Singularities [*]

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1 Introduction

This article recounts the rather wonderful interaction of topology and singularity theory which began to flower in the 1960's with the work of Hirzebruch, Brieskorn, Milnor and others. This interaction can be traced back to the work of Klein, Lefschetz and Picard, and also to the work of knot theorists at the beginning of this century. It continues to the present day, flourishing and expanding in many directions. However, this is not a survey article, but a history; the events of our time are harder to see in perspective, harder to marshal into coherent order, and their very multitude makes it impossible to recount them all. Hence this interaction is followed forward in only a few directions. [1]

The reader may get a sense of the current state of affairs in singularity theory by browsing in the conference proceedings [Orl83, LSE95]. The focus of this article is singularities of complex algebraic varieties. Real varieties are omitted. Also omitted from this account is the area of critical points of

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[1] That I have attempted to do this at all is due to the prodding of my conscience and a list suggested by W. Neumann of some recent areas where topology has had an effect on singularity theory. He added, though, that "the task becomes immense...other people would probably come up with almost disjoint lists." The randomness of my efforts here should be readily apparent, and my apologies to those whose work is not mentioned.

differentiable functions, work initiated by Thom, Mather, Arnold and others; a survey of this subject can be found in the books [AGZV85, AGZV88, Arn93].

When two areas interact, ideas flow in both directions. Ideas from topology have entered singularity theory, where algebraic problems have been understood as topological problems and solved by topological methods. (In fact, often the crudest invariants of an algebraic situation are topological.) Conversely, ideas of singularity theory have traveled in the reverse direction into topology. Algebraic geometry supplies many interesting examples both easily and not so easily understood, and these provide a convenient testing ground for topological theories.

2 Knots and singularities of plane curves

In the 1920's and 30's there was much activity in knot theory as the new tools of algebraic topology were being applied; the fundamental group of the knot complement was introduced, as were the Alexander polynomial, branched cyclic covers, the Seifert surface, braids, the quadratic form of a knot, linking invariants, and so forth. Many clearly-written wonderful papers were produced on these subjects.

At the same time in algebraic geometry there was interest in understanding complex algebraic surfaces, in particular by exhibiting them as branched covers of the plane. This method is the analogous to the method in one dimension lower of projecting a curve to a line. The discriminant locus in the latter case is a set of points and it is easy to understand the branching. For surfaces the branching is more complicated since the discriminant locus is a curve.

A method of examining the branching problem for surfaces was proposed by Wirtinger in Vienna, who gave some seminars on this subject beginning in 1905. He divided branch points into two types: At a smooth point of the discriminant curve, the branching group ("Verzweigungsgruppe") of the surface is cyclic, like that of a curve. These points were called "branch points of type I". Singular points of the discriminant curve were called "branch points of type II". He also worked out a simple example.

The classification and the example were recorded by his student Brauner in the beginning of his paper "On the geometry of functions of two complex variables" [Bra28]. Wirtinger's example is the smooth surface in $\mathbf{C}^3$ given by the equation

$$z^3 - 3zx + 2y = 0$$

When this is projected to the (x, y)-plane, the discriminant curve is

$$x^3 - y^2 = 0$$

There is one point in the surface over the origin in the (x, y)-plane, two points over the remaining points of the curve $x^3 - y^2 = 0$, and three points over the rest of the plane.

To understand the type II branching of the surface near the origin, a three-sphere $\mathbf{S}^3_r$ of radius r about the origin in the plane was mapped to real three-space by stereographic projection. The image of the intersection of this three-sphere with the discriminant curve was then exhibited as a trefoil knot Γ (Figure 1). It sufficed to understand the branching of the surface over $\Gamma \subset \mathbf{S}^3_r$. Let A_i for $i = 1, 2, 3$ be the branching substitution produced by travelling around the loop labelled A_i in the figure. The A_i must satisfy the (now well-known) Wirtinger relation

$$A_0^{-1} A_1 A_0 A_2^{-1} = 1$$

at the left-hand crossing point of the knot projection in the figure. The only possibility for the permutation of the sheets of the covering is thus $A_0 = (12)$, $A_1 = (23)$ and $A_2 = (13)$. Hence the branching group in the neighborhood of $(0, 0)$ is not cyclic (as it is for plane curves), but the symmetric group on three elements.

Brauner concluded "Wir haben aus obigem erkannt, dass es die topologischen Verhältnisse der Kurve Γ sind, welche dieses merkwürdiger Verhalten der Funktionen in der Umgebung der Verzweigungsstellen II. Art bedingen." [We thus have learned that the topological form of the curve Γ determines this remarkable behavior of the function in the neighborhood of a type II branch point.]

There are thus two problems, he said. The first is to determine the topology of the (discriminant) curve in the neighborhood of a singular point, i.e. the knot Γ. The second is to determine the group given by the Wirtinger relations (in modern terminology, the fundamental group of the complement of the knot Γ). These two problems were solved in his paper. He remarked

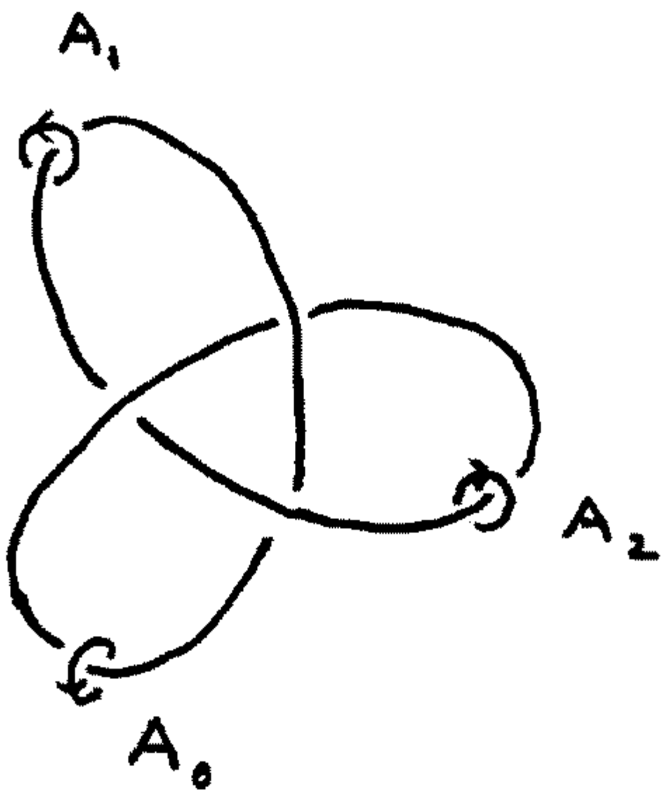

Figure 1: The discriminant intersected with the sphere

that there are three more problems. The first is to determine the branching group of a function locally in the neighborhood of a point. (This group is of course a quotient of fundamental group of the complement of the knot Γ.) Next, one should determine the global branching group of a function. Finally, given a group, is there a function which has this group as branching group? These problems, he said, will form the subject of two further papers [2].

He then continued with a systematic study of the links of curve singularities and their fundamental groups. He first looked at the curve

$$ax^n + by^m = 0$$

with the g.c.d $(n, m) = 1$, parameterizing it by setting $x = \alpha t^m$ and $y = \beta t^n$, where α and β were suitably chosen constants. He wrote the complex number t as $\rho e^{i\phi}$ with ρ and ϕ real and worked out parametric equations for the intersection of the curve with the sphere. Taking its image under the equations for stereographic projection, he observed that the image curve lay on a torus, winding n times in the direction of the meridian and m times in the direction of the equator, and hence was a torus knot.

He then went on to look at two such curves as above and described their linking. He then examined the curve parameterized by $x = t^m(a_m + ta_{m+1} + \ldots)$ and $y = t^n$ and showed that the link is a compound torus knot formed by taking torus knot on a small tube about the first torus knot and iterating this procedure. He also showed that only a finite number of terms (the characteristic pairs) in the (possibly infinite) power series parameterization of the curve determined the topological type of the knot. He continued by analyzing the case of curves with two branches. Brauner concluded by computing the fundamental group of the complement of these compound torus knots in terms of Wirtinger's generators and relations.

The next work in this area was done by Erich Kähler [Kah29] in Leipzig, who remarks at the beginning of his paper that "Obwohl die betreffenden Fragen zum grössten Teil bereits von Herrn Brauner beantwortet sind, habe ich mir erlaubt den Gegenstand auf dem etwas anschaulicheren Wege...darzustellen." [Although this question has been for the most part already answered by Mr. Branuner, I have allowed myself to explain it in a somewhat clearer fashion.]

[2] I do not know if these papers appeared.

Kähler replaced Brauner's sphere, the boundary of the "round" four-ball $\{|x|^2 + |y|^2 \leq r^2\}$ in $\mathbf{C}^2$ by the boundary of the "rectangular" four-ball $\{|x| \leq c_1\} \cap \{|y| \leq c_2\}$. This is a simplification since a curve tangent to the x-axis (say) intersects this boundary only in $\{|x| \leq c_1\} \cap \{|y| = c_2\}$, one of its two sides ($c_1 << c_2$). He noted that the two pieces of the boundary could be mapped easily into three-space where they formed a decomposition into two solid tori. He then looked at the curve $y = ax^{m/n}$ and observed that the image of the intersection of this curve with the boundary of the rectangular four-ball is obviously a torus knot or link. He then continued to obtain Brauner's results in easier fashion.

Thus the topological nature of the link could be computed from analytic data. The converse result, that the characteristic pairs could be determined from the topology of the knot, was proved simultaneously by Zariski at Johns Hopkins University and Werner Burau in Königsberg.

Zariski [Zar32] started with a singular point of the curve X and again derived a presentation of the local fundamental group of its complement. He then found a polynomial invariant $F(t)$ of this group which he later identified as the Alexander polynomial of the knot, and showed that the first Betti number of the k-fold branched cyclic cover of a punctured neighborhood of the origin in $\mathbf{C}^2$ with branch locus X is the number of roots of $F(t)$ which are k-th roots of unity. (This was later recognized to be a purely knot-theoretic result.)

Burau [Bur32], on the other hand, used Alexander's recent work to compute the Alexander polynomial of compound torus knots. He derived a recursive formula for these polynomials and showed that they were all distinct. He later treated the case when the polynomial had two branches at the origin, i.e. when the link had two components [Bur34].

A survey of the above work was given later by Reeve [Ree54], who also showed that the intersection number of two branches of a curve at the origin equals the linking number in the three-sphere of their corresponding knots. He gave two proofs. The first, following Lefschetz, notes that the algebraic intersection multiplicity of the curves is their topological intersection multiplicity, which is the linking number of their boundaries. The second proof uses Reidemeister's definition of linking number in terms of the knot projection.

Now let us move forward in time to the present. The computation of knot invariants of the link of a curve singularity becomes increasingly messy as the number of branches of the curve increases. A diagrammatic method for these computations (for the Alexander polynomial,the real Seifert form, the Jordan normal form of the monodromy and so forth) has been developed in [EN85].

The link of a singularity of a curve has a global analogue, the *link at infinity* K_∞ of a curve $X \subset \mathbf{C}^2$, which is defined to be the intersection of X with a sphere $\mathbf{S}^3_r$ of suitably large radius r. Neumann has shown that if the curve is a regular fiber of its defining equation (i. e. if the map is a locally trivial fibration near this value), then the topological type of the curve is determined by the knot type of $K_\infty \subset \mathbf{S}^3_r$. Also, Neumann and Rudolph have used these techniques to give topological proofs of a result of Abhyankar and Moh (that up to algebraic automorphism, the only embedding of $\mathbf{C}$ in $\mathbf{C}^2$ is the standard one) and similar results of Zaidenberg and Lin [Rud82, NR87, Neu89].

The knot type of the link of a singularity in higher dimensions has received some attention; see for instance [Dur75].

3 Three-manifolds and singularities of surfaces

It is useful at this point to introduce some terminology. An (affine) *algebraic variety* $X \subset \mathbf{C}^m$ is the zero locus of a collection of complex polynomials in m variables. If X is a hypersurface, and hence the zero locus of a single polynomial $f(x_1, x_2, \ldots, x_m)$, then a point p is *singular* if $\partial f/\partial x_1 = \ldots = \partial f/\partial x_m = 0$ at p. The set of nonsingular points is a complex manifold of dimension $m-1$. A point which is not singular is called *smooth.* (The definition of singular point for arbitrary varieties can be found, for example in [Mil68, Section 2], and similar results hold.)

If $p \in X \subset \mathbf{C}^m$, the *link* of p in X is defined to be

$$K = X \cap \mathbf{S}^{2m-1}_\epsilon$$

where $\mathbf{S}^{2m-1}_\epsilon$ is a sphere of sufficiently small radius ϵ about p in $\mathbf{C}^m$. If p is an isolated singularity of X, then the link is a compact smooth real manifold of dimension one less than the real dimension of X at p. Understanding the topology of the variety X near p is the same as understanding the topology of K and its embedding in the sphere; in fact, X is locally homeomorphic to a cone on K with vertex p [Mil68, 2.20]. (This fact is implicit in the work of Burau and Kähler, but not explicitly stated.) The *local fundamental group* of the singularity is the fundamental group of the link. This is particularly interesting for an isolated singular point of an algebraic surface (complex dimension two) where the link is a three-manifold.

Some time elapsed before the topological investigation of curve singularities chronicled in the first section was extended to higher dimensions. In the early 1960's the following result of Mumford confirmed a conjecture of Abhyankar [Mum61] (see also the Bourbaki talk of Hirzebruch [Hirb]):

Theorem. *If p is a normal point of a complex surface X with trivial local fundamental group, then p is a smooth point of X.*

The condition "normal" comes from the algebraic side of algebraic geometry; in particular it implies that the singularity is isolated and that its link is a connected space.

He proved this theorem by resolving the singularity, a technique which in the case of surfaces is old and essentially algorithmic. The process of resolution removes the singular point p from X and replaces it by a collection of smooth transversally-intersecting complex curves $E_1, \ldots, E_r$ so that the new space $\tilde{X}$ is smooth.

He showed that the link could be obtained from the curves E_i by a process called *plumbing*: The tubular neighborhood of E_i in $\tilde{X}$ is identified with a 2-disk bundle over the curve E_i. If E_i and E_j intersect in a point $q \in \tilde{X}$, the two-disk bundles over E_i and E_j are glued together by identifying a fiber over q in one with a disk in the base centered at q in the other. This makes a manifold with corners. If the corners are smoothed (so that the result looks rather like an plumbing elbow joint), the boundary is diffeomorphic to the link.

The *graph* of a resolution of a normal singularity of an algebraic surface is as follows: The i-th vertex corresponds to the curve E_i, labelled by the genus of E_i and the self-intersection $E_i \cdot E_i$. The i-th and j-th vertices are joined by an edge if $E_i \cdot E_j \neq 0$, and the edges are weighted by $E_i \cdot E_j$. The resolution graph thus determines the topological type of the link.

Mumford used Van Kampen's theorem and the plumbing description of the link to give a presentation of the local fundamental group of the singularity and thus prove the theorem.

The local fundamental group of a singularity of an algebraic surface turned out to be a a useful way to classify these singularities. For instance, Brieskorn [Bri68], using earlier work of Prill, showed that if the local fundamental group is finite, then the variety X is locally isomorphic to a quotient $\mathbf{C}^2/G$, where G is one of the well-known finite subgroups of $GL(2,\mathbf{C})$. He listed all such subgroups G together with the resolution graph of the minimal resolution of the corresponding singularity $\mathbf{C}^2/G$.

Wagreich [Wag72], inspired by work of Orlik [Orl70], used Mumford's presentation to find all singularities with nilpotent or solvable local fundamental group. Thus the local fundamental group became closely connected with the local analytic structure of the singularity.

Neumann showed that the topology of the link K determines the graph of the minimal resolution of the singularity. In fact, he showed that $\pi_1(K)$ determines this graph, except in a small number of cases [Neu81] .

Mumford's techniques in a global setting appeared later in work of Ramanujam [Ram71]:

Theorem. *A smooth complex algebraic surface which is contractible and simply connected at infinity is algebraically isomorphic to* $\mathbf{C}^2$.

Ramanujam showed this by compactifying the surface by a divisor with normal crossings, and then using the topological conditions to show that this divisor could be contracted to a projective line. He also showed that the condition of simple connectivity at infinity was essential by producing an example of a smooth affine rational surface X which is contractible but not algebraically isomorphic to $\mathbf{C}^2$. In fact, the intersection of X with a sufficiently large sphere is a homology three-sphere but not a homotopy three-sphere.

Ramanujam's result implies that the only complex algebraic structure on $\mathbf{R}^4$ is the standard one on $\mathbf{C}^2$, so that there are no "exotic" algebraic structures on the complex plane. The search for exotic algebraic structures thus continued in higher dimensions. Ramanujam remarked that the three-fold $X \times \mathbf{C}$ is diffeomorphic to $\mathbf{C}^3$ by the h-cobordism theorem. A cancellation theorem proved later had the corollary that $X \times \mathbf{C}$ is not algebraically isomorphic to $\mathbf{C}^3$. Hence there is an exotic algebraic structure on $\mathbf{C}^3$. Much work followed in this area; for the current state of affairs one can consult [Zai93], for example.

4 Exotic spheres

Brieskorn, who was spending the academic year 1965-66 at the Massachusetts Institute of Technology, investigated whether Mumford's theorem extended to higher dimensions. On September 28, 1965, he wrote in a letter to his doctoral advisor Hirzebruch that he had examined the three-dimensional variety

$$x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0$$

and its singularity at the origin. He explicitly calculated a resolution of the singular point, then used van Kampen's theorem to show that the link K of this singularity is simply-connected and the Mayer-Vietoris sequence to show that K is a homology 5-sphere. He concluded, using Smale's recent solution of the Poincaré conjecture in higher dimensions, that K is homeomorphic to $\mathbf{S}^5$. Hence Mumford's result did not extend to higher dimensions.

According to Hirzebruch [Hir87, C38], "Dieser Brief von Brieskorn war eine grosse Überraschung" [This letter from Brieskorn was a great surprise]. Later letters followed with more squared terms added to the equation above. Brieskorn's final result appeared in [Bri66b]: For odd $n \geq 3$, the link at the origin of

$$x_0^3 + x_1^2 + x_2^2 + \ldots + x_n^2 = 0 \tag{1}$$

is homeomorphic to the sphere S^{2n-1}.

The attention then shifted to the differentiable structure on this link. To describe the next events, we first need to recall the situation with non-standard or "exotic" differentiable structures on spheres. The first exotic sphere, a differentiable structure on $\mathbf{S}^7$ which is not diffeomorphic to the standard structure, had been discovered only ten years earlier by Milnor. Further investigations followed by Kervaire and Milnor [KM63]. By Smale's solution to the higher-dimensional Poincaré conjecture, it was sufficient to look at the set Θ_m of homotopy m-spheres (manifolds homotopy equivalent to the standard sphere S^m). The set Θ_m is an abelian group under connected sum, and Kervaire and Milnor showed that this group is finite ($m \neq 3$).

They also looked at the subgroup $bP_{m+1} \subset \Theta_m$ of homotopy spheres which are boundaries of parallelizable manifolds (manifolds with trivial tangent bundle), and showed that bP_{m+1} is trivial for m even, and finite cyclic for $m \neq 3$ odd.

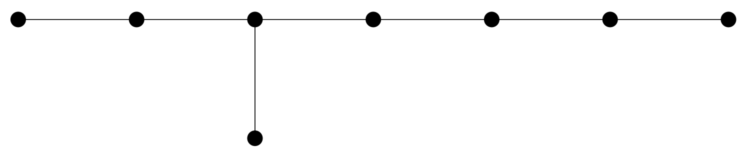

Figure 2: The E_8 graph

Its order could be computed as follows: If n is odd, the group bP_{2n} has order one or two. It is generated by the Kervaire sphere which is the boundary of the manifold constructed by plumbing two copies of the tangent disk bundle to S^n. The Kervaire sphere may or may not be diffeomorphic to the standard sphere; the first non-trivial group is bP_{10}. If $\Sigma \in bP_{2n}$ is the boundary of an $(n-1)$-connected parallelizable $2n$-manifold M, whether Σ is diffeomorphic to the standard sphere or the Kervaire sphere depends on the Arf invariant of a geometrically-defined quadratic form on M.

If $n \geq 4$ is even, the order of bP_{2n} can be calculated in terms of Bernoulli numbers. For example, there are 28 homotopy seven-spheres in $\Theta_7 = bP_8$. Also, the order of $\Sigma \in bP_{2n}$ can be calculated in terms of the signature of the intersection pairing on $H^n(M)$.

The construction of a generator of bP_{2n} for $n \geq 4$ even is once again bound up with singularity theory. In a preprint [Mil59] of January, 1959, Milnor had constructed a generator by plumbing according to an even unimodular matrix of rank and index eight. This matrix was not the well-known one associated to the E_8 graph (Figure 2), though, since its graph had a cycle. He then added a two-handle to make the boundary simply-connected and hence a homotopy sphere. Hirzebruch, however, was familiar with the E_8 matrix from his work on resolution of singularities of surfaces. He constructed a generator of the group bP_{2n} by plumbing copies of the tangent disk bundle to S^n according to the E_8 graph. (For more details, see [Hir87, C30], [Hirb, Hira, Mil64].)

At the same time in the fall of 1965 that Hirzebruch was receiving the letters from Brieskorn, he also received a letter from Klaus Jänich, another of his doctoral students, who was spending the year 1965-66 at Cornell. Jänich described his work on $(2n-1)$-dimensional $O(n)$-manifolds (manifolds with an action of the orthogonal group). In fact, he had classified $O(n)$-manifolds whose action had just two orbit types with isotropy groups $O(n-1)$ and $O(n-2)$, in particular showing that they were in one-to-one correspondence with the non-negative integers. (These results were also obtained by W. C.

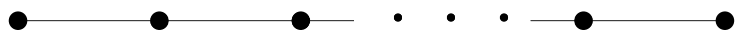

Figure 3: The A_k graph (k vertices)

Hsiang and W. Y. Hsiang.)

Hirzebruch noticed the connection between the research efforts of his two students and showed that the link of

$$x_0^d + x_1^2 + \ldots + x_n^2 = 0 \tag{2}$$

for $d \geq 2$, $n \geq 2$ is an $O(n)$-manifold as above with invariant d, the action being given by the obvious one on the last n coordinates. Since the boundary of the manifold constructed by plumbing copies of the tangent disk bundle of the n-sphere according to the A_{d-1} tree (Figure 3) also is an $O(n)$-manifold as above with invariant d, these manifolds are identical. Thus the link of the singularity (1) is the $(2n-1)$-dimensional Kervaire sphere; in particular for $n = 5$ it is an exotic 9-sphere.

These results were described in a manuscript "$O(n)$-Mannigfaltigkeiten, exotische Sphären, kuriose Involutionen" of March 1966. (This was not published, since it was supplanted by Hirzebruch's Bourbaki talk [Hira], and the detailed lecture notes [HM68] from his course in the winter semester 1966/67 at the University of Bonn.) In a letter [Hir87, C39] of March 29, 1966, Brieskorn reacted to the manuscript with "Klaus Jänich und ich hatten von diesem Zusammenhang unserer Arbeiten nichts bemerkt, und ich war vor Freude ganz ausser mir, wie Sie nun die Dinge zusammengebracht haben. Ein schöneres Zusammenspiel von Lehrern und Schülern–wenn ich das so sagen darf–kann man sich doch wirklich nicht denken." [Klaus Jänich and I had not noticed this connection between our work, and I was beside myself with joy to see how you had brought these together. A more beautiful cooperation of student and pupil can one hardly imagine, if I may say so myself.]

At this time the varieties

$$x_0^{a_0} + x_1^{a_1} + \ldots + x_n^{a_n} = 0 \tag{3}$$

($a_i \geq 2$) started to receive attention; they are now called "Brieskorn varieties", probably due to the influence of a chapter heading in Milnor's book [Mil68], although they were first examined in this context by Pham and

Figure 4: Milnor's sketch

Milnor as well. The corresponding $(2n-1)$-dimensional links

$$K(a_0, a_1, \ldots, a_n) = \{x_0^{a_0} + x_1^{a_1} + \ldots + x_n^{a_n} = 0\} \cap \mathbf{S}^{2n+1}$$

where $\mathbf{S}^{2n+1}$ is a sphere about the origin, are usually called "Brieskorn manifolds". (The radius of the sphere can be arbitrary since the equation is weighted homogeneous.)

Milnor, who was in Princeton, sent a letter in April of 1966 to John Nash at MIT describing a simple conjecture as to when $K(a_0, a_1, \ldots, a_n)$ is a homotopy sphere: Let $\Gamma(a_0, a_1, \ldots, a_n)$ be the graph with $n+1$ vertices labeled $0, 1, \ldots, , n$ and with two vertices i and j joined by an edge if the greatest common divisor (a_i, a_j) is bigger than 1.

Conjecture. For $n \geq 3$, the link $K(a_0, a_1, \ldots, a_n)$ is a homotopy $(2n-1)$-sphere if and only if the graph $\Gamma(a_0, a_1, \ldots, a_n)$ has

- at least two isolated points, or
- one isolated point and at least one connected component Γ' with an odd number of vertices such that the g.c.d. $(a_i, a_j) = 2$ for all $i \neq j \in \Gamma'$.

In the corner of Milnor's letter was a sketch (Figure 4) which has now become the standard picture of a map with an isolated critical point.

Brieskorn then chanced upon an article of Pham [Pha65] which dealt with exactly the variety (3) above. In fact, Pham was interested in calculating the ramification of certain integrals encountered in the interaction of elementary particles in theoretical physics. To do this he needed to generalize the Picard-Lefschetz formulas, so let us recall these.

Picard-Lefschetz theory can be summarized as follows (see for example [Arn93, 2.1]): Let

$$X_t = \{x_0^2 + x_1^2 + \ldots + x_n^2 = t\} \subset \mathbf{C}^{n+1}$$

$(n \geq 1)$. Then

1. The smooth variety X_t for $t \neq 0$ is homotopy equivalent to an n-sphere S^n. (In fact, it is diffeomorphic to the tangent bundle to S^n.)

2. The homology class of this n-sphere generates the kernel of the degeneration map $H_n(X_t) \to H_n(X_0)$, hence its name of *vanishing cycle*.

3. The self-intersection of the vanishing cycle is 2 if $n \equiv 0 \bmod 4$, -2 if $n \equiv 2 \pmod 4$ and 0 if $n \equiv 1, 3 \pmod 4$.

4. Starting at $t = 1$ in the complex plane, traveling once counterclockwise about the origin and returning to the starting point induces a smooth map called the *monodromy* of X_1 to itself. It is well-defined up to isotopy. Picard-Lefschetz theory gives a description of this map. For example, if $n = 1$ it is a Dehn twist about the one-dimensional vanishing cycle. Picard-Lefschetz theory also describes the induced maps $H_n(X_1) \to H_n(X_1)$ and $H_n(X_1, \partial X_1) \to H_n(X_1)$.

Pham generalized this situation to the case

$$X_t = \{x_0^{a_0} + x_1^{a_1} + \ldots + x_n^{a_n} = t\} \subset \mathbf{C}^{n+1}$$

and found

1. The smooth variety X_t for $t \neq 0$ is homotopy equivalent to a bouquet $S^n \vee S^n \vee \ldots \vee S^n$ of $(a_0 - 1)(a_1 - 1) \ldots (a_n - 1)$ n-spheres. (This was shown by retracting X_t to a join $Z_{a_0} * Z_{a_1} * \ldots * Z_{a_n}$ where Z_k denotes k disjoint points.)

2. The homology classes of these n-spheres generate the kernel of the map $H_n(X_t) \to H_n(X_0)$.

3. An explicit calculation of the intersection pairing on $H_n(X_t)$.

4. An explicit calculation of the monodromy action on $H_n(X_t)$. (This is induced by rotating each set of points Z_k.)

The article of Pham provided exactly the information Brieskorn needed. (He remarks [Bri66a] that "Für den Beweis von [diesen] Aussagen sind jedoch gewisse Rechnungen erforderlich, für die gegenwärtig keine allgemein brauchbare Methode verfügbar ist. Für den Fall der $K(a_0, a_1, \ldots, a_n)$ sind diese Rechnungen aber sämtlich in einem vor kurzem erschienen Artikel von Pham enthalten, und nur die Arbeit von Pham ermöglicht den so mühelosen Beweis unserer Resultate." [Certain calculations, for which there are no general methods at this time, are necessary for the proof of these results. In the case of $K(a_0, a_1, \ldots, a_n)$, however, these calculations are contained in an article of Pham which just appeared, and it is only Pham's work which makes possible such an effortless proof of our results.]) Brieskorn used it to prove a conjecture of Milnor from the preprint [Mil66b] about the characteristic polynomial of the monodromy [Bri66a, Lemma 4], [Mil68, Theorem 9.1]. He then used this to prove the conjecture above [Bri66a, Satz 1], [HM68, 14.5],[Hira, Section 2].

Brieskorn also noted that the link $K(a_0, a_1, \ldots, a_n)$, which is defined as $X_0 \cap \mathbf{S}^{2n+1}$, is diffeomorphic to $X_t \cap \mathbf{S}^{2n+1}$ for small $t \neq 0$. This is the boundary of the smooth $(n-1)$-connected manifold $X_t \cap \mathbf{D}^{2n+2}$; it is parallelizable since it has trivial normal bundle. Hence $K(a_0, a_1, \ldots, a_n) \in bP_{2n}$. The information in Pham's paper about the intersection form also led to a formula (derived by Hirzebruch) for the signature of $X_t \cap \mathbf{D}^{2n+2}$. Brieskorn concluded that the link of

$$x_0^{6k-1} + x_1^3 + x_2^2 + x_3^2 + \ldots + x_n^2 = 0 \tag{4}$$

for even $n \geq 4$ is k times the Milnor generator of bP_{2n-1} [Bri66a, HM68].

Through a preprint of Milnor [Mil66b], Brieskorn also learned of a recent result of Levine [Lev66] which showed how to compute the Arf invariant needed to recognize whether a link is the Kervaire sphere in terms of the higher-dimensional Alexander polynomial of the knot. The Alexander polynomial for fibered knots is the same as the characteristic polynomial of the monodromy on $H_n(F)$. Hence Brieskorn was able to show [Bri66a, Satz 2] that the link of

$$x_0^d + x_1^2 + x_2^2 + \ldots + x_n^2 = 0$$

for $n \geq 3$ odd is the standard sphere if $d \equiv \pm 1 \bmod 8$, and the Kervaire sphere if $d \equiv \pm 3 \bmod 8$, thus providing another proof of Hirzebruch's result that the link of the singularity (1) is the Kervaire sphere.

The explicit representation of all the elements of bP_{2n} by links of simple algebraic equations was rather surprising. It provided another way of thinking about these exotic spheres and led to various topological applications.

For example, Kuiper [Kui68] used them to obtain algebraic equations for all non-smoothable piecewise-linear manifolds of dimension eight. (PL manifolds of dimension less than eight are smoothable.) In fact, he started with the complex four-dimensional variety given by Equation (4) above with $n = 4$. This has a single isolated singularity at the origin. Its completion in projective space has singularities on the hyperplane at infinity, but adding terms of higher order to the equation eliminates these while keeping (analytically) the same singularity at the origin. This can be triangulated, giving a combinatorial eight-manifold which is smoothable except possibly at the origin. Since obstructions to smoothing are in one-to-one correspondence with the 28 elements of bP_8, the construction is finished.

Also, the high symmetry of the variety given by Equation (2) allowed the construction of many interesting group actions on spheres, both standard and exotic [Hira, Section 4], [HM68, Section 15]. The actions are the obvious ones: The cyclic group of order d acts by roots of unity on the first coordinate, and there is an involution acting on (any subset of) the remaining coordinates by taking a variable to its negative.

5 The Milnor fibration

About the same time as the above events were happening, Milnor proved a fibration theorem which turned out to be fundamental for much subsequent work. This theorem together with its consequences first appeared in the unpublished preprint [Mil66b], which dealt exclusively with isolated singularities. (A full account of this work was later published in the book [Mil68], where the results were generalized to non-isolated singularities. The earlier and somewhat simpler ideas can be found at the end of Section 5 of the book.)

Let $f(x_0, x_1, \ldots, x_n)$ for $n \geq 2$ be a complex polynomial with $f(0, \ldots, 0) = 0$ and an isolated critical point at the origin. Let $\mathbf{S}^{2n+1}_\epsilon$ be a sphere of suitably small radius ϵ about the origin in $\mathbf{C}^{n+1}$. As before, let $K = \{f(x_0, x_1, \ldots, x_n) = 0\} \cap \mathbf{S}^{2n+1}_\epsilon$ be the link of $f = 0$ at the origin. The main result of the preprint is the following *fibration theorem*:

Theorem. *The complement of an open tubular neighborhood of the link K in $\mathbf{S}^{2n+1}_\epsilon$ is the total space of a smooth fiber bundle over the circle $\mathbf{S}^1$. The fiber F has boundary diffeomorphic to K.*

The idea of the proof is as follows: If $\mathbf{D}^{2n+2}_\epsilon$ is the ball of radius ϵ about the origin and $\delta > 0$ is suitably small, then

$$f : f^{-1}(\mathbf{S}^1_\delta) \cap \mathbf{D}^{2n+2}_\epsilon \to \mathbf{S}^1_\delta$$

is a clearly a smooth fiber bundle with fiber

$$F' = \{f(x_0, x_1, \ldots, x_n) = \delta\} \cap \mathbf{D}^{2n+2}_\epsilon$$

The total space of this fibration is then pushed out to the sphere $\mathbf{S}^{2n+1}_\epsilon$ along the trajectories $p(t)$ of a suitably-constructed vector field. This vector field is constructed with the property that $|p(t)|$ is increasing along a trajectory, so that points eventually reach the sphere, and with the property that the argument of $f(p(t))$ is constant and $|f(p(t))|$ is increasing, so that the images of points in $\mathbf{C}$ travel out on rays from the origin. Thus Milnor's proof shows that F is diffeomorphic to F'. The proof also shows that F is parallelizable, since F' has trivial normal bundle.

The fiber F is now called the *Milnor fiber*. He then gives some facts which lead to the topological type of the fiber F and the link K:

(a) The pair $(F, \partial F)$ is $(n-1)$-connected.

(b) The fiber F has the homotopy type of a cell complex of dimension $\leq n$. In fact, it is built from the $2n$-disk by attaching handles of index $\leq n$.

These assertions follow from Morse theory. In fact, in a lecture at Princeton in 1957 (which was never published), Thom described an approach to the Lefschetz hyperplane theorems which was based on Morse theory. Thom's approach then inspired Andreotti and Frankel [AF59] (see also [Mil66a, Section 7]) to give another proof of Lefschetz's first hyperplane theorem which used Morse theory, but in a different way: The key observation is that given a n-dimensional complex variety $X \subset \mathbf{C}^m$ and a (suitably general) point $p \in \mathbf{C}^m - X$, then the function on X defined by $|x-p|^2$ for $x \in X$ has non-degenerate critical points of Morse index $\leq n$. Thus $H_k(X) = 0$ for $k > n$, which is equivalent to Lefschetz's first hyperplane theorem.

Assertion (b) follows since the function $|x|$ (or a slight perturbation of it) restricted to F' has critical points of index $\leq n$, and Assertion (a) follows since the function $-|x|^2$ on F' has critical points of index $\geq n$.

By (b), the complement $\mathbf{S}^{2n+1}_\epsilon - F$ has the same homotopy groups as $\mathbf{S}^{2n+1}_\epsilon$ through dimension $n-1$. Thus:

(c) The complement $\mathbf{S}_\epsilon^{2n+1} - F$ is $(n-1)$-connected.

By the Fibration Theorem, $\mathbf{S}_\epsilon^{2n+1} - F$ is homotopy equivalent to F. Thus

Proposition. *The fiber F has the homotopy type of a bouquet $S^n \vee \ldots \vee S^n$ of spheres.*

Fact (a) and the above proposition combined with the long exact sequence of a pair show the following:

Proposition. *The link K is $(n-2)$-connected.*

Milnor used the notation μ for the number of spheres in the bouquet of the first proposition and called it the "multiplicity" since it is the multiplicity of the gradient map of f. However, μ quickly became known as the *Milnor number*. The Milnor number has played a central role in the study of singularities. One reason is that it has analytical as well as a topological descriptions, for example:

$$\mu = dim_{\mathbf{C}} \mathbf{C}\{x_0, x_1, \ldots, x_n\}/(\partial f/\partial x_0, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$$

the (vector space) dimension of the ring of power series in $n+1$ variables divided by the Jacobian ideal of the function (see, for example, [Orl77]).

The fact that the Milnor number can be expressed in different ways is extremely useful. For example, the topological interpretation of μ was used by Le and Ramanujam to prove a result which became basic to the study of equisingularity: Suppose that $n \neq 2$. If a family of functions $f_t : \mathbf{C}^{n+1} \to \mathbf{C}$ depending on t with isolated critical points has constant Milnor number, then the differentiable type of the Milnor fibration of f_t is independent of t. The proof uses the topological interpretation of μ to produce a h-cobordism which is thus a product cobordism; hence the restriction $n \neq 2$ [LR76].

Results similar to the Fibration Theorem and the two propositions have now been obtained in many different situations: complete intersections, functions on arbitrary varieties, polynomials with non-isolated critical points, critical points of polynomials at infinity, and so forth; references to these results can be found in the books and conference proceedings cited at the beginning of this article. Also, there are now many different techniques for computing the Milnor number $\mu = \text{ rank } H_n(F)$, the characteristic polynomial of the monodromy $H_n(F) \to H_n(F)$, and the intersection pairing on $H_n(F)$.

The characteristic polynomial of the monodromy turned out to be cyclotomic, and a variety of proofs have appeared of this important fact: the geometric proof of Landman, geometric proofs of Clemens and Deligne-Grothendieck based on resolving the singularity, proofs based on the Picard-Fuchs equation by Breiskorn, Deligne and Katz, and analytic proofs using the classifying space for Hodge structures by Borel and Schmid. For a summary of these and the appropriate references, see [Gri73].

The Milnor number appears in another situation. To describe this we first return to Thom's original observation in his 1957 lecture, as recorded in [AF69]: Given an n-dimensional complex variety X in affine space and a suitably general linear function $f : X \to \mathbf{C}$, then $|f|^2$ has non-degenerate critical points of Morse index exactly n (except for the absolute minimum). This result is easily proved by writing the function in local coordinates. It forms the basis of Andreotti and Frankel's proof of the second hyperplane theorem of Lefschetz, which says that the kernel of the map on H_{n-1} from a hyperplane section of an n-dimensional projective variety to the variety is generated by vanishing cycles.

Thom's original observation was applied in the local context of singularities, where it leads to a basic result in the subject of polar curves relating the Milnor number of a singularity and a plane section. This result has both topological and analytic formulations [Tei73, p. 317], [Le73].

6 Brieskorn three-manifolds

The Brieskorn three-manifolds $K(a_0, a_1, a_2)$, the link of

$$x_0^{a_0} + x_1^{a_1} + x_2^{a_2} = 0$$

at the origin, have provided examples figuring in many topological investigations. For example, the local fundamental group of these singularities has proved interesting. As mentioned in Section 3, the surface singularities whose link have finite fundamental group are exactly the quotient singularities. If the surface is embedded in codimension one, and is hence the zero locus of a polynomial $f(x_0, x_1, x_2)$, then these singularities are the well-known *simple singularities*:

$$A_k : x_0^{k+1} + x_1^2 + x_2^2 = 0 \quad (k \geq 1)$$
$$D_k : x_0^{k-1} + x_0 x_1^2 + x_2^2 = 0 \quad (k \geq 4)$$

$$E_6 : x_0^4 + x_1^3 + x_2^2 = 0$$
$$E_7 : x_0^3 + x_0 x_1^3 + x_2^2 = 0$$
$$E_8 : x_0^5 + x_1^3 + x_2^2 = 0$$

These equations have appeared, and continue to appear, in many seemingly unrelated contexts [Dur79]. For example, V. Arnold showed that they are the germs of functions whose equivalence classes under change of coordinate in the domain have no moduli [AGZV85].

More general than Brieskorn polynomials is the class of weighted homogeneous polynomials: A polynomial $f(x_0, x_1, \ldots, x_n)$ is *weighted homogeneous* if there are positive rational numbers $a_0, a_1, \ldots, a_n$ such that

$$f(c^{1/a_0} x_0, c^{1/a_1} x_1, \ldots, c^{1/a_n} x_n) = c f(x_0, x_1, \ldots, x_n)$$

for all complex numbers c. (Weighted homogeneous polynomials probably first made their appearance in singularity theory in the book of Milnor [Mil68].) Brieskorn singularities are weighted homogeneous, with weights exactly the exponents.

The simple singularities are weighted homogeneous. Milnor [Mil68, p. 80] noted that their weights (a_0, a_1, a_2) satisfy the inequality $1/a_0 + 1/a_1 + 1/a_2 > 1$. He also remarked that the links of the *simple elliptic singularities*

$$\tilde{E}_6 : x_0^3 + x_1^3 + x_2^3 = 0$$
$$\tilde{E}_7 : x_0^2 + x_1^4 + x_2^4 = 0$$
$$\tilde{E}_6 : x_0^2 + x_1^3 + x_2^6 = 0$$

have infinite nilpotent fundamental group. In this case, the sum of the reciprocals of the weights is 1. He conjectured that if $1/a_0 + 1/a_1 + 1/a_2 \leq 1$, then the corresponding link had infinite fundamental group, and that this group was nilpotent exactly when $1/a_0 + 1/a_1 + 1/a_2 = 1$.

This conjecture was proved by Orlik [Orl70]. In fact, he and Wagreich [OW71] had already found an explicit form of a resolution for weighted homogeneous singularities using topological methods based on the existence of a $\mathbf{C}^*$ action, following earlier work by Hirzebruch and Jänich. They also noted that these links were Seifert manifolds [Sei32] and hence could use Seifert's work as well as earlier work by Orlik and others.

Topologists were interested in the question of which homology three-spheres bound contractible four-manifolds (c.f. [Kir78, Problem 4.2]). In fact, topological analogues (contractible four-manifolds which are not simply-connected at infinity) of the example of Ramanujam in Section 3 (a contractible complex surface which is not simply connected at infinity) had been found some ten years earlier by Mazur [Maz61] and Poenaru [Poe60]. As Mazur remarks, these examples provide a method of constructing many examples of odd topological phenomena.

It was known (see Milnor's conjecture in Section 4) that $K(a_1, a_2, a_3)$ is a homology three-sphere exactly when the integers a_1, a_2, a_3 are pairwise relatively prime. (As Milnor remarks in [Mil75], this result in this context of Seifert fiber spaces is already in [Sei32].) Links of Brieskorn singularities were particularly easy to study, since a resolution of the singularity exhibited the link as the boundary of a four-manifold, and data from the resolution provided a plumbing description of this manifold which then could be manipulated to eventually get a contractible manifold. For example, Casson and Harer [CH81] showed that the Brieskorn manifolds $K(2, 3, 13)$, $K(2, 5, 7)$ and $K(3, 4, 5)$ are boundaries of contractible four-manifolds. Much has now happened in this area as can be seen in Kirby's update of his problem list [Kir96].

Brieskorn three-manifolds and their generalizations also provided interesting examples of manifolds with a "geometric structure". Klein proved long ago that the links of the simple singularities listed above are of the form S^3/Γ, the quotient of the group of unit quarternions by a discrete subgroup.

Milnor [Mil75, Section 8] proved by a round-about method that the links of the simple elliptic singularities are quotients of the Heisenberg group by a discrete subgroups. He then showed that the links of Brieskorn singularities with $1/a_0 + 1/a_1 + 1/a_2 \leq 1$ are quotients of the universal cover of $SL(2, \mathbf{R})$ by discrete subgroups. (Similar results were obtained at the same time by Dolgachev.)

Thus many links admitted a locally homogeneous (any two points have isometric neighborhoods) Riemannian metric and hence provided nice examples of Thurston's eight geometries [Thu82]. These results were extended by Neumann [Orl83]. Later he and Scherk [NS87] found a more natural way of describing the connection between the geometry on the link and the complex analytic structure of the singularity in terms of locally homogeneous non-degenerate CR structures.

The three-dimensional Brieskorn manifolds have also been central examples in the study of the group Θ_3^H of homology three-spheres. This group is bound up with the question of whether topological manifolds can be triangulated. It was originally thought that this group might just have two elements. However, techniques from gauge theory were used to show that it is actually infinite and even infinitely generated. In particular the elements $K(2, 3, 6k - 1)$ for $k \geq 1$ have infinite order in this group, and are linearly independent. Brieskorn manifolds appear in this context because the corresponding singularities have easily computable resolutions, and hence the three-manifolds are boundaries of plumbed four-manifolds upon which explicit surgeries can be performed [FS90].

Also, the Casson invariant of some types of links of surface singularities in codimension one (including Brieskorn singularities) was proved to be 1/8 of the signature of the Milnor fiber [NW90].

7 Other developments

This last section recounts two developments which occurred outside the main stream of events as recounted in the previous sections. They are both applications of topology to algebraic geometry. The first is a theorem of Sullivan [Sul71]:

Theorem. *If K is the link of a point in a complex algebraic variety, then the Euler characteristic of K is zero.*

If the point is smooth or an isolated singular point, then the link is a compact manifold of odd dimension and hence has Euler characteristic zero. The surprising feature of this result is that is should be true for non-isolated singularities as well.

Sullivan discovered this result during his study of combinatorial Stiefel-Whitney classes. He recounts that initially it was clear to him that this result was true in dimensions one and two. He then asked Deligne if he knew of any counterexamples in higher dimensions, but the latter replied "almost immediately" with a proof based on resolving the singularity. Sullivan then deduced this result in another fashion: Since complex varieties have a stratification with only even-dimensional strata, the link has a stratification with only odd-dimensional strata. He then proved, by induction on the number strata, that a compact stratified space with only odd-dimensional strata has zero Euler characteristic.

Since real varieties are the fixed point set of the conjugation map acting on their complexification, this result has the following consequence for real varieties:

Corollary. *If K is the link of a point in a real algebraic variety, then the Euler characteristic of K is even.*

Sullivan remarks that the result for complex varieties follows from essentially "dimensional considerations", but that the corollary for real varieties is however "geometrically surprising". This result continues to form a basis for the investigation of the topology of real varieties.

The second result is one of Thom. Given a singularity of an arbitrary variety $X_0 \subset \mathbf{C}^m$, one can ask if it can be "smoothed" in its ambient space $\mathbf{C}^m$ in the sense that it can be made a fiber of a flat family $X_t \subset \mathbf{C}^m$ whose fibers X_t for small $t \neq 0$ are smooth. For example, a hypersurface singularity is smoothable in its ambient space since it is the zero locus of a polynomial and hence smoothed by nearby fibers of the polynomial.

The first example of a non-smoothable singularity was constructed by Thom (see [Har74]). In fact, Thom showed that the variety $X \subset \mathbf{C}^6$ defined by the cone on the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^5$ is not smoothable: If it were, the link $K^7 \subset \mathbf{S}^{11}$ of $X \subset \mathbf{C}^6$ at the origin would be null cobordant (as a manifold with complex normal bundle) in $\mathbf{S}^{11}$, but it is not. This is proved by a computation with characteristic classes. (The manifold K^7 is odd-dimensional and hence null cobordant, but not in $\mathbf{S}^{11}$.)

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