of $H_k(D^n, V)$, hence $0(n)$ is a group of isometries of the complete Riemannian manifold $\Omega^n(V, g)$. Now suppose A is a strongly elliptic differential operator of order $2k$, $A : C^\infty(D^n, \mathbf{R}^m) \to C^\infty(D^n, \mathbf{R}^m)$, such that $A(Tf) = T(Af)$ for all $T \in 0(n)$, for example $A = \Delta^k$ where Δ is the Laplacian

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Then $J : \Omega(V, g) \to \mathbf{R}$ defined by $J(f) = \frac{1}{2}\langle Af, f\rangle_0$ satisfies $J(Tf) = j$ for any $T \in 0(n)$. hence if f is a critical point of J so is Tf for any $T \in 0(n)$, and since non-degenerate critical points of J are isolated, $Tf = f$ if f is a non-degenerate critical point of J. But $Tf = f$ is equivalent to $R(f(x))$ being a function F of $\|x\|$ where R is the distance measured along the sphere $S^{m-1} = V$ of a point on V to the north pole. Moreover F will satisfy an ordinary differential equation of order $2k$. With a little computation one should be able to compute all the critical points and their indices and hence, via the Morse inequalities, get information about the homology groups of $\Omega^n(V, g)$ (which has the homotopy type of the nth loop space of S^{m-1}). Clearly the same sort of process will work whenever we can force a large degree of symmetry into the situation.

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ON THE DUNCE HAT

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The traditional dunce hat is a cone obtained from a triangle, *abc* say, by identifying the sides $ab = ac$. The *topological dunce hat* D, which is the subject of this paper, is defined by identifying all three sides $ab = ac = bc$. One way to visualise D is to embed it in 3-space by first making a traditional dunce hat and then, starting from $b = c$, sewing the generator $ab = ac$ onto the circular base bc.

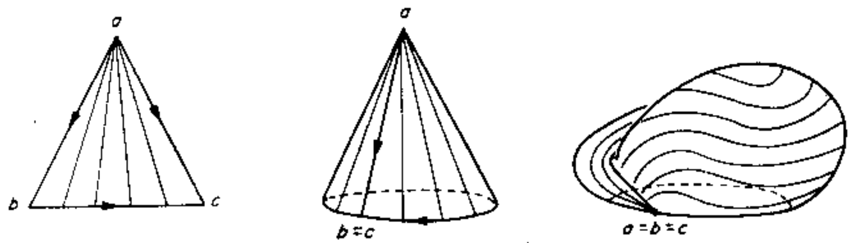

Fig. 1

D is remarkable because it is the simplest example of a polyhedron that is contractible, in the sense of homotopy, but not collapsible, in the sense of Whitehead [19]. Other examples are well known (see for instance [7]). To prove the contractibility of D is easy, because it is only necessary to check that the homotopy groups vanish; although to visualise the actual contraction is intriguingly difficult. To prove the non-collapsibility we merely observe there is nowhere to start collapsing: in any triangulation no 2-simplex has a free face.

Now the phenomenon of being contractible yet not collapsible pinpoints a primary source of difficulty in the study of manifolds of dimension $\geqslant 3$. In particular the phenomenon seems to be intimately connected with the Poincaré Conjecture in dimensions 3 and 4, which are the two unsolved dimensions. In general terms the difficulty is one of passing from finite structures (such as complexes) to *ordered* finite structures (such as handlebodies). Therefore in order to gain insight into the phenomenon it is worthwhile studying the simplest example in some detail: in this paper we analyse the dunce hat, and the manifolds of which it is a spine, from a geometrical point of view. We give ten theorems and five conjectures, which are related to the Poincaré Conjecture. For a more general approach see Curtis [5, 6] and Mazur [12].

Let I denote the unit interval. Although the dunce hat D is not collapsible, it transpires hat:

THEOREM (1). *$D \times I$ is collapsible.*
'his suggests:

CONJECTURE (1). *If K is a contractible 2-complex then $K \times I$ is collapsible.*
'he interest of the conjecture is:

THEOREM (2). *Conjecture 1 implies the 3-dimensional Poincaré Conjecture.*

Before proving Theorems 1 and 2 we recall some definitions, in particular the definition f collapsing, which will be our main tool. We shall work in the category of polyhedra and iecewise linear maps. For notation we use $\cong$ to denote homeomorphism. The n-sphere ıd n-ball are denoted by S^n and B^n. A *face* of B^n is an $(n-1)$-ball in its boundary. M is a manifold, denote by $\dot{M}$ the boundary and $\mathring{M}$ the interior.

COLLAPSING

We use a polyhedral definition of collapsing (as in [22]), which is a slight modification Whitehead's original definition [19]. Let X be a polyhedron and Y a subpolyhedron. ıere is an *elementary collapse from X to Y* if, for some n, there is an n-ball B^n with face $^{-1}$ such that

$$X = Y \cup B^n$$
$$B^{n-1} = Y \cap B^n.$$

e describe the elementary collapse from X to Y by saying *collapse across B^n onto B^{n-1}*, *collapse across B^n from B_*^{n-1}*, where B_*^{n-1} is the complementary face of B^n. Similarly scribe the elementary expansion from Y to X by saying *expand across B^n from B^{n-1}*, *expand across B^n onto B_*^{n-1}*. We say *X collapses to Y*, written $X \searrow Y$, (or *Y expands X*, written $Y \nearrow X$), if there is a sequence of elementary collapses

$$X = X_0 \searrow X_1 \searrow \dots \searrow X_n = Y.$$

Y is a point we call *X collapsible* and write $X \searrow 0$. In particular:

LEMMA (1). (Whitehead [19, Theorem 23, Corollary 1]). *A manifold is collapsible if ı only if it is a ball.*

The polyhedral definition of collapsing that we have given here is equivalent to the ıplicial definition: that is to say, given $X \searrow Y$, then we can find a triangulation of X, Y ı a (finer) sequence of elementary collapses in which each elementary collapse is across ımplex of X from a face. For a proof see [22, Theorem 4]. A corollary to this equival-e is:

LEMMA (2). *Let X, $Y \to X_*$, Y_* be a piecewise linear map that maps $X-Y$ homeo-phically onto X_*-Y_*. Then $X \searrow Y$ if and only if $X_* \searrow Y_*$.* See [16, Lemma 1], and a proof see [22, Chapter 7].

SPINES

Let M be a compact bounded polyhedral manifold. Define a spine X of M to be a subpolyhedron such that

$$\text{(i)} \quad M \searrow X.$$

Without loss of generality we can also presume that a spine fulfils two further conditions

$$\text{(ii)} \quad X \subset \text{interior of } M,$$
$$\text{(iii)} \quad \dim X < \dim M,$$

because we can first collapse away a collar from the boundary, and then collapse away all top dimensional simplexes of some triangulation. For example by Lemma (1) a manifold has a point spine if and only if it is a ball.

LEMMA (3). *Let $X_0, X_1, \dots, X_n$ be a sequence of polyhedra in the interior of a manifold M such that for each i, either $X_{i-1} \searrow X_i$ or $X_{i-1} \nearrow X_i$. If X_0 is a spine of M then so is X_n.* For a proof see [9, Corollary 2] or [22, Chapter 3].

Proof of Theorem (1). The definition of D as a triangle with the sides identified furnishes a cell decomposition $D = e_0 \cup e_1 \cup e_2$, where

$e_0 =$ *point*,

$e_1 =$ open 1-cell with both ends at e_0,

$e_2 =$ open 2-cell with boundary formula $e_1 e_1 e_1^{-1}$.

Let $\mathring{I}$ denote the interior of the unit interval. Then $D \times I$ has the cell decomposition

0-cells: $e_0 \times 0$, $e_0 \times 1$

1-cells: $e_1 \times 0$, $e_1 \times 1$, $e_0 \times \mathring{I}$

2-cells: $e_2 \times 0$, $e_2 \times 1$, $e_1 \times \mathring{I}$

3-cell: $e_2 \times \mathring{I}$.

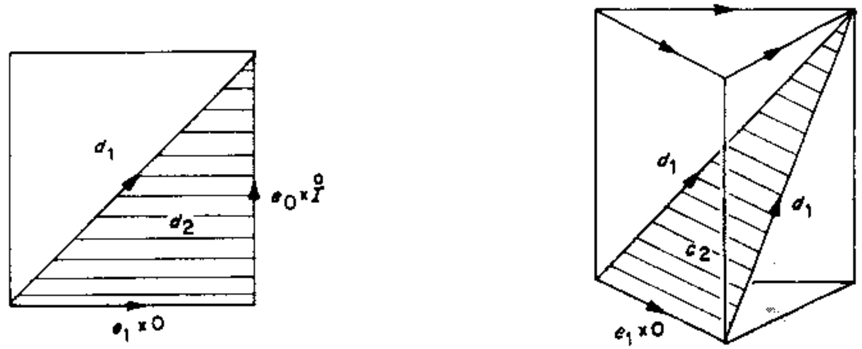

FIG. 2

ı the square $e_1 \times$ let $\dot{I}$ d_1 be the diagonal and d_2 be the triangle below the diagonal. Then

$$d_1 = \text{open 1-cell running from } e_0 \times 0 \text{ to } e_0 \times 1,$$

$$d_2 = \text{open 2-cell with boundary formula } (e_1 \times 0)(e_0 \times I)d_1^{-1}.$$

ı the triangular prism $e_2 \times$ let c_2 be the unique open 2-cell $\dot{I}$ that is a triangular cross ction with boundary formula $(e_1 \times 0)d_1\, d_1^{-1}$ (see Fig. 2).

'e can then collapse (see Fig. 3):

$D \times I \searrow (e_0 \times I) \cup (e_1 \times I) \cup c_2$, collapsing the prism $e_2 \times I$ onto its walls and cross-section c_2 from top and bottom, using Lemma (2);

$\searrow (e_0 \times I) \cup (e_1 \times 0) \cup d_1 \cup d_2 \cup c_2$, collapsing the top triangle of the square $e_1 \times I$ from the top, using Lemma (2);

$\searrow (e_0 \times \dot{I}) \cup (e_1 \times 0) \cup d_1 \cup c_2$, collapsing d_2 from the side $e_0 \times I$;

$\searrow 0$, because it is a disk.

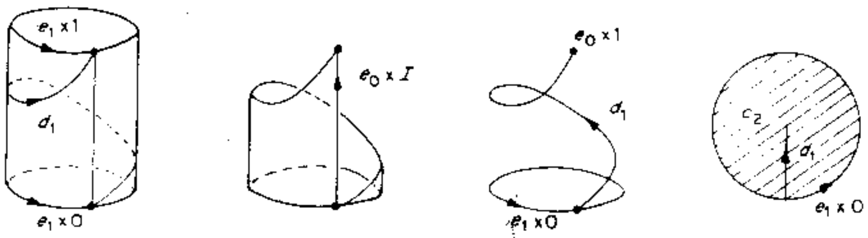

FIG. 3. The 3-cell is not shown, and the 2-cell c_2 is shown only in the last picture

Remark (1). The same trick can show that $K \times I$ is collapsible when K is a figure eight, ' b, with two disks attached by the formulae ab, $a^n b^{n+1}$, or by the formulae ab^n, ab^{n+1}. : I have been unable to discover whether or not $K \times I$ is collapsible if the disks are ıched by the formulae a^2b^3, a^3b^4. It may be that the conjecture is only true for disks n onto a figure eight when one of the disks is sewn on by a free generator of the free up on a, b.

Proof of Theorem (2). We use an argument that is well known, and I believe was oriılly due to Curtis (see for instance [4]). Let F^3 be a fake 3-sphere, i.e. a candidate for ıunterexample to the Poincaré Conjecture. Triangulate by Moise [13], and remove an n 3-simplex, leaving a fake 3-ball, M^3 say. M^3 collapses to a spine, K^2 say, which is tractible since M^3 is. Assuming Conjecture (1),

$$M^3 \times I \searrow K^2 \times I \searrow 0.$$

refore $M^3 \times I$ is a 4-ball by Lemma (1), and so $M^3 \subset (M^3 \times I)^{\cdot} = S^3$. But $\dot{M}^3 = S^2$, ıuse it is the boundary of the simplex removed, and so by the Schönflies Theorem [1]

M^3 is a true 3-ball. Therefore the original fake 3-sphere is a true 3-sphere after all. In other words Conjecture (1) implies the 3-dimensional Poincaré Conjecture.

In fact Conjecture (1) is stronger than the Poincaré Conjecture, because:

THEOREM (3). *There exist contractible 2-complexes that cannot be embedded in any 3-manifold. In fact there exist examples that can be embedded locally but not even immersed in any 3-manifold.*

Proof. The simplest example of a non-embeddable 2-complex is the cone on a graph that is non-embeddable in S^2. But this cannot be embedded locally because the vertex goes wrong. For a locally embeddable example let K be a figure eight, $a \vee b$, with two disks X, Y attached by the formulae a, $a^2b^2a^{-1}b^{-1}$, respectively. Then K is contractible because the fundamental group and Euler characteristic both vanish, and is not collapsible because there is no free edge. (One can show that $K \times I \searrow 0$, as in Theorem (1)). K is locally embeddable because in any triangulation the link of every vertex is embeddable in S^2: the link of the wedge point is shown in Fig. 4.

Suppose K were immersed in a 3-manifold. The second disk Y is a hexagon whose first two sides are to be identified with a. A neighbourhood of these two sides in the hexagon is identified into a Möbius strip, M say, with a as the central curve. But a also bounds the disk X. Therefore a must be an orientation reversing curve, because as we travel round it the three normals, one in X and two in M, perform an odd permutation. But a is homotopic to zero since it bounds the (immersed) disk X, and is therefore orientation preserving, a contradiction.

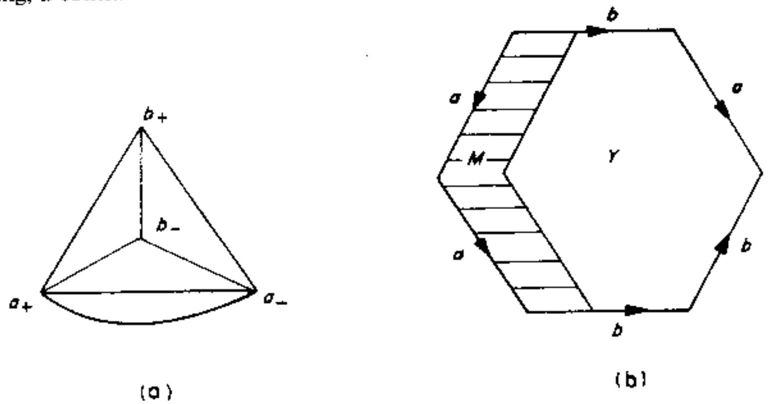

FIG. 4 (a) the link of the vertex; (b) the disk Y.

Remark (2). The examples of Theorem (3) suggest that we generalise to higher dimensions and ask the question: *if M^m is an m-manifold with a contractible 2-dimensional spine, is M^m a ball?* The answers are:

$m = 3$: Unknown. By Theorem (2) this is the same as the Poincaré Conjecture. If the spine is the dunce hat then the answer is yes by Theorem (1).

$m = 4$: No. The remarkable contractible 4-manifolds of Poénaru [14] and Mazur [11] have 2-spines; in fact in Theorem (5) we show that a spine of the latter is none other than the dunce hat. In Theorem (9) we give a criterion for a 4-manifold to have spine D, and in Theorem (8) we give a criterion for such a manifold to be a ball.

$m = 5$: Yes. Both Mazur [12] and Poénaru have proved this result independently. It is a corollary to Mazur's Non-stable Neighbourhood Theorem [12, Chapter 8]. But since the published proof of the latter contains gaps, and since Poénaru's proof is not yet published, and since both proofs are long, we give an elementary proof for the special case of the dunce hat in the next theorem.

$m \geq 6$: Yes. By Smale's Handlebody Theorem [15].

THEOREM (4). *If $m \neq 4$, then any m-manifold having D as a spine is a ball.*

Proof. Suppose M^m has D as a spine. If $m = 3$ then M is a ball by Theorem (1) and proof of Theorem (2). Therefore suppose $m \geq 5$. Choose triangulations of M, D and them by the same names. In particular the 0-cell e_0 of D will be a vertex of D. Let

$$B = \overline{st}(e_0, M)$$
$$L = lk(e_0, D).$$

B is an m-ball and L a 1-complex in $\dot{B}$. From the definition of D we see that L is a of circles, α and γ say, joined by an arc, $\beta = xy$ say.

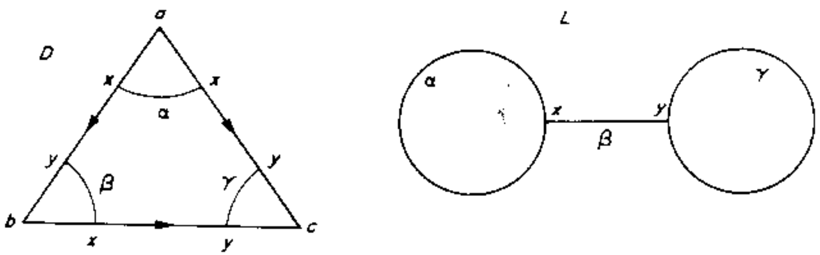

FIG. 5

dim $\dot{B} \geq 4$, we can span α by a disk, A say, in $\dot{B}$ not meeting L again. Therefore the e_0A is a 3-ball meeting D in $e_0\alpha$. Therefore $D \nearrow D \cup e_0A$ by expanding across e_0A A. Since a disk is collapsible (by two elementary collapses), there is a collapse $A \searrow x$, h we can lift conewise to a collapse $e_0A \searrow e_0x \cup A$. Therefore

$$D \nearrow D \cup e_0A \searrow \overline{D - e_0\alpha} \cup A$$
$$\searrow \overline{e_1 - e_0x} \cup A, \text{ collapsing across } \overline{e_2 - e_0\alpha} \text{ from } e_0x;$$
$$\searrow A, \quad \text{collapsing across } \overline{e_1 - e_0x} \text{ from } e_0;$$
$$\searrow x.$$

fore the point x is also a spine of M by Lemma (3), and so M is a ball by Lemma (1).

Remark (3). The above proof breaks down when $m = 4$, because then $\dot{B}$ is a 3-sphere, and L may be embedded in $\dot{B}$ so that the circles α and γ link, preventing the construction of the disk A. In fact this is exactly what happens in our next theorem (see Remark (4) and Fig. 8), and turns out to be essential in Theorem (8).

MAZUR'S CONTRACTIBLE 4-MANIFOLD [11, 21]

Form M^4 by attaching a 2-handle to the boundary $S^1 \times S^2$ of $S^1 \times B^3$ by the curve C as shown in Fig. 6.

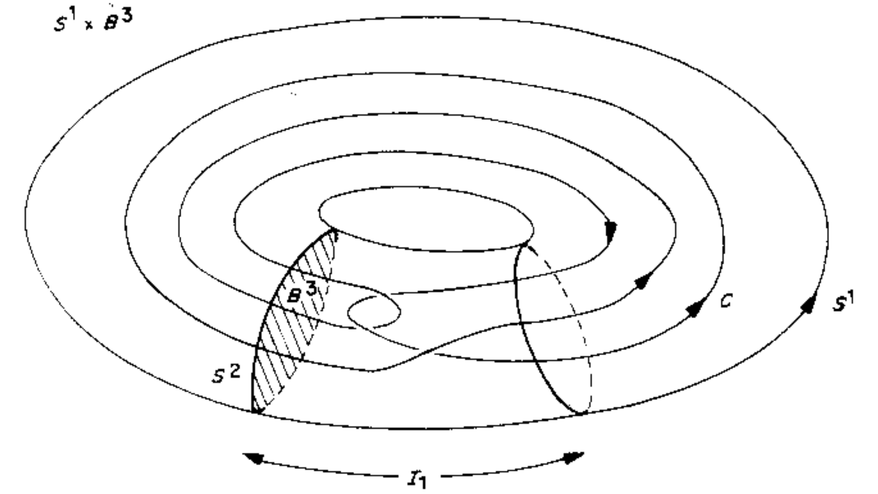

FIG. 6

More precisely, choose a tubular neighbourhood of C, that is to say an embedding $j: \dot{B}^2 \times B^2 \to S^1 \times S^2$ such that $j(\dot{B}^2 \times 0) = C$, where $0 \in \dot{B}^2$, and then define

$$M^4 = S^1 \times B^3 \cup_j B^2 \times B^2.$$

Up to isotopy there are a countable number of choices of neighbourhood, depending upon how many times we twist the tube as we go round C, relative to a chosen standard tube. Mazur made a particular choice of tube in order to compute $\pi_1(\dot{M}^4) \neq 0$ [11, page 227], by which he proved $M^4 \not\cong B^4$. Since his choice was algebraically the simplest, it is highly probable that π_1's are non-trivial and mutually distinct for all different choices of tube. M^4 is contractible because $M^4 \times I$ is a 5-ball (by Theorems (5) and (7) below).

THEOREM (5). *D is a spine of Mazur's manifold.*

Proof. It is not necessary to bother about the choice of tubular neighbourhood, because we show D is a spine in all cases. Divide S^1 into two arcs I_1, I_2 such that $I_1 \times S^2$ contains the guts of the curve C (see Fig. 6). Let $f: S^1 \to S^1$ be the map shrinking I_1 to

F

ɔoint. Let $p\colon S^1 \times S^2 \to S^1$ be the projection onto the first factor, and let g, h be the mpositions

$$C \xrightarrow{\subset} S^1 \times S^2 \xrightarrow{p} S^1 \xrightarrow{f} S^1, \qquad g = fp, \quad h = fp|C.$$

$M(g)$, $M(h)$ denote the mapping cylinders of g, h. We can extend the identity on $\times\, S^2$ to a homeomorphism $M(g) \cong S^1 \times B^3$ as follows. Let $g_1\colon I_1 \times S^2 \to fI_1$ and $I_2 \times S^2 \to S^1$ denote the restrictions of g. Regard the 4-ball $I_1 \times B^3$ as a cone on oundary, with vertex V say. Map $M(g_1)$ homeomorphically onto the subcone $V(\dot{I}_1 \times S^2)$, extend to a homeomorphism of $M(g_2)$ onto $V(\dot{I}_1 \times B^3) \cup I_2 \times B^3$. Therefore we write $M(g) = S^1 \times B^3$.

Meanwhile $M(h)$ is a subcylinder of $M(g)$, because $h = g|C$. Therefore $M(g) \searrow M(h)$, use a mapping cylinder collapses onto any subcylinder [19, Theorem 8]. The 2-handle ℓ^4 collapses to its core, B^2 say, which is a disk spanning C. Therefore

$$M^4 \searrow S^1 \times B^3 \cup_C B^2 \searrow M(h) \cup_C B^2.$$

$M(h) \cup B^2$ is none other than the dunce hat, as is seen from the picture of $M(h)$ in 7. Therefore M^4 has D as spine.

$M\ (h)$

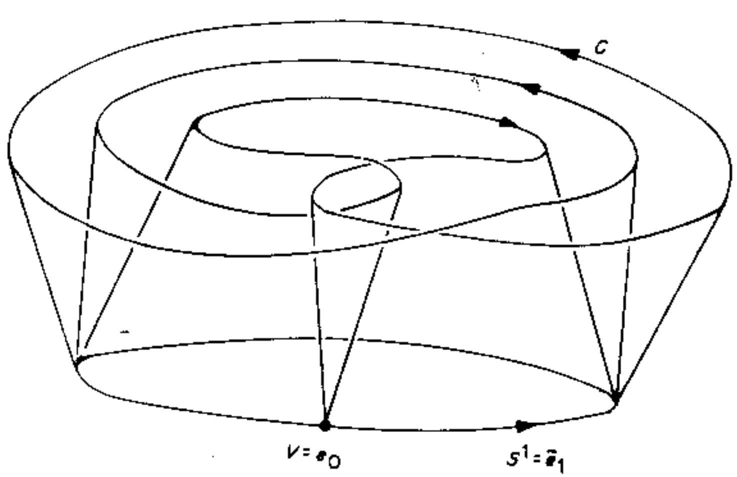

FIG. 7

mark (4). If the construction of Theorem (4) is made for Mazur's manifold, then L l to be embedded in $\dot{B}^4$ with the circles α and γ linked, as promised in Remark (3). . 8.

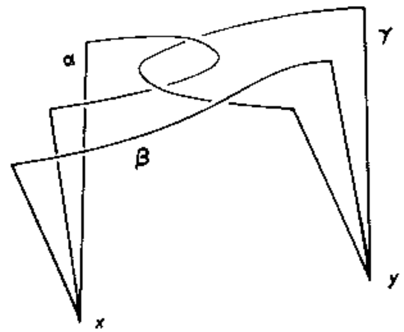

FIG. 8

THEOREM (6). *Suppose M^4 has a spine D and a non-simply connected boundary (for example Mazur's manifold). Then D is not contained in a ball in M^4.*

Proof. The proof is a variant of the proof of [11, Corollary 4]. Write $M = M^4$. Suppose there were a ball B such that $D \subset B \subset M$. We can assume D lies in the interior of B, otherwise replace B by a regular neighbourhood of B in M, which is a bigger ball. Choose a regular neighbourhood M_1 of D in the interior of B. Since M, M_1 are both regular neighbourhoods of D, there is a homeomorphism $h\colon M \to M_1$ keeping D fixed [9, Theorem 2]. Let $B_1 = hB$, $M_2 = hM_1$. Then

$$M \supset B \supset M_1 \supset B_1 \supset M_2 \supset D.$$

By the annulus theorem for regular neighbourhoods [9, Corollary 1], there are homeomorphisms

$$B - \mathring{B}_1 \cong S^3 \times I,$$

$$M - \mathring{M}_1 \cong M - \mathring{M}_2 \cong \dot{M} \times I.$$

Therefore in the commutative triangle induced by inclusion

$$\begin{array}{ccc} \pi_1(\dot{M}_1) & \longrightarrow & \pi_1(M - \mathring{M}_2) \\ & \searrow \quad \nearrow & \\ & \pi_1(B - \mathring{B}_1) & \end{array}$$

the top arrow is an isomorphism and the bottom group trivial. But this contradicts $\pi_1(\dot{M}) \neq 0$, and the theorem is proved.

Remark (6). The interest of Theorem (6) is that it focuses a difference between algebra (= homotopy) and geometry as follows. Recall that a polyhedron is called *q-connected* if the first q homotopy groups vanish, or, equivalently, if every subpolyhedron of dimension $\leqslant q$ can be shrunk to a point. Define a manifold M to be *geometrically q-connected* if every subpolyhedron of dimension $\leqslant q$ is contained in a ball. If $q \leqslant \dim M - 3$ then the two concepts are equivalent [20, Theorem 1]:

$$M \text{ is } q\text{-connected} \Leftrightarrow M \text{ is geometrically } q\text{-connected}.$$

But in codimension 2 the equivalence fails for *compact bounded* manifolds, because the M^4 of Theorem (6) is 2-connected but not geometrically 2-connected. Similarly the equivalence fails for *open* (non-compact without boundary) manifolds, because the interior of M^4 provides a counter-example; another counter-example is given by Whitehead's contractible 3-manifold [18], which is 1-connected but not geometrically 1-connected. It is unsolved whether or not the equivalence holds in codimension 2 for *closed* (compact without boundary) manifolds; in fact the conjecture that it does hold is equivalent to the Poincaré Conjecture in dimensions 3 and 4 (see [2, Theorem 1], [20, Theorem 2].)

The existence of non-trivial 4-manifolds with D as spine prompt the questions "can ve classify such manifolds?" and "do they all look like Mazur's example?". We tackle hese questions in the next three theorems.

THEOREM (7). *If M^4 has D as a spine then $M^4 \times I$ is a ball.*

COROLLARY (1). *M^4 can be embedded in S^4. In other words M^4 is homeomorphic to a gular neighbourhood of an embedding $D \subset S^4$.*

Proof. The theorem follows immediately either from Theorem (1) by collapsing $M^4 \times$ $\searrow D \times I \searrow 0$ and using Lemma (1), or from Theorem (4) by collapsing $M^4 \times I \searrow$ $\times I \searrow D$. The advantage of the second method is that it generalises to arbitrary ntractible 2-complexes by the Mazur [12]–Poénaru Theorem mentioned in Remark (3). ie corollary follows because $M^4 \subset (M^4 \times I)^{\cdot} = S^4$, and M^4 is a regular neighbourhood $D \subset S^4$ because D is a spine.

COROLLARY (2). *Two dunce hats can link in S^4.*

Proof (cf. Mazur [11, Corollary 4]). Choose M^4 as in Theorem (6). Then embed $\times \dot{I} \subset M^4 \times \dot{I} \subset (M^4 \times I)^{\cdot} = S^4$. If the dunce hats were unlinked we could enclose $\times 0$ in a ball, enclose $D \times 1$ in a regular neighbourhood N disjoint from the ball, and n isotope N onto $M^4 \times 1$ keeping the dunce hats fixed, because any two regular ghbourhoods are isotopic, [9, Theorem 3]. In the closure of the complement of N there ıld be a situation violating Theorem (6).

Remark (6). Theorem (7) shows that the problem of classifying 4-manifolds with ie D can be tackled by

(i) classifying isotopy classes of embeddings of D in S^4, and
(ii) determining when non-isotopic embeddings possess homeomorphic regular neighbourhoods.

hasten to add that we do not solve these problems, because (i) is already more complid than the classical knot problem. However (ii) affords a simplification, as is shown he next two theorems. First let us describe how badly D can be (tamely) embedded in r in a manifold of which it is a spine.

THE HANDLEBODY STRUCTURE OF M^4

Let $D \subset S^4$ be an arbitrary (tame) embedding. Any two regular neighbourhoods of D are homeomorphic keeping D fixed [9, Theorem 2], so we select a convenient one. Choose a triangulation T of S^4 containing D as a subcomplex and let $M^4 = N(D, T'')$, the closed simplicial neighbourhood of D in the second barycentric derived complex T'' of T. If $D = e_0 \cup e_1 \cup e_2$ is the cell-structure of D, then e_0 is a vertex and $\bar{e}_1$, a subcomplex of D, and so the derived neighbourhoods $N(e_0, T'')$, $N(\bar{e}_1, T'')$ are subcomplexes of M^4. Define

$$H_0 = N(e_0, T'')$$
$$H_1 = \overline{N(\bar{e}_1, T'') - H_0}$$
$$H_2 = \overline{M^4 - (H_0 \cup H_1)}$$

LEMMA (4). *$M^4 = H_0 \cup H_1 \cup H_2$ is a pseudo handlebody structure of M^4.*

Proof. We shall show that H_0 is a 0-handle, H_1 is a 1-handle attached to H_0, and H_2 a pseudo 2-handle attached to $H_0 \cup H_1$.

First observe that, for $i = 0, 1, 2$,

$$H_i \searrow H_i \cap e_i \searrow 0,$$

and so H_i is a 4-ball by Lemma (1). Therefore H_0 is a 0-handle. Since $H_0 \cap H_1 = \dot{H}_0 \cap \dot{H}_1$ collapses to two points, there is a homeomorphism

$$f : \dot{I} \times B^3 \xrightarrow{\cong} H_0 \cap H_1 \subset \dot{H}_0$$

which can be extended by the combinatorial annulus theorem [9, Corollary 1] to a homeomorphism

$$I \times B^3 \xrightarrow{\cong} \equiv H_1.$$

Therefore H_1 is a 1-handle attached to H_0 by f. Since $(H_0 \cup H_1) \cap H_2 = (H_0 \cup H_1)^{\cdot} \cap \dot{H}_2$ collapses to a circle, there is a homeomorphism

$$g : S^1 \times B^2 \xrightarrow{\cong} (H_0 \cup H_1) \cap H_2 \subset (H_0 \cup H_1)^{\cdot},$$

and so H_2 is a pseudo 2-handle attached to $H_0 \cup H_1$ by g.

However H_2 is not a true 2-handle in general, because no choice of g can be extended to a homeomorphism

$$B^2 \times B^2 \xrightarrow{\cong} H_2$$

if the solid torus $(H_0 \cup H_1) \cap H_2$ happens to be knotted in the 3-sphere $\dot{H}_2$.

THE EMBEDDING $D \subset M^4$

Continuing with the same notation, let $D_i = D \cap H_i$, $i = 0, 1, 2$. The embedding $D \subset M^4$ can be described in terms of the embeddings $D_i \subset H_i$, as follows.

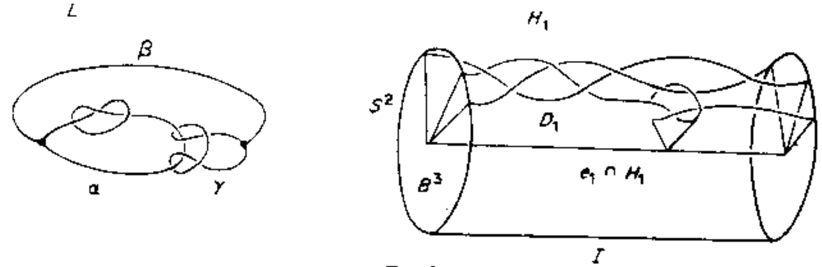

FIG. 9

(1) H_0 is a cone with vertex e_0 and base S^3, and D_0 is the subcone on $L \subset S^3$, where L ›nsists of two circles joined by an arc (see Fig. 5). The circles of L may be linked and ıotted in S^3 (see Fig. 9).

(2) H_1 is a tubular neighbourhood of the arc $e_1 \cap H_1$, and D_1 consists of three strips nning along the tube. Each strip has its inner edge on $e_1 \cap H_1$, its outer edge in the ›undary of the tube, and its ends at the ends of the tube. The strips may be braided, and :ally knotted at interior vertices of $e_1 \cap H_1$ (see Fig. 9). We say D_1 is *unknotted* in H_1 there is no braiding or local knotting; in other words when

$$(H_1, D_1) = I \times (\text{cone on } (S^2, 3 \text{ points})).$$

(3) H_2 is a regular neighbourhood of the disk $e_2 \cap H_2$, and the boundary of H_2 ıtains the boundary of the disk, but in general H_2 will not be a tubular neighbourhood the disk, because the disk may be locally knotted at interior vertices. We say D_2 is *:notted* in H_2 if there is no local knotting (there can be no global knotting because \ D_2; see [9, Corollary 5]); in other words when

$$(H_2, D_2) = D_2 \times (\text{disk, interior point}).$$

this case H_2 is a true 2-handle, not merely a pseudo handle.

Remark (7). There is yet a further complexity when discussing embeddings $D \subset S^4$, ıely the global knotting of the disk in (3) besides the local knotting. Once we restrict ntion to the regular neighbourhood M^4 this further complexity becomes irrelevant, we shall have cause to return to it in Theorem (10). Meanwhile:

CONJECTURE (2). *For any tame embedding $D \subset S^4$ the complement $S^4 - D$ is contractible.*

By Alexander duality the complement has trivial homology, so it is only a matter of ›ving it to be simply connected. The evidence for the conjecture is that it is true for special embeddings of Theorem (7), $D = D \times 0 \subset (M^4 \times I) = S^4$, because the plement $S^4 - D$ is homeomorphic to the interior of M^4, and it remains true if we add :s as follows (but I do not know if this covers all embeddings). Given $D \subset S^4$, choose tle ball $B^4 \subset S^4$ that meets D in an unknotted disk $B^2 = D \cap B^4$ (for example take ;tar of some vertex of the 2-cell of D where it is locally unknotted). Replace the un- :ted ball pair (B^4, B^2) by a knotted ball pair (B^4, B^2_*) having the same boundary, and ıe $D_* = (D - B^2) \cup B^2_*$.

LEMMA (5). *If D satisfies Conjecture 2 then so does D_*.*

Proof. Let $X = S^4 - D - \mathring{B}^4$ and $Y = B^4 - B^2_*$. Then $S^4 - D_* = X \cup Y$, and so we can use van Kampen's Theorem. X is contractible since it is a deformation retract of $S^4 - D$, and $X \cap Y$ is homeomorphic to the product of S^1 with an open disk. $\pi_1(Y)$ is generated by conjugates of the image of a generator of $\pi_1(X \cap Y)$, which is itself killed in $\pi_1(X)$. Therefore $\pi_1(X \cup Y) = 0$.

Remark (8). The purpose of the above description of the embedding $D \subset M^4$ was to illustrate the complexity involved in the statement "D is a spine of M^4". But it transpires in Theorem (8) that when we ask the next question "is M^4 a ball", only one detail is relevant, namely the linking of L in (1). And again when we come to "classify the M^4 with spine D" in Theorem (9) it transpires that we can choose a new spine D so as to eliminate all the complexities in (2) and (3), leaving only the knotting and linking of L in (1). Let us make precise definitions.

LOCAL NICENESS OF D

L consists of two circles α and γ joined by an arc β (see Fig. 5). Define an embedding $L \subset S^3$ to be *unlinked* if there exists an S^2 separating α and γ, and meeting β in one point (Fig. 8 and 9 show L linked). Notice that each circle may be separately knotted in its own hemisphere. We say that an embedding of D in a 4-manifold is *unlinked at the vertex* if, for some triangulation, the corresponding embedding $L \subset S^3$ is unlinked. By continuity the definition is independent of the triangulation.

We say that an embedding of D in a 4-manifold is *nice except at the vertex* if, for some triangulation, the handlebody structure of the second derived neighbourhood has the property that D_1, D_2 are unknotted in H_1, H_2, respectively; i.e. the complexities of (2) and (3) do not occur. Again the definition is independent of the triangulation, because the points where the local knotting of (2) and (3) occurs must be vertices of the triangulation, and the braiding of (2) is independent of the triangulation by continuity.

COROLLARY TO LEMMA (4). *If M^4 has a spine D that is nice except at the vertex then M^4 has a true handlebody structure, $M^4 = H_0 \cup H_1 \cup H_2$.*

Proof. By Theorem (7) there is a homeomorphism of M^4 onto the M^4 of Lemma (4). By definition D_2 in unknotted in H_2, and so H_2 is attached to $H_0 \cup H_1$ by an unknotted solid torus, and so is a true 2-handle.

THEOREM (8). *Let D be a spine of M^4. If D is unlinked at the vertex then M^4 is a ball.*

CONJECTURE (3). *The converse: if M^4 is a ball then D is unlinked at the vertex.*

Proof of the theorem. The unlinking of $L \subset S^3$ permits a refinement of the proof of Theorem (4). The proof of Theorem (4), as it stands, breaks down because α may be knotted, and so α may not span a disk. However we shall construct a 2-dimensional polyhedron A such that (the other symbols being the same as in Theorem (4)):

(i) $\alpha \subset A \subset S^3$;

(ii) $A \searrow x$;

(iii) $A \cap \beta = x$;

(iv) $A \cap \gamma = \phi$.

et B^3 be the hemisphere, given by the unlinking hypothesis, that contains α and does ıeet γ. Choose a (piecewise linear) homeomorphism $h: B^3 \to \Delta^3$ onto a simplex that /s the subarc $\beta \cap B^3$ onto a straight interval joining hx to a point on the boundary. $_*$ be the singular cone on $h\alpha$ with vertex hx, i.e. the union of all intervals joining hx ints of $h(\alpha - x)$, and let $A = h^{-1}A_*$. Then $A_* \searrow hx$ conewise, and $A_* \cap h(\beta \cap B^3) =$ Therefore A has the above four properties, which allow the proof of Theorem (7) ›n to follow the proof of Theorem (4), word for word.

Remark (9). Theorem (8) shows that if D is unlinked at the vertex, then the local ng at the vertex (of the circles α and γ) is irrelevant, from the point of view of the ›morphy type of the regular neighbourhood. On the other hand if D is linked at ›rtex, then the local knotting becomes important. For example if another knot is to the curve C in Mazur's example (Fig. 6) then $\pi_1(\dot{M}^4)$ changes, and so M^4 has ed. Similarly if D is linked at the vertex then the local knotting and braiding away he vertex affect M^4, but, as we observed in Remark (8) and will show in Theorem (9), complexity can be replaced by tieing a different knot at the vertex ,without changing

NORMAL FORM FOR M^4

'e now show that Mazur's example is indeed a prototype for all manifolds having a). Let C be a (tame) curve in $S^1 \times S^2$. Pick a base point $(x, y) \in S^1 \times S^2$ and call and $x \times S^2$ the first and second factors, respectively. Define the *algebraic index* λ o be the unique integer such that C is homotopic to λ times the first factor (we tly assume both curves to be oriented). Define the *geometric index* μ of C to be the umber of intersections that a curve ambient isotopic to C can have with the second Then by intersection theory, $|\lambda| \leqslant \mu$ and $\mu - \lambda$ is even. For example in Fig. 6, ınd $\mu = 3$.

MMA (6) (Gluck [8, Lemma 9.1]). *If C has geometric index* 1 *then it is isotopic to the* *tor.*

›of. The easiest way to see the proof is I believe due to Fox. Isotope C until it cuts ›nd factor once, and then slice along the second factor. Map the resulting $I \times S^2$ norphically onto a room minus an electric light bulb, the image of C being the which the bulb hangs from the ceiling. Although the flex looks knotted at first, e untied so as to hang straight.

EOREM (9). *Assume that M^4 is not a ball. Then the following three conditions are* *nt:*

(i) *M^4 has a spine D;*

(ii) *M^4 has a spine D that is nice except at the vertex;*

(iii) *M^4 is formed by attaching a 2-handle to $S^1 \times B^3$ by (some tubular neighbourhood of) a curve $C \subset S^1 \times S^2$ of algebraic index* 1 *and geometric index* 3.

Proof. (ii) implies (i) *a fortiori.* (iii) implies (ii) by Theorem (5), because the proof adapts immediately from the special case to the general case of an arbitrary curve of algebraic index 1 and geometric index 3. There remains to prove (i) implies (iii).

Let M^4 have spine D. By Theorem (7), Corollary (1), we can assume $M^4 = N(D, T'')$, the second derived neighbourhood of some embedding $D \subset S^4$, with respect to some triangulation T of S^4 containing D as a subcomplex. For this proof we use D ambiguously to denote both the dunce hat and this particular triangulation as a subcomplex of T. (We remark that the presence of S^4 in the proof is not significant.)

Let v be the barycentre of a 2-simplex of D. Then v is a vertex of T''. Define

the 4-ball,	$H_2 = \overline{st}(v, T'') \subset M^4$,
the 4-manifold,	$V = \overline{M^4 - H_2}$,
the disk,	$D_2 = D \cap H_2$,
and the curve,	$C = \dot{D}_2$.

Since $V \searrow \overline{D - D_2} \searrow e_0 \cup e_1 = S^1$, V is a regular neighbourhood of S^1 in S^4. Therefore $V \cong S^1 \times B^3$, because S^1 unknots in S^4. By a standard combinatorial argument (cf. [22, Chapter 9]) H_2 is a 2-handle with core D_2 attached to V by the neighbourhood $V \cap H_2$ of the curve C in $\dot{V}$. Therefore to complete the proof of Theorem (8), we only have to verify that the algebraic index λ and the geometric index μ of C are correct.

Now $M^4 = V \cup H_2$ has trivial homology since it is contractible, and so λ has to be ± 1 in order to kill the homology of V when attaching H_2, and we can choose $\lambda = +1$ by reorienting C if necessary. Therefore μ is odd. If $\mu = 1$, then by Lemma (6), H_2 is attached (up to isotopy) by the first factor of $S^1 \times S^2$, and so M^4 is collapsible and therefore a ball. Therefore $\mu \geqslant 3$, because by hypothesis we are assuming that M^4 is not a ball. Finally we must show $\mu \leqslant 3$.

Let w be the barycentre of a 1-simplex of D contained in e_1; choose three arcs in the dual 1-skeleton of D, with union U say, such that:

(i) each arc runs from v to w,

(ii) the arcs have disjoint interiors, and

(iii) $D - U$ is connected (see Fig. 10).

Then U is a 1-dimensional subcomplex of the first derived complex of D.

Let $W = U - st\,(v, U)$. Define

$$H_1 = N(W, T''), \qquad D_1 = D \cap H_1,$$

$$H_0 = \overline{V - H_1}, \qquad D_0 = D \cap H_0.$$

Then H_0, H_1 are 4-balls by Lemma (1), because both are manifolds and $H_1 \searrow W \searrow 0$, $H_0 \searrow D_0 \searrow 0$. Also

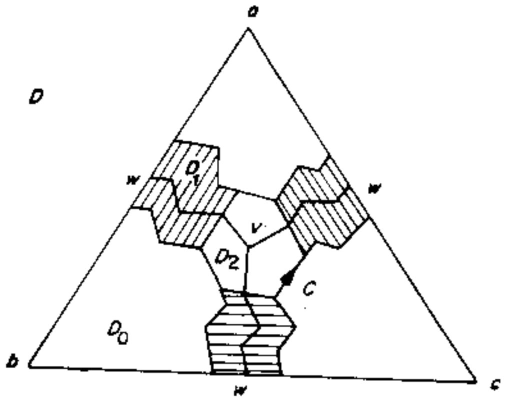

FIG. 10

because
$$H_0 \cap H_1 = \dot{H}_0 \cap \dot{H}_1 = N(D_0 \cap D_1, \dot{H}_1) \cong S^0 \times B^3,$$

$$D_0 \cap D_1 \cong S^0 \times (\text{cone on three points}) \searrow S^0.$$

Let S^2 denote the boundary of one of the components of $H_0 \cap H_1$. Then we can choose :he homeomorphism

$$V = H_0 \cup H_1 \xrightarrow{\cong} S^1 \times B^3$$

:o as to throw S^2 onto the second factor of the boundary $S^1 \times S^2$. Meanwhile

$$C \cap (H_0 \cap H_1) = C \cap (S^0 \times \dot{B}^3) = \dot{C} \cap (D_0 \cap D_1) = \text{six points},$$

f which three lie in each component. In other words C meets S^2 in three points. Therefore $\leqslant 3$, and the proof of Theorem (9) is complete.

Remark (10). By construction D_2 is unknotted in H_2. Therefore the decomposition $f^4 = H_0 \cup H_1 \cup H_2$ of the above proof is a true handlebody structure. Also one can iow that D_1 is unknotted in H_1 (using the methods of [9, Corollary 6]), because we iose U in the dual 1-skeleton of D, away from the vertices of T which are the only points here D can be locally knotted. Therefore to find the new spine that is nice except at the rtex it suffices to replace D_0 by a cone on $D \cap \dot{H}_0$ in the 4-ball H_0.

The following corollary has a bearing on Conjecture (2):

COROLLARY TO THEOREM (9). *Let $D \subset S^4$ be an arbitrary (tame) embedding. Then ere is another embedding of the dunce hat $D_* \subset S^4$ that is nice except at the vertex, and ere is a smooth embedding of the plane E^2 as a closed subset of $E^2 \times S^2$, such that*

$$S^4 - D \cong S^4 - D_* \cong E^2 \times S^2 - E^2.$$

Proof. $S^4 - D \cong S^4 -$(regular neighbourhood M^4 of D)

$\cong S^4 -$(any spine of M^4)

$= S^4 - D_*$, by Theorem (9),

$= (S^4 - S^1) - (D_* - S^1)$, where S^1 is the closure of the 1-cell of D_*,

$\cong E^2 \times S^2 - E^2$,

which can be given a smooth structure because it is locally unknotted.

A CANDIDATE FOR A FAKE 4-SPHERE

Let $D \subset S^4$ be an embedding. Let M^4 be a regular neighbourhood of D and let $V^4 = S^4 - M^4$.

CONJECTURE (4). $V^4 \times I = B^5$.

We cannot use the Mazur [12]–Poénaru Theorem to prove Conjecture (4) because in general V^4 may not have a 2-dimensional spine, only 3-spines. By the above corollary V^4 has a kind of 'negative' handlestructure

$$V^4 \cong B^2 \times S^2 - (\text{open 2-handle})$$

but this seems to be difficult to use. The evidence for Conjecture (4) is that the interior of $V^4 \times I$ is a 5-cell by a theorem of Stallings [17, Corollary (5.3)], and:

THEOREM (10) (Curtis [5, Theorem 1]).

(i) *Conjecture* 4 *implies Conjecture* 2;

(ii) *Conjecture* 2 *and the* 4-*dimensional Poincaré Conjecture imply Conjecture* (4).

Proof. (i) Conjecture (4) implies that V^4 is contractible, therefore $S^4 - D$, which is homeomorphic to $\dot{V}^4$, is also.

(ii) Let $F^4 = (V^4 \times I)^{\dot{}}$, or in other words the double of V^4. Conjecture (2) implies that V^4 is contractible and so F^4 is a homotopy sphere. We assume that the 4-dimensional Poincaré Conjecture says that F^4 is a combinatorial 4-sphere. Therefore we can glue a 5-ball onto $V^4 \times I$ and make a combinatorial manifold that is a homotopy 5-sphere. By the known 5-dimensional Poincaré Conjecture [15, 16, 20], this is a topological 5-sphere. Removing the ball again leaves a complementary (stellar) topological ball $V^4 \times I$ by Brown [3].

DEHN'S LEMMA IN 4 DIMENSIONS

In order to dispel the illusion that everything is known about manifolds with spine D, we conclude with an elementary conjecture, which would furnish a counter-example to Dehn's Lemma in simply connected bounded 4-manifolds, and show Irwin's Embedding Theorem [10] to be the best possible (with respect to codimension $\geqslant 3$).

CONJECTURE (5). *If M^4 has spine D and is not a ball, then there exists a curve in the boundary that cannot be spanned by a non-singular disk.*

Since M^4 is contractible, the curve can always be spanned by a singular disk, and any /o spanning disks are homotopic keeping the curve fixed. A suitable looking curve in [azur's example would be the first factor of $S^1 \times S^2$ (marked S^1 in Fig. (6)). It is easy span this S^1 by a locally unknotted disk with one self-intersection, but any attempt to move the singularity seems to repeat the original problem.

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INTERPOLATING MANIFOLDS FOR KNOTS IN S^3

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§1. INTRODUCTION

THE SEMI-LINEAR imbedding of a 1-sphere in the 3-sphere raises many questions both algebraic and topological in nature. We are particularly interested in the interplay between these questions. There are many theorems in the knot theory literature which make topological assumptions, and draw algebraic conclusions (of course the contrapositive statements of those theorems make algebraic assumptions, and draw topological conclusions). Examples of this type of theorem may be found in [1, 2, 3, 4 and 5], where the geometric assumptions involve: the existence of an alternating projection [1], the bounding of a locally flat disc in a half space in E^4 [2], the crookedness of a knot type [3], the minimal number of changes of overcrossings to undercrossings to unknot [4], the genus [5]. The algebraic conclusions involve: the Alexander polynomial [1, 2, 5], the minimal number of generators needed to generate the fundamental group of the complement of the knot [3], the minimal number of generators of the abelianized kernel of the homomorphism from the fundamental group of the complement of the knot onto Z_n [4]. Of course these results by no means exhaust the list of such theorems; left out, for example, are many partial results on the Smith problem (no non-trivial knot is the fixed point set of a periodic homeomorphism of S^3).

The question investigated here may be stated in a purely algebraic form, which is expressed as follows.

CONJECTURE A: *The fundamental group of the complement of a non-trivial polygonal knot in the 3-sphere is a non-trivial free product with amalgamation, and the amalgamating subgroup is free.*

The source of this conjecture is geometric, and will be explained in the next section. In fact the conjecture will be strengthened, and expressed in geometric form.

We have not been able to prove or disprove this conjecture. The main theorem presented here makes a geometric assumption (which may have other applications) in order to obtain the algebraic conclusion stated in the conjecture above.

The geometric aspects of the problem addressed here were also considerations of Aumann in [6], although his work was in a different direction, for he wished to prove the