{C} \longrightarrow \text{Mod-}\,\mathcal{E}(\mathcal{P})$$

denote the functor given by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P},Y)(P) = \mathrm{Hom}_{\mathcal{C}}(P,Y)$.

Note that if $\mathcal{P} = \{P\}$ has a single element, then $\mathcal{E}(\mathcal{P})$ is determined by the single differential graded ring $\mathrm{End}_{\mathcal{C}}(P) = \mathrm{Hom}_{\mathcal{C}}(P,P)$.

In [SS3, 6.1] it is established that there is a (projective) model structure on the category Mod-$\,\mathcal{E}(\mathcal{P})$ of $\mathcal{E}(\mathcal{P})$-modules: the weak equivalences are the maps which induce quasi-isomorphisms at each object and the fibrations are the epimorphisms (at each object).

Now we can state the differential graded analogue of Freyd's theorem; the difference is that here we have weak generators and a Quillen equivalence instead of strong generators and a categorical equivalence. A set of objects $\mathcal{P}$ in a stable model category $\mathcal{C}$ is a **set of (weak) generators** if the only localizing subcategory of $\mathrm{Ho}(\mathcal{C})$ which contains $\mathcal{P}$ is $\mathrm{Ho}(\mathcal{C})$. As mentioned above Theorem 4.2, when the elements of $\mathcal{P}$ are compact then they generate $\mathrm{Ho}(\mathcal{C})$ if and only if they can detect when an object is trivial; see [SS2, 2.2.1]. Note that a (possibly infinite) coproduct of a set of generators is still a generator, but is not necessarily compact.

Theorem 7.3. *Let $\underline{R}$ be a ring with many objects and $\mathcal{P}$ a set in $\mathcal{C}h_{\underline{R}}$ of bounded complexes of finitely generated projectives. If $\mathcal{P}$ is a set of (weak) generators for $\mathcal{C}h_{\underline{R}}$ then there is a Quillen equivalence*

$$\text{Mod-}\,\mathcal{E}(\mathcal{P}) \longrightarrow \mathcal{C}h_{\underline{R}}$$

in which the right adjoint is the functor $\mathrm{Hom}_{\underline{R}}(\mathcal{P},-)$.

Note that for every object $r \in \underline{R}$ there is a corresponding *'free module'* $F_r^{\underline{R}}$ given by $F_r^{\underline{R}}(s) = \underline{R}(s,r)$. A projective $\underline{R}$-module is *finitely-generated* if it is a direct summand of a module $\oplus_i F^{\underline{R}}_{r_i}$, where the sum is finite. And as usual, we denote the homotopy category of $\mathcal{C}h_{\underline{R}}$ by $\mathcal{D}_{\underline{R}}$. We need the following lemma:

Lemma 7.4. *The compact objects in $\mathcal{D}_{\underline{R}}$ are those complexes which are quasi-isomorphic to a bounded complex of finitely-generated projective $\underline{R}$-modules.*

Proof. This follows from [Kr, 5.3]. □

Proof of Theorem 7.3. Just as in the proof of Theorem 6.4, one can check that $\mathrm{Hom}_{\underline{R}}(P, -)$ takes fibrations and trivial fibrations in $\mathcal{C}h_{\underline{R}}$ to fibrations and trivial fibrations in $\mathcal{C}h$ for any bounded complex of projectives P. So the functor $\mathrm{Hom}_{\underline{R}}(\mathcal{P}, -)$ preserves fibrations and trivial fibrations. Thus, together with its left adjoint $- \otimes_{\mathcal{E}(\mathcal{P})} \mathcal{P}$, it forms a Quillen pair.

We proceed as in the proof of Theorem 6.4. The induced total derived functors are again exact functors of triangulated categories which commute with coproducts. Here we use the fact that each P is compact to show the right adjoint commutes with coproducts. The full subcategories for which the unit of the adjunction η or the counit of the adjunction ν are isomorphisms are localizing subcategories.

Note that for each object P in $\mathcal{P}$ there is a free $\mathcal{E}(\mathcal{P})$-module $F_P^{\mathcal{E}(\mathcal{P})}$ defined by $F_P^{\mathcal{E}(\mathcal{P})}(P') = \mathcal{E}(\mathcal{P})(P', P)$, and these generate the homotopy category of $\mathcal{E}(\mathcal{P})$-modules. For every $P \in \mathcal{P}$ the $\mathcal{E}(\mathcal{P})$-module $\mathrm{Hom}_{\underline{R}}(\mathcal{P}, P)$ is isomorphic to the free module $F_P^{\mathcal{E}(\mathcal{P})}$ by inspection, and $F_P^{\mathcal{E}(\mathcal{P})} \otimes_{\mathcal{E}(\mathcal{P})} \mathcal{P}$ is isomorphic to P since they represent the same functor on $\mathcal{C}h_{\underline{R}}$. Thus, η is an isomorphism on every free module and ν is an isomorphism on every object of $\mathcal{P}$. Since the free modules $F_P^{\mathcal{E}(\mathcal{P})}$ generate the homotopy category of $\mathcal{E}(\mathcal{P})$-modules and the objects of $\mathcal{P}$ generate $\mathcal{C}h_{\underline{R}}$, the localizing subcategories where η and ν are isomorphisms are the whole homotopy categories. This implies that the adjoint pair is a Quillen equivalence. □

Finally, we can write down a many-objects version of Theorem 4.2. If $\mathcal{P}$ is a set of (weak) generators with each element P a bounded complex of finitely generated projectives and $H_*\mathcal{E}(\mathcal{P})$ is concentrated in degree zero, then we call $\mathcal{P}$ a set of *tiltors*. The following theorem is a generalization of Keller's work [Kr, 8.2]:

Theorem 7.5. (Many-objects tilting theorem) *Theorem 4.2 holds when the rings R and S are replaced by rings-with-many-objects $\underline{R}$ and $\underline{S}$. The tilting complex is replaced by a set of tiltors $\mathbb{T}$ with $H_*\mathcal{E}(\mathbb{T}) \cong \underline{S}$.*

The proof is given below, but first we state some easy consequences:

Corollary 7.6. *Two rings-with-many-objects $\underline{R}$ and $\underline{S}$ are derived equivalent if and only if their associated model categories of chain complexes $\mathcal{C}h_{\underline{R}}$ and $\mathcal{C}h_{\underline{S}}$ are Quillen equivalent.*

Using Theorem 7.1 we get the following corollary as well. Given an abelian category $\mathcal{A}$ satisfying the hypotheses of Theorem 7.1, choose a set of small, projective, strong generators $\mathcal{P}$. Let $\underline{A} = \underline{\mathrm{End}}_{\mathcal{A}}(\mathcal{P})$ be the associated ring-with-many-objects. Freyd's theorem says that $\mathcal{A}$ is equivalent to Mod-$\underline{A}$, and so $\mathcal{C}h_{\mathcal{A}}$ is equivalent to $\mathcal{C}h_{\underline{A}}$. In particular, one gets a projective model structure on $\mathcal{C}h_{\mathcal{A}}$ by lifting the one on $\mathcal{C}h_{\underline{A}}$ across the equivalence; see [SS3, 6.1]. The next result is now an immediate consequence of Corollary 7.6.

Corollary 7.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be co-complete abelian categories with sets of small, projective, strong generators. Then $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent if and only* *if their associated model categories of chain complexes $\mathcal{C}h_{\mathcal{A}}$ and $\mathcal{C}h_{\mathcal{B}}$ are Quillen equivalent.*

Remark 7.8. Warning: Let $\mathcal{M}$ and $\mathcal{N}$ be two stable model categories whose underlying categories are abelian, with sets of small, strong, projective generators. The above corollary does *not* say that $\mathcal{M}$ and $\mathcal{N}$ are Quillen equivalent if and only if $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\mathcal{N})$ are triangulated equivalent. This statement is false; see [Sc], [DS]. Note in particular that it does not apply to the model category Mod-$\mathcal{R}$ where $\mathcal{R}$ is a DGA; the problem is that Ho(Mod-$\mathcal{R}$) is not the same as $\mathrm{Ho}(\mathcal{C}h_{\text{Mod-}\mathcal{R}})$.

Proof of Theorem D. Part (a) is the above corollary. Part (b) is immediate from (a) and Corollary 3.9. □

Proof of Theorem 7.5. The proof that condition (1) implies condition (2) and condition (2) implies condition (3) follows just as in Theorem 4.2.

Now assume condition (3) and fix a triangulated equivalence between $K_b(\text{proj-}\underline{R})$ and $K_b(\text{proj-}\underline{S})$. For s any object in $\underline{S}$, consider the module $F_s^{\underline{S}}$ as a complex concentrated in dimension zero; let T_s be its image in $K_b(\text{proj-}\underline{R})$. Since the objects in $\{F_s\}_{s\in\underline{S}}$ generate $K_b(\text{proj-}\underline{S})$, the objects in $\mathbb{T} = \{T_s\}_{s\in\underline{S}}$ generate $K_b(\text{proj-}\underline{R})$. But $K_b(\text{proj-}\underline{R})$ generates $\mathcal{D}_R$, so $\mathbb{T}$ generates $\mathcal{D}_R$ as well. By Lemma 7.4 the objects of $\mathbb{T}$ are compact in $\mathcal{D}_R$. Finally, we also have $H_*\mathcal{E}(\mathbb{T}) \cong H_*\mathcal{E}(\{F_s^{\underline{S}}\}) \cong \underline{S}$. So $\mathbb{T}$ is a set of tiltors.

If we are given a set of tiltors $\mathbb{T}$ for $\mathcal{C}h_{\underline{R}}$, then by Theorem 7.3 $\mathcal{C}h_{\underline{R}}$ is Quillen equivalent to the category of modules over the endomorphism category $\mathcal{E}(\mathbb{T})$. Since $H_*\mathcal{E}(\mathbb{T}) \cong \underline{S}$ is concentrated in dimension zero, $\mathcal{E}(\mathbb{T})$ is quasi-isomorphic to $\underline{S}$ by an extension of Lemma 6.6. Thus the categories of differential graded $\mathcal{E}(\mathbb{T})$-modules and differential graded $\underline{S}$-modules are Quillen equivalent by [SS3, 6.1] which generalizes Proposition 6.7:

$$\mathcal{C}h_{\underline{R}} \quad \simeq_Q \quad \text{Mod-}\mathcal{E}(\mathbb{T}) \quad \simeq_Q \quad \mathcal{C}h_{\underline{S}}$$

□

Appendix A. Proof of Proposition 3.6

Recall that $\mathcal{M}$ and $\mathcal{N}$ are pointed model categories, $(L, R)\colon \mathcal{M} \to \mathcal{N}$ is a Quillen equivalence, $\mathcal{U}$ is a complete Waldhausen subcategory of $\mathcal{M}$, and $\mathcal{V} = \overline{(L\mathcal{U})}$. (Note that L, being a left adjoint, must preserve the initial object). We must show that $\mathcal{V}$ is a complete Waldhausen subcategory of $\mathcal{M}$ and that the induced map $L\colon K(\mathcal{U}) \to K(\mathcal{V})$ is a weak equivalence.

For the remainder of this section, let F be a fibrant-replacement functor in $\mathcal{N}$ and let Q be a cofibrant-replacement functor in $\mathcal{M}$. Note that the functor $QRF\colon \mathcal{N} \to \mathcal{M}$ takes $\mathcal{V}$ into $\mathcal{U}$: for if $X \in \mathcal{V}$ then $X \simeq LA$ for some $A \in \mathcal{U}$, and then $QRFX \simeq QRFLA \simeq A$. Since $\mathcal{U}$ is complete and $A \in \mathcal{U}$, it follows that $QRFX \in \mathcal{U}$ as well.

Lemma A.1. *$\mathcal{V}$ is a complete Waldhausen subcategory of $\mathcal{N}$.*

Proof. The only point which takes work is axiom (iii) for Waldhausen categories. So if $A \rightarrowtail B$ and $A \to X$ are maps in $\mathcal{N}$ where A, B, $X \in \mathcal{V}$, we must show that the pushout $B \amalg_A X$ is also in $\mathcal{V}$.

Consider the maps $QRFA \to QRFB$ and $QRFA \to QRFX$. All the domains and codomains of these maps are in $\mathcal{U}$. Factor $QRFA \to QRFB$ as $QRFA \rightarrowtail Z \xrightarrow{\sim} QRFB$. Then $Z \in \mathcal{U}$ and so the pushout $P = Z \amalg_{QRFA} QRFX$ is also in $\mathcal{U}$, because $\mathcal{U}$ is a Waldhausen subcategory of $\mathcal{M}$. This pushout is weakly equivalent to the homotopy pushout (see [DwSp, 10]) of $Z \leftarrow QRFA \to QRFX$, because $QRFA \to Z$ is a cofibration and all the objects Z, $QRFA$, and $QRFX$ are cofibrant; see [Ho, 5.2.6]. Since $Z \xrightarrow{\sim} QRFB$, P is also weakly equivalent to the homotopy pushout of the diagram $QRFB \leftarrow QRFA \to QRFX$.

Finally, any left Quillen functor L preserves homotopy pushouts, in the sense that LP is weakly equivalent to the homotopy pushout of $LQRFB \leftarrow LQRFA \leftarrow LQRFX$. The latter homotopy pushout is weakly equivalent to the homotopy pushout of $B \leftarrow A \to X$, which in turn is just weakly equivalent to the pushout $B \amalg_A X$ (since $A \to B$ is a cofibration and all the objects A, B, X are cofibrant). So $B \amalg_A X$ is weakly equivalent to LP, and is therefore in $\overline{L\mathcal{U}}$. □

Let $w\mathcal{U}$ denote the subcategory consisting of all weak equivalences in $\mathcal{U}$, and write $N(w\mathcal{U})$ for the nerve of this category. The functor L induces a map $w\mathcal{U} \to w\mathcal{V}$.

Lemma A.2. *$NL\colon N(w\mathcal{U}) \to N(w\mathcal{V})$ is a weak equivalence of spaces.*

Proof. First of all, the functor $Q\colon \mathcal{M} \to \mathcal{M}$ maps $\mathcal{U}$ into itself (because $\mathcal{U}$ is complete), and comes equipped with a natural transformation $QX \to X$. This shows that the induced map $NQ\colon N\mathcal{U} \to N\mathcal{U}$ is homotopic to the identity [Se, 2.1]. Similarly, $NF\colon N\mathcal{V} \to N\mathcal{V}$ is homotopic to the identity.

The functor $QRF\colon \mathcal{N} \to \mathcal{M}$ maps $\mathcal{V}$ to $\mathcal{U}$, as was remarked prior to the previous lemma. There are natural transformations $LQRF \to LRF \to F$, and $Q \to QRL \to QRFL$. It follows that the compositions $NL \circ N(QRF)$ and $N(QRF) \circ NL$ are homotopic to the respective identity maps, and so are part of a homotopy equivalence. □

Let Δ_n denote the category consisting of n composable arrows $0 \to 1 \to \cdots \to n$. This may be given the structure of a *Reedy category* [Ho, 5.2.1] in which all the maps increase dimension. The category of diagrams $\mathcal{M}^{\Delta_n}$ has a corresponding *Reedy model structure* [Ho, 5.2.5] in which a map $X_\bullet \to Y_\bullet$ is a weak equivalence (respectively fibration) if and only if each $X_n \to Y_n$ is a weak equivalence (respectively fibration). A map is a cofibration if and only if all the maps $X_n \amalg_{X_{n-1}} Y_{n-1} \to Y_n$ are cofibrations. In particular, an object $X_\bullet$ is cofibrant if and only if the maps $X_{n-1} \to X_n$ are all cofibrations; by a simple recursion, this implies that all the X_i's are cofibrant as well.

Let $\mathcal{U}_n$ denote the full subcategory of $\mathcal{M}^{\Delta_n}$ consisting of cofibrant diagrams whose objects all belong to $\mathcal{U}$. It is easy to see that $\mathcal{U}_n$ is a complete Waldhausen subcategory of $\mathcal{M}^{\Delta_n}$. The functors (L, R) prolong to functors $(L, R)\colon \mathcal{M}^{\Delta_n} \to \mathcal{N}^{\Delta_n}$, and this is still a Quillen equivalence. We need the following:

Lemma A.3. *Any diagram in $\mathcal{V}_n$ is weakly equivalent to one in $L(\mathcal{U}_n)$—i.e., $\mathcal{V}_n = \overline{L\mathcal{U}_n}$.*

Proof. Let $X_\bullet = [X_0 \to \cdots \to X_n]$ be an object in $\mathcal{V}_n$. Then each X_i is in $\mathcal{V}$, and so $QRFX_i$ lies in $\mathcal{U}$ (as was shown above Lemma A.1). Consider the object $QRFX_\bullet = [QRFX_0 \to \cdots \to QRFX_n]$. This need not be cofibrant in $\mathcal{M}^{\Delta_n}$, but we can still take its cofibrant replacement—call this new object $C_\bullet$. Each C_i is a cofibrant object weakly equivalent to $QRFX_i$, and is therefore in $\mathcal{U}$; so $C_\bullet$ is in

$\mathcal{U}_n$. We have a sequence of maps $LC_\bullet \to LQRFX_\bullet \to FX_\bullet \leftarrow X_\bullet$, all of which are objectwise weak equivalences, and so $X_\bullet$ is weakly equivalent to an object in $L\mathcal{U}_n$. $\square$

The category $w(\mathcal{U}_n)$ is exactly the category $wF_n(\mathcal{U})$ defined in Section 3. So there is a 'forgetful' functor $wS_n(\mathcal{U}) \to w(\mathcal{U}_{n-1})$: in the notation of [Wa, 1.3] it sends an object $\{A_{ij}\}$ to the sequence $A_{01} \rightarrowtail A_{02} \rightarrowtail \cdots \rightarrowtail A_{0n}$. This functor is easily seen to be an equivalence of categories (see [Wa, bottom of p. 328]).

Proof of Proposition 3.6. Recall that $K(\mathcal{U})$ is defined as the geometric realization of a simplicial space $[n] \mapsto N(wS_n(\mathcal{U}))$. It is therefore enough to show that L induces weak equivalences $N(wS_n(\mathcal{U})) \to N(wS_n(\mathcal{V}))$ at each level. There is a commutative diagram

$$\begin{tikzcd} wS_n(\mathcal{U}) \arrow[r] \arrow[d] & wS_n(\mathcal{V}) \arrow[d] \\ w\mathcal{U}_{n-1} \arrow[r] & w\mathcal{V}_{n-1} \end{tikzcd}$$

and the vertical maps are equivalences of categories. So it suffices to show that the maps $N(w\mathcal{U}_n) \to N(w\mathcal{V}_n)$ are weak equivalences. But this follows from Lemma A.2 applied to the complete Waldhausen subcategories $\mathcal{U}_n$ and $\mathcal{V}_n = \overline{L\mathcal{U}_n}$ of $\mathcal{M}^{\Delta^n}$. $\square$

References

[BN] M. Bökstedt, A. Neeman, *Homotopy limits in triangulated categories* Compositio Math. **86** (1993), 209–234.

[B] F. Borceux. *Handbook of categorical algebra II, Categories and structures*, Cambridge University Press, 1994.

[DS] D. Dugger and B. Shipley, *Stable module categories and differential graded rings*, in preparation.

[DK80] W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), 427–440.

[DwSp] W. G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology (Amsterdam), North-Holland, Amsterdam, 1995, pp. 73–126.

[F] P. Freyd, *Abelian categories*, Harper and Row, New York, 1964.

[G] P. Gabriel, *Des catégories abeliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[Hi] V. Hinich, *Homological algebra of homotopy algebras*, Comm. Algebra **25** (1997), no. 10, 3291–3323.

[Hi] P. Hirschhorn, *Model categories and their localizations*, 2002 preprint.

[Ho] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, **63**, American Mathematical Society, Providence, RI, 1999, xii+209 pp.

[HSS] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), 149–208.

[Kr] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.

[Ky] G. M. Kelly, *Basic concepts of enriched category theory*, Cambridge Univ. Press, Cambridge, 1982, 245 pp.

[N1] A. Neeman, *K-theory for triangulated categories I(A): Homological functors*, Asian J. Math. Vol 1, no. 2 (1997), 330–417.

[N2] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, **148** Princeton University Press, Princeton, NJ, 2001, viii+449.

[Q1] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, **43**, Springer-Verlag, 1967.

[Q2] D. G. Quillen, Higher algebraic K-theory. I. *Algebraic K-theory, I: Higher K-theories* (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., **341**, Springer, Berlin 1973.

[R1] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), 436–456.

[R2] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), 37–48.

[Se] G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes tudes Sci. Publ. Math. **34** (1968), 105–112.

[SS1] S. Schwede and B. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. **80** (2000), 491-511.

[SS2] S. Schwede and B. Shipley, *Stable model categories are categories of modules*, to appear in Topology. `http://www.math.purdue.edu/~bshipley/`

[SS3] S. Schwede and B. Shipley, *Equivalences of monoidal model categories*, Preprint, 2002. `http://www.math.purdue.edu/~bshipley/`

[Sc] M. Schlichting, *A note on K-theory and triangulated categories*, to appear in Inv. Math. `http://www.math.uiuc.edu/~mschlich/`

[S] B. Shipley, *An algebraic model for rational S^1-equivariant homotopy theory*, Quart. J. Math **53** (2002), 87–110.

[Sr] V. Srinivas, *Algebraic K-theory*, Progress in Mathematics **90**, Birkhauser, Boston, 1996.

[TT] R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 247–435, Progr. Math., 88, Birkhuser Boston, Boston, MA, 1990.

[Wa] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), 318–419, Lecture Notes in Math., **1126**, Springer, Berlin, 1985.

[We] C. Weibel, *An introduction to homological algebra* Cambridge Studies in Advanced Mathematics, **38**, Cambridge University Press, Cambridge, 1994, xiv+450 pp.

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: `ddugger@math.uoregon.edu`

Department of Mathematics, Purdue University, W. Lafayette, IN 47907, USA
E-mail address: `bshipley@math.purdue.edu`