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Periodicity of Branched Cyclic Covers*

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Let $K \subset S^{2n+1}$ be a simple fibered knot, that is, an embedding of an $(n-2)$-connected $(2n-1)$-manifold K in the $(2n+1)$-sphere whose complement fibers over S^1 with $(n-1)$-connected fibers. (See 1.5 for the precise definition.) In this paper we examine the periodicity, in k, of the smooth k-fold cyclic covers K_k of S^{2n+1} branched along K.

For K the trefoil knot in S^3, Fox [6, p. 192] discovered a homological periodicity of period 6, namely that $H_*(K_{k+6}) \simeq H_*(K_k)$ for all $k \geqq 2$. A second, apparently unrelated, example of such periodicity occurs in the links of Brieskorn singularities, where the link of the singularity $z_0^3 + z_1^2 + \ldots + z_n^2 + z_{n+1}^{6l-1}$ for odd $n \geqq 3$ is diffeomorphic to a connected sum of $(-1)^{(n+1)/2} l$ copies of the Milnor sphere [1, p. 13]. We will show that these are special cases of the same phenomenon, namely a periodicity in the homology, homeomorphism, and diffeomorphism type of K_k when the manifold K is a rational homology sphere and the fibered knot $K \subset S^{2n+1}$ has periodic monodromy.

In the first section we make precise the idea of a smooth branched cover. This has already been done in [5] for topological and simplicial branched covers; the analogous results in the smooth case are straightforward.

Section 2 proves the well-known result that the link of $f(z_0, \ldots, z_n) - z_{n+1}^k$ is the smooth k-fold cyclic cover of S^{2n+1} branched along the link of $f(z_0, \ldots, z_n)$, and hence a branched cyclic cover of a fibered knot.

In § 3 we show that the k-fold cyclic cover K_k is an $(n-1)$-connected $(2n+1)$-manifold and compute $H_n(K_k)$ directly in terms of the monodromy of $K \subset S^{2n+1}$. The proof uses the transfer homomorphism. An immediate corollary is that if K is a rational homology sphere and the monodromy is of finite order d, then $H_*(K_{k+d}) \simeq H_*(K_k)$. Since the trefoil knot is a fibered knot with monodromy of period 6, this explains the first example above. (Periodicity of $H_1(K_k)$ for arbitrary knots $K \subset S^3$ is examined in [8] and generalized in [25].)

The periodicity results of §§ 4 and 5 are only for odd $n \neq 1$, 3, or 7. To obtain the homeomorphism periodicity, in § 4 we must use other results about K_k, namely that it itself can be embedded in S^{2n+3} as a fibered knot and that its invariants are computable in terms of the invariants of the original knot $K \subset S^{2n+1}$. Given this, it is easy to prove the theorem of § 3 again. Furthermore, K_k is thus

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seen to be a stably-parallelizable highly-connected manifold, so that certain classification results can be applied. In particular, to show periodicity of the homeomorphism type it is only necessary to show that a linking form, and its associated quadratic form, are periodic.

To get diffeomorphism periodicity, it only remains to compute a signature. In § 5 we compute this from the Seifert pairing of the knot $K \subset S^{2n+1}$, and prove a special case of a signature periodicity theorem due to Neumann [17]. Our main result, Theorem 5.3, then follows. In addition, we give another derivation of the formula for the signature of Brieskorn singularities.

Since the results of Sections 4 and 5 depend upon classification theorems for highly-connected manifolds, similar results are probably true for n even, and $n = x$ or 7. In § 6 we present an example with n even.

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§ 1. Smooth Branched Covers

Let M be a smooth m-dimensional manifold, with $m \geqq 2$, and K a smooth compact codimension-two submanifold, both without boundary. Let $X = M - K$, and suppose that $\tilde{X} \to X$ is a finite covering. The composite map $\tilde{X} \to X \subset M$ is then a *spread*, and has a unique *completion* $p: \tilde{M} \to M$ [5]; $\tilde{X}$ is embedded as a dense subset of $\tilde{M}$ and p extends the above map. Let $\tilde{K} = p^{-1}(K)$. Let D^2 be the closed unit disk in $\mathbb{R}^2 = \mathbb{C}$, and let $\mu_k: D^2 \to D^2$ for non-zero integers k be defined, in complex notation, by

$$\mu_k(z) = \begin{cases} z^k, & \text{if } k > 0 \\ \bar{z}^{-k}, & \text{if } k < 0. \end{cases}$$

Proposition 1.1. *There is a smooth manifold structure for $\tilde{M}$ such that*

(i) *$p | \tilde{X}$ is a local diffeomorphism, and*

(ii) *$\tilde{K}$ is a smooth submanifold of $\tilde{M}$, and there are smooth closed tubular disk neighborhoods $\tilde{T}$ of $\tilde{K}$ and T of K such that the map p takes $\tilde{T}$ to T, preserves fibers, and is (up to a rotation) of the form μ_k on each fiber, where the integer k depends on the fiber.*

Any two differentiable structures for $\tilde{M}$ satisfying (i) *and* (ii) *are diffeomorphic. Furthermore, if M and K are oriented, then $\tilde{M}$ and $\tilde{K}$ may be oriented such that*

(iii) *$p | \tilde{X}$ and $p | \tilde{K}$ are orientation preserving, and k is positive (with respect to the natural orientation of the fibers of the tubular neighborhoods).*

We call $\tilde{M}$ the *smooth completion*. Note in particular that p is smooth. G. Bredon has pointed out although $\tilde{M}$ has a differentiable structure, it does not have a natural differentiable structure, as does $\tilde{X}$. The difficulty is that the lift of a diffeomorphism may not be a diffeomorphism. For example, let $M = \tilde{M} = \mathbb{C}$ with coordinate $z = (x, y)$, let $p(z) = z^2$, let f be the diffeomorphism of M defined by $f(x, y) = (x, 2y)$, and let $\tilde{f}$ be the unique lift of f with $\tilde{f}(1, 0) = (1, 0)$. The map $\tilde{f}$ is homogeneous of degree one. If it were differentiable at the origin, it would have to be linear, and hence the identity; since this clearly is not true, $\tilde{f}$ is not differentiable. Thus an atlas for M does not lift to an atlas for $\tilde{M}$. We will use the orthogonal structure on a tubular neighborhood of K in M to circumvent this difficulty; note that rotations and reflections do lift.

Proof. We first show existence.

The local homeomorphism $p|\tilde{X}$ gives $\tilde{X}$ a differentiable structure satisfying (i). Choose a closed tubular disk neighborhood T of K consisting of all points at distance $\leqq 1$ in the normal bundle of K in M with some smooth Riemannian metric. Let $T^* = T - K$, and let $\pi: T^* \to K$ be the bundle projection. The map p restricted to $\tilde{T}^* = p^{-1}(T^*)$ is a finite cover of the total space of this bundle. We claim that $\tilde{T}^*$ is the total space of a smooth punctured two-disk bundle over a covering $\tilde{K}$ of K, and that the map p factors into a bundle map p'' over $\tilde{K}$ of the form μ_k (up to a rotation) on each fiber, followed by a pull-back map p' induced by the covering $\tilde{K} \to K$. Assuming this, $\tilde{T} - \tilde{T}^*$ is exactly the zero section $\tilde{K}$ of the bundle, since $\tilde{M}$ is a complete spread, so $\tilde{T}$ is the total space of a smooth two-disk bundle over $\tilde{K}$. The differentiable structure on K determines one on its cover $\tilde{K}$. Let $\tilde{T}$ have the differentiable structure determined by its bundle atlas; this restricts to the previously given differentiable structure on $\tilde{T}^*$. Thus (ii) is satisfied.

We establish the above claim. The spaces K and $\tilde{T}^*$ may be assumed connected, for otherwise we apply the following argument to each component. Let $q: \tilde{K} \to K$ be the covering projection with $q_{\#}\pi_1(\tilde{K})$ equal to $(\pi p)_{\#}\pi_1(\tilde{T}^*)$ in $\pi_1(K)$. Let $\bar{\pi}: \bar{T}^* \to \tilde{K}$ be the pull-back over q of the bundle $\pi: T^* \to K$, and let $p': \bar{T}^* \to T^*$ be the induced map. The map p' is a covering projection, and is trivial when restricted to a fiber of the bundle $\bar{\pi}$. There is a covering projection $p'': \tilde{T}^* \to \bar{T}^*$ with $p = p'p''$. The inverse image $\tilde{F}$ under p'' of a fiber $\bar{F}$ of $\bar{\pi}$ is again connected, since $(\bar{\pi}p'')_{\#}\pi_1(\tilde{T}^*) = \pi_1(\tilde{K})$, so $p''_{\#}\pi_1(\tilde{T}^*, \tilde{F}) = \pi_1(\bar{T}^*, \bar{F})$. [In general, for any covering projection $f: \tilde{Y} \to Y$ with $\tilde{Y}$ path connected, A a path connected subset of Y, and $\tilde{A} = f^{-1}(A)$, the path components of $\tilde{A}$ are in one-to-one correspondence with the cosets of $f_{\#}\pi_1(\tilde{Y}, \tilde{A})$ in $\pi_1(Y, A)$.] It is now easy to make $\tilde{T}^*$ into the total space of a bundle with base space $\tilde{K}$, and to see that p' and p'' have the required properties.

Next we show uniqueness. Suppose there is another differentiable structure for $\tilde{M}$ satisfying (i) and (ii) with respect to tubular neighborhoods $\tilde{T}'$ of $\tilde{K}$ and T' of K. By uniqueness of tubular neighborhoods, there is a diffeomorphism of M to itself, isotopic to the identity, that restricts to a bundle equivalence of T to T'. The unique lift of this map is a diffeomorphism of the differentiable structures on $\tilde{M}$.

It remains to show part (iii). The fibers of the bundle $\pi: T^* \to K$ are oriented by the orientations of M and K. $\tilde{M}$ and $\tilde{K}$ are orientable, and oriented by making the local diffeomorphisms $p|\tilde{X}$ and $p|\tilde{K}$ orientation preserving. Then the fibers of the bundle $\tilde{T}^* \to \tilde{K}$ are oriented, and k is positive. This completes the proof of Proposition 1.1.

Now suppose that M is the m-sphere S^m with its usual orientation, $m \geqq 2$, and that K has l oriented components. As before let $X = S^m - K$. By duality, $H_1(X)$ is free abelian of rank l, generated by small oriented circles about each component of K. Let β map $H_1(X)$ to the integers by taking each such generator to 1. Pick a positive integer k and let G be the kernel of the map $\pi_1(X) \to H_1(X) \xrightarrow{\beta} \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$. Let $q: X_k \to X$ be the connected covering with $q_{\#}\pi_1(X_k) = G$.

Definition 1.2. The smooth completion, denoted $p: K_k \to S^m$, of this cover is the *k-fold cyclic cover of S^m branched along K.*

Note that p restricted to $p^{-1}(K)$ is bijective, and that the covering $q: X_k \to X$ is *regular* [that is, $q_{\#}\pi_1(X_k)$ is a normal subgroup of $\pi_1(X)$]. The group of deck transformations of X_k over X is $\mathbb{Z}/k\mathbb{Z}$. Furthermore, the normal bundle T of K in S^m is trivial, and, when K is connected, K_k is independent of the choice of orientation of K. Also, K_1 is simply S^m. The following lemma will be used to recognize the k-fold cyclic cover.

Lemma 1.3. *Let K be a compact oriented codimension-two submanifold of S^m, and let p be a map of a smooth oriented manifold $\tilde{M}$ to S^m. Let $X = S^m - K$, let $\tilde{X} = p^{-1}(X)$, and suppose that $p|\tilde{X}: \tilde{X} \to X$ is a regular covering. Let $\tilde{K} = p^{-1}(K)$, and suppose that $\tilde{K}$ is an oriented submanifold, and that $p|\tilde{K}$ is bijective. Furthermore, suppose that conditions* (i), (ii), *and* (iii) *of Proposition 1.1 are satisfied, for a fixed k. Then $p: \tilde{M} \to S^m$ is the smooth k-fold cyclic cover of S^m branched along K.*

Proof. Let G be as above. First we show that $(p|\tilde{X})_{\#}\pi_1(\tilde{X}) = G$. Since the cover $\tilde{X} \to X$ is regular, the group of deck transformations of $\tilde{X}$ over X is the same as the group of deck transformations over a boundary of a tubular neighborhood of a connected component of K, which is $\mathbb{Z}/k\mathbb{Z}$, by (ii) and the fact that $p|\tilde{K}$ is bijective. The subgroup $(p|\tilde{X})_{\#}\pi_1(\tilde{X})$ is also the kernel of the usual map of $\pi_1(X)$ to the group of deck transformations, and this kernel is the same as G, by (iii). Thus the cover $\tilde{X} \to X$ is equivalent to the cyclic cover $X_k \to X$. $\tilde{M} \to S^m$ is a topological completion of the cover $\tilde{X} \to X$, and hence homeomorphic to K_k, since topological completions are unique. The smoothness structure for $\tilde{M}$ satisfies (i) and (ii), so $\tilde{M}$ is diffeomorphic to K_k. This completes the proof.

The k-fold branched cyclic cover may be constructed as follows: According to [4, proof of Lemma 2.2], there is a closed tubular disk neighborhood T of K, a smooth map $\phi: S^m - K \to S^1$ representing the element of $H^1(X)$ determined by the map β above, and a smooth bundle equivalence α of T to the trivial bundle $K \times D^2$ such that $\phi\alpha^{-1}|\{x\} \times (D^2 - 0)$ is the obvious projection to S^1, for all $x \in K$. (In [4], $\phi\alpha^{-1}|\{x\} \times S^1$ is a rotation of S^1 depending on x. Replacing α by $(1 \times \sigma) \circ \alpha$, where $\sigma(x, -)$ is the linear extension to D^2 of the inverse of this rotation, changes it to the identity.) Let $\pi: K^2 \times D^2 \to D^2$ be the projection to the second factor. Define a continuous map $\bar{\phi}: S^m \to D^2$ by setting

$$\bar{\phi}(x) = \begin{cases} \phi(x), & \text{if } x \in \overline{S^m - T} \\ \pi\alpha(x), & \text{if } x \in T. \end{cases}$$

The smooth k-fold cyclic cover K_k of S^m branched along K is then a fiber product

$$\begin{array}{ccc} K_k & \longrightarrow & S^m \\ \downarrow & & \downarrow{\scriptstyle \bar{\phi}} \\ D^2 & \xrightarrow{\mu_k} & D^2 \end{array}$$

This fiber product is a smooth manifold, since it is the union of the smooth fiber products $S^1 \times_{S^1} (S^m - K)$ and $D^2 \times_{D^2} T$. It is the k-fold branched cyclic cover by Lemma 1.3.

We take this fiber product to be the definition of the k-fold branched cyclic cover if $K \subset S^1$ is an empty knot (see below).

If ϕ has no critical points, by Ehresmann's theorem (since $\phi|\partial T$ never has critical points) ϕ is the projection map of a smooth fiber bundle, so S^m has an open book structure with binding K [11, Definition 2.2]. With some additional connectivity assumptions, K becomes a simple fibered knot[1] [3]:

Definition 1.5. Let $n \geqq 0$. A *simple fibered knot* $K \subset S^{2n+1}$ is an embedding of an $(n-2)$-connected $(2n-1)$-manifold K in S^{2n+1} (where K is empty if $n=0$) together with a smooth fiber bundle $\phi: S^{2n+1} - K \to S^1$ that has the following properties:

(i) There is a closed tubular disk neighborhood T of K and a smooth bundle equivalence α of T to the trivial bundle $K \times D^2$ such that $\phi\alpha^{-1}|\{x\} \times (D^2 - 0)$ is the obvious projection to S^1, for all $x \in K$;

(ii) the fiber $F = \phi^{-1}(1)$ [whose closure is a $2n$-manifold with boundary K, by (i)] is $(n-1)$-connected; and

(iii) S^{2n+1} and K are oriented.

Note that an orientation of K orients S^1 and conversely, in the presence of an orientation of S^{2n+1}. A fibered knot with $n=0$ is an *empty knot*; its *degree* is the degree of $\phi: S^1 \to S^1$.

2. Links of Singularities as Branched Covers

Let $f(z_0, \ldots, z_n)$, for $n \geqq 0$, be a complex polynomial vanishing at the origin with an isolated singularity at that point. The *link* K of the singularity is the intersection of $f^{-1}(0)$ with a sphere S_ε^{2n+1} of suitably small radius ε about the origin. $K \subset S_\varepsilon^{2n+1}$ is a simple fibered knot [16]. The polynomial $f(z_0, \ldots, z_n) - z_{n+1}^k$ also has an isolated singularity at the origin in $\mathbb{C}^{n+2}$; let its link N be the intersection of the zero locus of the polynomial with a sphere S_ε^{2n+3} of the same radius. The singularity $f(z_0, \ldots, z_n) + z_{n+1}^k$ is analytically isomorphic to $f(z_0, \ldots, z_n) - z_{n+1}^k$, and hence has the same link. The following result is well-known ([9, 17]); the proof is a refined version of [17], and due to Neumann.

Proposition 2.1. *N is the smooth k-fold cyclic cover of S_ε^{2n+1} branched along K.*

We will need a combined version of [16, Theorem 2.10 and Lemma 5.9]:

Lemma 2.2. *Let $f(z)$, with $z = (z_0, \ldots, z_n)$, be a polynomial as above. Then there is a disk D_ε^{2n+2} of suitably small radius ε about the origin 0 and a smooth vector field v on $D_\varepsilon^{2n+2} - \{0\}$ such that the Hermitian inner product $\langle v(z), z\rangle$ has positive real part, $\langle v(z), \operatorname{grad} \log f(z)\rangle = 1$ for z with $f(z) \neq 0$, and $v(z)$ is tangent to $f^{-1}(0)$ when $f(z) = 0$.*

The first condition makes $|z|$ increase along integral curves of v. The second condition keeps the argument of $f(z)$ constant and increases $|f(z)|$ when $f(z) \neq 0$, and the third condition keeps z in $f^{-1}(0)$ when $f(z) = 0$.

Proof of Lemma 2.2. By Lemmas 2.9 and 4.3 of [16], there is a D_ε^{2n+2} such that $f^{-1}(0)$ intersects all spheres of smaller radius transversally, and z and $\operatorname{grad} \log f(z)$

[1]These are called "fibered knots" in [3]. The cumbersome terminology "simple highly-connected fibered knot" would be more accurate.

are either linearily independent over $\mathbb{C}$ or $\operatorname{grad}\log f(z) = \lambda z$ for some non-zero complex number λ with $|\arg\lambda| < \pi/4$, for all z in $D_\varepsilon^{2n+2} - f^{-1}(0)$. It suffices to construct v locally, combine with a partition of unity, and normalize.

At a point z_0 with $f(z_0) \neq 0$, either z_0 and $\operatorname{grad}\log f(z_0)$ are linearly independent over $\mathbb{C}$, in which case the equations $\langle v(z), z\rangle = 1$ and $\langle v(z), \operatorname{grad}\log f(z)\rangle = 1$ may be solved simultaneously for $v(z)$ in a neighborhood of z_0, or $\operatorname{grad}\log f(z_0) = \lambda_0 z_0$, in which case $\langle v(z), \operatorname{grad}\log f(z)\rangle = 1$ may be solved; then $\langle v(z), z\rangle$ is some function $\lambda(z)$ with $\lambda(z_0) = \lambda_0$, so $\operatorname{Re}\lambda(z) > 0$ in a neighborhood of z_0.

At a point $z_0 \neq 0$ with $f(z_0) = 0$, the map $(f(z), |z|): \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} \to \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$ is regular in a neighborhood of z_0 [since $f^{-1}(0)$ intersects all sufficiently small spheres transversally] and hence is the projection to the first three factors in suitable real local coordinates $u_1, \ldots, u_{2n+2}$ about z_0. In these local coordinates, $u_1 \dfrac{\partial}{\partial u_1} + u_2 \dfrac{\partial}{\partial u_2} + \dfrac{\partial}{\partial u_3}$ is a suitable vector field.

Proof of Proposition 2.1. Set $z = (z_0, \ldots, z_n)$ and $x = z_{n+1}$. Choose $\tau > 0$ small and set

$$K = \{z \in \mathbb{C}^{n+1}\colon\ |z|^2 = \tau^2 \quad \text{and} \quad f(z) = 0\}$$

$$S = \{z \in \mathbb{C}^{n+1}\colon\ |z|^2 = \tau^2\}$$

$$S' = \{z \in \mathbb{C}^{n+1}\colon\ |z|^2 + |f(z)|^2 = \tau^2\}$$

$$N = \{(z, x) \in \mathbb{C}^{n+1} \times \mathbb{C}\colon\ f(z) = x^k \quad \text{and} \quad |z|^2 + |x|^2 = \tau^2\}$$

$$N' = \{(z, x) \in \mathbb{C}^{n+1} \times \mathbb{C}\colon\ f(z) = x^k \quad \text{and} \quad |z|^2 + |x|^{2k} = \tau^2\}\,.$$

Thus S is a sphere of radius τ, K and N are links as in the beginning of this section, and S' and N' are a "stretched" sphere and link respectively. For small τ, K is a smooth submanifold of each of these, by [16, Corollary 2.8]. Let v be the vector field on $\mathbb{C}^{n+1}$ of the lemma. Using the vector field $(v(z), k^{-1}x)$ on $\mathbb{C}^{n+1} \times \mathbb{C} - \{0\} \times \mathbb{C}$, we push S' out to S and N out to N', fixing K. [This vector field preserves the argument of $f(z) - x^k$ and is tangent to $f(z) = x^k$.] Thus the pairs (S, K) and (S', K) are diffeomorphic, as are the pairs (N, K) and (N', K).

Let $p\colon N' \to S'$ be defined by $p(z, x) = z$. We use Lemma 1.3 to show that this is a smooth k-fold cyclic cover branched along K. The map $p|(N' - K)$ is a local diffeomorphism, and this covering is regular since its group of deck transformations is $(z, x) \mapsto (z, e^{2\pi i l/k}x)$, for $0 \leqq l < k$. Let

$$T = \{z \in S'\colon |f(z)| \leqq \delta^k\}$$

for suitably small $\delta > 0$ be a tubular neighborhood of K in S'. Choose $\pi\colon T \to K$ such that $(\pi, f)\colon T \to K \times D^2_{\delta^k}$ is a smooth trivialization. Let

$$\tilde{T} = \{(z, x) \in N'\colon |x| \leqq \delta\}$$

be a tubular neighborhood of K in N', and similarly choose $\tilde{\pi}\colon \tilde{T} \to K$ such that $(\tilde{\pi}, x)\colon \tilde{T} \to K \times D^2_\delta$ is a smooth trivialization. Then $p(\tilde{T}) = T$ and p takes the fiber coordinate x into the fiber coordinate $z = x^k$. By multiplying (z, x) by a constant if necessary, we may take $\delta = 1$; then clearly (ii) and (iii) of Proposition 1.1 are satisfied.

3. Homological Periodicity

Let $K \subset S^{2n+1}$ be a simple fibered knot, $n \geqq 1$. Recall that the *monodromy* automorphism h of $H_n(F)$ is defined by the action of the positive generator of $\pi_1(S^1)$ on the fiber $F = \phi^{-1}(1)$ of the fiber bundle $\phi: S^{2n+1} - K \to S^1$. Let $p: K_k \to S^{2n+1}$ be the smooth k-fold cyclic cover of S^{2n+1} branched along K. We will compute the homology of K_k in terms of the monodromy.

Theorem 3.1. *K_k is an $(n-1)$-connected $(2n+1)$-manifold, and*

$$H_n(F) \xrightarrow{I+h+\ldots+h^{k-1}} H_n(F) \to H_n(K_k) \to 0$$

is exact.

Let $i: F \to S^{2n+1} - K$ be the inclusion, let $\tilde{K} = p^{-1}(K)$, and let $\tilde{i}: F \to K_k - \tilde{K}$ be an embedding with $p \circ \tilde{i} = i$.

Suppose $n > 1$. The above exact sequence should be compared with the Wang sequence of the fibration ϕ:

$$0 \to H_{n+1}(S^{2n+1} - K) \to H_n(F) \xrightarrow{I-h} H_n(F) \xrightarrow{i_*} H_n(S^{2n+1} - K) \to 0$$

and the fact that $H_n(S^{2n+1} - K) \simeq H_{n-1}(K)$. The Wang sequence of the fibration of $K_k - \tilde{K}$ is

$$0 \to H_{n+1}(K_k - \tilde{K}) \to H_n(F) \xrightarrow{I-h^k} H_n(F) \xrightarrow{\tilde{i}_*} H_n(K_k - \tilde{K}) \to 0\,.$$

When K (and hence $\tilde{K}$) is an integral homology sphere, the inclusion map induces an isomorphism $H_n(K_k - \tilde{K})$ to $H_n(K_k)$ by Poincaré and Alexander duality, and $I - h^k$ has the same cokernel as $I + h + \ldots + h^{k-1}$ (since $I - h$ is invertible over $\mathbb{Z}$), so the theorem follows easily in this case. Similar remarks apply in the case $n = 1$. The difficulty lies in proving the theorem for general K.

Corollary 3.2. *Suppose K is a rational homology sphere and $h^d = I$ for some d. Then*

(i) $H_n(K_d) \simeq H_n(F)$.

(ii) $H_*(K_{k+d}) \simeq H_*(K_k)$, for all $0 < k$.

(iii) $H_*(K_{d-k}) \simeq H_*(K_k)$, for all $0 < k < d$.

Proof. (i) $0 = I - h^d = (I-h)(I + h + \ldots + h^{d-1})$. Since $I - h$ is invertible over the rational numbers (by the above Wang sequence), the latter factor is 0 over the rationals, and hence 0 over the integers, since $H_n(F)$ is free.

(ii) $I + \ldots + h^{d-1} + h^d + \ldots + h^{d+k} = 0 + h^d(I + h + \ldots + h^{k-1}) = I + h + \ldots + h^{k-1}$.

(iii) The cokernel of $I + h + \ldots + h^{d-k-1}$ is the same as the cokernel of $(I + h + \ldots + h^{d-k-1})h^k = h^k + \ldots + h^{d-1} = -(I + h + \ldots + h^{k-1})$. Q.E.D.

The proof of 3.1 uses the *transfer homomorphism.* There are two ways of defining it.

The first is: Let $p: Y \to Z$ be a finite regular covering projection of topological spaces, possibly branched as in § 1. Define a chain map $C_*(Z) \to C_*(Y)$ by taking each singular simplex c of Z to $c_1 + \ldots + c_k$, the sum of all singular simplices c_i of Y with $p \circ c_i = c$. The induced map $t: H_*(Z) \to H_*(Y)$ is the transfer homomorphism. The same definition applies to the relative groups if p is a map of pairs.

The second way of defining the transfer is: Let $p: Y \to Z$ be as in the first definition, but suppose further that Y and Z are oriented manifolds without boundary. Let $D_Y: H_c^*(Y) \to H_*(Y)$ be the Poincaré duality isomorphism of Y, and let D_Z be that of Z. The transfer homomorphism is then $D_Y p^* D_Z^{-1}$. The same definition works if $p:(Y, \partial Y) \to (Z, \partial Z)$ is a map of manifolds with boundary.

It is well known that these two definitions coincide. This may be proved by showing on the chain level that both maps when followed by p_* are equal, and that both maps are unaltered when followed by g_*, for all deck transformations g of Y over Z.

Proof of Theorem 3.1. Let $S = S^{2n+1}$, and $N = K_k$.

The transfer map $t': H_*(S, S-K) \to H_*(N, N-\tilde{K})$ is an isomorphism: Let T be a closed tubular disk neighborhood of K as in § 1, and let $\tilde{T} = p^{-1}(T)$. The diagram

$$\begin{tikzcd}
H_*(\tilde{T}, \partial\tilde{T}) \arrow[r] & H_*(N, N-\tilde{K}) \\
H_*(T, \partial T) \arrow[r] \arrow[u, "t_T"] & H_*(S, S-K) \arrow[u, "t'"]
\end{tikzcd}$$

commutes, where the horizontal arrows are excision isomorphisms, and the vertical arrows transfer maps. But, using the second definition, t_T is a composite of isomorphisms

$$H_*(T, \partial T) \xrightarrow{D_T^{-1}} H^*(T) \xrightarrow{(p|\tilde{T})^*} H^*(\tilde{T}) \xrightarrow{D_{\tilde{T}}} H_*(\tilde{T}, \partial\tilde{T}).$$

The Wang sequences for $N-\tilde{K}$ and $S-K$ show that $\tilde{H}_l(N-\tilde{K}) = \tilde{H}_l(S-K) = 0$ for $l < n$. Consider the ladder of exact sequences, where the vertical arrows are transfer maps, and $j: N-\tilde{K} \to N$ is the inclusion:

$$\begin{tikzcd}
\arrow[r] & H_{l+1}(N, N-\tilde{K}) \arrow[r] & H_l(N-\tilde{K}) \arrow[r, "j_*"] & H_l(N) \arrow[r] & H_l(N, N-\tilde{K}) \arrow[r] & {} \\
\arrow[r] & H_{l+1}(S, S-K) \arrow[r] \arrow[u, "t'"] & H_l(S-K) \arrow[r] \arrow[u, "t"] & H_l(S) \arrow[r] \arrow[u, "t''"] & H_l(S, S-K) \arrow[r] \arrow[u, "t'"] & {}
\end{tikzcd}$$

Since t' is an isomorphism, $\tilde{H}_l(N) = 0$ for $l < n$, and the sequence

$$H_n(S-K) \xrightarrow{t} H_n(N-\tilde{K}) \xrightarrow{j} H_n(N) \longrightarrow 0$$

is exact.

Suppose $n > 1$. The Van Kampen theorem shows that $\pi_1(N)$ is trivial, so N is $(n-1)$-connected. Consider the following diagram, whose columns are Wang sequences and whose top row is the above sequence:

$$\begin{tikzcd}
0 & 0 & & \\
H_n(S-K) \arrow[r, "t"] \arrow[u] & H_n(N-\tilde{K}) \arrow[r, "j_*"] \arrow[u] & H_n(N) \arrow[r] & 0 \\
H_n(F) \arrow[r, "1+h+\dots+h^{k-1}"] \arrow[u, "i_*"] & H_n(F) \arrow[u, "\tilde{i}_*"] & & \\
H_n(F) \arrow[r, "1"] \arrow[u, "1-h"] & H_n(F) \arrow[u, "1-h^k"] & &
\end{tikzcd}$$

Let g be a positive generator of the group of deck transformations of $N-\tilde{K}$ over $S-K$. For all $c\in H_n(F)$ and all integers l, $g_*^l\tilde{i}_*c=\tilde{i}_*h^lc$. Using the first definition of the transfer, $t(i_*c)=g_*^0\tilde{i}_*c+\ldots+g_*^{k-1}\tilde{i}_*c=\tilde{i}_*h^0c+\ldots+\tilde{i}_*h^{k-1}c$, so the top square commutes. A diagram chase now shows that

$$H_n(F)\xrightarrow{I+h+\ldots+h^{k-1}}H_n(F)\xrightarrow{j_*\tilde{i}_*}H_n(N)\to 0$$

is exact.

A similar argument works when $n=1$.

4. Homeomorphism Periodicity

As in the previous section, let $K\subset S^{2n+1}$ be a simple fibered knot and let K_k be the smooth k-fold cyclic cover of S^{2n+1} branched along K. In § 3 we showed that K_k is an $(n-1)$-connected $(2n+1)$-manifold. Furthermore we concluded that if the monodromy is of period d, and if K is a rational homology sphere, then $H_*(K_{k+d})\simeq H_*(K_k)$ for all k. In this section we extend the homological periodicity for odd n to a homeomorphism periodicity. This necessitates calculating the linking pairing on the torsion subgroup of $H_n(K)$ and using classification results for highly-connected manifolds, which in turn requires that we know considerably more about K_k.

The *linking pairing* lk of K_k on the torsion subgroup of $H_n(K_k)$ is defined in the classical way [21, § 77].

When K is a homology sphere and $n>1$, it is possible to compute this in an elementary fashion reminiscent of Seifert [20]: According to § 3, the sequence of integral homology groups

$$H_n(F)\xrightarrow{I-h^k}H_n(F)\xrightarrow{\tilde{i}_*}H_n(K_k)\to 0$$

is exact. Let $[c]$ denote the homology class of a chain c. Suppose y is an integral n-cycle on F with $\tilde{i}_*[y]$ torsion, and x is a rational n-cycle on F with $(I-h^k)[x]$ integral and $\tilde{i}_*(I-h^k)[x]$ torsion of order r. Let $\hbar$ be a geometric map of F inducing the homological monodromy h. Transporting x around the fiber bundle gives a rational $(n+1)$-chain X on K_k with $\partial X=(I-\hbar^k)x$. Since rx is an integral chain, so is rX, and $\partial(rX)=r(I-\hbar^k)x$. The linking number of $[(I-\hbar^k)x]=\tilde{i}_*(I-h^k)[x]$ with $\tilde{i}_*[y]$ is by definition $1/r$ times the intersection number of X and y in K_k mod1, i.e., the intersection number of X and y in K_k mod1, which is seen to be the negative of the intersection number $S([x],[y])$ in F mod1. Thus one obtains the formula

$$lk(\tilde{i}_*(I-h^k)[x],\tilde{i}_*[y])=-S([x],[y])\ \mathrm{mod}\,1\,,$$

which determines the linking pairing on K_k. It is then easy to prove periodicity in this case. However, if K is not a homology sphere, this approach does not work. Instead we will compute the linking pairing of K_k from the intersection form of a manifold F_k which it bounds.

Recall that our fibered knot $K\subset S^{2n+1}$ has a unimodular integral bilinear *Seifert form* θ defined on $H_n(F)$, where $F=\phi^{-1}(1)$ is a fiber of the fibration $\phi:S^{2n+1}-K\to S^1$. For the precise definition, which we will not need, see [3, § 2].

The monodromy h of § 3 may be calculated from this as $(-1)^{n+1}\theta^{-1}\theta'$ (where $'$ denotes transpose). Since the fibration of the complement $S^{2n+1}-K$ is trivial in a tubular neighborhood of K, the map $I-h$ induces a *variation map* $var: H_n(F, \partial F) \to H_n(F)$. Let $\bar{F}$ be the closure of the fiber F in S^{2n+1}. The integral bilinear *intersection form* S on $H_n(\bar{F})$ is defined by evaluating the cup product of the dual classes on the orientation class v of $\bar{F}$. In terms of the Seifert form, $S=\theta+(-1)^n\theta'$. (See for example [11, Lemma 2.1]. There is some confusion about the sign; we are following the conventions of [3] rather than [11].) The variation map is an isomorphism, for $\theta(x, \mathrm{var}(\alpha\cap v))=\theta(x,(I-h)(\alpha\cap v))=\theta(x,\alpha\cap v)+(-1)^n\theta(\alpha\cap v, x)= S(x,\alpha\cap v)=(-1)^n S(\alpha\cap v, x)=(-1)^n\alpha(x)$, for all $\alpha\in H^n(F,\partial F)$ and $x\in H_n(F)$ [11, § 2]. Thus the matrix of *var* with respect to a basis of $H_n(F)$ and the corresponding Lefschetz-dual basis of $H_n(F,\partial F)$ is the inverse of the matrix of $(-1)^n\theta$.

We let Λ_k be the $(k-1)$-dimensional Seifert form of the empty knot of degree $k\geqq 1$; with respect to a basis $e_1,\dots,e_{k-1}$ of $\mathbb{Z}^{k-1}$ the bilinear form Λ_k has matrix (a). The monodromy $h_k=-\Lambda_k^{-1}\Lambda_k'$ of this fibered knot then has matrix (b), the companion matrix of the polynomial $t^{k-1}+\dots+t+1$ [3, § 1.3].

$$\begin{bmatrix} 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots \; 1 & -1 \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & & & -1 \\ 1 & 0 & & \vdots \\ & \ddots \; 1 & 0 & -1 \\ & & 1 & -1 \end{bmatrix}.$$

(a) (b)

We need the following result on the branched cyclic cover K_k.

Proposition 4.1. *K_k embeds in a natural way in S^{2n+3} as a fibered knot. The fiber F_k of the fibration has $H_{n+1}(F_k)\simeq H_n(F)\otimes\mathbb{Z}^{k-1}$. With respect to this isomorphism, the monodromy of $K_k\subset S^{2n+3}$ is $h\otimes h_k$, and the Seifert form is $(-1)^{n+1}\theta\otimes\Lambda_k$.*

It follows, using the Lefschetz duality isomorphism, that $H_{n+1}(F_k,\partial F_k)\simeq H_{n+1}(F_k)^*\simeq(H_k(F)\otimes\mathbb{Z}^{k-1})^*\simeq H_n(F)^*\otimes(\mathbb{Z}^{k-1})^*\simeq H_n(F,\partial F)\otimes\mathbb{Z}^{k-1}$, where the isomorphism $(\mathbb{Z}^{k-1})^*\simeq\mathbb{Z}^{k-1}$ is determined by the above choice of basis of $\mathbb{Z}^{k-1}$ and the dual basis in $(\mathbb{Z}^{k-1})^*$. By the above comments, the variation of $K_k\subset S^{2n+3}$ is $\mathrm{var}\otimes\Lambda_k^{-1}$ with respect to this isomorphism.

The proposition and its consequences are proved in [11] by geometric arguments. [The assumption in the case $n=2$ that every homology class in $H_2(F)$ may be represented by a combinatorially embedded two-sphere is unnecessary.] For links of hypersurface singularities as in § 2, the statement about the monodromy is proved in [19], and the more general assertion about the Seifert form in [7, 12] and [18]. Generalizations of Proposition 4.1 have been announced in [12] and [17].

In particular, note that K_k is an $(n-1)$-connected $(2n+1)$-manifold, and is stably parallelizable (which we did not know before). The map $1\otimes h_k$ is induced by a generating deck transformation of F_k over D^{2n+2}.

A consequence of 4.1 is that the link of the singularity $z_0^{a_0}+\dots+z_n^{a_n}$ has Seifert form $(-1)^{n(n+1)/2}\Lambda_{a_0}\otimes\dots\otimes\Lambda_{a_n}$.

Next we reprove Theorem 3.1. Let H be the monodromy of $K_k \subset S^{2n+3}$, and consider $I-H$. The corresponding transformation $I-h\otimes h_k$ of $H_n(F)\otimes\mathbb{Z}^{k-1}$ has $(k-1)\times(k-1)$ block matrix

$$\begin{bmatrix} I & & & & h \\ -h & I & & & h \\ & \ddots & \ddots & & \vdots \\ & & -h & I & h \\ & & & -h & I+h \end{bmatrix}.$$

By multiplying the first column by $-h$ and adding it to the last, then multiplying the second by $-(h+h^2)$ and adding it to the last, and so forth, up to the $(k-2)^{\text{nd}}$ column, we transform this matrix into a lower triangular matrix with I along the diagonal, except for the bottom right-hand corner, which contains $I+h+\ldots+h^{k-1}$. This reproves Theorem 3.1, since the Wang sequence of the fibration $S^{2n+3}-K_k \to S^1$ exhibits $H_n(K_k)$ as the cokernel of $I-H$.

The rest of this section is concerned with the linking pairing of K_k. Wall [24, p. 274] refines this by a quadratic form when the torsion subgroup of $H_n(K_k)$ contains two-torsion, and $n \neq 1$, 3, or 7 is odd. In our case, K_k is the boundary of the n-connected manifold F_k, so both these forms may be computed indirectly: Let u and u' be torsion classes in $H_n(K_k)$ with $ru=r'u'=0$. Consider the long exact sequence

$$H_{n+1}(F_k) \xrightarrow{i_*} H_{n+1}(F_k, \partial F_k) \xrightarrow{\partial} H_n(K_k) \to 0$$

of the pair $(F_k, \partial F_k)$. The last term is zero since F_k is n-connected. Let $\partial y = u$, $\partial y' = u'$, $i_* x = ry$, and $i_* x' = r'y'$. Then the linking number of u and v is $(rr')^{-1} \cdot S(x, x') \bmod 1$, where S is intersection pairing on $H_{n+1}(F_k)$. The refining quadratic form is similarly determined mod2 ([2, 24]).

Let $\varrho = \sum_{j=0}^{k-2} \sum_{i=j}^{k-2} h^i \otimes h_k^j$ and let $(I\otimes e_1)x = x\otimes e_1$.

Lemma 4.2. *The following diagram commutes, and has exact horizontal rows:*

$$\begin{tikzcd}
H_{n+1}(F_k) \arrow[r] \arrow[d, "\simeq"] & H_{n+1}(F_k, \partial F_k) \arrow[r, "\partial"] \arrow[d, "\simeq"'] & H_n(K_k) \arrow[r] \arrow[ddd, "\simeq"', "(-1)^{n+1}I"] & 0 \\
H_n(F)\otimes\mathbb{Z}^{k-1} \arrow[r] \arrow[dr, "I-h\otimes h_k"] & H_n(F, \partial F)\otimes\mathbb{Z}^{k-1} \arrow[d, "\simeq"', "\mathrm{var}\otimes\Lambda_k^{-1}"] & & \\
H_n(F)\otimes\mathbb{Z}^{k-1} \arrow[u, "\varrho"] \arrow[r, "(I+h+\ldots+h^{k-1})\otimes I"'] & H_n(F)\otimes\mathbb{Z}^{k-1} & & \\
H_n(F) \arrow[u, "I\otimes e_1"] \arrow[r, "I+h+\ldots+h^{k-1}"] & H_n(F) \arrow[u, "I\otimes e_1"] \arrow[r, "\tilde{i}_*"] & H_n(K_k) \arrow[r] & 0
\end{tikzcd}$$

Proof. A direct computation shows that the lower left triangle commutes. In order to see that the large right-hand rectangle commutes, it is necessary to recall some of the geometry of the situation. In [11], maps $\Sigma: H_n(F) \to H_{n+1}(F_k)$ and $\mathcal{T}: H_n(F, \partial F) \to H_{n+1}(F_k, \partial F_k)$ are constructed. The maps Σ and $\mathcal{T}$ suspend homology classes, and have the properties $\partial\mathcal{T}=\tilde{i}_*$ var and Var$\mathcal{T}=(-1)^{n+1}\Sigma$ var, where Var is the variation for K_k. Thus $\tilde{i}_*=\tilde{i}_*$ var var$^{-1}=\partial\mathcal{T}$ var$^{-1}=(-1)^{n+1}\cdot$ ∂Var$^{-1}\Sigma$ var var$^{-1}=(-1)^{n+1}\partial$Var$^{-1}\Sigma$. This is equivalent to commutativity of the right-hand rectangle. The rest of the lemma follows from the above remarks.

Corollary 4.3. *The linking and quadratic forms of $H_n(K_k)$ are determined by the bilinear form $[\,,\,]_k$ on $H_n(F)$ defined by setting $[x,y]_k$ equal to the intersection number of $\varrho(x\otimes e_1)$ and $\varrho(y\otimes e_1)$ in $H_{n+1}(F_k)$ (or $H_n(F)\otimes\mathbb{Z}^{k-1}$).*

Proof. This follows at once from Lemma 4.2: Let u and v be torsion classes in $H_n(K_k)$ of order r and s respectively, and let x' and y' in $H_n(F)$ be such that $\tilde{i}_*x'=u$ and $\tilde{i}_*y'=v$. There are x and y in $H_n(F)$ with $(I+h+\ldots+h^{k-1})x=rx'$ and $(I+h+\ldots+h^{k-1})y=sy'$. Now referring to the top of the diagram, $lk(u,v)=(1/rs)[x,y]_k$. The proof for the quadratic form is similar.

One can compute that

$$[x,y]_k=\theta((\textstyle\sum_{i=0}^{k-2}(k-i-1)h^i)x,y)+\theta((\sum_{i=0}^{k-2}(k-i-1)h^i)y,x),$$

where θ is the Seifert pairing of $K\subset S^{2n+1}$. It is not clear how this formula is related to the simple one in the beginning of this section, and it is tricky to prove periodicity using it. Hence we will write $[\,,\,]_k$ another way. We have

$$[x,y]_k=LD_k(\varrho(x\otimes e_1),(\mathrm{var}^{-1}\otimes\Lambda_k)(y+hy+\ldots+h^{k-1}y)\otimes e_1),$$

where $LD_k: H_{n+1}(F_k)\times H_{n+1}(F_k,\partial F_k)\to\mathbb{Z}$ is the Lefschetz duality pairing on F_k. This can be evaluated on $H_n(F)\otimes\mathbb{Z}^{k-1}\times H_n(F,\partial F)\otimes\mathbb{Z}^{k-1}$ as well. Now $\Lambda_k e_1=e_1$, so since the $\mathbb{Z}^{k-1}$ factors have dual bases, only the first component of $\varrho(x\otimes e_1)$ contributes, and the above expression equals

$$LD_k((x+hx+\ldots+h^{k-2}x)\otimes e_1,\mathrm{var}^{-1}(y+hy+\ldots+h^{k-1}y)\otimes e_1).$$

Thus

$$[x,y]_k=LD(x+hx+\ldots+h^{k-2}x,\mathrm{var}^{-1}(y+hy+\ldots+h^{k-1}y)),$$

where $LD: H_n(F)\times H_n(F,\partial F)\to\mathbb{Z}$ is the Lefschetz duality pairing on F.

Now suppose that $h^d=I$ for some d and that K is a rational homology sphere. Thus $1+h+\ldots+h^{d-1}=0$, as in the proof of 3.2 (i), so we have the following extension of 3.2 (ii):

Lemma 4.4. *The linking and quadratic forms of K_{k+d} are isomorphic to those of K_k, for all $k\geqq 1$.*

For the example at the end of § 5 the linking on K_k is the negative of the linking on K_{d-k}. We do not know if Corollary 3.2 (iii) always extends this way.

Theorem 4.5. *Let $K\subset S^{2n+1}$ for odd $n\neq 1$, 3, or 7 be a fibered knot. Suppose that the monodromy has period d and K is a rational homology sphere. Then K_{k+d} is homeomorphic to K_k, for all $k\geqq 1$.*

Proof. This follows immediately from 4.4 and the fact for odd $n\neq 1$, 3, or 7 that homology groups, linking pairing, and quadratic form classify stably-parallelizable $(n-1)$-connected $(2n+1)$-manifolds up to homeomorphism (in fact, "almost diffeomorphism") [2, 24].

5. Diffeomorphism Periodicity

In this section we extend the homeomorphism periodicity of the smooth k-fold cyclic cover K_k of a simple fibered knot $K \subset S^{2n+1}$, with n odd not 1, 3, or 7, to a diffeomorphism periodicity. To find the differentiable structure of K_k, we only need calculate the signature of the manifold F_k of § 4 which it bounds. The main idea is that the Seifert pairing of the empty knot is diagonalizable as sesquilinear form over the complex numbers.

First some algebraic preliminaries. Let V be a finite-dimensional complex vector space. Recall that a map $\Sigma: V \times V \to \mathbb{C}$ is *sesquilinear* if it is additive in each factor, $\Sigma(ax, y) = a\Sigma(x, y)$, and $\Sigma(x, by) = \bar{b}\Sigma(x, y)$ for all complex numbers a, b and all $x, y \in V$. Two such forms Σ_1 and Σ_2 are *isomorphic*, written $\Sigma_1 \simeq \Sigma_2$, if there is an isomorphism of the underlying vector spaces preserving the forms.

Suppose Σ is non-singular. The automorphism h of V with matrix $\pm \Sigma^{-1}\bar{\Sigma}'$ is an isometry with respect to Σ [that is, $\Sigma(hx, hy) = \Sigma(x, y)$ for all $x, y \in V$] which depends only on the isomorphism class of Σ. (Here $'$ denotes transpose. Either sign may be chosen.) If x and y are eigenvectors of h with eigenvalues a and b satisfying $a\bar{b} \neq 1$, then $\Sigma(x, y) = 0$.

The Seifert pairing θ of a fibered knot is an example of a sesquilinear form, after extension of coefficients from $\mathbb{Z}$ to $\mathbb{C}$.

Let Λ_k be the Seifert form of the empty knot of degree k and $h_k = -\Lambda_k^{-1}\bar{\Lambda}_k'$ its monodromy (§ 4). The automorphism h_k acts on the basis $e_1, \ldots, e_{k-1}$ by $e_i \to e_{i+1}$ for $1 \leqq i \leqq k-2$, and $e_{k-1} \mapsto -(e_1 + \ldots + e_{k-1})$. Its eigenvalues are $\eta, \ldots, \eta^{k-1}$, where η is some primitive kth root of unity. Hence the underlying complex vector space is the direct sum of one-dimensional eigenspaces, and as sesquilinear form Λ_k splits orthogonally on this direct sum. For any kth root of unity $\omega \neq 1$, the vector $x_\omega = e_1 + \bar{\omega}he_1 + \ldots + \bar{\omega}^{k-1}h^{k-1}e_1$ is an eigenvector of h with eigenvalue ω. Since $\Lambda_k(x_\omega, x_\omega) = k(1-\omega)$, Λ_k is isomorphic to a diagonal form:

$$\Lambda_k \simeq \langle 1-\eta, 1-\eta^2, \ldots, 1-\eta^{k-1} \rangle .$$

For any sesquilinear form Σ and any complex number $\omega \neq 1$ of unit norm we define an *associated Hermitian form* $\Sigma_\omega = (1-\omega)\Sigma + (1-\bar{\omega})\bar{\Sigma}'$. (See [15, § 25] and [22]. However, we will give this form geometric meaning.) This process is well-defined on isomorphism classes and commutes with direct sums. Clearly $((1-\omega)\Sigma)_{-1} = 2\Sigma_\omega \simeq \Sigma_\omega$. If Σ is isomorphic to a Seifert pairing θ, the Hermitian form Σ_{-1} is isomorphic to the symmetrized Seifert pairing $\theta + \theta'$; in particular their signatures are equal.

Here is an application of the above ideas. By 4.1, the link of the singularity $z_0^{a_0} + \ldots + z_n^{a_n} = 0$ at the origin has Seifert form

$$\theta = (-1)^{n(n+1)/2} \Lambda_{a_0} \otimes \ldots \otimes \Lambda_{a_n} \simeq (-1)^{n(n+1)/2} \oplus \langle (1-\eta_0^{j_0}) \ldots (1-\eta_n^{j_n}) \rangle ,$$

where the η_i are primitive a_ith roots of unity and the sum is over all $0 < j_k < a_k$ for $k = 0, \ldots, n$. For even n, the intersection pairing on the fiber F of the fibration is $\theta + \theta'$, and its eigenvalues are positive, negative, or zero according as the real part of $(-1)^{n/2}(1-\eta_0^{j_0}) \ldots (1-\eta_n^{j_n})$ is positive, negative, or zero. From this the usual formula for the signature of $\theta + \theta'$ can be derived ([1, Satz 3]. Start on p. 12, third line from the bottom.)

We apply this next to branched cyclic covers. Let $K\subset S^{2n+1}$ for n odd be a simple fibered knot with Seifert pairing θ. According to the beginning of § 4, the smooth k-fold cyclic cover K_k of S^{2n+1} branched along K is boundary of a parallelizable manifold F_k with intersection pairing

$$\begin{aligned}\theta\otimes\Lambda_k+(\theta\otimes\Lambda_k)' &\simeq(\theta\otimes\Lambda_k)_{-1}\simeq((1-\eta)\theta)_{-1}\oplus\ldots\oplus((1-\eta^{k-1})\theta)_{-1}\\ &\simeq\theta_\eta\oplus\ldots\oplus\theta_{\eta^{k-1}}\end{aligned}$$

as Hermitian forms. We have proved the following result.

Proposition 5.1. *The intersection pairing of F_k is isomorphic, as Hermitian form, to $\theta_\eta\oplus\ldots\oplus\theta_{\eta^{k-1}}$.*

Geometrically this decomposition corresponds to the eigenspace decomposition of $H_{n+1}(F_k)\otimes\mathbb{C}$ with respect to the covering translation of F_k over D^{2n+2} (see [11, § 4]). The computation of the intersection pairing on $H_{n+1}(F_k)\otimes\mathbb{C}$ and its decomposition into Hermitian forms as above has been independently obtained by S. Capell and J. Shaneson.

Define σ_k for $k\geqq 1$ to be the signature σ of the intersection pairing on $H_{n+1}(F_k)$. When $k=2$ this is the usual (Murasugi) signature; also $\sigma_1=0$. The above proposition shows that

$$\sigma_k=\sigma(\theta_\eta)+\ldots+\sigma(\theta_{\eta^{k-1}})\,.$$

Thus σ_k depends only on the isotopy class of $K\subset S^{2n+1}$, by results of Levine (see, for instance, [3]). This also may been seen as follows:

Remark. These signatures may actually be defined for any codimension-two submanifold K of S^{2n+1} with n odd (see also [13]): Let $L\subset D^{2n+2}$ be an oriented compact connected manifold with ∂L, the transversal intersection of L and S^{2n+1}, equal to K. Let L_k be the k-fold cyclic cover of D^{2n+2} branched along L, and define σ_k to be the signature $\sigma(L_k)$ of the middle-dimensional cup product pairing on L_k. This is independent of the choice of L: Let $L'\subset D^{2n+2}$ be another spanning manifold. Then $L\cup(-L')\subset D^{2n+2}\cup D^{2n+2}=S^{2n+2}$ is a knot; choose a spanning manifold $M\subset D^{2n+3}$ with $\partial M=M\cap S^{2n+2}=L\cup(-L')$. Let L'_k be the k-fold cyclic cover of D^{2n+2} branched along L', and M'_k the k-fold cyclic cover of D^{2n+3} branched along M. Then $\partial M_k=L\cup(-L'_k)$, so $0=\sigma(L_k\cup(-L'_k))=\sigma(L_k)-\sigma(L'_k)$. See also [17, (iv), p. 980].

The following result is a special case of a signature periodicity theorem announced by Neumann [17, pp. 978, 980].

Theorem 5.2. *Let $K\subset S^{2n+1}$ for odd $n\geqq 1$ be a simple fibered knot. Suppose there is a natural number d such that $\zeta^d=1$ for all roots ζ of the characteristic polynomial of the monodromy* (§ 3) *with $|\zeta|=1$. Then $\sigma_{k+d}=\sigma_k+\sigma_{d+1}$, for all $k\geqq 1$.*

The hypotheses of 5.2 are of course satisfied if the monodromy is periodic of period d. They are also satisfied for links of isolated hypersurface singularities (§ 2), since the eigenvalues of the monodromy are well known to be roots of unity in that case.

Proof. Let θ be the Seifert pairing of $K\subset S^{2n+1}$. For $\omega\neq 1$ with $|\omega|=1$, the Hermitian form $\theta_\omega=(1-\omega)\theta+(1-\bar\omega)\bar\theta'=(1-\omega)\theta(I-\bar\omega\theta^{-1}\bar\theta')$ is unimodular unless $\det(I-\bar\omega\theta^{-1}\bar\theta')=\det(I-\bar\omega h)=0$, where h is the monodromy of $K\subset S^{2n+1}$. By assumption, such ω satisfy $\omega^d=1$.

Partition $S^1-\{1\}$ into sets $A_1,\ldots,A_d$, $B_1,\ldots,B_{d-1}$, where $A_l=\{e^{2\pi i t}:(l-1)/d<t<l/d\}$ and $B_l=\{e^{2\pi i l/d}\}$. As in [15, § 25], $\sigma(\theta_\omega)$ is constant for ω in a fixed A_l.

By Proposition 5.1, $\sigma_k=\sigma(\theta_\eta)+\ldots+\sigma(\theta_{\eta^{k-1}})$, where η is a primitive kth root of unity. The number of $(k+d)$th roots of unity lying in some A_l is one more than the number of kth roots in A_l, and the number of $(k+d)$th roots in some B_l is the same as the number of kth roots in B_l. If ω is a primitive $(d+1)^{\text{st}}$ root of unity, then $\omega^l\in A_l$. Thus $\sigma_{d+k}=\sigma_k+\sigma_{d+1}$, for all $k\geqq 1$.

This argument may be generalized to non-fibered knots. Finally we have the diffeomorphism periodicity theorem:

Theorem 5.3. *Let $K\subset S^{2n+1}$ for odd $n\neq 1$, 3, or 7 be a simple fibered knot. Suppose that the monodromy has period d and K is a rational homology sphere. Then K_{k+d} is diffeomorphic to $(\sigma_{d+1}/8)\Sigma\,\#\,K_k$, for all $k\geqq 1$.*

Here $(\sigma_{d+1}/8)\Sigma$ denotes the connected sum of $\sigma_{d+1}/8$ copies of the Milnor sphere Σ, and σ_{d+1} is as in 5.2.

Proof. By Theorem 4.5, the linking and quadratic forms of K_k and K_{k+d} are isomorphic. Hence [2, Theorem 8.1], K_{k+d} is diffeomorphic to $\frac{1}{8}(\sigma_{k+d}-\sigma_k)\Sigma\,\#\,K_k$, and the result follows from Theorem 5.2.

We close with the example of the introduction. Let $n\geqq 1$ be odd and let $K\subset S^{2n+1}$ be the link of the singularity $z_0^3+z_1^2+\ldots+z_n^2=0$. Thus K is a trefoil knot if $n=1$, and a "suspended" trefoil knot if $n>1$. The fiber F has $H_n(F)$ of rank2, the Seifert pairing θ is $(-1)^{(n-1)/2}\Lambda_3$, and the monodromy $h=\Lambda_3^{-1}\Lambda_3'$ has period 6. By 3.2, $H_n(K_{k+6})\simeq H_n(K_k)$. One easily calculates that

$$\begin{array}{ll} H_n(K_1)\simeq 0 & H_n(K_4)\simeq \mathbb{Z}_3 \\ H_n(K_2)\simeq \mathbb{Z}_3 & H_n(K_5)\simeq 0 \\ H_n(K_3)\simeq \mathbb{Z}_2\oplus\mathbb{Z}_2 & H_n(K_6)\simeq \mathbb{Z}\oplus\mathbb{Z}\,. \end{array}$$

This is Fox's example. The linking can be calculated as in the beginning of § 4.

To simplify signs, assume $(n-1)/2$ even. Then $\theta_\omega=(1-\omega)\Lambda_3+(1-\bar{\omega})\bar{\Lambda}_3'$ is isomorphic to the diagonal matrix $\langle 2-\omega-\bar{\omega}, 1-\omega-\bar{\omega}\rangle$, so $\sigma(\theta_\omega)=1+\sigma(\langle 1-\omega-\bar{\omega}\rangle)$. Thus $\sigma(\theta_\omega)$ is 0 if ω belongs to the smaller component of $S^1-\{e^{2\pi i/6}, e^{-2\pi i/6}\}$, and is 2 if ω is in the larger component. Also, $\sigma(\theta_{e^{2\pi i/6}})=\sigma(\theta_{e^{-2\pi i/6}})=1$. Hence

$$\begin{array}{lll} \sigma_1=0 & \sigma_4=6 & \sigma_7=8 \\ \sigma_2=2 & \sigma_5=8 & \\ \sigma_3=4 & \sigma_6=8 & \end{array}$$

and $\sigma_{k+6}=\sigma_k+8$ for all $k\geqq 1$.

K_k is the link of the singularity $z_0^3+z_1^2+\ldots+z_n^2+z_{n+1}^k=0$ by 2.1. In particular the link of $z_0^3+z_1^2+\ldots+z_n^2+z_{n+1}^{6l-1}$ for odd $n\geqq 3$ is diffeomorphic to a connected sum of $(-1)^{(n+1)/2}l$ copies of the Milnor sphere.

For another approach to this example, see [2, § 11.3].

6. An Example with n Even

Let $K\subset S^{2n+1}$ for even $n\geqq 2$ be the link of the singularity $z_0^2+z_1^2+\ldots+z_n^2=0$, and let K_k for $k\geqq 1$ be its smooth branched k-fold cyclic cover, the link of $z_0^2+\ldots$

$\ldots + z_n^2 + z_{n+1}^k = 0$. We let Σ denote the $(2n+1)$-dimensional Kervaire sphere and T the tangent S^n-bundle to S^{n+1}. The symbol $\approx$ means "is diffeomorphic to."

Proposition 6.1. *K_{k+8} is diffeomorphic to K_k, and*

$K_1 \approx S^{2n+1}$	$K_5 \approx \Sigma$
$K_2 \approx T$	$K_6 \approx T \# \Sigma$
$K_3 \approx \Sigma$	$K_7 \approx S^{2n+1}$
$K_4 \approx (S^n \times S^{n+1}) \# \Sigma$	$K_8 \approx S^n \times S^{n+1}$.

The result is well known for odd k [1].

The original proof for all k used "mod2 reduction," a general 2-adic computation technique [2, § 11.5]. We prove it below by a simple change of basis. For a geometric approach using link manifolds and "band passing", see [10]. Recall also that $\Sigma \approx S^{2n+1}$ at least if $n = 0$, 2, and 6, and that $T \# \Sigma \approx T$. (See for instance Wall's theorem below.)

Again our results depend on classification theorems for highly-connected manifolds. Suppose m is odd and that F is a stably-parallelizable $(m-1)$-connected $2m$-manifold with $(m-2)$-connected boundary. We measure more carefully the geometric self-intersection of homology classes in $H_m(F)$ by means of a quadratic form $\psi: H_m(F)/2H_m(F) \to \mathbb{Z}_2$ defined by Wall [23]: If $x \in H_m(F)$ is represented by a smoothly embedded sphere, let $[x]$ be the image of x in $H_m(F)/2H_m(F)$ and define $\psi([x])$ to be the characteristic element of the normal bundle to x. This characteristic element is in the kernel of the map $\pi_{m-1}(\mathrm{SO}_n) \to \pi_{m-1}(\mathrm{SO})$, which is isomorphic to $\mathbb{Z}_2$ for odd m not 1, 3, or 7, and 0 otherwise. According to Wall [23], two such manifolds F_1 and F_2 are diffeomorphic if and only if their intersection forms S_1 and S_2 and their self-intersection forms ψ_1 and ψ_2 are simultaneously congruent, that is, there is an isomorphism $f: H_m(F_1) \to H_m(F_2)$ with $S_1(x, y) = S_2(fx, fy)$ and $\psi_1([x]) = \psi_2([fx])$ for all $x, y \in H_m(F_1)$. (Of course the self-intersection forms are unnecessary if $m = 1, 3$, or 7.)

If F is the fiber of a simple fibered knot $K \subset S^{2n+1}$ with Seifert form θ, then $\psi([x])$ is $\theta(x, x)$ reduced mod2 [14].

Proof of Proposition 6.1. Let $m = n+1$. By 4.1, the Seifert form θ of $K_k \subset S^{2m+1}$ is $(-1)^{(m+1)/2} \Lambda_k$ with respect to the basis $e_1, \ldots, e_{k-1}$ of § 4. To simplify notation, we will assume that the sign$(-1)^{(m+1)/2}$ is $+1$. We compute the intersection form $\theta + \theta'$ and the self-intersection form $\theta(x, x)$ mod2 on the fiber of the fibered knot $K_k \subset S^{2m+1}$.

Introduce new basis elements $e'_{2i-1} = e_1 + e_3 + \ldots + e_{2i-1}$ and $e'_{2i} = e_{2i}$, for $i = 1, 2, \ldots$. Then

$$\psi([e'_i]) = \begin{cases} 0, & \text{for} \quad i = 3, 7, 11, \ldots \\ 1, & \text{otherwise}. \end{cases}$$

Let H be the 2-dimensional skew-symmetric form defined on $\mathbb{Z}x \oplus \mathbb{Z}y$ by $H(x, y) = -H(y, x) = 1$.

Case 1. k is odd. The intersection form in the new basis is isomorphic to $H \oplus \ldots \oplus H$ and is unimodular. The quadratic form ψ also splits, and its Arf invariant $c(\psi)$ may be easily computed to be

$$c(\psi)=\begin{cases}0, & k\equiv 1,7\ (\mathrm{mod}\,8)\\ 1, & k\equiv 3,5\ (\mathrm{mod}\,8).\end{cases}$$

Case 2. k is even. The intersection form in the new basis is isomorphic to $H\oplus\ldots\oplus H\oplus\langle 0\rangle$. The quadratic form ψ splits as $\psi'\oplus\eta$, where ψ' is non-singular and η is singular and one-dimensional. Furthermore,

$$c(\psi')=\begin{cases}0, & k\equiv 0,2\ (\mathrm{mod}\,8)\\ 1, & k\equiv 4,6\ (\mathrm{mod}\,8).\end{cases}$$

The linear functional η is determined by

$$\eta(e'_{k-1})=\begin{cases}0, & k\equiv 0,4\ (\mathrm{mod}\,8)\\ 1, & k\equiv 2,6\ (\mathrm{mod}\,8).\end{cases}$$

We find $(m-1)$-connected $2m$-manifolds with the same invariants. $S^m\times S^m$ minus a small disk has boundary S^{2m-1}, intersection form H on $\mathbb{Z}\oplus\mathbb{Z}$, and non-singular self-intersection form on $\mathbb{Z}_2\oplus\mathbb{Z}_2$ of Arf invariant 0. The Kervaire sphere Σ is boundary of a plumbed $2m$-manifold with intersection form H and self-intersection form of Arf invariant 1. $D^m\times S^m$ has boundary $S^{m-1}\times S^m$, intersection form $\langle 0\rangle$ on $\mathbb{Z}$, and singular self-intersection form η on $\mathbb{Z}_2$ determined by $\eta(1)=0$. Lastly, T, the tangent S^{m-1} bundle to S^m, is boundary of the corresponding disk bundle, which has intersection form $\langle 0\rangle$ and singular self-intersection form η determined by $\eta(1)=1$, for $n\neq 1$, 3, or 7, and 0 otherwise. Taking connected sums of these $2m$-manifolds to match the invariants of the fiber of $K_k\subset S^{2m+1}$ proves the proposition.

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