MATH. SCAND. 26 (1970), 5–13

ON HOMOTOPY INVARIANCE OF THE TANGENT BUNDLE I

JOHAN L. DUPONT

1. Introduction.

A well-known result of M. F. Atiyah [2] states that if M and M' are compact q-dimensional oriented differentiable manifolds and $f: M \to M'$ is an orientation preserving homotopy equivalence, then $f^*\tau'$ is stably fibre homotopy equivalent to τ. Here τ and τ' denote the tangent sphere bundles of M and M' respectively. The problem to be studied in this note is, whether the word "stably" can be cancelled in the above statement.

This was partly done by W. A. Sutherland [16] and we follow the line of his paper. Only we use a method of "least indeterminacy" introduced by W. Browder [5] to define an invariant $b(\xi)$ for certain $(q-1)$-dimensional sphere bundles ξ over M^q, q odd and different from $1,3,7$. This invariant is a substitute for the Euler class in the even case. Unfortunately I am *not* able to show that $b(\xi)$ only depends on ξ except in the case, where $\tau(M^q)$ is stably homotopy trivial, in which case this is a consequence of the solution of the Hopf invariant one problem. Therefore, this paper only gives new information for $q=2^i-1$; but nevertheless I still hope to solve the general case by the same method.

At last we remark that $b(\tau)=\chi^*(M^q)$, the semi-characteristic introduced by M. Kervaire [8].

2. Stably equivalent bundles over a manifold.

$\mathrm{SH}(n)$ is the space of maps $S^{n-1} \to S^{n-1}$ of degree $+1$, and $B_n=\mathrm{BSH}(n)$ is the classifying space, defined by J. Stasheff [14], for oriented $(n-1)$-dimensional sphere bundles.

Consider a q-dimensional manifold M and an embedded disk $D^q \subset M$ with boundary S^{q-1}. According to J. Milnor [10, § 8] the triad

After this was written it was pointed out to me that the problem was solved in general by different methods by René Benlian and John Wagoner in C. R. Acad. Sci. Paris Série A–B 265 (1967), A 207–A 209.

Received June 27, 1968.

$(M \setminus D^\circ, S^{q-1}, \emptyset)$ can be given a "self-indexing" Morse function with no points of index q. So $N = M \setminus D^\circ$ has the homotopy type of a $(q-1)$-dimensional CW-complex L, and M has the homotopy type of $K = L \cup_\theta e^q$, where $\theta\colon S^{q-1} \to N$ is the injection, and e^q denotes a q-cell.

Pinching $S^{q-1} \subset M$ defines a map $c\colon M \to M \vee S^q$, which defines an action of $\pi_q(X)$ on $[M,X]$ for any space X (see P. Hilton [6, chapter XV]): If $v \in [M,X]$ and $\mu \in \pi_q(X)$, we denote the composite

$$M \xrightarrow{c} M \vee S^q \xrightarrow{v \vee \mu} X \vee X \xrightarrow{\nabla} X$$

by $v^\mu \in [M,X]$. This action has the property that if $v_1, v_2 \in [M,X]$ such that v_1 and v_2 restricted to N are homotopic, then there exists $\mu \in \pi_q(X)$ such that $v_2 = v_1{}^\mu$ in $[M,X]$.

Furthermore let $u\colon N \to X$; W. D. Barcus and M. G. Barratt [3, § 2] have defined a map $\alpha_u\colon \pi_1(X^N, u) \to \pi_q(X)$ such that if $v\colon M \to X$ is an extension of u, then $v^\mu = v$ iff $\mu \in \operatorname{Im} \alpha_u$.

Now consider two stably equivalent $(q-1)$-dimensional oriented sphere bundles ξ_1, ξ_2 over M^q.

Proposition 2.1. *Let v_1 and v_2 be the classifying maps from M into B_q. Then $v_2 = v_1{}^{\mu_0}$ in $[M, B_q]$, where $\mu_0 \in \pi_q(B_q)$ is in the kernel of*

$$j_* : \pi_q(B_q) \to \pi_q(B_{q+1}) .$$

Here $j\colon B_q \to B_{q+1}$ is the natural inclusion.

Proof. Restricting to N, v_1 and v_2 become homotopic, because N has the homotopy type of a $(q-1)$-dimensional complex. Therefore, $v_2 = v_1{}^\mu$, where $\mu \in \pi_q(B_q)$. Without loss of generality we can assume v_1 and v_2 restricted to N to equal a map $u\colon N \to B_q$.

Then $j_* v_2 = (j_* v_1)^{j_*\mu}$, and according to the q-dimensionality of M, $j_* v_2 = j_* v_1$, because ξ_2 and ξ_1 are stably equivalent. So $j_* \mu \in \operatorname{Im} \alpha_{ju}$. It follows easily (e.g. by using Theorem 22 in Spanier [11, chapter VII, § 6]) that

$$j_\# : \pi_1(B_q{}^N, u) \to \pi_1(B_{q+1}^N, ju) \quad \text{is onto} .$$

Using this,

$$j_* \mu = \alpha_{ju}(z) ,$$

where $z = j_\#(x)$ for some $x \in \pi_1(B_q{}^N, u)$. Let $\mu' = \alpha_u(x)$ and $\mu_0 = \mu - \mu' \in \pi_q(B_q)$. Then

$$v_2 = v_1{}^\mu = (v_1{}^\mu)^{-\mu'} = v_1{}^{\mu - \mu'} = v_1{}^{\mu_0} .$$

Finally

$$j_* \mu' = j_* \alpha_u(x) = \alpha_{ju} j_\#(x) = j_* \mu ,$$

so $j_* \mu_0 = 0$.

The following proposition is a well-known consequence of the solution of the Hopf invariant one problem:

PROPOSITION 2.2. *The kernel of* $j_*: \pi_q(B_q) \to \pi_q(B_{q+1})$ *is cyclic generated by* τ_q, *the map classifying the tangent bundle of* S^q.

$$\operatorname{Ker} j_* = \begin{cases} \mathsf{Z} & q \ \ even \\ 0 & q=1,3,7 \\ \mathsf{Z}_2 & q \ \ odd \quad q \neq 1,3,7 \,. \end{cases}$$

Returning to the problem stated in § 1, let $\xi_1 = \tau$, $\xi_2 = j^* \tau'$. If v_1 and v_2 are the classifying maps, $v_2 = v_1^{\mu_0}$, $\mu_0 \in \ker j_*$. So in case $q=1,3,7$ it is trivial that ξ_1 and ξ_2 are equivalent. Now consider the even case. Letting $E(v_i)$ denote the Euler class of ξ_i evaluated on the orientation class, it is easily shown that $E(v_2) = E(v_1) + E(\mu_0)$ for $v_2 = v_1^{\mu_0}$. Hence in the present case $E(\mu_0) = 0$ and consequently $\mu_0 = 0$, so also in this case τ and $f^*(\tau')$ are fibre homotopy equivalent.

3. Definition of $b(\xi)$ in the stably trivial case.

In this section M is a q-dimensional manifold, q odd and different from 1,3,7, and we assume $\tau(M)$ to be stably fibre homotopy trivial. Let ξ denote a stably fibre homotopy trivial bundle over M. This means that for k large there exists a Thom map

$$\eta : S^{2q+k} \to T(\nu_M) \cong \Sigma^{k+q}(M_+) \cong T(\xi + k) ,$$

that is, a map inducing isomorphism by taking $H_{2q+k}(\cdot, \mathsf{Z})$. Let

$$U_\xi : T(\xi) \to K(\mathsf{Z}_2, q)$$

denote the map representing the Thom class. Put

$$\delta = \Sigma^k U_\xi \circ \eta : S^{2q+k} \to \Sigma^k K(\mathsf{Z}_2, q) .$$

DEFINITION 3.1. $b(\xi) = Sq_\delta^{q+1}(\Sigma^k \iota) \in H^{2q+k}(S^{2q+k}, \mathsf{Z}_2) = \mathsf{Z}_2$.

Here $\iota \in H^q(K(\mathsf{Z}_2, q))$ and $b(\xi)$ is defined as the functional Steenrod square. (See W. Browder [5, § 1].)

PROPOSITION 3.2. $b(\xi)$ *is independent of the choice of* k *and*

$$\eta : S^{2q+k} \to T(\xi + k) .$$

PROOF. Suspending η we evidently get the same $b(\xi)$, and thus it is sufficient to prove the last part of the proposition.

Let $\eta' : S^{2q+k} \to T(\xi+k)$ be another map of degree one. Then $\eta = \eta' + \gamma$ in $\pi_{2q+k}(\Sigma^k T(\xi))$, where $\gamma : S^{2q+k} \to \Sigma^k T(\xi)$. Consider the cofibration

$$T(\xi_{|N}) \xrightarrow{i} T(\xi) \xrightarrow{p} S^{2q} .$$

Using the covariant Puppe sequence for stable homotopy, we see that $\gamma = (\Sigma^k i) \circ \gamma'$ for k large, where $\gamma' : S^{2q+k} \to T(\xi_{|N})$. In fact $(\Sigma^k p) \circ \gamma$ has degree 0.

$\xi_{|N}$ is trivial and the trivialisation defines a map $g : T(\xi_{|N}) \to S^q$. Let $l : S^q \to K(\mathbf{Z}_2, q)$ be such that $l^* \iota = \sigma_q$, the generator of $H^q(S^q)$. Then

$$U_\xi \circ i = l \circ g : T(\xi_{|N}) \to K(\mathbf{Z}_2, q) .$$

Putting

$$\delta = \Sigma^k U_\xi \circ \eta, \qquad \delta' = \Sigma^k U_\xi \circ \eta', \qquad \beta = (\Sigma^k g) \circ \gamma' ,$$

we have

$$\delta = \delta' + (\Sigma^k l) \circ \beta .$$

Hence by Lemma 1.6 in Browder [5]

$$Sq_\delta{}^{q+1} \iota = Sq_{\delta'}{}^{q+1} \iota + Sq^{q+1}_{\Sigma^k l \circ \beta} \iota .$$

According to Adams [1]

$$Sq^{q+1}_{\Sigma^k l \circ \beta} \iota = Sq_\beta{}^{q+1} (\Sigma^k \sigma_q) = 0$$

when q is odd $\neq 1, 3, 7$.

Proposition 3.3. *Let ξ_1, ξ_2 be stably fibre homotopy trivial bundles over M, and ζ a third such over S^q. Assume the classifying maps v_1, v_2 and μ_0 satisfy $v_2 = v_1{}^{\mu_0}$.*

Then $b(\xi_2) = b(\xi_1) + b(\zeta)$.

Proof. If $c : M \to M \vee S^q$ is the pinching map and $\xi_1 \vee \zeta$ denotes the bundle over $M \vee S^q$, which is ξ_1 over M and ζ over S^q, then $c^*(\xi_1 \vee \zeta) = \xi_2$ and so there is a natural map

$$h : T(\xi_2 + k) \to T((\xi_1 + k) \vee (\zeta + k)) .$$

This is equivalent to a cofibration with cofibre S^{2q+k}, which we for convenience write

$$T(\xi_2 + k) \xrightarrow{h} T((\xi_1 + k) \vee (\zeta + k)) \xrightarrow{p} S^{2q+k} .$$

By the Thom isomorphism theorem,

$$H_{2q+k}(T(\xi_2 + k), \mathbf{Z}) = \mathbf{Z}(d_2) ,$$
$$H_{2q+k}((\xi_1 + k) \vee (\zeta + k)) = \mathbf{Z}(d_1) \oplus \mathbf{Z}(d_0) ,$$

and

$$h_* d_2 = d_1 + d_0 .$$

Consider

$$\begin{array}{ccccc} S^{2q+k} & \xrightarrow{\Delta} & S^{2q+k} \vee S^{2q+k} & \xrightarrow{\eta_1 \vee \eta_0} & T(\xi_1+k) \vee T(\zeta+k) \\ {\scriptstyle\gamma}\big\downarrow & & & & \big\downarrow{\scriptstyle g} \\ & & \longrightarrow & & T((\xi_1+k) \vee (\zeta+k)) . \end{array}$$

Here Δ is the usual pinching map.

$$\eta_1 : S^{2q+k} \to T(\xi_1+k), \qquad \eta_0 : S^{2q+k} \to T(\zeta+k)$$

are maps of degree one, and g is induced by the map

$$M \cup S^q \to M \vee S^q .$$

Under this map g the bottom cells of the two Thom complexes are identified. Further

$$\gamma_* \sigma_{2q+k} = d_1 + d_0 ,$$

so $p \circ \gamma$ has degree 0. Again by the covariant Puppe sequence there exists a map

$$\eta_2 : S^{2q+k} \to T(\xi_2+k)$$

such that $h \circ \eta_2 = \gamma$. Clearly η_2 has degree one. The proposition follows from an easy computation using the commutative diagram:

$$\begin{array}{ccccc} S^{2q+k} \vee S^{2q+k} & \xrightarrow{\eta_1 \vee \eta_0} & \Sigma^k T(\xi_1) \vee \Sigma^k T(\zeta) & \xrightarrow{\Sigma^k U_{\xi_1} \vee \Sigma^k U_\zeta} & \Sigma^k K(\mathbf{Z}_2,q) \vee \Sigma^k K(\mathbf{Z}_2,q) \\ {\scriptstyle\Delta}\big\uparrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow \\ & \xrightarrow{h} & \Sigma^k T(\xi_1 \vee \zeta) & \xrightarrow[\Sigma^k U_{\xi_1 \vee \zeta}]{} & \Sigma^k K(\mathbf{Z}_2,q) \\ & & & & \big\uparrow \\ S^{2q+k} & \xrightarrow[\eta_2]{} & T(\xi_2+k) & \xrightarrow[\Sigma^k U_{\xi_2}]{} & \end{array}$$

PROPOSITION 3.4. *Let M be a q-dimensional compact differentiable manifold, q odd $\neq 1,3,7$. If the tangent sphere bundle τ is stably fibre homotopy trivial, then*

$$b(\tau) = \chi^*(M) \equiv \sum_{i=0}^{(q-1)/2} \dim H^i(M, \mathbf{Z}_2) \pmod 2 .$$

PROOF. Choosing a tubular neighborhood of the diagonal in $M \times M$ we get a map

$$j : M \times M \to T(\tau)$$

by pinching everything outside. Let $\sigma_M \in H^q(M, \mathbf{Z}_2)$ be the generator.

Following E. Thomas [17, § 4] we choose a basis $\{\alpha_1,\ldots,\alpha_d,\beta_d,\ldots,\beta_1\}$ for $H^*(M,\mathbf{Z}_2)$ such that if

$$\deg \alpha_i + \deg \beta_j = q ,$$

then

$$\alpha_i \cup \beta_j = \delta_{ij}\sigma_M ,$$

where

$$d \equiv \chi^*(M) \quad (\mathrm{mod}\, 2) .$$

Let $t\colon M\times M \to M\times M$ be the transposition map, and put

$$A = \sum_{i=1}^{d} \alpha_i \otimes \beta_i \in H^q(M\times M) .$$

Then

$$A \cup t^*A = d\sigma_{M\times M} \quad \text{and} \quad j^*U_\tau = A + t^*A .$$

Now let $g\colon S^{q+k} \to \Sigma^k M_+$ be a map of degree one. The map

$$g \wedge g : S^{2q+2k} \to \Sigma^{2k}(M\times M_+)$$

defines an $\boldsymbol{S}$-orientation of $M\times M$, in the sense of Browder [5, § 1] ($\boldsymbol{S}$ is the sphere cospectrum). This defines an operation

$$\psi : H^q(M\times M) \to \mathbf{Z}_2 .$$

The map

$$\eta : S^{2q+2k} \xrightarrow{g\wedge g} \Sigma^{2k}(M\times M_+) \xrightarrow{\Sigma^{2k}j} \Sigma^{2k}(T(\tau))$$

has clearly degree one. By Theorem 1.4 in Browder [5], it follows that

$$\begin{aligned} b(\tau) = \psi(j^*U_\tau) &= \psi(A+t^*A) \\ &= \psi(A) + \psi(t^*A) + (g\wedge g)^*(A\cup t^*A) . \end{aligned}$$

From the commutativity of the diagram

$$\begin{array}{ccc} S^{2q+2k} & \xrightarrow{g\wedge g} & \Sigma^{2k}(M\times M_+) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \Sigma^{2k}t} \\ S^{2q+2k} & \xrightarrow[g\wedge g]{} & \Sigma^{2k}(M\times M_+) \end{array}$$

it follows that $\psi(A)=\psi(t^*A)$. Hence

$$b(\tau) = (g\wedge g)^*(A\cup t^*A) = d\sigma_{2q+2k} .$$

COROLLARY 3.5. *Let ζ be a stably fibre homotopy trivial $(q-1)$-dimensional sphere bundle over S^q. $\tau_q = \tau(S^q)$. Then*

$$\zeta = \tau_q \Leftrightarrow b(\zeta) = 1, \qquad q \text{ odd} \neq 1,3,7 .$$

PROOF. $b(\tau_q)=1$ according to Proposition 3.4. It follows from Proposition 2.2 that either ζ is trivial or $\zeta=\tau_q$. If ζ is trivial, $b(\zeta)=0$ according to Proposition 3.3.

Using b in stead of E in the concluding remarks of § 2, we get

THEOREM 3.6. *Consider M, M' q-dimensional oriented compact differentiable manifolds, q odd $\neq 1,3,7$. Assume the stable fibre homotopy classes of the tangent sphere bundles are trivial. Let $f\colon M \to M'$ be an orientation preserving homotopy equivalence.*

Then $f^*\tau'$ and τ are fibre homotopy equivalent.

In fact, τ is fibre homotopy trivial iff $\chi^*(M)=0$.

Finally we remark that, if we use BSO(n) instead of BSH(n) in Section 2, then we can prove in the same way

THEOREM 3.7. *Consider M, M' and $f\colon M \to M'$ satisfying the conditions of Theorem 3.6. Let τ and τ' denote the tangent q-plane bundles of M and M' respectively, and assume further τ and $f^*\tau'$ to be stably isomorphic. Then τ and $f^*\tau'$ are isomorphic.*

Especially we get the following corollary implicitly proved in G. Bredon and A. Kosinski [4].

COROLLARY 3.8. *Let M and M' be π-manifolds and $f\colon M \to M'$ a homotopy equivalence. Then $f^*\tau'$ and τ are isomorphic.*

In fact, for q odd $\neq 1,3,7$ τ is trivial iff $\chi^*(M)=0$.

4. Remarks concerning the general case.

In this section we will explain the difficulty in the general case. First we recall some notation of W. Browder [5]: We assume q odd.

$$v_{q+1}: \; B_n \to K(\mathsf{Z}_2, q+1)$$

represents the Wu class v_{q+1}. Consider the fibration

$$B_n\langle v_{q+1}\rangle \xrightarrow{\pi} B_n$$

induced by v_{q+1} from the path fibration over $K(\mathsf{Z}_2, q+1)$ with fibre

$$\Omega K(\mathsf{Z}_2, q+1) \;=\; K(\mathsf{Z}_2, q)$$

and let $\bar{\gamma}_n = \pi^*(\gamma_n)$ denote the pull back of the universal sphere bundle over B_n. Further $Y_n = T(\bar{\gamma}_n)$ defines a Wu spectrum, and $\{X_n\}$ is the dual Wu cospectrum.

Now consider M^q a compact differentiable oriented manifold with normal bundle ν and let ξ be an oriented $(q-1)$-dimensional sphere bundle over M^q. Choose a bundle ξ' such that $\xi+\xi'$ is trivial (such one exists according to M. Spivak [13] or C. T. C. Wall [18]), and assume that the classifying map $\varphi: M \to B_n$ (n large) for $\nu+\xi'$ is given a specific lifting φ' through π. Then $\varphi=\pi\varphi'$ and $\nu+\xi'=\varphi'^*(\bar{\gamma}_n)$. This defines maps $T(\nu+\xi') \to Y_n$ and thus dual maps

$$X_{-2q-k} \xrightarrow{g_k} \Sigma^k T(\xi)$$

for k large such that

$$g_{k*}: H_{2q+k}(X_{-2q-k}, \mathsf{Z}) \to H_{2q+k}(\Sigma^k T(\xi), \mathsf{Z})$$

is an isomorphism. Such a system of maps we call an $\boldsymbol{X}$-orientation for ξ.

Let $U_\xi \in H^q(T(\xi), \mathsf{Z}_2)$. Assume $g_k^*(\Sigma^k U_\xi)=0$. (Using S-duality, see E. Spanier [12], this is seen to be equivalent to the following condition: If $i_1+\ldots+i_s=q$, then $w_{i_1}(\nu+\xi')\cup\ldots\cup w_{i_s}(\nu+\xi')=0$. This is clearly fullfilled if ξ is stably equivalent to τ.) Consider the map

$$\delta = \Sigma^k U_\xi \circ g_k : X_{-2q-k} \to \Sigma^k K(\mathsf{Z}_2, q)$$

and define

$$b_g(\xi) = Sq_\delta^{q+1}(\Sigma^k \iota) \in H^{2q+k}(X_{-2q-k}) = \mathsf{Z}_2 .$$

As in Browder [5] the indeterminacy is 0. A priori $b_g(\xi)$ might depend on the orientation g_k. In fact it does for $q=1,3,7$.

In turn the orientation depends on the following choices:

I a) ν and the trivialization of $\nu+\tau$.
  b) ξ' and the trivialization of $\xi+\xi'$.
II The lifting φ' of φ.

It turns out that II is not very serious, and the problem concerning I can be reduced to the following:

Let $\psi: M\times S^{k-1} \to M \times S^{k-1}$ be a fibre-homotopy equivalence. This induces a map

$$\alpha = T(id \oplus \psi): T(\xi+k) \to T(\xi+k),$$

and $g_k'=\alpha\circ g_k$ defines a new orientation for ξ.

If $b_{g'}(\xi)=b_g(\xi)$, then Section 3 goes through with minor changes, and proves the conclusions of Theorems 3.6 and 3.7 without the assumption on the stable fibre-homotopy class to be trivial. However, as pointed out by the referee there are examples where $\xi^{\tau_q}=\xi$ contradicting Proposition 3.3 but not 3.6 and 3.7. Is this the only case? Or stated in another way: If there is a map

$$X_{-2q-k} \to T(\xi + k_{|N})$$

with a non-zero functional Sq^{q+1}, is it true then that $\xi^{\tau_q} = \xi$? We will discuss this in a later paper.

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UNIVERSITY OF AARHUS, DENMARK

MATH. SCAND. 26 (1970), 200–220

ON HOMOTOPY INVARIANCE OF THE TANGENT BUNDLE II

JOHAN L. DUPONT

1. Introduction.

This paper is a subsequence of the paper [5], in which the following problem is considered.

Let M and M' be oriented compact, differentiable manifolds, let $f\colon M \to M'$ be a homotopy equivalence preserving orientation, and denote the tangent sphere bundles τ and τ' respectively. Is it true then, that τ and $f^*\tau'$ are fibre homotopy equivalent?

This is actually shown by R. Benlian and J. Wagoner [3]; but here we will prove it by the simple method developed in [5]. As kindly pointed out to me by C. T. C. Wall, this method also applies to define the unstable tangent sphere fibration for a Poincaré complex which is necessary for developing a theory for embedding and surgery of Poincaré complexes.

Finally I also want to thank M. F. Atiyah, W. Browder and W. Sutherland for interesting remarks on the note [5] which made this paper possible.

2. Sphere fibrations.

In this section we will study more closely the "action" defined in [5, § 2]. The results of this section are closely related to the work of James and Thomas [7], Rutter [8] and Barcus and Barratt [2]. In particular our Corollary 2.3 and Proposition 2.7 are reformulations of Theorem 1.8 in James and Thomas [7]. (Compare the remark following our Definition 4.6.)

As usual $H(n)$ denotes the space of homotopy equivalences of S^{n-1}, $SH(n)$ denotes the component of $H(n)$ consisting of maps of degree $+1$, and $F(n)$ denotes the subspace of $SH(n+1)$ consisting of basepoint preserving maps. There is a natural inclusion of $SH(n)$ in $F(n)$ by means of unreduced suspension.

Received January 31, 1969.

A Hurewicz fibration with fibre a homotopy $n-1$ sphere is called a $n-1$ sphere fibration. According to J. Stasheff [12] there is a space $B_n = BSH(n)$ classifying oriented $n-1$ sphere fibrations over CW-complexes, such that homotopy classes of the classifying maps are in one-to-one correspondence with equivalence classes of sphere fibrations under orientation preserving fibre homotopy equivalence.

If ξ_1 and ξ_2 are sphere fibrations over a space X, $\xi_1 + \xi_2$ denotes the fibrewise join of ξ_1 and ξ_2. If ξ_1 and ξ_2 are sphere fibrations over X_1 and X_2 respectively and $p_i\colon X_1 \times X_2 \to X_i$ are the projections, we put

$$\xi_1 \times \xi_2 = (p_1{}^* \xi_1) + (p_2{}^* \xi_2) .$$

The trivial $k-1$ sphere fibration is simply denoted by k.

Analogously there are spaces $BF(n)$ classifying pairs (ξ, s) consisting of an oriented n sphere fibration ξ and a section s. Homotopy classes of the classifying maps are in one-to-one correspondence with equivalence classes of pairs under section and orientation preserving fibre homotopy.

If ξ is an oriented sphere fibration and s and s' are homotopic sections, then the pairs (ξ, s) and (ξ, s') are clearly equivalent.

For any CW-complex X the natural map

$$[X, BF(n)] \to [X, BSH(n+1)]$$

corresponds to forgetting the section, and the map

$$[X, BSH(n)] \to [X, BF(n)]$$

corresponds to the map sending ξ to the pair $(\xi + 1, s_1)$, where s_1 is the section which is constantly 1.

LEMMA 2.1. *Let ξ be a q sphere fibration over a q-dimensional finite CW-complex X.*

Any section s of ξ gives rise to a $q-1$ sphere fibration ξ', such that $(\xi' + 1, s_1)$ and (ξ, s) are equivalent pairs. The equivalence class of ξ' only depends on the homotopy class of s.

PROOF. According to James [6], the map

$$j_*\colon \pi_i(SH(q)) \to \pi_i(F(q))$$

is an isomorphism for $i < 2(q-2)$ and an epimorphism for $i = 2(q-2)$. This, together with an easy calculation for $q = 2, 3$, implies that j_* is an isomorphism for $i \leqq q-1$. Hence the map

$$[X, BSH(q)] \to [X, BF(q)]$$

is bijective for X at most q-dimensional. This proves the lemma.

Especially consider $X = L \cup e^q$, where L is a $(q-1)$-dimensional complex (according to Wall [15] this is the case for a q-dimensional Poincaré complex), and let $\xi = \xi_0 + 1$, where ξ_0 is a $q-1$ sphere fibration.

By obstruction theory any section s of ξ is homotopic over L to the trivial section s_1 which is constantly 1. Extending this homotopy to X (Strøm [13]) we conclude that any homotopy class of sections of $\xi_0 + 1$ is representable by a section which is trivial over L. Trivializing ξ_0 over e^q, s defines a map

$$(e^q, S^{q-1}) \to (S^q, *)$$

of a certain degree $d(s)$.

Later in this section we will see that for ξ_0 oriented, $d(s)$ depends only on the homotopy class of s, and thus $d(s)$ determines this uniquely. (For ξ_0 non-orientable the homotopy class of s is determined by the mod 2 degree.)

For any integer d let g_d denote the composite map

$$X \xrightarrow{c} X \vee S^q \xrightarrow{1 \vee f_d} X \vee S^q,$$

where c is the pinching map and $f_d \colon S^q \to S^q$ is of degree d. Further let ξ_d denote the fibration

$$\xi_d = g_d{}^*(\xi_0 \vee \tau_q),$$

where τ_q is the tangent sphere bundle of S^q. If $v_0 \colon X \to B_q$ is classifying for ξ_0 and $\mu_0 \colon S^q \to B_q$ is classifying for τ_q, then in the notation of [5, Section 2] $v_0{}^{d\mu_0}$ is classifying for ξ_d. Clearly there is a natural equivalence

$$\xi_d + 1 = g_d{}^*((\xi_0 + 1) \vee (\tau_q + 1)) \cong g_d{}^*((\xi_0 + 1) \vee (q+1)) \cong \xi_0 + 1$$

which we denote by γ_d. Under this the constant section of $\xi_d + 1$ defines a section $\sigma(d)$ of $\xi_0 + 1$ of degree d. In fact the constant section of $\tau_q + 1$ over S^q has degree one with respect to the obvious trivialization.

Using Lemma 2.1 we clearly have

Proposition 2.2. *For any section s of $\xi_0 + 1$,*

$$\xi' = \xi_{d(s)} = g^*_{d(s)}(\xi_0 \vee \tau_q)$$

is the unique fibration such that

$$(\xi' + 1, s_1) \quad \textit{and} \quad (\xi_0 + 1, s)$$

are equivalent pairs.

Proof. In fact $\sigma(d(s))$ and s are homotopic sections of ξ_0+1.

An equivalence α of a fibration ξ with itself is called an *automorphism* of ξ. For any automorphism α of the fibration ξ_0+1, where ξ_0 is a sphere fibration over an arbitrary space X, we define the section $s_\alpha=\alpha\circ s_1$ of ξ_0+1. Here again s_1 denotes the trivial section, and clearly $s_{id}=s_1$.

We now obtain in the special case of $X=L\cup e^q$:

Corollary 2.3. *For q odd, we have $v_0^{\mu_0}=v_0$ iff there is an automorphism α of ξ_0+1 such that $d(s_\alpha)$ is odd.*

Proof. According to [5, Proposition 2.2], we have $v_0^{\mu_0}=v_0^{d\mu_0}$ for d odd. Hence $v_0=v_0^{\mu_0}$ iff $v_0=v_0^{d\mu_0}$ or equivalently $\xi_0\simeq\xi_d$ for some odd integer d.

If $\beta\colon \xi_0\to\xi_d$ is an equivalence, then the composite equivalence

$$\gamma_d\circ(\beta+1)\colon\ \xi_0+1\to\xi_0+1$$

defines the section $s_{\gamma_d\circ(\beta+1)}=\sigma(d)$ of degree d.

Conversely, if $\alpha\colon \xi_0+1\to\xi_0+1$ has $d(s_\alpha)=d$, then (ξ_0+1,s_1) and (ξ_0+1,s_α) are equivalent pairs, and hence we conclude from Proposition 2.2 that

$$\xi_0 \simeq \xi_{d(s_\alpha)} = \xi_d\,.$$

Turning to the general case of a $q-1$ sphere fibration ξ_0 over an arbitrary space X, we consider the Thom complex $T(\xi_0)$. This is defined as the mapping cone on the projection map, and it is easily seen to be homeomorphic to the space $\xi_0+1/s_1(X)$, in such a way that the inclusion $X\to T(\xi_0)$ in the mapping cone corresponds to the section s_{-1} of ξ_0+1 which is constantly -1.

When ξ_0 is oriented, the Thom class

$$U_{\xi_0}\in H^q(\xi_0+1,\mathsf{Z})$$

is the unique class which restricted to the fibre is the generator and which satisfies $s_1{}^*U_{\xi_0}=0$.

Definition 2.4. For any section s of ξ_0+1, put

$$d(s) = s^*U_{\xi_0}\in H^q(X,\mathsf{Z})$$

and for α an automorphism, put

$$\chi(\alpha) = d(s_\alpha)\in H^q(X,\mathsf{Z})\,.$$

As an example the equivalence α induced by multiplication by -1 in the trivial part 1 of ξ_0+1, has $\chi(\alpha)=e(\xi_0)$, the Euler class of ξ_0.

PROPOSITION 2.5. *For orientation preserving automorphisms α and β of ξ_0+1 we have*

$$\chi(\alpha\circ\beta) = \chi(\alpha)+\chi(\beta)\ .$$

PROOF. Put $u=U_{\xi_0}$ for short and denote the projection for ξ_0+1 by p. Obviously

$$\alpha^*u = u + p^*(d(s_\alpha))\ .$$

Hence

$$\begin{aligned} s^*_{\alpha\circ\beta}u = (\alpha\circ s_\beta)^*u &= s_\beta{}^*u+s_\beta{}^*p^*(d(s_\alpha)) \\ &= d(s_\beta) + d(s_\alpha)\ . \end{aligned}$$

For X a q-dimensional Poincaré complex Definition 2.4 agrees with the previously defined degree. In fact for any integer d, the degree of $\sigma(d)$ is d.

Notice that we could also have defined d and χ mod 2 for any sphere fibration. Then Proposition 2.5 is valid for all automorphisms.

In view of Corollary 2.3 only the mod 2 degree is essential for our purpose. We will thus restrict to $\mathbf{Z}_2$ coefficients in all cohomology groups for the rest of this paper, unless otherwise specified.

DEFINITION 2.6. Let ξ be a sphere fibration over a space X with base point x_0, and consider an automorphism α of ξ. Denoting the unit interval by I, consider $\xi\times I$ with the identifications

$$\begin{aligned} (x,1) &\sim (\alpha x,0) \quad \text{for} \quad x\in\xi\ , \\ (x,t) &\sim (x,t') \quad \text{for} \quad x\in\xi_{x_0} \text{ and } t,t'\in I\ . \end{aligned}$$

This defines a fibration denoted ξ_α over $X\times S^1/x_0\times S^1$.

Denote the Euler class by e, the suspension of X by ΣX, the suspension homomorphism by Σ, and the natural map of $X\times S^1/x_0\times S^1$ onto ΣX by j. We then have

PROPOSITION 2.7. *For any automorphism α of $\xi=\xi_0+1$, where ξ_0 is a $q-1$ sphere fibration, we have*

$$e(\xi_\alpha) = j^*\Sigma(\chi(\alpha))\ .$$

PROOF. The Euler class of ξ_α is the image under the transgression of the generator of $H^q(S^q,+)$. The transgression is the additive relation

$$H^q(S^q,+) \xrightarrow{\delta} H^{q+1}(\xi_\alpha, S^q) \xleftarrow{p^*} H^{q+1}(X \times S^1, x_0 \times S^1)\ ,$$

where p is the projection.

Obviously $e(\xi_0+1)=0$, so $e(\xi_\alpha)$ is in the image of j^*. Consider the commutative diagram with exact columns:

$$\begin{array}{ccccc}
H^q(S^q,+) & \xrightarrow{\delta} & H^{q+1}(\xi_\alpha, S^q) & \xleftarrow{p^*} & H^{q+1}(X\times S^1, x_0\times S^1) \\
\uparrow{\scriptstyle i^*} & & \uparrow & & \uparrow{\scriptstyle j^*} \\
H^q((\xi_0+1)\times 0, s_1(X)\times 0) & \xrightarrow{\delta} & H^{q+1}(\xi_\alpha, (\xi_0+1)\times 0) & \xleftarrow{p^*} & H^{q+1}(\Sigma X) \\
 & & \uparrow{\scriptstyle \delta'} & & \uparrow{\scriptstyle 0} \\
 & & H^q(\xi_0+1, S^q) & \xleftarrow{p^*} & H^q(X)\ .
\end{array}$$

It is easy to see that the lower p^* is an isomorphism, and hence $\delta'=0$.

By definition $i^*U_{\xi_0}$ is the generator of $H^q(S^q,+)$. Hence $(j^*)^{-1}e(\xi_\alpha)$ is the image of U_{ξ_0} under the additive relation on the middle row. Now

$$p\colon\ \xi_\alpha/(\xi_0+1)\times 0 \to \Sigma X$$

has a right inverse s_0 defined by

$$s_1\times \mathrm{id}\colon\ X\times I \to (\xi_0+1)\times I\ .$$

That is, $(j^*)^{-1}e(\xi_\alpha)$ is the image of U_{ξ_0} under the map

$$H^q((\xi_0+1)\times 0,\ s_1(X)\times 0) \to H^{q+1}(\xi_\alpha/(\xi_0+1)\times 0) \xrightarrow{s_0^*} H^{q+1}(\Sigma X)\ .$$

Define a space F as the quotient space of $(X\times I)\cup(\xi_0+1)$ with the identifications

$$(x,1) \sim s_\alpha(x) \quad \text{for } x\in X\ .$$

There is a map of triples

$$(F, (\xi_0+1)\cup X\times 0, X\times 0) \to (\xi_\alpha, (\xi_0+1)\times 0, s_1(X)\times 0)$$

defined by sending (x,t) to (s_1x,t). Hence we have the commutative diagram

$$\begin{array}{ccc}
H^q(\xi_0+1, s_1(X)) & \xrightarrow{\delta} & H^{q+1}(\xi_\alpha, (\xi_0+1)\times 0) \\
\downarrow & & \downarrow{\scriptstyle s_0^*} \\
H^q(\xi_0+1) & \xrightarrow{\delta} & H^{q+1}(\Sigma X)\ .
\end{array}$$

Here the lower δ is the connecting homomorphism for the pair (C_{s_α}, ξ_0+1), where $C_{s_\alpha}=F/X\times 0$ is the mapping cone on s_α. This proves Proposition 2.7.

We conclude this section with a lemma concerning homotopy of automorphisms. If ξ is a $n-1$ sphere fibration over X with base point x_0 and α is an automorphism of ξ, we have defined the fibration ξ_α over $X \times S^1/x_0 \times S^1$, the restriction of which to $X \times 0$ is ξ. Hence fibre homotopy classes of automorphisms of ξ are in one-to-one correspondence with homotopy classes of maps

$$X \times S^1/x_0 \times S^1 \to BH(n)$$

the restriction of which to $X \times 0$ is the classifying map for ξ.

LEMMA 2.8. *Let ξ be a q sphere fibration over a finite q-dimensional CW-complex, and α an automorphism of $\xi + k$, $k > 0$.*

Then α is homotopic to an automorphism of the form $\alpha' + id$, where α' is an automorphism of ξ.

PROOF. The map $BH(q+1) \to BH(q+k+1)$ is a $q+1$ equivalence according to James [6]. Hence the lemma follows from Spanier [9, Chapter 7, § 6, Theorem 22].

3. S-duality.

We shall need some simple lemmas concerning S-duality of Thom complexes. We refer to the papers of Atiyah [1], Spivak [11] and Wall [15] for the following fact:

Let M denote a q-dimensional Poincaré complex, with $(k-1)$-dimensional normal sphere fibration ν. If ξ and η are $n-1$ and $m-1$ sphere fibrations over M such that $\xi + \eta$ is trivial, then the diagonal $\Delta \colon M \to M \times M$ induces a map of Thom complexes

$$T(\nu + n + m) \to T(\nu + \eta) \wedge T(\xi) .$$

The composite with a Thom map

$$S^{q+k+n+m} \to T(\nu + n + m) \to T(\nu + \eta) \wedge T(\xi)$$

is a S-duality for $T(\nu + \eta)$ and $T(\xi)$.

PROPOSITION 3.1. *Let $i \colon M_1 \hookrightarrow M_2$ be an embedding of a closed manifold in another. Denote the normal bundle of M_1 and M_2 by ν_1 and ν_2 respectively, and the normal bundle of i by ν_0. Then the dual map of*

$$T(i^*\nu_2) \to T(\nu_2)$$

is the map

$$(M_2)_+ \to T(\nu_0)$$

which collapses everything outside a tubular neighbourhood of M_1 in M_2.

COROLLARY 3.2. *Let M be a closed manifold with normal bundle ν and angent bundle τ. Then the map*

$$T(\nu+\nu) \to T(\nu \times \nu)$$

induced by the diagonal $M \to M \times M$ is the dual of the map

$$(M \times M)_+ \to T(\tau)$$

which collapses everything outside a tubular neighbourhood of the diagonal.

PROOF OF PROPOSITION 3.1. Let N be a tubular neighbourhood of M_1 in M_2 with boundary $\dot{N}$. Clearly

$$T(\nu_1) \;=\; T(\nu_{2|N})/T(\nu_{2|\dot{N}})\,.$$

Embedding M_2 in S^n, for n large, the proposition follows from the commutative diagram

$$\begin{array}{ccccc}
 & \nearrow & T(\nu_1) & \xrightarrow{\;f_1\;} & T(\nu_0) \wedge T(\nu_{2|M_1}) \\
 & & \| & & \downarrow \\
S^n & \longrightarrow & T(\nu_{2|N})/T(\nu_{2|\dot{N}}) & \xrightarrow{\;f_2\;} & T(\nu_0) \wedge T(\nu_2) \\
 & & & & \uparrow \\
 & \searrow & T(\nu_2) & \xrightarrow{\;f_3\;} & ((M_2)_+) \wedge T(\nu_2)\,.
\end{array}$$

Here f_1, f_2 and f_3 are induced by the diagonals $M_1 \to M_1 \times M_1$, $N \to N \times M_2$ and $M_2 \to M_2 \times M_2$ respectively.

Now let M denote an arbitrary Poincaré complex with normal sphere fibration ν, and let ξ and η be sphere fibrations such that $\xi+\eta$ is trivial.

LEMMA 3.3. *If α and β are automorphisms of ξ and η respectively, such that the automorphism $\alpha+\beta$ of $\xi+\eta$ is fibre homotopic to the identity, then*

$$T(1+\beta)\colon\; T(\nu+\eta) \to T(\nu+\eta)$$

is the dual of

$$T(\alpha)\colon\; T(\xi) \to T(\xi)\,.$$

LEMMA 3.4. *For any automorphism α of ξ, there is an automorphism α' of the trivial $k-1$ sphere fibration for some $k>0$, such that $\alpha+\mathrm{id}$ and $\mathrm{id}+\alpha'$ are fibre homotopic automorphisms of $\xi+k$.*

LEMMA 3.5. *For any automorphism α of ξ there is an automorphism β of $\eta+k$, for some k, such that $\alpha+\beta$ is fibre homotopic to the identity.*

PROOFS. The proof of Lemma 3.3 is trivial. Adding η to ξ it suffices to prove Lemmas 3.4 and 3.5 for ξ trivial.

For ξ trivial the stable fibre homotopy class of α corresponds to a map $\Sigma M \to BH$, where $BH = \lim BH(n)$. Lemma 3.5 now follows by well-known arguments from the fact that $[\Sigma M, BH]$ is a group in one and only one way.

Finally 3.4 follows from 3.5.

For later reference we finally state without proof the following well-known fact.

LEMMA 3.6. *For M an n-dimensional Poincaré complex with normal $k-1$ sphere fibration ν, the composite map*

$$H^i(M) \xrightarrow{D} H_{n+k-i}(T(\nu)) \xrightarrow{\Phi} H_{n-i}(M)$$

of the S-duality homomorphism D and the Thom isomorphism Φ equals the Poincaré duality homomorphism. That is, $\Phi \circ D$ is cap product with the orientation class $[M]$.

4. Definition of $b(\xi)$.

We recall the notation of [5, § 4].

Assume q odd. The map

$$v_{q+1}\colon\ B_n \to K(\mathsf{Z}_2, q+1)$$

represents the Wu class v_{q+1}. Consider the fibration

$$\pi\colon\ B_n\langle v_{q+1}\rangle \to B_n$$

induced by v_{q+1} from the path fibration over $K(\mathsf{Z}_2, q+1)$ with fibre $\Omega K(\mathsf{Z}_2, q+1) = K(\mathsf{Z}_2, q)$. Put $\bar{\gamma}_n = \pi^* \gamma_n$, where γ_n is the universal $n-1$ sphere fibration over B_n. Then $Y_n = T(\bar{\gamma}_n)$ defines a Wu spectrum in the sense of Browder [4]. $\{X_n\}$ is the dual Wu cospectrum.

Now consider M a q-dimensional compact differentiable oriented manifold with normal bundle ν, and let ξ be any oriented $q-1$ sphere fibration over M. Choose a fibration η such that $\xi+\eta$ is trivial, and choose a lifting φ' through π of the classifying map φ for $\nu+\eta$.

Clearly $\nu+\eta = (\varphi')^*(\bar{\gamma}_n)$. This defines maps

$$T(\nu+\eta) \to Y_n$$

and thus dual maps

$$g_k\colon X_{-2q-k} \to \Sigma^k T(\xi)$$

for k large, such that

$$g_{k*}\colon H_{2q+k}(X_{-2q-k}, \mathbf{Z}) \to H_{2q+k}(\Sigma^k T(\xi), \mathbf{Z})$$

is an isomorphism. We say that g_k has degree one.

A system $g=\{g_k\}$ of maps constructed in this way is called an X-orientation for ξ.

In the following all homology and cohomology have $\mathbf{Z}_2$ coefficients.

Definition 4.1. Let $U_\xi \in H^q(T(\xi))$ be the Thom class. For a fixed orientation g of ξ satisfying

$$g_{k*}(\Sigma^k U_\xi) = 0$$

define the composite map

$$\delta = \Sigma^k h \circ g_k,$$

where $h\colon T(\xi) \to K(\mathbf{Z}_2, q)$ represents U_ξ, and put

$$b_g(\xi) = Sq_\delta^{q+1}(\Sigma^k \iota) \in H^{2q+k}(X_{-2q-k}) = \mathbf{Z}_2 .$$

Here Sq_δ^{q+1} is the functionalized Sq^{q+1} on δ. As in Browder [4] it is clear that the indeterminacy is 0, and that $b_g(\xi)$ is independent of k.

Lemma 4.2. *Let ξ be stably equivalent to a SO sphere bundle. Then*

$$g_k^*(\Sigma^k U_\xi) = 0$$

if

$$w_{i_1}(\nu+\eta) \cup \ldots \cup w_{i_s}(\nu+\eta) = 0 \quad \textit{for} \quad i_1+\ldots+i_s = q .$$

Proof. Here w_i denotes the ith Stiefel-Whitney class. Since U_ξ is the bottom class of $T(\xi)$, by S-duality

$$g_k^*(\Sigma^k U_\xi) = 0$$

iff

$$T(\varphi')_*\colon H_{q+n}(T(\nu+\eta)) \to H_{q+n}(T(\bar{\gamma}))$$

is zero. Now

$$\pi_*\colon H_q(B_n\langle v_{q+1}\rangle) \to H_q(B_n)$$

is injective. Hence we only need to see that

$$\varphi_*\colon H_q(M) \to H_q(B_n)$$

is zero. When φ factors through $BSO(n)$, this is clearly fulfilled when the Stiefel–Whitney numbers of $\nu+\eta$ are zero.

REMARK. The condition of 4.2 is fulfilled for q odd and ξ stably equivalent to τ, because $w_i(\nu+\nu) \neq 0$ only for i even.

A similar necessary and sufficient criterion in general needs the structure of $H^*(BSH, \mathbf{Z}_2)$. This is calculated by J. Milgram.

When ξ is X-orientable, the orientation depends on the following choices:

I a) ν and the Thom map for $T(\nu)$.
  b) η and the trivialization of $\xi+\eta$.
II The lifting φ' of φ.

First let us examine the choices according to I:

If ν' is equivalent to ν and η' is equivalent to η, a choice of equivalences β_1 and β_2 respectively defines the S-duality

$$S^N \to T(\nu+\eta) \wedge T(\xi+k) \to T(\nu'+\eta') \wedge T(\xi+k)$$

where the last map is $T(\beta_1+\beta_2)\wedge \mathrm{id}$. With respect to this S-duality an orientation $T(\nu'+\eta') \to T(\bar{\gamma}_n)$ defines the same X-orientation for $T(\xi+k)$ as the composite map

$$T(\nu+\eta) \xrightarrow{T(\beta_1+\beta_2)} T(\nu'+\eta') \to T(\bar{\gamma}_n)$$

does with respect to the original S-duality.

Another choice of ν' and η' thus amounts to a change of the S-duality

$$S^N \to T(\nu+\eta+\xi+k) \to T(\nu+\eta) \wedge T(\xi+k) \tag{4.1}$$

by automorphisms of ν and η.

Also, fixing ν and η, another choice of trivialization of $\xi+\eta$ just changes the S-duality map (4.1) by an automorphism of $\eta+\xi$.

Finally, according to Theorem 3.5 in Wall [15], another choice of Thom map changes the S-duality map (4.1) by an automorphism of ν.

Hence in all cases, a different choice according to I just changes the S-duality map (4.1) by an automorphism of $\nu+\eta+\xi+k$. Choosing η of sufficiently large dimension, it follows from Lemma 3.4 that this automorphism can be assumed to be of the form $id+\beta+\mathrm{id}$, where β is an automorphism of η only.

In this way we conclude from Lemma 3.3 that a different choice according to I is equivalent to

I′ Replace the orientation

$$g_k\colon X_{-2q-k} \to T(\xi+k)$$

by the orientation

$$g_k' = T(\alpha) \circ g_k ,$$

where $T(\alpha): T(\xi + k) \to T(\xi + k)$ is induced by an automorphism α of $\xi + k$.

LEMMA 4.3. *If $b_g(\xi)$ is independent of the choices* I, *it is also independent of the choices* II, *and hence independent of the choice of X-orientation for ξ.*

PROOF. If $\varphi': M \to B_n\langle v_{q+1}\rangle$ is a lifting of φ, the other lifting is homotopic to the composite φ'':

$$M \xrightarrow{c} M \vee S^q \xrightarrow{\varphi' \vee \iota} B_n\langle v_{q+1}\rangle \vee K(\mathbf{Z}_2, q) \xrightarrow{\nabla} B_n\langle v_{q+1}\rangle .$$

Here c is the pinching map, and ∇ the map folding $K(\mathbf{Z}_2, q)$ onto the fibre of π. Since $\nabla^* \bar{\gamma}_n$ is trivial over $K(\mathbf{Z}_2, q)$,

$$T(\nabla^* \bar{\gamma}_n) = T(\bar{\gamma}_n) \vee \Sigma^n(K(\mathbf{Z}_2, q)) .$$

Taking the dual it is clear that $\Sigma^n(K(\mathbf{Z}_2, q))$ gives no contribution to the functionalized Sq^{q+1}.

We now consider the change of orientation originating from I'. According to Lemma 2.8, we can assume that the automorphism α of $\xi + k$ (ξ a $q-1$ sphere fibration) is of the form $\alpha' + \mathrm{id}$, where α' is an automorphism of $\xi + 1$.

THEOREM 4.4. *Let ξ_0 be an X-orientable $q-1$ sphere fibration over M^q, q odd, and let α be an automorphism of $\xi_0 + 1$. Further choose an X-orientation g of ξ_0 and let g' denote the orientation defined by*

$$g_k' = T(\alpha + \mathrm{id}) \circ g_k$$

for k large. Then

$$b_g(\xi_0) - b_{g'}(\xi_0) = \chi(\alpha) .$$

COROLLARY 4.5. *The number $b_g(\xi_0)$ depends on the choice of X-orientation, iff every $q-1$ sphere fibration which is stably equivalent to ξ_0, automatically is equivalent to ξ_0.*

PROOFS. Corollary 4.5 clearly follows from Theorem 4.4, Corollary 2.3 and Definition 2.4.

For the proof of Theorem 4.4 it suffices, according to Proposition 2.7, to show that

$$b_g(\xi_0) - b_{g'}(\xi_0) = e(\xi_\alpha) .$$

In the stable track group $\{T(\xi_0), T(\xi_0)\}$ put

$$\gamma = T(\alpha) - \mathrm{id} .$$

Use the Puppe sequences for the cofibrations

$$S^q \to T(\xi_0) \xrightarrow{j} T(\xi_0)/T(\xi_{0|*})$$

and

$$T(\xi_{0|N}) \xrightarrow{i} T(\xi_0) \to S^{2q},$$

where N is homotopy equivalent to a $(q-1)$-dimensional complex, and $*$ is the base point of M. We then get a factorization of γ through j and i, that is, there is a stable element

$$\eta\colon\ T(\xi_0)/T(\xi_{0|*}) \to T(\xi_{0|N})$$

such that $\gamma = i \circ \eta \circ j$. It is easy to see that if γ is represented by the map

$$\gamma_k\colon\ \Sigma^k T(\xi_0) \to \Sigma^k T(\xi_0),$$

then

$$Sq_{\gamma_k}^{q+1}(\Sigma^k U_{\xi_0})$$

is well defined with zero indeterminacy, and furthermore

$$Sq_{\gamma_k}^{q+1}(\Sigma^k U_{\xi_0}) = b_g(\xi_0) - b_{g'}(\xi_0).$$

Put $T = \Sigma^k T(\xi_0)$ and $f = T(\alpha + \mathrm{id})$, where f is a map of ΣT into itself. Define $M_f = \Sigma T \times I$ with identifications

$$(x,1) \sim (f(x),0) \quad \text{and} \quad (*,t) \sim (*,t')$$

for $x \in \Sigma T$ and $t,t' \in I$. Clearly

$$M_f = \Sigma^k T(\xi_\alpha).$$

On the other hand, f is homotopic to the map

$$\Sigma T \xrightarrow{\Delta} \Sigma T \vee \Sigma T \xrightarrow{\mathrm{id} \vee \Sigma\gamma_k} \Sigma T \vee \Sigma T \xrightarrow{\nabla} \Sigma T,$$

where Δ is the pinching map and ∇ the folding map. Hence M_f is homotopy equivalent to $\Sigma T \times I$, with the identifications

$$(x,t,1) \sim \begin{cases} (x,2t,0) & \text{for} \quad t \leqq \frac{1}{2} \\ (\gamma_k x, 2t-1, 0) & \text{for} \quad t \geqq \frac{1}{2} \end{cases}$$

and $(*,s) \sim (*,s')$, where $x \in T$, $s,s' \in I$ and t is in the interval defining ΣT.

Let Y be the subspace of points with coordinates $(x,t,0)$ satisfying $t \geqq \frac{1}{2}$ or coordinates $(x,\frac{1}{2},s)$ satisfying $0 \leqq s \leqq 1$. Obviously Y is homeomorphic to ΣT. The image of the set

$$\{(x,t,s) \mid t \leqq \tfrac{1}{2}\}$$

in M_f/Y is homotopy equivalent to the space $\Sigma T \times S^1/(*) \times S^1$ whereas the image of the set

$$\{(x,t,s) \mid t \geqq \tfrac{1}{2}\}$$

is homotopy equivalent to $C_{\Sigma\gamma_k}$, the mapping cone on $\Sigma\gamma_k$.

In this way M_f/Y is homotopy equivalent to the space

$$\Sigma T \times S^1/(*) \times S^1 \cup C_{\Sigma\gamma_k},$$

where the base of the cone is $\Sigma T \times 0$ in $\Sigma T \times S^1$. Denoting the projection $M \times S^1/(*) \times S^1 \to M$ by π, we have

$$\Sigma T \times S^1/(*) \times S^1 = T(\pi^*(\xi_0 + k + 1)) .$$

There is a unique class

$$u \in H^{q+k+1}(M_f/Y)$$

such that the restriction to $\Sigma T \times S^1/(*) \times S^1$ is the bottom class. Let p be the natural map $M_f \to M_f/Y$. Then p^*u is the bottom class of

$$M_f = \Sigma^k(T(\xi_\alpha))$$

and

$$p^*\colon H^{2q+k+2}(M_f/Y) \to H^{2q+k+2}(M_f)$$

is the sum map $Z_2 \oplus Z_2 \to Z_2$. Now

$$Sq^{q+1}(\Sigma^k U_{\xi_\alpha}) = p^* Sq^{q+1} u .$$

In order to calculate $Sq^{q+1}u \in Z_2 \oplus Z_2$ we restrict to $T(\pi^*(\xi_0 + k + 1))$ and $C_{\Sigma\gamma_k}$ respectively.

Clearly Sq^{q+1} is zero in $T(\pi^*(\xi_0 + k + 1))$ so as an element in Z_2

$$\begin{aligned} Sq^{q+1}(\Sigma^k U_{\xi_\alpha}) &= Sq^{q+1}(i^*u) \\ &= Sq^{q+1}_{\Sigma\gamma_k}(\Sigma^{k+1} U_{\xi_0}), \end{aligned}$$

where $i\colon C_{\Sigma\gamma_k} \to M_f/Y$ is the inclusion. On the other hand

$$Sq^{q+1}(\Sigma^k U_{\xi_\alpha}) = \Sigma^k U^2_{\xi_\alpha} = \Sigma^k \Phi(e(\xi_\alpha)) ,$$

where

$$\Phi\colon H^*(M \times S^1/(*) \times S^1) \to H^*(T(\xi_\alpha))$$

is the Thom isomorphism. This ends the proof of Theorem 4.4.

Definition 4.6. If $b_g(\xi_0)$ is independent of the choice of X-orientation, we write $b(\xi) = b_g(\xi)$ for any $q-1$ sphere fibration, which is stably equivalent to ξ_0.

Remark. Theorem 4.4 shows that $b_g(\xi_0)$ is not independent of the choice of orientation, precisely in case there is an automorphism α of ξ_0+1 satisfying $\chi(\alpha) \neq 0$ or equivalently $e(\xi_\alpha) \neq 0$.

Now $e(\xi_\alpha)=w_{q+1}(\xi_\alpha)$, and the collection of stable fibrations over $M \times S^1/(*) \times S^1$ represented by ξ_α, where α is any automorphism of ξ_0+1, is the same as the collection of stable fibrations of the form $\pi^*\xi_0+\eta$, where π is the projection onto M and η is induced from a fibration over ΣM.

Hence $b(\xi_0)$ is not well defined iff there is a sphere-fibration η over ΣM satisfying

$$w_{q+1}(\pi^*\xi_0+\eta) = \sum_{i=0}^{q+1} \pi^* w_i(\xi_0) \cup w_{q+1-i}(\eta) \neq 0 .$$

This is the criterion of James and Thomas [7] saying that there is only one $q-1$ sphere fibration which is stably equivalent to ξ_0.

5. The invariance theorem.

We are now in the position to prove the following theorem.

THEOREM 5.1. *Let M and M' be closed q-dimensional differentiable manifolds with tangent sphere bundles τ and τ' respectively. If $f: M \to M'$ is an orientation preserving homotopy equivalence, then τ and $f^*\tau'$ are fibre homotopy equivalent.*

PROOF. This theorem is proved in [5] for q even and $q=1,3,7$, and according to Atiyah [1], τ and $f^*\tau'$ are at least stably equivalent. We know from Lemma 4.2 that τ is X-orientable in the sense of Definition 4.1. Hence we conclude from Corollary 4.5 that either τ and $f^*\tau'$ are in fact equivalent, or the invariant $b(\xi)$ is well defined for $q-1$ sphere fibrations which are stably equivalent to τ.

The theorem now follows as in [5] from the following two lemmas. Using the notation of [5] we have for q odd different from $1,3,7$:

LEMMA 5.2. *Let ξ_1 and ξ_2 be $q-1$ sphere fibrations over M with classifying maps v_1 and v_2 respectively, and let ζ be a stably trivial $q-1$ sphere fibration over S^q with classifying map μ.*

If $v_2=v_1{}^\mu$, then $b(\xi_2)=b(\xi_1)+b(\zeta)$, whenever $b(\xi_1)$ is defined and independent of orientation.

PROOF. Let η be a fibration such that $\eta+\xi_1$ is trivial, and choose an X-orientation of ξ_1 originating from a classifying map

$$\varphi\colon M \to B_n\langle v_{q+1}\rangle$$

for $\nu+\eta$.

Consider the commutative diagram

$$\begin{array}{ccccc} M \cup (*) & \xrightarrow{i} & M \cup S^q & \xrightarrow{\varphi \cup j} & B_n\langle v_{q+1}\rangle \cup (*) \\ \downarrow & & & & \downarrow \\ M & & \xrightarrow{\varphi} & & B_n\langle v_{q+1}\rangle , \end{array}$$

where i is the inclusion and j the collapsing map. Taking Thom complexes on the appropriate fibrations, we get the dual homotopy commutative diagram for k large:

$$\begin{array}{ccccc} X_{-2q-k} \vee S^{2q+k} & \xrightarrow{g_k \vee t} & T(\xi_1+k) \vee T(q+k) & \xrightarrow{r_1} & T((\xi_1+k) \vee (q+k)) \\ \uparrow \Delta & & & & \uparrow c_1 \\ X_{-2q-k} & & \xrightarrow{g_k} & & T(\xi_1+k) . \end{array}$$

Here Δ is the map which splits the top cell into two, and t is the inclusion of S^{2q+k} in

$$T(q+k) = S^{2q+k} \vee S^{q+k} .$$

The natural map $M \cup S^q \to M \vee S^q$ induces the map r_1 of Thom complexes

$$r_1\colon T(\xi_1+k) \vee T(q+k) \to T((\xi_1+k) \vee (q+k))$$

which identifies the bottom cells. The pinching map $c\colon M \to M \vee S^q$ induces

$$\bar{c}_1\colon T(\xi_1+k) \to T((\xi_1+k) \vee (q+k)) .$$

Analogously there are induced maps

$$r\colon T(\xi_2+k) \vee T(\zeta+k) \to T((\xi_2+k) \vee (\zeta+k)) ,$$
$$\bar{c}\colon T(\xi_2+k) \to T((\xi_1+k) \vee (\zeta+k)) .$$

The fact that ζ is stably trivial, shows that there is an equivalence α between ξ_1+k and ξ_2+k, such that there are the commutative diagrams

$$\begin{array}{ccc} T(\xi_1+k) & \xrightarrow{\bar{c}_1} & T((\xi_1+k) \vee (q+k)) \\ \downarrow T(\alpha) & & \downarrow \\ T(\xi_2+k) & \xrightarrow{\bar{c}} & T((\xi_1+k) \vee (\zeta+k)), \end{array}$$

$$\begin{array}{ccc} T(\xi_1+k) \vee T(q+k) & \to & T(\xi_1+k) \vee T(\zeta+k) \\ \downarrow r_1 & & \downarrow r \\ T((\xi_1+k) \vee (q+k)) & \to & T((\xi_1+k) \vee (\zeta+k)). \end{array}$$

Clearly $g_k' = T(\alpha) \circ g_k$ defines an X-orientation for ξ_2 in such a way that we have the commutative diagram

$$\begin{array}{ccccc} X_{-2q-k} \vee S^{2q+k} & & \longrightarrow & & T(\xi_1+k) \vee T(\zeta+k) \\ \uparrow \Delta & & & & \downarrow r \\ X_{-2q-k} & \xrightarrow{g_k'} & T(\xi_2+k) & \xrightarrow{\bar{c}} & T((\xi_1+k) \vee (\zeta+k)). \end{array}$$

The lemma now follows by an easy calculation as in [5, § 3].

Lemma 5.3. *Let $f\colon M \to M'$ be an orientation preserving homotopy equivalence of oriented q-manifolds with tangent sphere bundles τ and τ' respectively. If $b(\tau)$ is well defined, we have*

$$b(\tau) = b(\tau') .$$

Proof. Let $A \in H^q(M \times M)$ denote the element defined in the proof of [5, Proposition 3.4]. Also let

$$j\colon M \times M_+ \to T(\tau)$$

denote the map collapsing everything outside a tubular neighbourhood. Finally consider the twisting map

$$t\colon M \times M \to M \times M .$$

We know that $j^*U = A + t^*A$. The normal bundle $\nu \times \nu$ of $M \times M$ clearly satisfies $v_{q+1}(\nu \times \nu) = 0$. Accordingly we can find a map

$$\varphi\colon M \times M \to B_n\langle v_{q+1}\rangle$$

classifying $\nu \times \nu$. Obviously

$$\varphi \circ \Delta\colon M \to M \times M \to B_n\langle v_{q+1}\rangle$$

classifies $\nu + \nu$ over M. Hence we conclude from Corollary 3.2 that the corresponding X-orientation for τ is the composite $\Sigma^k j \circ g_k$, where

$$g_k\colon X_{-2q-k} \to \Sigma^k(M \times M_+)$$

is an X-orientation for $M \times M$ in the sense of Browder [4, § 1]. Hence $b(\tau)$ is the functionalized Sq^{q+1} on the map $\Sigma^k h \circ g_k$, where

$$h\colon M \times M \to K(\mathbf{Z}_2, q)$$

represents $A+t^*A$. Clearly

$$\Sigma^k(f\times f)\circ g_k\colon X_{-2q-k}\to M'\times M'$$

is an X-orientation for $M'\times M'$. If

$$h'\colon M'\times M'\to K(\mathsf{Z}_2,q)$$

represents the analogous element

$$A'+t^*A'\in H^q(M'\times M')\,,$$

we obviously have

$$(f\times f)^*(A'+t^*A')=A+t^*A\,,$$

and thus

$$h'\circ(f\times f)=h\colon M\times M\to K(\mathsf{Z}_2,q)\,.$$

Hence $b(\tau')$ is also the functionalized Sq^{q+1} on the map

$$\Sigma^k h'\circ\Sigma^k(f\times f)\circ g_k=\Sigma^k h\circ g_k\,.$$

This ends the proof of Lemma 5.3 and hence of Theorem 5.1.

Analogously using $BSO(n)$ instead of $BSH(n)$ we have the following theorem.

THEOREM 5.4. *Let $f\colon M\to M'$ be a homotopy equivalence of oriented q-manifolds with tangent q-plane bundles τ and τ' respectively. If $f^*\tau'$ and τ are stably isomorphic (as SO-bundles) then they are automatically isomorphic (as $SO(q)$-bundles).*

As a consequence of Theorem 5.1 we have according to Sutherland [14, Corollary 3.4]:

COROLLARY 5.5. *Let M and M' be oriented q-manifolds which are oriented homotopy equivalent and suppose $k\leqq\frac{1}{2}(q-1)$. Then M admits a k-field iff M' does.*

6. Connection with the semi-characteristic.

In this section we will show that under certain circumstances $b(\tau)=\chi^*(M)$, the semi-characteristic of M. This is defined by the formula

$$\chi^*(M)=\sum_{i=0}^{\frac{1}{2}(q-1)}\dim H^i(M,\mathsf{Z}_2)\bmod 2\,.$$

First we use $B_n=BO(n)$ for defining an X-orientation

$$g_k\colon X_{-2q-k}\to\Sigma^k(M\times M_+)$$

for M an arbitrary q-dimensional manifold. We assume q odd. Let ψ denote the operation introduced by Browder [4, § 1],

$$\psi\colon \operatorname{Ker}(g_k{}^*)^{q+k} \to \mathbf{Z}_2 .$$

Using the notation of Lemma 5.3 we have

$$b_g(\tau) = \psi(A + t^*A) .$$

LEMMA 6.1. *If* $[M]=0$ *in the non-oriented bordism ring, then*

$$\Sigma^k A \in \operatorname{Ker}(g_k{}^*)^{q+k} .$$

PROOF. Arguing as in the proof of Lemma 4.2 and using Lemma 3.6, we have to show that $\varphi_*(A \cap [M \times M]) = 0$, where $\varphi\colon M \times M \to B_n$ is the classifying map for $\nu \times \nu$, and $[M \times M]$ is the orientation class of $M \times M$. This is equivalent to show that

$$A \cup w_{i_1}(\nu \times \nu) \cup \ldots \cup w_{i_s}(\nu \times \nu) = 0$$

for all $i_1, \ldots, i_s$ satisfying $i_1 + \ldots + i_s = q$. Here of course w_i denotes the ith Stiefel–Whitney class. Now

$$A = \sum_{i=1}^{d} \alpha_i \otimes \beta_i$$

where $\{\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d\}$ is a basis for $H^*(M)$ satisfying

$$\alpha_i \cup \beta_j; = \delta_{ij} \sigma_M$$

for $\deg \alpha_i + \deg \beta_j = q$. Here σ_M denotes the top class of $H^*(M)$.

SUBLEMMA 6.2. *Let* $x, y \in H^*(M)$ *satisfy* $\deg x + \deg y = q$. *Then*

1) $(x \otimes y) \cup A \neq 0 \Leftrightarrow x \cup y \neq 0$ *for* $\deg x > \deg y$,
2) $(x \otimes y) \cup A = 0$ *for* $\deg x < \deg y$.

SUBPROOF. 1) Assume $\deg x > \deg y$. Write x and y as a sum of β_i's and α_i's respectively. Then

$$(x \otimes y) \cup A \neq 0$$

iff, for an odd number of times, x contains β_i and y contains α_i, iff $x \cup y \neq 0$.

2) is trivial.

Lemma 6.1 follows from the sublemma and the fact that

$$w_i(\nu \times \nu) = \sum_{j=0}^{i} w_j(\nu) \otimes w_{i-j}(\nu) .$$

We now use Theorem 1.4 in Browder [4] and the fact that

$$A \cup t^*A = \chi^*(M)\sigma_{M\times M} ,$$

where $\sigma_{M\times M}$ is the top class of $M \times M$, to conclude that

$$b_g(\tau) = \psi(A) + \psi(t^*A) + \chi^*(M) .$$

THEOREM 6.3. *If* $[M]=0$ *in the non-oriented bordism ring, then there is an* X*-orientation* g *for* τ, *such that*

$$b_g(\tau) = \chi^*(M) .$$

PROOF. We want to show that for some X-orientation for $M \times M$ it happens that $\psi(A)=\psi(t^*A)$.

Analogously to the construction of $B_n\langle v_{q+1}\rangle$ let

$$B_n' = B_n\langle v_{(q+1)/2}, \ldots , v_{q+1}\rangle$$

denote the total space of the fibration

$$\pi' \colon B_n' \to B_n$$

which kills the Wu classes $v_{(q+1)/2}, \ldots , v_{q+1}$. Put $\gamma_n'=(\pi')^*\gamma_n$, $Y_n'=T(\gamma_n')$, and denote the corresponding dual cospectrum by $X'=\{X_n'\}$. Clearly the Whitney sum map

$$B_n \times B_n \to B_n$$

lifts to a map

$$B_n' \times B_n' \to B_{2n}\langle v_{q+1}\rangle .$$

Hence the corresponding map of Thom complexes gives rise to a dual map of degree one:

$$h_k \colon X_{-2q-2k} \to X'_{-q-k} \wedge X'_{-q-k} .$$

Clearly the normal bundle ν of M^q has a classifying map $\varphi \colon M \to B_n'$. The map induced on Thom complexes defines a map

$$f_k \colon X'_{-q-k} \to \Sigma^k(M_+) .$$

Hence the composite map $(f_k \wedge f_k)\circ h_k$ defines an X-orientation for $M \times M$. We can thus use

$$X'_{-q-k} \wedge X'_{-q-k}$$

for computing the functionalized Sq^{q+1}, just we know that $\Sigma^k A$ (and $\Sigma^k(t^*A)$) goes to zero under $f_k \wedge f_k$. In that case $\psi(A)=\psi(t^*A)$, because the twisting map of $X'_{-q-k}\wedge X'_{-q-k}$ into itself has degree one.

Arguing as in the proof of Lemma 6.1 and Lemma 4.2, we need to require that

$$A \cup \big((w_{i_1}(\nu) \cup \ldots \cup w_{i_l}(\nu)) \otimes (w_{i_{l+1}}(\nu) \cup \ldots \cup w_{i_m}(\nu))\big) = 0$$

whenever $i_1 + \ldots + i_m = q$. According to the Sublemma 6.2, this is the case precisely when all Stiefel–Whitney numbers are 0. This ends the proof of Theorem 6.3.

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UNIVERSITY OF AARHUS, DENMARK