polyedral homology manifold.

3) Other examples of aspherical manifolds for Theorems 1 and 2 are obtainable as follows : in the proofs, replace the degree one map $T^k \to S^k$ by a map $f : K \to S^k$ inducing an isomorphism on integral homology, where K is a finite aspherical polyhedron of dimension k (K and f exist by [Ma]). The manifold Q will then be a thickening of K with $\tau_Q = a \circ f$, which exists in the stable range.

4) By obstruction theory, if K is a complex of dimension 4, any map $K \to BG$ which lifts through BTOP admits a lifting through BPL. Therefore, it is not possible to assert that the manifolds M of Theorem 2 are not homotopy equivalent to closed PL-manifolds. But if a homotopy equivalence $f : M' \to M$ existed with M' a closed PL-manifold, then f would yeld a homotopy equivalence between aspherical closed manifolds which is not homotopic to a homeomorphism. This would be a negative answer to a question of A. Borel.

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Homology with simplicial coefficients

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§1. Introduction

1.1 Summary. This paper deals with *homology of simplicial sets over a fixed simplicial set*, say L, and in particular with the *simplicial coefficient systems* for such homology.

A simplicial coefficient system (over L) consists of a collection of abelian groups and homomorphisms between them indexed by the simplices of L and the simplicial operators between them, i.e., it is an abelian group object in the category of simplicial sets over L. If all the homomorphisms between the abelian groups are actually isomorphisms, then such a simplicial coefficient system is essentially a *local coefficient system* in the usual sense.

It turns out that, given a simplicial coefficient system A and a weak (homotopy) equivalence $K \to K'$ of simplicial sets over L, the induced map on homology $H_*(K;A) \to H_*(K';A)$ need *not* be an isomorphism unless either *the structure map* $A \to L$ *is a fibration of simplicial sets* or *both of the structure maps* $K \to L$ *and* $K' \to L$ *are so.* This suggests calling a map $A \to A'$ between simplicial coefficient systems a *weak equivalence* whenever it induces an isomorphism $H_*(K;A) \cong H_*(K;A')$ for every simplicial set K over L for which the structure map $K \to L$ is a fibration, and asking whether, given any simplicial coefficient system A, there exists a weak equivalence $A \to A'$ such that the structure map of A' is a fibration (and every weak equivalence $K \to K'$ of simplicial sets over L thus induces an isomorphism $H_*(K;A') \cong H_*(K';A')$). We give a positive answer to this question by showing that *the category* $\mathbf{ab}/L$ *of simplicial coefficient systems over* L *admits a closed simplicial model category structure in the sense of Quillen* in which the weak equivalences are as above and in which the fibrant objects are exactly those simplicial coefficient systems for which the structure map is a fibration.

In the remainder of the paper we compare the model categories $\mathbf{ab}/L$ and $\mathbf{ab}/L'$ for weakly equivalent L and L' and we observe that, for connected L, the weak equivalence classes of the simplicial coefficient systems over L are in a natural 1−1 correspondence with the weak equivalence classes of the simplicial modules over the loop group GL of L (and hence with the weak equivalence classes of non-negatively graded differential modules over the chains on GL).

1.2 Organization of the paper. After fixing some notation and terminology (in §2), we define (in §3) the *homology* $H_*(K;A)$ of a simplicial set K over L with simplicial coefficients A, and obtain some of its basic properties. *Weak equivalences* betweeen simplicial coefficient systems then are introduced in §4, where we also give a positive answer to the question which was raised in 1.1. In §5, we establish the *closed simplicial model category structure* on $\mathbf{ab}/L$. The proof is more difficult than one would expect and requires a *Bousfield cardinality argument.* The remaining two sections are devoted to the results which were mentioned at the end of 1.1.

1.3 Application. The arguments which establish the closed simplicial model category structure on $\mathbf{ab}/L$ will be used in [**3**,§6] to obtain closed simplicial model category structures *on the category of abelian group objects* over a fixed *simplicial diagram of simplicial sets* and *on the category of abelian group objects* over a fixed *small simplicial category.* An understanding of these structures is necessary for our study of Hochschild-Mitchell cohomology [**3**].

§2. Notation, terminology, etc.

We will use among others the following notation, terminology and results:

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2.1 Simplicial sets. As usual ([5] and [2, Ch. VIII]) S will denote the category of *simplicial sets*; for every integer $n \geq 0$, $\Delta[n] \in S$ will be the *standard n-simplex* (i.e., the simplicial set freely generated by a single n-simplex i_n), $\dot{\Delta}[n] \subset \Delta[n]$ will be its subcomplex spanned by the faces of i_n and, for every pair of integers (k,n) with $0 \leq k \leq n$, $V[n,k] \subset \Delta[n]$ will be the subcomplex spanned by the faces $d_j i_n$ with $j \neq k$. If $L \in S$ and $x \in L$ is an n-simplex, then $\Delta[x] : \Delta[n] \to L \in S$ will denote the unique map which sends i_n to x and $\dot{\Delta}[x] : \dot{\Delta}[x] \to L \in S$ and $V[x,k] : V[n,k] \to L \in S$ will be the restrictions of $\Delta[x]$ to $\dot{\Delta}[n]$ and $V[n,k]$, respectively.

2.2 The over category S/L. For $L \in S$, we write S/L for its *over category* (which has as objects the maps $K \to L \in S$). An object $(K \to L) \in S/L$ will often be denoted by K alone, without its *structure map* $K \to L$. To avoid confusion we therefore use $\times_L$ for the *product* in S/L.

2.3 A model category structure for S/L. The category S/L admits a closed simplicial model category structure [6, Ch.II] in which the simplicial structure is the obvious one and in which the fibrations, the cofibrations and the weak equivalences are induced by those of S [2, Ch.VIII]. Thus (2.2) an object $K \in S/L$ is fibrant iff its structure map $K \to L \in S$ is a fibration (in S).

2.4 Abelian group objects in S/L. An abelian group object in S/L consists of an object $(f : K \to L) \in S/L$ together with a *multiplication* map $m : K \times_L K \to K$, a *unit* map $u : L \to K$ and an *inverse* map $i : K \to K$ in S/L satisfying the usual abelian group axioms. These abelian group objects in S/L form an abelian category which we denote by $\mathbf{ab}/L$.

2.5 A pair of adjoint functors $S/L \leftrightarrow \mathbf{ab}/L$. The forgetful functor $U : \mathbf{ab}/L \to S/L$ has as left adjoint the functor $Z_L : S/L \to \mathbf{ab}/L$ which sends an object $K \in S/L$ (2.2) to the object $Z_L K$, consisting of the disjoint union of the free abelian groups on the inverse images (in K) of the simplices of L.

Using this pair of adjoint functors, one can assign to each object $A \in \mathbf{ab}/L$ its *simplicial resolution* $(Z_L U)^{*+1} A$ which is the simplicial object over $\mathbf{ab}/L$ which, in dimension n, consists of $(Z_L U)^{n+1} A$ and which has the property that

$$\pi_0 (Z_L U)^{*+1} A \cong A \text{ and } \pi_i (Z_L U)^{*+1} A = 0 (i > 0).$$

2.6 Homotopy categories. If **C** is a closed model category, then [6, Ch. 1, §1] ho(**C**) will denote its homotopy category, i.e., the category obtained by formally inverting all weak equivalences.

§3. Homology with simplicial coefficients

In this section, we define homology with simplicial coefficients and prove some of its basic properties. First some

3.1 Preliminaries. Let (2.1) $L \in S$ and (2.4) $A, A' \in \mathbf{ab}/L$. Then one can form the *tensor product* $A \otimes A' \in \mathbf{ab}/L$ which assigns to every simplex of L the tensor product of its inverse images in A and A', and note that, for $K, K' \in S/L$, *there is a natural isomorphism* (2.5) $Z_L K \otimes Z_L K' \cong Z_L (K \times_L K')$.

Another useful construction assigns to an object $A \in \mathbf{ab}/L$ the simplicial abelian group $\bigoplus_L A$ which, in dimension n, consists of the direct sum of the inverse images (in A) of the n-simplices of L. For $K \in S/L$, *the simplicial abelian group* $\bigoplus_L Z_L K$ *is just the free simplicial abelian group on the simplices of* K.

Now we can define

3.2 Homology with simplicial coefficients. Given $K \in S/L$ and $A \in \mathbf{ab}/L$, *the homology* $H_*(K; A)$ of K *with simplicial coefficients* A will be

$$H_*(K; A) = \pi_* \bigoplus_L (Z_L K \otimes A)$$

and this definition readily implies:

3.3 Proposition. *If $K, K' \in S/L$, then $H_*(K; Z_L K')$ is just the ordinary integral homology of $K \times_L K'$.*

3.4 Proposition. *Let $A \in \mathbf{ab}/L$ and $K \in S/L$ and let $K_1, K_2 \subset K$ be subcomplexes. Then there is a natural long exact (Mayer-Vietoris) sequence*

$$\to H_n(K_1 \cap K_2; A) \to H_n(K_1; A) \oplus H_n(K_2; A) \to H_n(K_1 \cup K_2; A) \to H_{n-1}(K_1 \cap K_2; A) \to$$

3.5 Proposition. *Let $K \in S/L$ and let $0 \to A'' \to A \to A' \to 0$ be a short exact sequence in $\mathbf{ab}/L$. Then there is a natural long exact sequence*

$$\cdots \to H_n(K; A'') \to H_n(K; A) \to H_n(K; A') \to H_{n-1}(K; A'') \to \cdots$$

Less obvious is

3.6 Proposition. *Let $A \in \mathbf{ab}/L$ and let $g : K \to K' \in S/L$ be a weak equivalence (2.3). Then g induces an isomorphism $H_*(K; A) \cong H_*(K'; A)$ whenever one of the following conditions is satisfied:*

(1) *the structure maps $K \to L$ and $K' \to L$ are both fibrations (in S) or,*
(2) *the structure map $A \to L$ is a fibration (in S).*

Proof: If $A = Z_L K''$ for some $K'' \in S/L$, then part (1) is an easy consequence of 3.3 and the general case now follows readily from (2.5) and the existence of simplicial resolutions.

Part (2) is proved in a similar manner using the following lemma.

3.7 Lemma. *If the structure map of $K \in S/L$ is a fibration (in S), then so is the structure map of $Z_L K$.*

Proof: Given a pair of integers (k,n) with $0 \leq k < n$ (resp. $0 < k \leq n$), an n-simplex $x \in L$ and an $(n-1)$-simplex $y_k \in Z_L K$ over $d_k x$ such that $d_i y_k = 0$ for $i < k$ (resp. $k < i$), a careful calculation (which uses the fact that the structure map $K \to L \in S$ is a fibration) yields an n-simplex $y \in Z_L K$ over x such that $d_k y = y_k$ and $d_i y = 0$ for $i < k$ (resp. $k < i$). The rest of the proof now is straightforward.

§4. Weak equivalences between simplicial coefficient systems

Next we discuss the notion of weak equivalence between simplicial coefficient systems which was mentioned in 1.1 and give a positive answer (4.6) to the question which was raised there. We start with the definition of

4.1 Weak equivalences between simplicial coefficient systems. A map $A \to A' \in \mathbf{ab}/L$ will be called a *weak equivalence* if, for every fibrant (2.3) object $K \in S/L$, it induces an isomorphism $H_*(K; A) \cong H_* K; A')$.

Using 3.4 and 3.6, one then readily shows

4.2 Proposition. *Let $A, A' \in \mathbf{ab}/L$ be such that (2.5) $UA, UA' \in S/L$ are fibrant. Then a map $A \to A' \in \mathbf{ab}/L$ is a weak equivalence iff the underlying map $UA \to UA' \in S/L$ is a weak equivalence (2.3).*

One also has

4.3 Proposition. *A map $A \to A' \in \mathbf{ab}/L$ is a weak equivalence if the underlying map $UA \to UA' \in S/L$ is a weak equivalence as well as a fibration.*

Proof: The map $A \to A'$ fits into a short exact sequence $0 \to A'' \to A \to A' \to 0$ such that the underlying map of the obvious map $A'' \to L \in \mathbf{ab}/L$ (recall that L is the zero object of $\mathbf{ab}/L$) is a weak equivalence as well as a fibration. By 4.2 the map $A'' \to L \in \mathbf{ab}/L$ is a weak equivalence and the desired result now readily follows from 3.5.

Furthermore 3.3 implies

4.4 Proposition. *If a map $K \to K' \in \mathbf{S}/L$ is a weak equivalence, then so is the induced map (2.3) $Z_L K \to Z_L K' \in \mathbf{ab}/L$.*

Applying this to the maps $V[x,k] \to \Delta[x] \in \mathbf{S}/L$ (2.1), one can construct as follows

4.5 The extension functor $E : \mathbf{ab}/L \to \mathbf{ab}/L$. For $A \in \mathbf{ab}/L$, let $EA \in \mathbf{ab}/L$ be determined by the push out diagram

$$\begin{array}{ccc} \coprod Z_L V[x,k] & \longrightarrow & \coprod Z_L \Delta[x] \\ \downarrow & & \downarrow \\ A & \longrightarrow & EA \end{array}$$

in which the sums are taken over all 4-tuples (k,n,x,g), where k and n are integers such that $0 \le k \le n$, x is an n-simplex of L and g is a map $g : Z_L V[x,k] \to A \in \mathbf{ab}/L$. Then 3.5 implies that the map $A \to EA \in \mathbf{ab}/L$ *is a weak equivalence* and hence *so is the resulting map*

$$A \to E^\infty A = \varinjlim E^n A \in \mathbf{ab}/L\ .$$

This last statement immediately provides a positive answer to the question which was raised in 1.1 as one has, almost by definition:

4.6 Proposition. *For every object $A \in \mathbf{ab}/L$, the structure map of $E^\infty A$ is a fibration of simplicial sets, i.e. $UE^\infty A \in \mathbf{S}/L$ is fibrant.*

We end with observing that the above results also readily imply the following *characterization of weak equivalences in* $\mathbf{ab}/L$.

4.7 Proposition. *A map $A \to A' \in \mathbf{ab}/L$ is a weak equivalence iff the induced map $UE^\infty A \to UE^\infty A' \in \mathbf{S}/L$ is a weak equivalence.*

4.8 Proposition. *Let $P \to L \in \mathbf{S}$ be a path fibration (i.e., a fibration such that (i) the induced map $\pi_0 P \to \pi_0 L$ is an isomorphism and (ii) each component of P is contractible). Then a map $A \to A' \in \mathbf{ab}/L$ is a weak equivalence iff the induced map $H_*(P;A) \to H_*(P;A')$ is an isomorphism.*

§5. A model category structure for $\mathbf{ab}/L$

The preceding results suggest

5.1 Proposition. *The category $\mathbf{ab}/L$ admits a closed simplicial model category structure [6, Ch. II] in which the simplicial structure is the obvious one, the weak equivalences are as in 4.1 and a map $X \to Y$ is a trivial fibration (i.e., a fibration as well as a weak equivalence) whenever the underlying map $UX \to UY \in \mathbf{S}/L$ is so (2.3).*

This, of course, implies the following rather formal

5.2 Definition of cofibrations and fibrations in $\mathbf{ab}/L$.

(1) The *cofibrations* in $\mathbf{ab}/L$ are the maps which have the left lifting property [6, Ch. I, §5] with respect to the maps $X \to Y$ for which the underlying map $UX \to UY \in \mathbf{S}/L$ is a trivial fibration.
(2) The *fibrations* in $\mathbf{ab}/L$ are the maps which have the right lifting property [6, Ch. I,§5] with respect to the trivial cofibrations (i.e., the cofibrations which are weak equivalences).

A more useful description of the cofibrant objects and the cofibrations is

5.3 Proposition.

(1) *An object $A \in \mathbf{ab}/L$ is cofibrant iff it is free (i.e. iff the inverse image in A of each simplex in L is a free abelian group).*
(2) *A map $A \to B \in \mathbf{ab}/L$ is a cofibration iff it is relatively free (i.e., it fits into a short exact sequence $0 \to A \to B \to C \to 0$ in which C is free).*

Proof: A map in $\mathbf{ab}/L$ clearly has a trivial fibration in $\mathbf{S}/L$ as underlying map iff it has the right lifting property with respect to all inclusions (2.1) $Z_L\dot{\Delta}[x] \to Z_L\Delta[x] \in \mathbf{ab}/L$. In view of 5.2(i) and the small object argument of [6, Ch. II, §3], this implies that *the cofibrations in* $\mathbf{ab}/L$ *are the retracts of the maps $A \to B \in \mathbf{ab}/L$ which admit (possibly transfinite) factorizatons*

$$A = A_1 \to \cdots \to A_s \to A_{s+1} \to \cdots \to \varinjlim{}^s A_s = B$$

in which each map $A_s \to A_{s+1}$ is obtained by pushing out an inclusion $Z_L\dot{\Delta}[x] \to Z_L\Delta[x]$ and in which, for every limit ordinal t involved, $A_t = \varinjlim{}^{s<t} A_s$. The desired result now follows readily.

For fibrations one can, in general, do no better than 5.2(ii). However, for fibrant objects and fibrations between them, one has:

5.4 Proposition.

(1) *An object $Y \in \mathbf{ab}/L$ is fibrant iff the underlying object $UY \in \mathbf{S}/L$ is fibrant (i.e., the structure map $Y \to L \in \mathbf{S}$ is a fibration).*
(2) *Let $X,Y \in \mathbf{ab}/L$ be fibrant. Then a map $X \to Y \in \mathbf{ab}/L$ is a fibration iff the underlying map $UX \to UY \in \mathbf{S}/L$ is a fibration.*

5.5 Corollary. *Let $X,Y \in \mathbf{ab}/L$ be fibrant. Then (4.2) a map $X \to Y \in \mathbf{ab}/L$ is a weak equivalence iff the underlying map $UX \to UY \in \mathbf{S}/L$ is a weak equivalence.*

Proof: In view of 4.4 and 5.2(ii) a fibration $X \to Y \in \mathbf{ab}/L$ has the right lifting property with respect to the maps $Z_L V[x,k] \to Z_L\Delta[x]$ and hence its underlying map $UX \to UY \in \mathbf{S}/L$ is also a fibration. It thus remains to show that a map $X \to Y \in \mathbf{ab}/L$, for which the underlying map $UX \to UY \in \mathbf{S}/L$ is a fibration between fibrant objects, has the right lifting property with respect to all trivial cofibrations in $\mathbf{ab}/L$. Because UX and UY are fibrant (in $\mathbf{S}/L$), a commutative diagram in $\mathbf{ab}/L$

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in which the map $A \to B$ is a trivial cofibration, admits a factorization (4.4)

$$\begin{array}{ccccc} A & \longrightarrow & E^\infty A & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & E^\infty B & \longrightarrow & Y \end{array}$$

in which the map $E^\infty A \to E^\infty B$ is a trivial cofibration. Moreover (4.5) $UE^\infty A$ and $UE^\infty B$ are fibrant objects of $\mathbf{S}/L$ and hence (4.2) the underlying map $UE^\infty A \to UE^\infty B \in \mathbf{S}/L$ is a weak equivalence. Using this fact it is now not difficult to obtain a lifting $E^\infty B \to X$ which, composed with the map $B \to E^\infty B$, yields the desired lifting $B \to X$.

We end with observing that the pair of adjoint functors $Z_L : \mathbf{S}/L \leftrightarrow \mathbf{ab}/L : U$ behaves as expected, i.e., (4.4 and 5.3-5).

5.6 Proposition. *The functor $Z_L : \mathbf{S}/L \to \mathbf{ab}/L$ preserves cofibrations and weak equivalences and the functor $U : \mathbf{ab}/L \to \mathbf{S}/L$ preserves fibrations and weak equivalences between fibrant objects.*

It thus remains to give a

5.7 Proof of 5.1. One has to verify axioms CM1-5 of [2, Ch. VIII, §2]. This is straightforward, except for axioms CM4(ii) and CM5(ii).

To verify CM4(ii), it suffices to show: if a map $X \to Y \in \mathbf{ab}/L$ is a trivial fibration, then so is the underlying map $UX \to UY \in \mathbf{S}/L$. By the small object argument [6, Ch.II, §3] the map $X \to Y$ admits a factorization $X \to X' \to Y$ in $\mathbf{ab}/L$ such that the underlying map $UX' \to UY \in \mathbf{S}/L$ is a trivial fibration and the map $X \to X' \in \mathbf{ab}/L$ is a cofibration. In view of 4.3, the map $X' \to Y \in \mathbf{ab}/L$ is a weak equivalence and, because the map $X \to Y \in \mathbf{ab}/L$ is a weak equivalence, it follows that the map $X \to X' \in \mathbf{ab}/L$ is a trivial cofibration. As the map $X \to Y \in \mathbf{ab}/L$ is also a fibration,one can apply CM4(i) and deduce that it is a retract of the map $X' \to Y \in \mathbf{ab}/L$ and that its underlying map therefore is also a trivial fibration.

Finally to verify CM5(ii) one uses the Bousfield argument [1, §11], i.e., one combines the small object argument [6, Ch. II, §3] with the observation that proposition 4.8 readily implies

5.8 Lemma. *Let c be an infinite cardinal number at least as large as the number of simplices in L. Then a map in $\mathbf{ab}/L$ is a fibration iff it has the right lifting property with respect to all trivial cofibrations $A \to B \in \mathbf{ab}/L$ in which the number of simplices in B is at most c.*

§6 Dependence of $\mathbf{ab}/L$ on L

Our aim in this section is to show

6.1 Proposition. *Let $g : L \to L' \in \mathbf{S}$ be a weak equivalence. Then g induces an equivalence of categories $ho(\mathbf{ab}/L) \cong ho(\mathbf{ab}/L')$ (2.6) and hence a 1-1 correspondence between the weak equivalence classes of the simplicial coefficient systems over L and the ones over L'.*

To prove this we consider

6.2 A pair of adjoint functors $g_* : \mathbf{ab}/L \leftrightarrow \mathbf{ab}/L' : g^*$. Given a map $g : L \to L' \in \mathbf{S}$, the *pull back functor* $g^* : \mathbf{ab}/L' \to \mathbf{ab}/L$ has as left adjoint the *push out functor* $g_* : \mathbf{ab}/L \to \mathbf{ab}/L'$ which "takes direct sums, over the simplices of L which have the same image in L', of their inverse images", and which clearly has the property that $\bigoplus_{L'} g_*A = \bigoplus_{L} A$ for every object $A \in \mathbf{ab}/L$.

Moreover, one readily verifies:

6.3 Proposition. *The left adjoint $g_* : \mathbf{ab}/L \to \mathbf{ab}/L'$ preserves cofibrations and weak equivalences and the right adjoint $g^* : \mathbf{ab}/L' \to \mathbf{ab}/L$ preserves fibrations and weak equivalences between fibrant objects.*

The desired result now follows immediately from [6, Ch. I, §4, Th. 3] and

6.4 Proposition. *Let $g : L \to L' \in \mathbf{S}$ be a trivial cofibration (i.e., a weak equivalence which is 1-1). Then, for every cofibrant object $A \in \mathbf{ab}/L$ and every fibrant object $A' \in \mathbf{ab}/L'$, a map $A \to g^*A' \in \mathbf{ab}/L$ is a weak equivalence iff its adjoint $g_*A \to A' \in \mathbf{ab}/L'$ is so.*

Proof: As g is 1-1, for every object $A \in \mathbf{ab}/L$ the adjunction map $A \to g^*g_*A \in \mathbf{ab}/L$ is an isomorphism and hence a map $A \to B \in \mathbf{ab}/L$ is a weak equivalence iff the induced map $g_*A \to g_*B \in \mathbf{ab}/L'$ is so. Moreover, because g is a weak equivalence, the adjunction map $g_*g^*A' \to A' \in \mathbf{ab}/L'$ is (in view of 3.6(ii) and 6.2) a weak equivalence for every fibrant object $A' \in \mathbf{ab}/L'$. The proposition now readily follows.

§7. Simplicial modules over the loop group GL of L

We end with showing that (7.4), for $L \in \mathbf{S}$ connected, *the weak equivalence classes of the simplicial coefficient systems over L are in a natural 1-1 correspondence with the weak equivalence classes of the simplicial modules over the loop group GL of L* (or more precisely, its integral group ring ZGL).

First we recall from [6] the existence of

7.1 A closed model category structure for simplicial modules over GL. Let $L \in \mathbf{S}$ be connected and have a base point, let GL be its *loop group* [4] (which is a free simplicial group which has the homotopy type of the loops on L) and let $\mathbf{M}_{GL}$ denote the category of *simplicial (left) modules over the integral group ring ZGL of GL*. Then [6, ch. II, §6] $\mathbf{M}_{GL}$ *admits a closed simplicial model category structure* in which the simplicial structure is the obvious one and in which a map is a weak equivalence or a fibration whenever the underlying map in $\mathbf{S}$ is a weak equivalence or a fibration.

Next we observe the existence of

7.2 A pair of adjoint functors $h : \mathbf{ab}/L \leftrightarrow \mathbf{M}_{GL} : k$. Let $EL \to L \in \mathbf{S}$ be the *path fibration* of [4], which is a principal fibration with group GL (i.e., [5, §18] GL acts freely on EL from the right and $EL/GL = L$). Then it is not difficult to see that the induced map $Z_L EL \to L \in \mathbf{S}$ is a principal fibration with group ZGL. Thus one can consider the functor $h : \mathbf{ab}/L \to \mathbf{M}_{GL}$ which sends an object $A \in \mathbf{ab}/L$ to the object $\bigoplus_L (Z_L EL \otimes A) \in \mathbf{M}_{GL}$ and the functor $k : \mathbf{M}_{GL} \to \mathbf{ab}/L$ which sends an object $M \in \mathbf{M}_{GL}$ to the object $M \otimes_{ZGL} Z_L EL \in \mathbf{ab}/L$. A straightforward calculation then yields that these functors form a pair of adjoint functors $h : \mathbf{ab}/L \leftrightarrow \mathbf{M}_{GL} : k$ and that

7.3 Proposition. *Both of the functors h and k preserve weak equivalences and both adjunction maps $hk \to id$ and $id \to kh$ are natural weak equivalences.*

7.4 Corollary. *The functor h induces an equivalence of categories $ho(\mathbf{ab}/L) \cong ho(\mathbf{M}_{GL})$ (2.6) and hence a 1-1 correspondence between the weak equivalence classes of the simplicial coefficient systems over L and the weak equivalence classes of the simplicial modules over GL.*

7.5 Remark: The weak equivalence classes of the simplicial coefficient systems over L are also in natural 1-1 correspondence with the *weak equivalence classes of the differential graded modules over the normalized chains on GL, which are trivial in negative dimensions*. This follows immediately from 7.4 and the fact that [5, §22 and §29] the normalization functor N gives rise to a functor $N : \mathbf{M}_{GL} \to \mathbf{m}_{GL}$ (where $\mathbf{m}_{GL}$ denotes the category of these differential graded modules), which induces an equivalence of categories $\text{ho}(MGL) \cong \text{ho}(\mathbf{m}_{GL})$. One proves this by observing that

(1) the category $\mathbf{m}_{GL}$ admits a closed model category structure in which a map is a weak equivalence whenever it induces an isomorphism on homology and a fibration whenever it is onto in positive demensions, and

(2) the functor $N : \mathbf{M}_{GL} \to \mathbf{m}_{GL}$ and its right adjoint satisfy the conditions of [6, Ch. I, §4, Th.3].

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