Math. Ann. 232, 85—98 (1978)

The Signature of Smoothings of Complex Surface Singularities

Alan H. Durfee*

Department of Mathematics, Columbia University, New York, NY 10027, USA

Let $f:(\mathbb{C}^3, 0)\to(\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. For $\varepsilon>0$ suitably small and δ yet smaller, the space $V'=f^{-1}(\delta)\cap D_\varepsilon$ (where D_ε denotes the closed disk of radius ε about 0) is a real oriented four-manifold with boundary whose diffeomorphism type depends only on f. It has been proved that V' has the homotopy type of a wedge of two-spheres; the number μ of two-spheres is readily computable. Recently an interesting formula for μ was given in terms of analytic invariants of a resolution of the singularity at 0 of the complex surface $f^{-1}(0)$ [13]. This formula is proved by applying the Riemann-Roch theorem to the projective completions of $f^{-1}(0)$ and $f^{-1}(\delta)$, then canceling terms coming from the parts away from the origin. The purpose of this paper is to find a similar formula for the signature of the intersection pairing on the two-dimensional homology of the manifold V', using the Hirzebruch signature theorem instead of the Riemann-Roch theorem.

Various other signature formulas are known, in higher dimensions as well as in dimension two. For $f(x, y, z)$ of the form $g(x, y)+z^2$, the intersection pairing of V' is the same as a symmetrized Seifert matrix of the compound torus link $\{g(x,y)=0\}\cap S^3_\varepsilon$. There is a simple formula for the signature of the symmetrized Seifert matrix of a compound link of one component [20]; hence if $g^{-1}(0)$ is irreducible, there is a simple formula for the signature of V' in terms of the Puiseux pairs of g. If $g^{-1}(0)$ has several branches at the origin, it is possible to find a Seifert matrix for the link defined by g and compute the signature [17], but this process is tedious for all but the simplest links. Formulas also exist for the signature when $f(x,y,z)$ is of the type $x^a+y^b+z^c$ [10], or when f is weighted homogeneous [22]. There is in addition a formula for the signature in terms of mixed Hodge structure [23].

The *genus* (or *geometric genus*) of the singularity $f^{-1}(0)$ (assumed Stein) is the dimension of $H^1(\tilde{V}, \mathcal{O}_{\tilde{V}})$, where $\tilde{V}$ is a resolution of $f^{-1}(0)$. By combining the formulas for the signature and the number μ it is possible to show that twice the genus of the singularity $f^{-1}(0)$ is equal to the number of positive plus the number of

* Research partially supported by National Science Foundation grant MPS 72-05055 A03

zero eigenvalues of the intersection pairing (Proposition 3.1). This result provides a direct link between analytic and topological data of the germ f. It is used, for instance, to extend to the case of complete intersections Arnold's classification of germs with respect to the signature of the intersection pairing, and obtain results on the local fundamental group (Proposition 3.3).

It is possible to give a local proof of the formula for the signature of V' using the concept of the signature defect. More generally, the local proof makes it possible to give a formula for the signature of any smoothing V' of a complex analytic germ V with an isolated normal singularity, provided that the holomorphic tangent bundle of V' is topological trivial. This will be true if V is a complete intersection, and is probably true if V is Gorenstein. The general formula for the signature, Theorem 1.7, is the main result of this paper. Sections 2 and 3 contain examples and applications to complete intersections, and the proof of the theorem is given in Section 4. The corresponding formula for μ has been conjectured by Laufer (given here as Conjecture 1.8), and similar methods using the integrality of the Todd genus enable it to be proved modulo 12. Thus the general theme of this paper is to compute invariants of a deformation of V from a resolution of V.

Section 5 contains some conjectures on the size of the signature, its relation to degenerating germs, and a necessary condition for smoothing Gorenstein singularities. An application to conjugate singularities is also given (Proposition 3.4).

It is possible to obtain a similar formula for the signature in dimensions greater than two, but its computational value seems limited.

All homology and cohomology is with integer coefficients unless otherwise indicated.

I thank H. Laufer for helpful conversations. V. Arnold has informed me that the results of Section 2 have been obtained independently by I. Dolgachev and M. Reid.

1. The Main Theorem

Let $\mathscr{V}$ be the germ of a normal two-dimensional complex analytic space with at most an isolated singularity. Let $\mathscr{S}$ be the germ of the complex numbers at the origin. A *smoothing* of $\mathscr{V}$ is the germ of a three-dimensional complex analytic set $\mathscr{W}$ and a flat map $f:\mathscr{W}\to\mathscr{S}$ with $f^{-1}(0)$ isomorphic to the germ $\mathscr{V}$ and $f^{-1}(s)$ nonsingular for $s\neq 0$. The germ $\mathscr{W}$ has at most an isolated singularity [7, Satz 2.3].

Embed $\mathscr{W}$ in $\mathbb{C}^n$ (for some n) with the singularity of $\mathscr{V}$ at the origin 0, and choose $\varepsilon>0$ such that all spheres about 0 of radius less than or equal to ε intersect $\mathscr{W}$ and $\mathscr{V}$ transversally [27, Theorem 8.5], [15, p. 17]. Let D_ε be the closed disk of radius ε about 0, and set

$$V=\mathscr{V}\cap D_\varepsilon .$$

The diffeomorphism type of the boundary ∂V of V is independent of the choice of ε, and V is homeomorphic to the cone over ∂V [15, p. 18].

We forget temporarily about the smoothing. Let

$$\pi:\tilde{V}\to V$$

be a resolution of the singularity of V. (See for instance [12].) $\tilde{V}$ is nonsingular, π is a proper surjective analytic map whose restriction $\tilde{V}-\pi^{-1}(0)\to V-\{0\}$ is an analytic isomorphism, and $\pi^{-1}(V-\{0\})$ is dense in $\tilde{V}$. Although V has boundary and is not strictly speaking an analytic set, this causes no difficulty. Note that $\pi^{-1}(0)$ is a deformation retract of $\tilde{V}$.

The *genus* p of the singularity of V is defined as

$$p=\dim_{\mathbb{C}} H^1(\tilde{V}, \mathcal{O}_{\tilde{V}}).$$

This number is independent of the resolution, and may alternatively be defined as the dimension of the stalk at the origin of the sheaf $R^1\pi_*\mathcal{O}_{\tilde{V}}$ on V[5]. (Note that V is Stein.)

We may write the compact complex one-dimensional *exceptional locus* $E=\pi^{-1}(0)$ as the union of its irreducible components, say s of them:

$$E=E_1\cup\ldots\cup E_s.$$

E is connected since V is normal. Without loss of generality, we may assume that the resolution is *good*, that is, that the curves E_i are non-singular, intersect transversally, and that no three meet at a point. V has a minimal good resolution, in the sense that any other good resolution factors through the minimal good resolution, and hence may be obtained from it by blowing up points. The matrix $\{E_i\cdot E_j\}$, where $E_i\cdot E_j$ is defined as the number of points of intersection of E_i and E_j for $i\neq j$, and the self-intersection of E_i for $i=j$ (the first Chern class of the normal bundle to E_i in $\tilde{V}$), is known to be negative definite [12, p. 49]. Let

$$h=\operatorname{rank} H_1(E).$$

Next we discuss complex vector bundles. We let c_i denote the i^{th} Chern class, and τ denote holomorphic tangent bundle. The following lemma is well known.

Lemma 1.1. *Let ξ be a two-dimensional complex bundle over a CW complex X with $H^i(X)=0$ for $i>3$. The following conditions are equivalent:*

(i) *ξ is topologically trivial.*
(ii) *ξ is stably trivial.*
(iii) *The first Chern class $c_1(\xi)$ is zero.*
(iv) *The second exterior power $\Lambda^2\xi$ is a topologically trivial line bundle.*

Proof. Condition (i) implies (ii) which implies (iii). Conditions (iii) and (iv) are equivalent since the first Chern class of $\Lambda^2\xi$ equals the first Chern class of ξ. Finally, (iii) implies (i) since two-dimensional complex bundles over X are topologically isomorphic if and only if their first Chern classes are equal: Isomorphism classes of bundles over X are in one-to-one correspondence with homotopy classes $[X, BU_2]$ of maps of X into the classifying space BU_2 of complex two-plane bundles. Now $H^i(X)=0$ for $i>3$, and $\pi_1(BU_2)=\pi_3(BU_2)=0$, while $\pi_2(BU_2)$ is infinite cyclic. Hence [21, 8.4.3], the set $[X, BU_2]$ is isomorphic to $H^2(X)$ under the correspondence which assigns to each bundle its first Chern class. This proves the lemma.

Definition 1.2. The germ V is *Gorenstein* if there is a nowhere-zero holomorphic two-form on its regular points $V-\{0\}$. It is *numerically Gorenstein* if the holomorphic tangent bundle to $V-\{0\}$ is topologically trivial.

Hypersurface singularities as in Section 2 are Gorenstein, since the form

$$(dx \wedge dy)/(\partial f/\partial z) = (dy \wedge dz)/(\partial f/\partial x) = (dz \wedge dy)/(\partial f/\partial y)$$

is nowhere zero on $V - \{0\}$. Gorenstein singularities are numerically Gorenstein by Lemma 1.1.

Let V be a germ as above with resolution $\tilde{V} \to V$, and consider the exact sequence

$$H^2(\tilde{V}, \partial\tilde{V}) \xrightarrow{j^*} H^2(\tilde{V}) \to H^2(\partial\tilde{V})$$

where j is the inclusion map. The abelian groups $H^2(\tilde{V}, \partial\tilde{V})$ and $H^2(\tilde{V})$ are free and of the same rank; j^* is injective since $\{E_i \cdot E_j\}$ is negative definite. There is a unique class $c \in H^2(\tilde{V}, \partial\tilde{V}) \otimes \mathbb{Q} \cong H^2(\tilde{V}, \partial\tilde{V}; \mathbb{Q})$ such that $j^*c = c_1(\tau\tilde{V})$. The *canonical class* $K \in H_2(\tilde{V}; \mathbb{Q})$ is defined to be the Lefschetz dual of $-c$. The class K can be written as $\sum n_i E_i$, where the n_i are rational numbers.

Lemma 1.3. *V is numerically Gorenstein if and only if the n_i above are integers.*

Proof. The bundle $\tau(V - \{0\})$ is topologically trivial $\Leftrightarrow$ its restriction $\tau(V - \{0\})|\partial V$ to ∂V is topologically trivial $\Leftrightarrow \tau(\tilde{V})|\partial\tilde{V}$ is topologically trivial $\Leftrightarrow c_1(\tau(\tilde{V})|\partial\tilde{V}) = 0$ (by Lemma 1.1) $\Leftrightarrow c$ is integral $\Leftrightarrow$ the n_i are integers.

For example, a germ V whose minimal good resolution has exceptional set a curve E of genus zero and $E^2 < -2$ is not numerically Gorenstein. The following lemma is well known.

Lemma 1.4 (Adjunction formula). *The rational numbers n_i are uniquely determined by the equations*

$$-K \cdot E_i = E_i^2 + \chi(E_i)$$

for $i = 1, \ldots, s$, where $\chi(E_i)$ denotes the Euler characteristic of E_i.

Proof. Let $\alpha_i : E_i \to \tilde{V}$ be the inclusions, and let ν denote holomorphic normal bundle. Since $j^*c = c_1(\tau\tilde{V})$, for each i we have

$$\alpha_i^* j^* c = c_1(\alpha_i^* \tau\tilde{V}) = c_1(\nu E_i) + c_1(\tau E_i).$$

Hence

$$(\alpha_i^* j^* c)[E_i] = c_1(\nu E_i)[E_i] + c_1(\tau E_i)[E_i]$$

for each i, where $[E_i]$ denotes the orientation class of E_i. It is an exercise to show that the dual of this equation is the same as the equation of the lemma. Uniqueness follows since the matrix $\{E_i \cdot E_j\}$ is negative definite. This completes the proof.

The self-intersection of K is defined to be the number

$$K^2 = \sum_{i,j} n_i n_j E_i \cdot E_j.$$

Since the matrix $\{E_i \cdot E_j\}$ is negative definite, we know that $K^2 \leqq 0$, and that $K^2 = 0$ implies $K = 0$.

We return to the smoothing of V. Choose $\delta>0$ such that $f^{-1}(s)$ intersects the sphere of radius ε transversally for all $|s|<2\delta$. This is possible since the map f is flat, and hence a product near any point of ∂V. Set

$$V'=f^{-1}(\delta)\cap D_{\varepsilon}.$$

Then V' is a real four-manifold with boundary whose diffeomorphism type is independent of the choice of ε, δ, and the embedding of $\mathscr{W}$ in $\mathbb{C}^n$. It is oriented by the complex structure on its interior. Furthermore, ∂V and $\partial V'$ are isotopic in ∂D_{ε}, and thus in particular diffeomorphic.

Lemma 1.5. *Let V and V' be as above. If the holomorphic tangent bundle $\tau V'$ of V' is topologically trivial, then V is numerically Gorenstein.*

This follows easily, since $V-\{0\}$ deformation retracts to ∂V. Smoothings of hypersurface and complete intersection singularities have $\tau V'$ topologically trivial (cf. Section 2).

Conjecture 1.6. *Let V' be the smoothing of a Gorenstein singularity V. Then $\tau V'$ is topologically trivial.*

The signature $\sigma(M)$ of an arbitrary real oriented four-manifold M with or without boundary is defined as follows: There is a symmetric bilinear *intersection pairing* (,) on $H_2(M;\mathbb{R})$ defined by setting

$$(x,y)=(x'\cup y')[M]$$

where x' and y' in $H^2(M,\partial M;\mathbb{R})$ are Lefschetz duals to x and y in $H_2(M;\mathbb{R})$, and $[M]\in H_4(M,\partial M;\mathbb{R})$ is the orientation class. This bilinear form may be diagonalized, with diagonal entries $+1$, 0, and -1. The *signature* $\sigma(M)$ of M is the signature of this bilinear form, namely, the number of positive minus the number of negative diagonal entries.

Theorem 1.7. *Let $\mathscr{V}$ be a germ of a normal two-dimensional complex analytic space with an isolated singularity, and let $f:\mathscr{W}\to\mathscr{S}$ be a smoothing of $\mathscr{V}$. Construct V' as above and assume that the holomorphic tangent bundle to V' is topologically trivial. Let χ and σ be the Euler characteristic and signature of V' respectively. Then*

$$\chi\equiv K^2+1-h+s \pmod{12}$$

and

$$\sigma=-\tfrac{1}{3}(2(\chi-1)+K^2+s+2h).$$

Note that χ and σ are invariants of the smoothing. Examples and applications of this result are given in Sections 2 and 3, and its proof in Section 4.

Conjecture 1.8 [13]. *Under the same hypotheses as Theorem 1.7,*

$$\chi=K^2+1-h+s+12p.$$

Laufer has proved this conjecture for complete intersections. The same method of proof works for smoothings that may be put in a projective family with no other singularities, for instance as in Example 5.1.

2. Complete Intersections

Let $\mathscr{V}$ be the germ of a two-dimensional complete intersection with at most an isolated singularity. Such a $\mathscr{V}$ is normal [1, p. 435]. The versal deformation of $\mathscr{V}$ is well understood [25, Theorem 8.1]; in particular $\mathscr{V}$ has a unique smoothing up to diffeomorphism. In fact, let $f:(\mathbb{C}^{k+2},0)\to(\mathbb{C}^k,0)$ for $k\geqq 1$ be the germ of an analytic function with $f^{-1}(0)$ isomorphic to $\mathscr{V}$. Let $\mathscr{S}$ be the germ of a line through the origin 0 in $\mathbb{C}^k$ such that $f^{-1}(s)$ is nonsingular for s in $\mathscr{S}-\{0\}$, and set $\mathscr{W}=f^{-1}(\mathscr{S})$. Then $f:\mathscr{W}\to\mathscr{S}$ is a smoothing of $\mathscr{V}$.

Construct V and V' as in Section 1. The diffeomorphism type of the smooth four-manifold V' depends only on $\mathscr{V}$. The space V' has topologically trivial holomorphic tangent bundle by Lemma 1.1, since its holomorphic normal bundle is topologically trivial. It is shown in [15, Theorem 6.5] and [8, Satz 1.7] that V' has the homotopy type of a bouquet of two-spheres. Let

$$\mu=\dim H_2(V')$$

be the *Milnor number* of $\mathscr{V}$. For $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$, this may be computed by the formula

$$\mu=\dim_{\mathbb{C}}\mathbb{C}\{x,y,z\}\Big/\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right) \tag{1}$$

[15, p. 115] or by various other methods. The signature σ of V' is defined as in § 1. Both μ and σ are thus invariants of the germ $\mathscr{V}$. The following result is an immediate corollary of Theorem 1.7.

Corollary 2.1. $\sigma=-\frac{1}{3}(2\mu+K^2+s+2h)$.

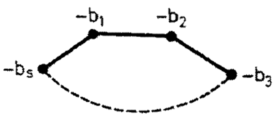
Furthermore, Conjecture 1.8 is true for complete intersections:

Theorem 2.2 [13]. $\mu=K^2-h+s+12p$.

The remainder of this section gives a number of examples. It is often convenient to represent the configuration of the E_i in the resolution of V (cf. Section 1) by a dual graph: Each curve E_i is represented by a vertex p_i weighted by the integer E_i^2 and two vertices p_i and p_j are joined by $E_i\cdot E_j$ lines.

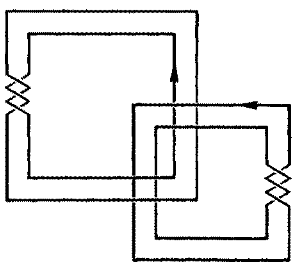
Example 2.3. Let $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ be the germ of a rational double point [5, p. 135; 4, § 3.1]. The minimal good resolution $\tilde{V}$ of $f^{-1}(0)$ has dual graph a Dynkin diagram of type A_s for $s\geqq 1$, D_s for $s\geqq 4$, E_6, E_7, or E_8, where all vertices represent curves of genus zero and self-intersection -2. Using the adjunction formula 1.4, it is easy to compute that the divisor K is zero for each of the above, so $K^2=0$. [Conversely, if a germ has $K=0$, then the dual graph of its minimal good resolution must be one of the types listed: If $K=0$, then $E_i^2+\chi(E_i)=0$ for each i by the adjunction formula, so each E_i has genus 0, and $E_i^2=-2$. Since the matrix $\{E_i\cdot E_j\}$ is negative definite, the assertion follows by standard arguments.] Clearly $h=0$. Furthermore, $\mu=s$, as may be checked by (1), or Theorem 2.2. Thus $\sigma=-\mu$, that is, the intersection pairing is negative definite. Of course this is well known; according to [24], $\tilde{V}$ is actually diffeomorphic to V'.

Example 2.4. Suppose the exceptional set in a minimal good resolution of the germ of a complete intersection $\mathscr{V}$ consists of two or more nonsingular curves $E_1, \ldots, E_s$ of genus zero intersecting in a circular chain, so that the dual graph is



(The fact that $\mathscr{V}$ is a complete intersection imposes conditions on the b_i; see Proposition 3.3D.) Clearly $K = -(E_1 + \ldots + E_s)$, so $K^2 = \sum(2 - b_i)$. $\mathscr{V}$ is minimally elliptic [14], so $p = 1$. Hence $\mu = 11 + \sum(3 - b_i)$, and $\sigma = 3 - \mu$. For example, the function $f(x, y, z) = (x^2 + y^3)(x^3 + y^2) + z^2$ (whose type is $Y_{5,5}$ in the notation of Arnold) has resolution whose dual graph is

Hence $\mu = 11$ and $\sigma = -8$. This is the signature of the two-component link defined by $\{(x, y) \in \mathbb{C}^2 : (x^2 + y^3)(x^3 + y^2) = 0\} \cap S_\varepsilon^3$:



Example 2.5. Suppose that $f(x, y, z) = g(x, y) + z^2$, where $g^{-1}(0)$ is the union of three cusps of the form $u^2 + v^3$ with distinct tangent lines at the origin. The minimal good resolution consists of two rational curves E_1 and E_2 intersecting at three points, with $E_1^2 = E_2^2 = -4$. The divisor K is $-2(E_1 + E_2)$. Using [2] we find $\mu = 28$. Hence $\sigma = -18$. This is the signature of the link $\{g(x, y) = 0\} \cap S_\varepsilon^3$, which has three components, where each component is a trefoil knot linking each other component four times.

Example 2.6. Let $f(x, y, z) = x^k + y^k + z^k$. By (1), $\mu = (k-1)^3$. The singularity of $f^{-1}(0)$ may be resolved by blowing up the origin in $\mathbb{C}^3$; E is a single curve of genus $\frac{1}{2}(k-1)(k-2)$ with $E^2 = -k$. Furthermore, $K = (k-2)E$. Thus $\sigma = -\frac{1}{3}(k-1)(k^2 + k - 3)$.

3. Applications to Complete Intersections

This section contains applications of the results in Section 2. As in that section, let $\mathscr{V}$ be the germ of a two-dimensional complete intersection with at most an isolated singularity. Let σ_+, σ_0, and σ_- be the number of positive, zero, and negative entries in the diagonalization of the intersection pairing of V'. Hence the Milnor number is

$$\mu=\sigma_+ +\sigma_0+\sigma_- \tag{1}$$

and the signature is

$$\sigma=\sigma_+ -\sigma_-. \tag{2}$$

Note that σ_0 equals the invariant h of Section 1, since σ_0 equals the rank of $H_1(\partial V')$, which equals h by the exact sequence of the pair $(\tilde{V}, \partial\tilde{V})$ and the fact that the matrix $\{E_i \cdot E_j\}$ is negative definite. The genus p of $\mathscr{V}$ is defined as in Section 1.

Proposition 3.1. $2p=\sigma_0+\sigma_+$.

Thus p, an analytic invariant of the resolution, depends only on the topology of V'. The fact that $\sigma_0+\sigma_+$ is an even number had already been obtained using the theory of mixed Hodge structure [23, Proposition 4.14], answering a question in [4, §3.2.12] for dimension two.

Proof. Solving (1) and (2) for σ_+ yields

$$\sigma_+ =\tfrac{1}{2}(\sigma+\mu-\sigma_0),$$

so

$$\sigma_+ +\sigma_0=\tfrac{1}{2}(\sigma+\mu+\sigma_0).$$

Substituting the expression for σ given in Corollary 2.1 and replacing σ_0 by h yields

$$\sigma_+ +\sigma_0=\tfrac{1}{6}(\mu+h-s-K^2).$$

Then substituting the expression for μ given in Theorem 2.2 gives the desired formula.

A germ $\mathscr{V}$ of a singularity of a complete intersection is *rational* if $p=0$ [5], and *minimally elliptic* if $p=1$ [14].

Corollary 3.2. *The singularity of the complete intersection $\mathscr{V}$ is rational if and only if $\sigma=-\mu$, and minimally elliptic if and only if $\sigma_0+\sigma_+=2$.*

The first part of this corollary was already known from the combined work of Artin, Brieskorn, and Tjurina. (See the survey article [6].) The second part may be used to extend the classification of hypersurface germs with small σ_0 and σ_+ [4, §3.1] to the case of complete intersections. Results on the local fundamental group π of the germ $\mathscr{V}$ also fit into this framework. (The local fundamental group is by definition the fundamental group of $V-\{0\}$, which is isomorphic to the fundamental group of ∂V.) Parts (A) and (B) of the following proposition are well known, and are included for comparison. (Again see [6], for instance.)

Proposition 3.3. *Let E be the exceptional set in the minimal good resolution of the germ of a two-dimensional complete intersection $\mathscr{V}$ with at most an isolated singularity.*

A. *The following statements are equivalent.*

(i) $\sigma_-=\sigma_0=\sigma_+=0$.

(ii) *π is trivial.*

(iii) *E is a point (so $\mathscr{V}$ is nonsingular).*

B. *The following statements are equivalent.*

(i) $\sigma_0=\sigma_+=0$.

(ii) π *is finite.*

(iii) *The dual graph of* E *is of type* $A_k(k\geqq 1)$, $D_k(k\geqq 4)$, E_6, E_7, *or* E_8 *(so* $\mathscr{V}$ *is a rational double point).*

C. *The following statements are equivalent.*

(i) $\sigma_0=2$ *and* $\sigma_+=0$.

(ii) π *is infinite nilpotent.*

(iii) E *is a elliptic curve with* $-4\leqq E^2\leqq -1$.

D. *The following statements are equivalent.*

(i) $\sigma_0=\sigma_+=1$.

(ii) π *is infinite solvable and not nilpotent.*

(iii) E *is a circular chain as in Example 2.4 with* $\sum(b_i-2)\leqq 4$ *and either*

(a) $s\geqq 2$ *with all* $b_i\geqq 2$ *and at least one* $b_i\geqq 3$, *or*

(b) $s=2$ *with* $b_1=1$ *and* $b_2\geqq 5$.

Proof. First we show that (i) implies (iii) in both (C) and (D). By Corollary 3.2, $p=1$; since $\mathscr{V}$ is a complete intersection, it is minimally elliptic [14, Theorem 3.10]. A minimally elliptic singularity has the property that every connected proper subset of the exceptional set is the exceptional set of a rational singularity [14, Theorem 3.4]. Now $\sigma_0=h$, as above, so $\sigma_0=2$ implies that E must be an elliptic curve, and $\sigma_0=1$ implies that E must be as in Example 2.4. A germ of a minimally elliptic singularity is a complete intersection if and only if $K^2\geqq -4$ [14, Theorems 3.4(2) and 3.13]. The conditions on E^2 in (C) and on the b_i in (D) are precisely those to make $\{E_i\cdot E_j\}$ negative definite and insure that the germ is a complete intersection. This shows that (i) implies (iii); the reverse implication is obvious.

Next we show that (ii) and (iii) are equivalent. The paper [26] lists up to homeomorphism all resolutions $\tilde{V}$ such that $\partial\tilde{V}$ has nilpotent or solvable fundamental group. The only such $\tilde{V}$ whose corresponding V is a complete intersection are as in (iii) of (C) and (D). This completes the proof of Proposition 3.3.

Equations for the germs in (C) are listed in [26, p. 66–67]; these have $\mu=11+E^2$. Equations for the germs in (D) are given in [14] and [11].

Finally, we note that Corollary 2.1 shows that the signature is an algebraic, and not just a topological, invariant. For example:

Proposition 3.4. *Let* τ *be a (discontinuous) automorphism of* $\mathbb{C}$. *Let* $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ *be the germ of an analytic function with an isolated critical point at* 0, *and let* f^τ *be the germ whose coefficients are the image under* τ *of the coefficients of* f. *Then the signature of* f^τ *equals the signature of* f.

Proof. The Milnor numbers μ of f and f^τ are equal, since they may be calculated from (1) of Section 2. The resolution of $(f^\tau)^{-1}(0)$ is obtained from the resolution of $f^{-1}(0)$ by applying τ, so the terms K^2, s, and h on the right-hand side of Corollary 2.1 are the same. Hence the signatures of f and f^τ are the same.

4. The Proof of Theorem 1.7

The Hirzebruch signature theorem does not hold for four-manifolds M with boundary. Instead, one defines the signature defect of a framed three-manifold N by taking the difference of the L-polynomial and the signature of some M with $\partial M=N$

[9, 3.1]. This is independent of the choice of M. In our case, the three-manifold ∂V is diffeomorphic to both $\partial \tilde{V}$ and $\partial V'$. Hence the signature defect of ∂V can be computed in two different ways, and setting these equal yields the equation for the signature of V'. The congruence for the Euler characteristic of V' is proved in the same way, using the well-known Adams e invariant in place of the signature defect.

Let N be a real three-dimensional manifold, and assume that its tangent bundle (which is trivial) plus a trivial line bundle has the structure of a two-dimensional complex bundle which is trivial. (Such manifolds N with stable bundles occur in complex cobordism theory. However, it is necessary here to assume that the unstable bundles have a complex structure.) Assume that there is a four-manifold M with $\partial M = N$ and a complex structure on tangent bundle to M that restricts to the given complex structure on the bundle over N. Let X be the four-dimensional CW complex obtained from M by collapsing N to a point, and let $p: M \to X$ be the projection. Choose a trivialization α of the complex bundle over N. This gives a complex bundle ξ on X with $p^*\xi$ isomorphic to the tangent bundle of M with its complex structure.

The group $H_4(X)$ is infinite cyclic and generated by a class $[X]$ with $p_*[M] = [X]$, where $[M]$ is the orientation class of M. We let $\chi(M)$ denote the Euler characteristic of M and $\sigma(M)$ its signature.

The *signature defect* d_σ of the framed manifold N is by definition the rational number

$$d_\sigma(N, \alpha) = \tfrac{1}{3}(c_1^2(\xi) - 2c_2(\xi))[X] - \sigma(M)$$

where $c_i(\xi)$ denotes the i^{th} Chern class of the bundle ξ, and its *Todd defect* d_T (usually called the *complex Adams e invariant*) is by definition the element

$$d_T(N, \alpha) = \tfrac{1}{12}(c_1^2(\xi) + c_2(\xi))[X]$$

of $\mathbb{Q}/\mathbb{Z}$. We also define the *Euler defect* d_χ of the framed manifold N to be the integer

$$d_\chi(N, \alpha) = c_2(\xi)[X] - \chi(M).$$

(It is here that we must not stabilize the bundles.)

These invariants are independent of the choice of M; we sketch the proof for $d_\chi(N, \alpha)$. The class $c_2(\xi)$ equals the Euler class $\chi(\xi_R)$ of the underlying real bundle ξ_R. Choose a fixed smooth manifold M' whose boundary is N with the opposite orientation, and let ξ' be the real bundle on $X' = M'/N$ obtained from the framing on N. Let W be the smooth manifold $M \cup_N M'$ and let τ be its tangent bundle. Then $(\chi(\xi_R)[X] - \chi(M)) + (\chi(\xi')[X'] - \chi(M')) = (\chi(\xi_R)[X] + \chi(\xi')[X']) - (\chi(M) + \chi(M'))$ $= \chi(\tau)[W] - \chi(W) = 0$ by additivity of the Euler class and Euler characteristic and the fact that the Euler class of the tangent bundle of a smooth oriented manifold evaluated on its orientation class is its Euler characteristic. Thus $\chi(\xi_R)[X] - \chi(M)$ is independent of the choice of M. The same argument together with the fact that $c_1^2(\xi) - 2c_2(\xi)$ equals the first Pontrjagin class of the underlying real bundle shows that $d_\sigma(N, \alpha)$ is independent of the choice of M [9, § 3.1]. The integrality of the Todd genus shows that $d_T(N, \alpha)$ is independent of the choice of M; the manifold M' exists since the complex cobordism group is zero in dimension three.

It is not hard to see that the decomposable Chern number $c_1^2(\xi)[X]$ is independent of the choice of frame α. We will not need this fact, though. Let $-N$ be N with the opposite orientation. Note that $d_\sigma(-N,\alpha)=-d_\sigma(N,\alpha)$, but that $d_\chi(-N,\alpha)\neq -d_\chi(N,\alpha)$. Note also that if one vector of the 2-frame α is always an outward-pointing normal, then $d_\chi(N,\alpha)=0$. (Cf. [9, p. 222]: This vector field may be extended to a vector field on M and on X; its number of zeros is $\chi(M)$ by a theorem of Hopf, and is $\chi(\xi_R)[X]$ by obstruction theory.) Harder results from [16, § 3] show that the rational number $c_1^2(\xi)[X]-2\chi(M)$ and the element $\frac{1}{12}(c_1^2(\xi)[X]+\chi(M))$ of $\mathbb{Q}/\mathbb{Z}$ are independent of both M and the choice of frame α. It is also possible to prove Theorem 1.7 this way, thus avoiding the Euler defect.

Proof of Theorem 1.7. Choose a framing of the complex tangent bundle to V', and let α' be the restriction of this frame to $\partial V'$. There is a diffeomorphism of $\partial V'$ to ∂V, and of ∂V to $\partial\tilde{V}$. Let $\tilde{\alpha}$ be the corresponding framing over $\partial\tilde{V}$. Thus

$$d(\partial V',\alpha')=d(\partial\tilde{V},\tilde{\alpha})$$

for $d=d_\sigma$, d_T, and d_χ. Let

$$X'=V'/\partial V'$$

with bundle ξ' obtained as above from the framing α', and let

$$\tilde{X}=\tilde{V}/\partial\tilde{V}$$

with bundle $\tilde{\xi}$ obtained from the framing $\tilde{\alpha}$. The bundle ξ' is trivial.

By the definition of d_σ,

$$d_\sigma(\partial\tilde{V},\tilde{\alpha})=\tfrac{1}{3}(c_1^2(\tilde{\xi})-2c_2(\tilde{\xi}))[\tilde{X}]-\sigma(\tilde{V})$$

and

$$d_\sigma(\partial V',\alpha')=-\sigma(V')$$

since ξ' is trivial. Combining,

$$\sigma(V')=\sigma(\tilde{V})-\tfrac{1}{3}(c_1^2(\tilde{\xi})-2c_2(\tilde{\xi}))[\tilde{X}]. \tag{1}$$

Similarly, $d_T(\partial V',\alpha')=0$ since ξ' is trivial, so

$$(c_1^2(\tilde{\xi})+c_2(\tilde{\xi}))[\tilde{X}]\equiv 0 \pmod{12}. \tag{2}$$

We analyze individual terms in these equations. First,

$$\sigma(\tilde{V})=-s, \tag{3}$$

the negative of the number of components in the exceptional set, since E is a deformation retract of $\tilde{V}$ and the matrix $\{E_i\cdot E_j\}$ is negative definite.

Also

$$\begin{aligned} c_2(\tilde{\xi})[\tilde{X}] &= d_\chi(\partial\tilde{V},\tilde{\alpha})+\chi(\tilde{V})\\ &= d_\chi(\partial V',\alpha')+\chi(\tilde{V})\\ &= -\chi(V')+\chi(\tilde{V}) \end{aligned} \tag{4}$$

by the definition of the Euler defect, and the fact that ξ' is trivial. Finally, we claim that

$$c_1^2(\tilde{\xi})[\tilde{X}] = K^2. \tag{5}$$

Let $j: (\tilde{V}, \emptyset) \to (\tilde{V}, \partial\tilde{V})$ be inclusion and let $p: \tilde{V} \to \tilde{X}$ be projection. The homology class K is defined to be the dual of the unique cohomology class $-c$ in $H^2(\tilde{V}, \partial\tilde{V})$ satisfying $j^*c = c_1(\tau\tilde{V})$. Thus

$$K^2 = c^2[\tilde{V}]$$

where $[\tilde{V}] \in H_4(\tilde{V}, \partial\tilde{V})$ is the orientation class. Let $q^*: H^2(\tilde{X}) \to H^2(\tilde{V}, \partial\tilde{V})$ be the map induced by projection; q^* is an isomorphism. Now $c = q^*c_1(\xi)$, since the map j^* is injective and $j^*c = c_1(\tau\tilde{V}) = p^*c_1(\tilde{\xi}) = j^*q^*c_1(\tilde{\xi})$. Thus

$$c^2[\tilde{V}] = q^*c_1^2(\xi)[\tilde{V}].$$

Finally, it is not hard to see that

$$q^*c_1^2(\xi)[\tilde{V}] = c_1^2(\xi)[\tilde{X}].$$

This proves (5). Substituting (3) through (5) into (1) and (2) proves the theorem.

5. Remarks

This section contains conjectures and further examples.

Example 5.1. Let $\mathscr{V}$ be the germ of a surface with a normal singularity whose minimal good resolution has exceptional set a curve E of genus 1 and self-intersection $-k$, where $k \geqq 1$. It is shown in [18] that $\mathscr{V}$ is smoothable if and only if $k \leqq 9$, and that the smooth fiber V' is obtained as follows: Embed E in $\mathbb{CP}^2$ as a cubic, blow up $9-k$ points in general position on E to get a surface M, and remove an open tubular neighborhood T in M of the proper transform (the non-exceptional curve) of E. Then V' is $M-T$. We can calculate the signature σ of V' directly: Since $\sigma(\mathbb{CP}^2)=1$, we have $\sigma(M) = 1-(9-k) = k-8$. The closure $\bar{T}$ of T in M has signature 1 since $E^2 = +9$. Thus by Novikov additivity, $\sigma(V') = \sigma(M) - \sigma(\bar{T}) = k-9$. The Euler characteristic χ of V' may be calculated similarly: Since $\chi(\mathbb{CP}^2)=3$, we have $\chi(M) = 3+(9-k) = 12-k$. Now $\chi(\bar{T})=0$ since it is a circle bundle, so by additivity, $\chi(V') = \chi(M) - \chi(\bar{T}) = 12-k$. ($\mathscr{V}$ is Gorenstein and minimally elliptic, so this agrees with Conjectures 1.6 and 1.8 and Theorem 1.7.)

Conjecture 5.2. *Let $\mathscr{V}$ be a germ of a normal two-dimensional complex analytic space with an isolated singularity, and suppose $\mathscr{V}$ is smoothable. Let σ be the signature of the smoothing V' as in Section 1. Then $\sigma \geqq 0$.*

When $\mathscr{V}$ is Gorenstein, this combined with Conjectures 1.6 and 1.8 and Theorem 1.7 would yield a necessary condition for smoothing in terms of the resolution, namely, that $s + K^2 + 8p \leqq 0$. Conjectures 5.2 is true for smoothings of rational singularities which have simultaneous resolution (so $\tilde{V}$ is homeomorphic to V'), Example 5.1, and examples in [11]. It is also true for hypersurfaces $\mathscr{V}$ defined as the zero locus of an equation of the form $g(x,y)+z^2$. (The signature of V' is then the same as the signature of a symmetrized Seifert matrix of the link $\{g(x,y)=0\} \cap S_\varepsilon^3$. If g is irreducible, Conjecture 5.2 follows easily from a formula for the signature of compound knots [20]. If g is reducible, there is a Dynkin diagram for the intersection pairing of g as in [3]. The subspace spanned by the δ_i corresponding to double points is negative definite, and the number of such δ_i is greater than $\mu/2$ by Lemma 4 of that paper.) It is also conjectured in [13] that the Euler characteristic χ of the smoothing satisfies $\chi \geqq 1$.

The rest of this section is concerned with germs $\mathscr{V}$ of complete intersections with at most an isolated singularity, as in Section 2.

Conjecture 5.3. *Let $\mathscr{V}$ be the germ of a complete intersection. Then $6p \leqq \mu$, with equality only when $\mu = 0$.*

For example, the germ of Proposition 3.3 (C) with $E^2 = -4$ has $\mu = 7$ and $p = 1$. (Is this germ the non-trivial complete intersection of smallest Milnor number μ?) Consider also Example 2.6, where $\mu = (k-1)^3$ and $p = \frac{1}{6}k(k-1)(k-2)$. Here $6p \leqq \mu$, and the limit of $\mu/6p$ as k approaches infinity is 1. The germ $\mathscr{V}$ is nonsingular if and only if $\mu = 0$; for $\mathscr{V}$ a hypersurface, this is shown in [15, Corollary 7.3]. Recently it has been shown that Conjecture 5.3 is true for singularities that are non-degenerate in the sense of Kouchnirenko [19]. The inequality $6p \leqq \mu$ is equivalent to $\sigma \leqq -(2p + \sigma_0)$. Hence Conjecture 5.3 implies Conjecture 5.2 for complete intersections.

Suppose that f and g are germs of analytic functions as in Section 2. The germ f is said to *topologically degenerate* to the germ g if there is a family of germs f_t depending analytically on the complex parameter t with $f_1 = f$ and $f_0 = g$, and f_t of constant Milnor number μ for $t \neq 0$. Tables of degeneracy in the case of hypersurfaces are found in [4]. It is easy to show that if f topologically degenerates to g, then the Milnor number of f is less than or equal to the Milnor number of g. The following conjecture is verified by all examples I know, and would imply Conjecture 5.2 for complete intersections.

Conjecture 5.4. *If the germ f topologically degenerates to the germ g, the signature of f is greater than or equal to the signature of g.*

Here is a consequence of this conjecture:

Corollary 5.5. *Assume Conjecture 5.4 is true. If the germ f topologically degenerates to the germ g, then the genus p (see Section 1) of $f^{-1}(0)$ is less than or equal to the genus of $g^{-1}(0)$.*

Proof. By Proposition 3.1, $2p = \sigma_0 + \sigma_+$. Assuming $\sigma \leqq 0$, it is not hard to see that the number $\sigma_0 + \sigma_+$ is exactly the dimension of a maximal subspace restricted to which the intersection form is zero. Since the intersection form of f maps isometrically into the intersection form of g [24], this number cannot decrease.

References

1. Abhyankar, S.: Local analytic geometry. New York: Academic Press 1964
2. A'Campo, N.: La fonction zêta d'une monodromie. Comment. Math. Helvetici **50**, 233—248 (1975)
3. A'Campo, N.: Le groupe de monodromie du déploiement des singularités isolées de courbes planes I. Math. Ann. **213**, 1—32 (1975)
4. Arnold, V.: Remarks on the stationary phase method and Coxeter numbers. Russ. Math. Surveys **28**, 19—48 (1973)

5. Artin, M.: On isolated rational singularities of surfaces. Amer. J. Math. **88**, 129—136 (1966)
6. Durfee, A.: Fifteen characterizations of rational double points and simple critical points. To appear, L'Enseignement Math.
7. Grauert, H., Kerner, H.: Deformationen von Singularitäten Komplexer Räume. Math. Ann. **153**, 236—260 (1964)
8. Hamm, H.: Lokale topologische Eigenschaften komplexer Räume. Math. Ann. **191**, 235—252 (1971)
9. Hirzebruch, F.: Hilbert modular surfaces. L'Enseignement Math. **19**, 183—281 (1973)
10. Hirzebruch, F., Mayer, K.: $O(n)$-Mannigfaltigkeiten, exotische Sphären und Singularitäten. Lecture Notes in Mathematics 57. Berlin, Heidelberg, New York: Springer 1968
11. Karras, U.: Deformations of cusp singularities. In: Several complex variables, pp. 37—44. Proceedings of Symposia in Pure Math., Vol. 30. Providence, R. I.: Amer. Math. Soc. 1977
12. Laufer, H.: Normal two-dimensional singularities. Annals of Mathematical Studies 71. Princeton: Princeton University Press 1971
13. Laufer, H.: On μ for surface singularities. In: Several complex Variables, pp. 45—49. Proceedings of Symposia in Pure Math., Vol. 30. Providence, R.I.: Amer. Math. Soc. 1977
14. Laufer, H.: On minimally elliptic singularities. To appear, Amer. J. Math.
15. Milnor, J.: Singular points of complex hypersurfaces. Ann. Math. Studies **91**. Princeton: Princeton University Press 1968
16. Morita, S.: Almost complex manifolds and Hirzebruch invariants for isolated singularities in complex spaces. Math. Ann. **211**, 245—260 (1974)
17. Murasugi, K.: On a certain numerical invariant of link types. Trans. Amer. Math. Soc. **117**, 387—422 (1965)
18. Pinkham, H.: Deformations of algebraic varieties with G_m action. Thesis, Harvard University, 1974; also Astérisque, No. 20
19. Randell, R.: On some numerical invariants of singularities (preprint)
20. Shinohara, Y.: On the signature of knots and links. Trans. Amer. Math. Soc. **156**, 273—285 (1971)
21. Spanier, E.: Algebraic topology. New York: McGraw-Hill 1966
22. Steenbrink, J.: Intersection form for quasi-homogeneous singularities. University of Amsterdam report 75-09 (1975)
23. Steenbrink, J.: Mixed Hodge structure on the vanishing cohomology (revised version), Report 76-06, University of Amsterdam
24. Tjurina, G.: The topological properties of isolated singularities of complex spaces of codimension one. Izvestija **2**, 557—571 (1968)
25. Tjurina, G.: Locally semi-universal flat deformations of isolated singularities of complex spaces. Izvestija **33**, 967—999 (1969)
26. Wagreich, P.: Singularities of complex surfaces with solvable local fundamental group. Topology **11**, 51—72 (1972)
27. Whitney, H.: Local properties of analytic varieties. In: Differential and Combinatorial Topology (Morse symposium), pp. 205—244. Princeton: Princeton University Press 1965

Received January 31, 1977, and in revised form July 1, 1977