Self-intersections of Immersions and Steenrod Operations

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Abstract

We present a geometric formula which leads to relations between the Koschorke and Sanderson self-intersection operations in the bordism of immersions [7], and the cobordsim Steenrod operations of tom Dieck [13]. The proof uses techniques of Vogel [14] to analyse the double-point immersion of a certain non-self-transverse immersion.

1 Introduction

Let $f\colon M^{n-k} \looparrowright N^n$ be a codimension k, self-transverse immersion of closed manifolds. (All manifolds and maps between them are assumed smooth, and closed means compact and boundaryless). The *r-fold self intersection set of f* is defined as

$$N_r(f) = \{n \in N \mid |f^{-1}(n)| \geq r\}.$$

By the self-transversality of f, this is the image of an immersion

$$\psi_r(f)\colon \Delta_r(f)^{n-rk} \looparrowright N^n$$

of codimension rk, the so-called *r-fold self-intersection immersion* of f. In this way one obtains self-intersection operations in the bordism of immersions. These operations ψ_r have been studied by Koschorke and Sanderson ([7]), Vogel ([14]), Eccles and Asadi-Golmankhaneh [1] and others. In particular, one has the identification due to Wells [15] (and later generalised in [14] and [7]), of bordism groups of immersions as stable homotopy of Thom complexes. Then the ψ_r are induced by the *James-Hopf maps*

$$h^r\colon QX \to QD_rX$$

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employed in the proof of the stable splitting of QX ([4], [10]), where $QX = \lim_{l\to\infty} \Omega^l \Sigma^l X$ and $D_r X$ is the r-th extended power of a pointed space X.

The purpose of this paper is to present a formula relating the above self-intersection operations and tom Dieck's Steenrod operations in Cobordism theory [13]. Any f as above represents a cobordism class $[f] \in \mathsf{MO}^k(N)$ by Quillen's geometric description of cobordism [8]. We shall give an alternative description of the class $\mathcal{P}^k[f]$, where $\mathcal{P}^k\colon \mathsf{MO}^k(N) \to \mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N)$ is the internal Steenrod operation of [13]. The universal principal $\mathbb{Z}_2$-bundle $c\colon S^\infty \to \mathbb{R}P^\infty$ may be viewed as a union of immersions of closed manifolds $c_l\colon S^l \to \mathbb{R}P^l$. Form the Cartesian product immersion $c_l \times f\colon S^l \times M \looparrowright \mathbb{R}P^l \times N$. We show (Theorem 6.2) that the double point self-intersection immersion $\psi_2(c_l \times f)$ is in fact an immersion of two disjoint manifolds in $\mathbb{R}P^l \times N$. One of these is the product immersion $c \times \psi_2(f)$. The other immersion represents $\mathcal{P}^k[f]$ after passing to the limit over l.

Mapping our formula to $\mathbb{Z}_2$-cohomology, where the internal operation $\mathcal{P}^k\colon H^k(N;\mathbb{Z}_2) \to H^{2k}(\mathbb{R}P^\infty \times N)$ completely determines the classical Steenrod squares, we obtain the following interesting result. Let us say that the immersion f *represents* both the homology class $f_*[M] \in H_{n-k}(N;\mathbb{Z}_2)$ and its Poincaré dual cohomology class in $H^k(N;\mathbb{Z}_2)$.

Corollary 6.7. *Suppose the cohomology class $\alpha \in H^k(N;\mathbb{Z}_2)$ is represented by an immersion $f : M^{n-k} \looparrowright N^n$. Then the element $\mathcal{P}^k(\alpha) \in H^{2k}(\mathbb{R}P^\infty \times N)$ is obtained, by passage to the limit, from classes represented by the immersions $\psi_2(c_l \times f)$.*

The proof of the main result uses methods introduced by Vogel (in [14]) which allow us to replace the bordism class of an immersion f with the bordism class of a 'spreading' of f. Roughly speaking one extends f to an immersion of its normal disc bundle. This spreading is self-transverse by virtue of having zero codimension. Thus we have a way of analysing the self-intersection immersions of a non-self-transverse immersion, by looking at the self-intersections of the spreading. This has proved fruitful in the case of $c \times f$, and it is hoped will yield further results.

In §2 we discuss Steenrod operations from a general viewpoint, before specialising to give the geometric construction of the operations in MO-cobordism, and relating these to the classical case of $\mathbb{Z}_2$-cohomology. In §3 and §4 we examine in detail the functors given by bordism of immersions and bordism of spreadings, as outlined in Vogel [14]. These sections necessarily mirror each other, as we shall find out in §5 that these functors are naturally isomorphic. In §6 we use the technical results of §5 to prove our main result, and tie this in with the constructions of §2.

2 Steenrod Operations in Cobordism

Steenrod operations are cohomology operations arising from higher homotopy commutativity properties of the product in a multiplicative generalised cohomology theory. They were discovered in the case of $\mathbb{Z}_p$-cohomology by Steenrod [11]; analogous operations were found to exist in the cobordism cohomology theories by tom-Dieck [13], and in K-theory by Atiyah [3]. We are mainly interested in cobordism, where Quillen's geometric interpretation of cobordism classes as equivalence classes of suitably oriented proper maps of manifolds [8] allows for a particularly nice geometric construction of the so-called Steenrod-tom Dieck operations.

2.1 Generalities on Steenrod Operations

Let r be a positive integer and let G be a subgroup of the symmetric group S_r. Let EG be a contractible space with a free right G-action. The group G acts on the left of the r-fold smash product $X^{\wedge r}$ of any pointed space X, by permutation of the factors.

Definition 2.1. *The quotient space*

$$EG \ltimes_G X^{\wedge r} := \frac{EG \times_G X^{\wedge r}}{EG \times_G \{*\}}$$

is denoted by $D_G X$, where $\times_G$ means we are factoring out by the diagonal action of G. The r-th extended power of X *is the space $D_{S_r} X = D_r X$.*

Note that any point $e \in EG$ gives a map $i = i_e \colon X^{\wedge r} \to D_G X$ by setting $i([x_1, \ldots, x_r]) = [e, x_1, \ldots, x_r]$.

Let E be a commutative ring spectrum, and let $\widetilde{\mathsf{E}}^n(X)$ denote the n-th *reduced* cohomology group of X in the cohomology theory represented by E.

Definition 2.2. *For any positive integer d, an* external Steenrod operation of type (G, d) in $\widetilde{\mathsf{E}}^*$ *is a family $P = (P^{nd} \mid n \in \mathbb{Z})$ of natural transformations*

$$P^{nd} \colon \widetilde{\mathsf{E}}^{nd}(X) \to \widetilde{\mathsf{E}}^{rnd}(D_G X)$$

with the additional property that the composition

$$i^* \circ P^{nd} \colon \widetilde{\mathsf{E}}^{nd}(X) \to \widetilde{\mathsf{E}}^{rnd}(X^{\wedge r})$$

is the r-fold exterior product $x \mapsto x^{\wedge r}$.

Thus there is a sense in which these operations 'extend the operation of raising x to the r-th power'. One may give additional axioms for the operations, from which a myriad of properties may be derived (see for example [13], [11]). One may also define internal operations, using the *generalised diagonal map*

$$\triangle_G\colon\ BG \wedge X = EG \ltimes_G X \to D_G X$$

which maps $[e, x]$ to $[e, x, \ldots, x]$ (here $BG := EG/G$).

Definition 2.3. *An external Steenrod operation of type (G, d) in $\widetilde{\mathsf{E}}^*$ gives an* internal Steenrod operation *$(\mathcal{P}^{nd} \mid n \in \mathbb{Z})$ by setting*

$$\mathcal{P}^{nd} = \triangle_G^* \circ P^{nd}\colon\ \widetilde{\mathsf{E}}^{nd}(X) \to \widetilde{\mathsf{E}}^{rnd}(BG \wedge X).$$

Now let ζ be a vector bundle of dimension k with projection $p\colon\ E(\zeta) \to B(\zeta)$.

Definition 2.4. *The rk-dimensional vector bundle*

$$1 \times_G p^{(r)}\colon\ EG \times_G E(\zeta)^{(r)} \to EG \times_G B(\zeta)^{(r)}$$

will be denoted by $\mathcal{S}_G\zeta$, and the bundle $\mathcal{S}_{S_r}\zeta = \mathcal{S}_r\zeta$ will be called the r-th extended power of ζ*.*

Proposition 2.5. *There is a homeomorphism*

$$T\mathcal{S}_G\zeta \approx D_G T\zeta,$$

where T denotes the Thom space of a bundle. □

Proposition 2.6. *Let $P = (P^{nd} \mid n \in \mathbb{Z})$ be a Steenrod operation of type (G, d) in $\widetilde{\mathsf{E}}^*$, and let $t \in \widetilde{\mathsf{E}}^{nd}(T\zeta)$ be a Thom class for the nd-dimensional bundle ζ. Then $P^{nd}(t) \in \widetilde{\mathsf{E}}^{rnd}(D_G T\zeta)$ is a Thom class for $\mathcal{S}_G\zeta$.*

Proof. Let $i\colon\ T\zeta^{\wedge r} \to D_G T\zeta$ be the map induced by the inclusion of a point in EG. By Definition 2.2,

$$i^* \circ P^{nd}(t) = t^{\wedge r} \in \mathsf{E}^{rnd}(T\zeta^{\wedge r}),$$

which is a Thom class for $\zeta^{(r)}$. The inclusion of a compactified fibre of $\mathcal{S}_G\zeta$ factors as

$$S^{rnd} = (S^{nd})^{\wedge r} \to T\zeta^{\wedge r} \xrightarrow{i} D_G T\zeta.$$

The Proposition follows by the definition of a Thom class. □

The construction of such operations comes from higher homotopy commutativity properties of the product in E. For simplicity consider first the case when E is an Ω-spectrum, meaning that its structure maps $\sigma_n\colon \Sigma\mathsf{E}_n \to \mathsf{E}_{n+1}$ are adjoint to homeomorphisms $\overline{\sigma}_n\colon \mathsf{E}_n \to \Omega\mathsf{E}_{n+1}$. Then a cohomology class $\alpha \in \widetilde{\mathsf{E}}^{nd}(X)$ is represented by a map $f\colon X \to \mathsf{E}_{nd}$. The construction of D_GX given above extends to give a functor from the category of pointed spaces and maps to itself; thus we have a map

$$D_Gf\colon D_GX \to D_G\mathsf{E}_{nd}.$$

Now suppose that for each $n \in \mathbb{Z}$ we have a map $\xi_{nd}\colon D_G\mathsf{E}_{nd} \to \mathsf{E}_{rnd}$. Then we have a naturally defined element $P^{nd}(\alpha) \in \widetilde{\mathsf{E}}^{rnd}(D_GX)$, represented by the composition $\xi_{nd} \circ D_Gf$, and this gives a Steenrod operation of type (G,d) in $\widetilde{\mathsf{E}}^*$.

Example 2.7. *Take E to be the Eilenberg-Maclane spectrum $\mathsf{H}\mathbb{Z}_p$ with p prime, and $G = \mathbb{Z}_p \leq S_p$. Maps $\xi_n\colon D_{\mathbb{Z}_p}K(\mathbb{Z}_p,n) \to K(\mathbb{Z}_p,pn)$ may be constructed for all n using the commutativity of the $\times$-product. This gives an external operation of type $(\mathbb{Z}_p,1)$ in $\widetilde{\mathsf{H}\mathbb{Z}}_p^*$.*

In the case $G = \mathbb{Z}_2$, the familiar Steenrod squares $Sq^i\colon \widetilde{H}^n(X;\mathbb{Z}_2) \to \widetilde{H}^{n+i}(X;\mathbb{Z}_2)$ are obtained as follows. The corresponding internal operation applied to an element $\alpha \in \widetilde{H}^n(X;\mathbb{Z}_2)$ yields an element $\mathcal{P}^n(\alpha) \in \widetilde{H}^{2n}(\mathbb{R}P^\infty \wedge X;\mathbb{Z}_2)$. By the Künneth Theorem,

$$H^*(\mathbb{R}P^\infty \wedge X;\mathbb{Z}_2) \cong \mathbb{Z}_2[w] \otimes H^*(X;\mathbb{Z}_2),$$

where $w \in \widetilde{H}^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$ is the Euler class of the canonical line bundle. The action of the Sq^i on α are determined by the formula

$$\mathcal{P}^n(\alpha) = \sum_{i\geq 0} w^{n-i} \otimes Sq^i(\alpha).$$

Now we turn to a set of examples where E is not an Ω-spectrum, the external Steenrod-tom Dieck operations in the various cobordism theories. Let Γ denote one of the classical infinite Lie groups O, U, Sp, SO or SU, and let $\mathsf{M}\Gamma$ denote the associated Thom spectrum. Let d be a positive integer, equal to 1 when $\Gamma = O$ or SO, 2 when $\Gamma = U$ or SU, and 4 when $\Gamma = Sp$. Let G be a subgroup of S_r when $\Gamma = O, U$ or Sp and a subgroup of the alternating group A_r when $\Gamma = SO$ or SU. In [13], T. tom Dieck proves the following.

Proposition 2.8. *There exists a (unique) external Steenrod operation $P = (P^{nd} \mid n \in \mathbb{Z})$ of type (G,d) in $\widetilde{\mathsf{M}\Gamma}^*$.*

The construction of these operations is similar to that given above, but involves slightly more (a Thom isomorphism). The vital point again is the existence of a map $\xi_{nd}\colon D_G\mathsf{M}\Gamma_{nd} \to \mathsf{M}\Gamma_{rnd}$ for each $n \in \mathbb{Z}$, which is deduced from the orientability of the relevant extended powers of universal bundles.

Example 2.9. *Take $\Gamma = SO$, $d = 1$ and $G \leq A_r$. Let $\widetilde{\gamma}_n$ denote the universal $SO(n)$-bundle, with fibres $\mathbb{R}^n$ and Thom space $MSO(n)$. For every natural number n the extended power bundle $\mathcal{S}_G\widetilde{\gamma}_n$ admits a classifying bundle map*

$$\overline{\xi}_n\colon\ \mathcal{S}_G\widetilde{\gamma}_n \to \widetilde{\gamma}_{rn},$$

since an even permutation of the factors of $\widetilde{\gamma}_n^{(r)}$ preserves the product orientation. Passing to Thom spaces and using Proposition 2.5 we get maps $\xi_n\colon D_GMSO(n) \to MSO(rn)$ which yield a Steenrod operation of type $(G,1)$ in $\widetilde{\mathsf{MSO}}^*$.

As a supplementary example we mention that the power operations in K-theory of Atiyah [3] are external Steenrod operations of type $(S_r, 2)$.

It should be clear from these examples that the Steenrod operations in a cohomology theory $\widetilde{\mathsf{E}}^*$ exist by virtue of certain extra structure on the ring spectrum E. This structure has been abstracted and codified in the notion of a H_∞^d ring spectrum, by May and others [5]. They give a construction of the extended power spectrum $D_r\mathsf{E}$ of a ring spectrum E. Then E is an H_∞^d ring spectrum if there is a sequence of maps of spectra $\xi_{r,i}\colon D_r\Sigma^{di}\mathsf{E} \to \Sigma^{rdi}\mathsf{E}$ satisfying various compatibility relations.

2.2 Geometric Construction of the Operations

Having used the structure of the spectrum E to construct external Steenrod operations in $\widetilde{\mathsf{E}}^*$, we may ask for a geometrical construction corresponding to this homotopical one. This is complicated in the classical case of $\mathsf{H}\mathbb{Z}_p$, where elements of $\widetilde{H}^*(X;\mathbb{Z}_p)$ are represented by cocycles on singular simplices in X. However the situation is much simpler for the Steenrod-tom Dieck powers in cobordism, thanks to Quillen's geometric interpretation of cobordism [8]. Here we may give a simple and revealing construction of the operations in terms of proper maps of manifolds. We shall demonstrate this in the simplest case of MO-cobordism, which is also closest to the classical case of $\mathbb{Z}_2$-cohomology. For a thorough account of geometric cobordism, see Dold [6].

Let us assume that X is an n-manifold without boundary. It is well known that the *unreduced* bordism group $\mathsf{MO}_{n-k}(X)$ consists of bordism classes of singular $(n-k)$-manifolds in X, that is, bordism classes of pairs (M, f) where M^{n-k} is a closed manifold and $f\colon M \to X$ a continuous map. In fact such a pair (M, f) also represents a class in the *unreduced* cobordism group $\mathsf{MO}^k(X)$, by the following procedure.

We have an embedding $z\colon X = X \times \{0\} \hookrightarrow X \times \mathbb{R}^K$ for every $K \geq 0$, and if K is chosen sufficiently large the composition

$$M^{n-k} \xrightarrow{f} X^n \xhookrightarrow{z} X \times \mathbb{R}^K$$

will be homotopic to an embedding which is unique up to isotopy. This embedding $f'\colon M \hookrightarrow X \times \mathbb{R}^K$ has a tubular neighbourhood which is a $(k+K)$-dimensional bundle. Hence the Pontrjagin-Thom collapse gives a map from $\Sigma^K X_+ = (X \times \mathbb{R}^K)_+$ to $MO(k+K)$, which represents a class in

$$\mathsf{MO}^k(X) = \lim_{K\to\infty} [\Sigma^K X_+, MO(k+K)].$$

Now consider the set of all pairs (M, f) where M is an $(n-k)$-manifold without boundary (not necessarily compact), and $f\colon M \to X$ is a proper map (meaning $f^{-1}(C)$ is compact whenever C is compact in X). One may put a suitable relation of *cobordism* on this set in such a way that the following is true (see [8], Proposition 1.2).

Proposition 2.10. *The above procedure describes a well-defined map from the resulting set of cobordism classes to* $\mathsf{MO}^k(X)$*, and this map is an isomorphism.*

Thus we have a way of thinking of classes in $\mathsf{MO}^k(X)$ as being represented by proper maps of codimension-k manifolds to X. These ideas are already evident in Atiyah's paper [2], where a Poincaré Duality isomorphism $\mathsf{MO}_{n-k}(X) \cong \mathsf{MO}^k(X)$ is demonstrated for closed n-manifolds X. Atiyah's isomorphism is given by the identity on representatives.

Addition in MO^* is given, as in the dual bordism theory, by disjoint union of manifolds, and the external $\times$-product is given by Cartesian product. The contravariance is given (at least on the category of smooth manifolds) by a pull-back of transverse representatives. Given an element $[(M, f)] \in \mathsf{MO}^k(X_1)$ and $g\colon X_2 \to X_1$ a map of manifolds, there is a smooth map g' which is homotopic to g, proper, and transverse to $f\colon M \to X_1$. Then the map δ in the pull-back diagram

$$\begin{tikzcd} X_2 \times_{X_1} M \arrow[r] \arrow[d, "\delta"'] & M \arrow[d, "f"] \\ X_2 \arrow[r, "g'"] & X_1 \end{tikzcd}$$

is proper, and this gives a homomorphism $g^*\colon \mathsf{MO}^k(X_1) \to \mathsf{MO}^k(X_2)$ by setting $g^*[(M,f)] = [(X_2 \times_{X_1} M, \delta)]$.

Armed with this functoriality we may define an internal product

$$\mathsf{MO}^k(X) \times \mathsf{MO}^l(X) \xrightarrow{\cup} \mathsf{MO}^{k+l}(X)$$

by setting $[(M_1,f_1)] \cup [(M_2,f_2)] = \triangle^*[(M_1 \times M_2, f_1 \times f_2)]$ where $\triangle\colon X \to X \times X$ is the diagonal map. These ideas will be explored in more detail in §3, in the setting of the bordism groups of immersions.

We now proceed to the geometric construction of the Steenrod-tom Dieck powers in $\widetilde{\mathsf{MO}}^*$, restricting our attention to the simplest case where $G = \mathbb{Z}_2 = S_2$, giving operations of type $(\mathbb{Z}_2, 1)$. We may take as $E\mathbb{Z}_2$ the infinite sphere $S^\infty = \bigcup_l S^l$, where the non-trivial element of $\mathbb{Z}_2$ acts by the antipodal map. The quotient space under this action is $\mathbb{R}P^\infty$. We shall describe, for each $k \in \mathbb{Z}$, a natural transformation

$$P^k\colon \mathsf{MO}^k(X) = \widetilde{\mathsf{MO}}^k(X_+) \to \widetilde{\mathsf{MO}}^{2k}(D_2X_+) = \mathsf{MO}^{2k}(S^\infty \times_{\mathbb{Z}_2} X \times X).$$

Let $[f] \in \mathsf{MO}^k(X)$ be represented by a proper map $f\colon M^{n-k} \to X^n$ of manifolds. For every $l \in \mathbb{N}$, the group $\mathbb{Z}_2$ acts freely on the product $S^l \times M \times M = S^l \times M^{(2)}$ by $(v, m_1, m_2) \mapsto (-v, m_2, m_1)$, giving a quotient manifold $S^l \times_{\mathbb{Z}_2} M^{(2)}$. The map

$$\lambda_l(f) := 1 \times_{\mathbb{Z}_2} f^{(2)}\colon S^l \times_{\mathbb{Z}_2} M^{(2)} \to S^l \times_{\mathbb{Z}_2} X^{(2)}$$

which extends $f \times f$ is a proper codimension $2k$ map of manifolds, and so represents an element $[\lambda_l(f)] \in \mathsf{MO}^{2k}(S^l \times_{\mathbb{Z}_2} X^{(2)})$. One may easily check that this gives a well-defined natural transformation

$$P^k_l\colon \mathsf{MO}^k(X) \to \mathsf{MO}^{2k}(S^l \times_{\mathbb{Z}_2} X^{(2)}),$$
$$[f] \mapsto [\lambda_l(f)]$$

for every $k \in \mathbb{Z}$, $l \in \mathbb{N}$.

Next we note that when $l \leq m$ are natural numbers there is an embedding

$$\jmath^m_l := \imath^m_l \times_{\mathbb{Z}_2} 1^{(2)}\colon S^l \times_{\mathbb{Z}_2} X^{(2)} \hookrightarrow S^m \times_{\mathbb{Z}_2} X^{(2)},$$

coming from the usual embedding $\imath^m_l\colon S^l \hookrightarrow S^m$ of spheres. The induced homomorphisms

$$(\jmath^m_l)^*\colon \mathsf{MO}^{2k}(S^m \times_{\mathbb{Z}_2} X^{(2)}) \to \mathsf{MO}^{2k}(S^l \times_{\mathbb{Z}_2} X^{(2)})$$

form a direct system. To determine its inverse limit, we use the following lemma (see Rudyak [9], Corollary III.4.17).

Lemma 2.11. *Let Y be a CW-complex filtered by finite sub-complexes Y_n, $n \in \mathbb{N}$. If E is a spectrum such that $\pi_i(\mathsf{E})$ is a finite Abelian group for all i, then the map*

$$\mathsf{E}^k(Y) \to \lim_{\leftarrow} \mathsf{E}^k(Y_n)$$

is an isomorphism for all k.

In particular, the well known fact $\pi_*(\mathsf{MO}) = \mathbb{Z}_2[x_2, x_4, x_5, \ldots]$ (first proved in [12]) gives that $\pi_i(\mathsf{MO})$ is finite Abelian, and we have

$$\lim_{\leftarrow} \mathsf{MO}^{2k}(S^l \times_{\mathbb{Z}_2} X^{(2)}) \cong \mathsf{MO}^{2k}(S^\infty \times_{\mathbb{Z}_2} X^{(2)}).$$

We may also verify easily that $(j_l^m)^* \circ P_m^k = P_l^k$ whenever $l \leq m$. This allows us to define a map

$$P^k \colon \mathsf{MO}^k(X) \to \mathsf{MO}^{2k}(S^\infty \times_{\mathbb{Z}_2} X^{(2)})$$

by the universal property of limits.

Proposition 2.12. *The operation $(P^k \mid k \in \mathbb{Z})$ described above agrees with that mentioned in Proposition 2.8.*

Proof. This is essentially Satz 4.1 of [13]. □

The corresponding internal operation $(\mathcal{P}^k \mid k \in \mathbb{Z})$ is obtained as follows. For each $l \in \mathbb{N}$ there is a generalised diagonal map of manifolds

$$\triangle_{\mathbb{Z}_2}^l \colon \mathbb{R}P^l \times X = S^l \times_{\mathbb{Z}_2} X \xrightarrow{1 \times_{\mathbb{Z}_2} \triangle} S^l \times_{\mathbb{Z}_2} X^{(2)}.$$

We may define a natural transformation

$$\mathcal{P}_l^k \colon \mathsf{MO}^k(X) \to \mathsf{MO}^{2k}(\mathbb{R}P^l \times X)$$

by setting $\mathcal{P}_l^k[f \colon M \to X] = (\triangle_{\mathbb{Z}_2}^l)^*[\lambda_l(f)]$. Thus $\mathcal{P}_l^k[f]$ is represented by the map $\xi_l(f) \colon \Sigma(f) \to \mathbb{R}P^l \times X$ in the following pullback diagram

$$\begin{tikzcd}
\Sigma(f) \arrow[r] \arrow[d, "\xi_l(f)"'] & S^l \times_{\mathbb{Z}_2} M^{(2)} \arrow[d, "\lambda_l(f)"] \\
\mathbb{R}P^l \times X \arrow[r, "(\triangle_{\mathbb{Z}_2}^l)'"] & S^l \times_{\mathbb{Z}_2} X^{(2)},
\end{tikzcd}$$

where $(\triangle_{\mathbb{Z}_2}^l)'$ is homotopic to $\triangle_{\mathbb{Z}_2}^l$ and transverse to $\lambda_l(f)$. The induced maps $(\triangle_{\mathbb{Z}_2}^l)^*$ form a morphism of direct systems, and so passing to the inverse limit we obtain the internal operations

$$\mathcal{P}^k \colon \mathsf{MO}^k(X) \to \mathsf{MO}^{2k}(\mathbb{R}P^\infty \times X).$$

2.3 From Cobordism to Cohomology

The universal Thom class $\mu\colon \mathsf{MO} \to \mathsf{H}\mathbb{Z}_2$ gives multiplicative natural transformations from MO-(co)homology to $\mathsf{H}\mathbb{Z}_2$-(co)homology, which we shall also denote by μ. In the case of homology, μ is the so-called *Steenrod-Thom homomorphism* which takes the bordism class $[f\colon M^{n-k} \to X] \in \mathsf{MO}_{n-k}(X)$ to $f_*[M] \in H_{n-k}(X;\mathbb{Z}_2)$, where $[M] \in H_{n-k}(M;\mathbb{Z}_2)$ is the fundamental class of M. In the case of cohomology, μ maps Thom classes to Thom classes, in the following sense.

Proposition 2.13. *Let ζ be a vector bundle of dimension k, and let $t \in \widetilde{\mathsf{MO}}^k(T\zeta)$ be a cobordism Thom class. Then $\mu(t) \in \widetilde{H}^k(T\zeta;\mathbb{Z}_2)$ is a Thom class in ordinary cohomology.*

We have seen in section §2.1 that there are external Steenrod operations of type $(\mathbb{Z}_2, 1)$ in both $\widetilde{\mathsf{MO}}^*$ and $\widetilde{\mathsf{H}\mathbb{Z}_2}^*$. Denoting both of these operations by $P = (P^k \mid k \in \mathbb{Z})$, we have the following result which says that μ commutes with them.

Proposition 2.14. *The following diagram commutes for all pointed spaces X and all $k \in \mathbb{Z}$.*

$$\begin{tikzcd}
\widetilde{\mathsf{MO}}^k(X) \arrow[r, "P^k"] \arrow[d, "\mu"'] & \widetilde{\mathsf{MO}}^{2k}(D_2X) \arrow[d, "\mu"] \\
\widetilde{H}^k(X;\mathbb{Z}_2) \arrow[r, "P^k"] & \widetilde{H}^{2k}(D_2X;\mathbb{Z}_2)
\end{tikzcd}$$

Proof. By naturality, it suffices to prove that

$$\mu \circ P^k(t) = P^k \circ \mu(t),$$

where $t \in \widetilde{\mathsf{MO}}^k(MO(k))$ is the Thom class represented by the identity map $1\colon MO(k) \to MO(k)$. But by Propositions 2.6 and 2.13, both of these are the unique Thom class in $\widetilde{H}^{2k}(D_2MO(k);\mathbb{Z}_2)$ of the second exterior power $\mathcal{S}_2\gamma_k$ of the universal $O(k)$-bundle. □

3 Bordism of Immersions

In this section we introduce the monoid $\mathcal{I}(N;\zeta)$ of bordism classes of immersions of closed manifolds in a connected manifold N which admit a strict vector bundle morphism from their normal bundle to ζ. This describes a homotopy bifunctor in N and ζ. Our exposition is based on the paper [14].

3.1 Definitions

Let N^n be a connected manifold without boundary of dimension n, and let ζ be a k-dimensional vector bundle over a space X which has the homotopy type of a closed manifold. Consider all triples of the form (M^{n-k}, f, v), where M is a closed $(n-k)$-manifold, $f\colon M \looparrowright N$ is an immersion with normal bundle $\nu(f)$, and $v\colon \nu(f) \to \zeta$ is a bundle map which is isomorphic on the fibres (we shall say that f has a ζ-structure on its normal bundle).

Definition 3.1. *Two such triples (M_0, f_0, v_0) and (M_1, f_1, v_1) are* bordant *if there is a triple (W^{n-k+1}, F, V), where W is a compact manifold, $F\colon W \looparrowright N \times I$ is an immersion transverse to $N \times \partial I$ such that the squares in the following diagram are pullback squares*

$$\begin{tikzcd} M_0 \arrow[r] \arrow[d, "f_0"'] & W \arrow[d, "F"] & M_1 \arrow[l] \arrow[d, "f_1"] \\ N \times \{0\} \arrow[r] & N \times I & N \times \{1\} \arrow[l] \end{tikzcd}$$

and $V\colon \nu(F) \to \zeta$ is a bundle map restricting to v_i over M_i for $i = 0, 1$.

Bordism is an equivalence relation, and the resulting set of bordism classes $\mathcal{I}(N;\zeta)$ has a commutative monoid structure arising from the disjoint union of immersions. In fact this is an Abelian group if $k = dim(\zeta) \geq 1$, hence we shall often refer to it as a bordism group. The class of the empty immersion provides a zero element.

3.2 Functoriality

The above construction of $\mathcal{I}(N;\zeta)$ can be used to define a homotopy bifunctor to the category of commutative monoids, as we shall now explain.

Let $\mathcal{C}_k$ denote the category of k-dimensional vector bundles and strict vector bundle morphisms, by which we mean those vector bundle maps which map each fibre by a vector space isomorphism. It is clear how a morphism $\eta\colon \zeta \to \xi$ in this category induces a homomorphism

$$\eta_*\colon \mathcal{I}(N;\zeta) \to \mathcal{I}(N;\xi)$$

by composition. Bundle homotopic maps give the same induced map. Fixing the manifold N, we have a covariant homotopy functor $\mathcal{I}(N;-)$ from $\mathcal{C}_k$ to the category CMon of commutative monoids.

The functoriality in N is given by a pullback construction on transverse representatives. Let $\mathcal{D}$ denote the category of smooth manifolds without boundary and smooth, proper immersions. Suppose we have a triple (M, f, v) representing the class $[f] \in \mathcal{I}(N;\zeta)$, and let $g\colon Q \looparrowright N$ be a morphism in $\mathcal{D}$. We may choose a representative $f'\colon M \looparrowright N$ of $[f]$ which is regularly homotopic to $f\colon M \looparrowright N$ and transverse to g as a map to N. We then form the pullback square.

$$\begin{tikzcd} Q \times_N M \arrow[r, "\rho"] \arrow[d, "\delta"'] & M \arrow[d, "f'"] \\ Q \arrow[r, "g"] & N \end{tikzcd}$$

The manifold

$$Q \times_N M = \{(q, m) \in Q \times M \mid g(q) = f'(m)\}$$

is compact since M is compact and g is proper. The map δ is an immersion with normal bundle isomorphic to $\rho^*\nu(f') \cong \rho^*\nu(f)$, hence admits a bundle map $\overline{\rho}\colon \nu(\delta) \to \nu(f)$. Then we may set

$$g^*[f] = [(Q \times_N M, \delta, v \circ \overline{\rho})] \in \mathcal{I}(Q;\zeta).$$

It is straightforward to check that this construction is well-defined and functorial, and that regularly homotopic immersions of Q in N give the same map $\mathcal{I}(N;\zeta) \to \mathcal{I}(Q;\zeta)$. In fact, modulo some easy but tedious checking of details, we have proved the following.

Proposition 3.2. *The construction $(N, \zeta) \to \mathcal{I}(N;\zeta)$ gives a covariant homotopy bifunctor*

$$\mathcal{I}(-;-)\colon \mathcal{D}^{\mathsf{opp}} \times \mathcal{C}_k \to \mathsf{CMon}.$$

3.3 Products

There is an *external product*

$$\mathcal{I}(N;\zeta) \times \mathcal{I}(N';\zeta') \xrightarrow{\times} \mathcal{I}(N \times N';\zeta \times \zeta')$$

given by the Cartesian product of representatives. This is distributive over the addition. We may also obtain an *internal product*

$$\mathcal{I}(N;\zeta) \times \mathcal{I}(N;\zeta') \xrightarrow{\cup} \mathcal{I}(N;\zeta \times \zeta')$$

by setting $[f] \cup [g] = \triangle^*([f] \times [g])$, where $\triangle\colon N \hookrightarrow N \times N$ is the diagonal embedding.

3.4 The self-intersection operations $\psi_r\colon \mathcal{I}(-;\zeta) \to \mathcal{I}(-;\mathcal{S}_r\zeta)$

Let $f\colon M^{n-k} \looparrowright N^n$ be a self-transverse immersion of a closed manifold M in a manifold without boundary N, and let $r \geq 1$ be an integer. We denote by $\mathcal{F}(M;r)$ the open submanifold of the r-fold Cartesian product $M^{(r)}$ which consists of ordered r-tuples of *distinct* points in M. The restriction

$$f^{(r)}|\colon \mathcal{F}(M;r) \looparrowright N^{(r)}$$

of the r-fold product of f is an immersion transverse to the diagonal embedding $\triangle\colon N \hookrightarrow N^{(r)}$, by the self-transversality of f. When we pull back,

$$\begin{tikzcd} \overline{\Delta}_r(f) \arrow[r] \arrow[d, "\overline{\psi}_r(f)"'] & \mathcal{F}(M;r) \arrow[d, "f^{(r)}|"] \\ N \arrow[r, "\triangle"] & N^{(r)} \end{tikzcd}$$

we therefore obtain a closed submanifold

$$\overline{\Delta}_r(f) = \{(m_1,\ldots,m_r) \in \mathcal{F}(M;r) \mid f(m_1) = \ldots f(m_r)\} \subseteq \mathcal{F}(M;r)$$

which is in fact a compact manifold of dimension $n - rk$. This manifold admits a free action of the symmetric group S_r which permutes the factors. The immersion $\overline{\psi}_r(f)$ is equivariant under this action, so we may factor out by the action to obtain the *r-fold self-intersection immersion*

$$\psi_r(f)\colon \Delta_r(f) := \overline{\Delta}_r(f)/S_r \looparrowright N$$

$$[m_1,\ldots,m_r] \mapsto f(m_1) = \ldots = f(m_r)$$

of the self-transverse immersion f.

Proposition 3.3. *Suppose the immersion $f\colon M \looparrowright N$ has a ζ-structure on its normal bundle. Then $\psi_r(f)\colon \Delta_r(f) \looparrowright N$ has a $\mathcal{S}_r\zeta$-structure on its normal bundle (see Definition 2.4).*

Proof. We may take as ES_r the space $\mathcal{F}(\mathbb{R}^\infty;r)$. Fix once and for all an embedding $\rho\colon \nu(f) \hookrightarrow \mathbb{R}^\infty$. The normal fibre of $\psi_r(f)$ above the point $[m_1,\ldots,m_r] \in \Delta_r(f)$ is the unordered direct sum of the normal fibres of f at the points $m_1,\ldots,m_r$. Hence we may define a strict vector bundle morphism $\mathcal{S}_r(v)\colon \nu(\psi_r(f)) \to \mathcal{S}_r\zeta$ by setting

$$\mathcal{S}_r(v)[x_1,\ldots,x_r] = [(\rho(x_1),\ldots,\rho(x_r)),(v(x_1),\ldots,v(x_r))].$$

Note that $\mathcal{S}_r(v)$ is independent of the choice of ρ up to bundle homotopy, since such a ρ is unique up to isotopy. $\square$

Since any bordism class $[f] \in \mathcal{I}(N;\zeta)$ contains a self-transverse immersion, and any two such representatives are bordant via a self-transverse bordism, we have the following definition.

Definition 3.4. *There are operations*

$$\psi_r \colon \mathcal{I}(-;\zeta) \to \mathcal{I}(-;\mathcal{S}_r\zeta),$$

defined for each $r \geq 1$ by setting

$$\psi_r[(M,f,v)] = [(\Delta_r(f'), \psi_r(f'), \mathcal{S}_r(v'))],$$

where (M', f', v') is a self-transverse representative of $[f]$. These self-intersection operations *are natural transformations of set valued cofunctors.*

4 Bordism of spreadings

Let N_+ be the one-point compactification of the manifold N, and let $T\zeta$ be the Thom space of the bundle ζ. In proving the homotopy classification

$$\mathcal{I}(N;\zeta) \cong [N_+, T\zeta]_S,$$

which generalises an earlier result of Wells [15], Vogel introduces an auxillary functor given by the bordism of spreadings. For pointed compact spaces X and Y he describes a monoid $\mathcal{J}(X;Y)$, the so-called 'bordism group of spreadings of type Y in X'. This construction is functorial in both X and Y, and there is a natural isomorphism of monoids (see §5)

$$\Theta \colon \mathcal{I}(N;\zeta) \xrightarrow{\cong} \mathcal{J}(N_+;T\zeta).$$

Then Vogel goes on to show that the functor $\mathcal{J}(-;T\zeta)$ is classified by the space $QT\zeta = \lim_{l\to\infty} \Omega^l\Sigma^l T\zeta$. The key observation for us is that the bordism class of an immersion f with normal bundle $\nu(f)$ is determined by the classifying map of $\nu(f)$ and the bordism class of a spreading of f (an extension of f to an immersion of its normal disc bundle $D\nu(f)$).

4.1 Definitions

Let (X, A) be a pair of compact topological spaces, and let Y be a pointed compact space with base point y.

Definition 4.1. *A* spreading of type Y in (X, A) *is a triple* (K, α, β)*, where* K *is a compact topological space,* $\alpha\colon K \to X$ *is a continuous map, and* $\beta\colon (K, \alpha^{-1}(A)) \to (Y, y)$ *is a continuous map such that the restriction of* α *to* $K - \beta^{-1}(y)$ *is a local homeomorphism.*

We may put a bordism relation on the class of all such triples as follows.

Definition 4.2. *Two spreadings* (K_0, α_0, β_0) *and* (K_1, α_1, β_1) *of type* Y *in* (X, A) *are* bordant *if there is a spreading* (L, Ψ, Φ) *of type* Y *in* $(X \times I, A \times I)$ *such that the squares in the following diagram are pullback squares,*

$$\begin{tikzcd} K_0 \arrow[r] \arrow[d, "\alpha_0"'] & L \arrow[d, "\Psi"] & K_1 \arrow[l] \arrow[d, "\alpha_1"] \\ X \times \{0\} \arrow[r] & X \times I & X \times \{1\} \arrow[l] \end{tikzcd}$$

and $\Phi|_{K_i} = \beta_i$ *for* $i = 0, 1$.

Now let X and Y be pointed, compact spaces with base points x and y. We denote the set of bordism classes of spreadings of type Y in (X, x) by $\mathcal{J}(X; Y)$, and note that it has a commutative monoid structure arising from the disjoint union of spreadings, for which the class of the empty spreading acts as a zero element (compare §3.1).

4.2 Functoriality

Let $\mathsf{CTop}_\bullet$ denote the category of pointed, compact topological spaces, and pointed maps. We have the following analogue of Proposition 3.2.

Proposition 4.3. *The construction* $(X, Y) \to \mathcal{J}(X; Y)$ *gives a homotopy bifunctor*

$$\mathcal{J}(-,-)\colon \mathsf{CTop}_\bullet^{opp} \times \mathsf{CTop}_\bullet \to \mathsf{CMon}.$$

Proof. It is clear that a pointed map $t : Y_1 \to Y_2$ induces a homomorphism

$$t_*\colon \mathcal{J}(X; Y_1) \to \mathcal{J}(X; Y_2)$$

by sending $[(K, \alpha, \beta)]$ to $[(K, \alpha, t \circ \beta)]$, and that a homotopy of such maps gives a bordism of the induced spreadings. This takes care of covariance in the second argument. For contravariance in the first, suppose we have a map $\phi\colon X_1 \to X_2$ of pointed compact spaces, and a spreading (K, α, β) of type Y in X_2. We can form the pullback square

$$\begin{tikzcd} X_1 \times_{X_2} K \arrow[r, "\phi'"] \arrow[d, "\alpha'"'] & K \arrow[d, "\alpha"] \\ X_1 \arrow[r, "\phi"] & X_2 \end{tikzcd}$$

and set $\phi^*[(K, \alpha, \beta)] = [(X_1 \times_{X_2} K, \alpha', \beta \circ \phi')]$. One verifies that this gives a well-defined monoid homomorphism depending only on the homotopy class of ϕ, and the Proposition is proved. □

4.3 Products

Let (X, x), (Y, y), (X', x') and (Y', y') be compact based topological spaces. The smash products $X \wedge X'$ and $Y \wedge Y'$ are also compact with base points $x \wedge x'$ and $y \wedge y'$ respectively. There are obvious quotient maps $p\colon X \times X' \to X \wedge X'$ and $q\colon Y \times Y' \to Y \wedge Y'$ (so that $x \wedge x' = p(X \vee X')$ and $y \wedge y' = q(Y \vee Y')$).

Definition 4.4. *Given a spreading (K, α, β) of type Y in X, and a spreading (K', α', β') of type Y' in X', their* smash product *is the triple*

$$(K \times K', p \circ (\alpha \times \alpha'), q \circ (\beta \times \beta')),$$

which is a spreading of type $Y \wedge Y'$ in $X \wedge X'$.

One may check that this gives a well-defined smash product pairing on bordism classes

$$\mathcal{J}(X; Y) \times \mathcal{J}(X'; Y') \to \mathcal{J}(X \wedge X'; Y \wedge Y').$$

4.4 The self-intersection operations $\Psi_r\colon \mathcal{J}(-; Y) \to \mathcal{J}(-; D_r Y)$

We now introduce certain operations in the bordism of spreadings, which take the form of natural transformations of set-valued co-functors

$$\Psi_r\colon \mathcal{J}(-; Y) \to \mathcal{J}(-; D_r Y).$$

Let (K, α, β) be a spreading of type (Y, y) in (X, x). As in §3.4, let $\mathcal{F}(K; r)$ denote the configuration space of ordered r-tuples of distinct points in K.

We can form the pull-back square

$$\begin{tikzcd}
\overline{\Delta}_r(\alpha) \arrow[r] \arrow[d, "\overline{\Psi}_r(\alpha)"'] & \mathcal{F}(K;r) \arrow[d, "\alpha^{(r)}|"] \\
X \arrow[r, "\triangle"] & X^{(r)}
\end{tikzcd}$$

Note that the space

$$\overline{\Delta}_r(\alpha) = \{(k_1, \ldots, k_r) \in \mathcal{F}(K;r) \mid \alpha(k_1) = \ldots = \alpha(k_r)\}$$

admits a free action of S_r and is compact, hence the quotient space

$$\Delta_r(\alpha) := \overline{\Delta}_r(\alpha)/S_r$$

is also compact. The equivariant map $\overline{\Psi}_r(\alpha)$ induces a map $\Psi_r(\alpha)\colon \Delta_r(\alpha) \to X$.

Now let $\rho\colon K \to \mathbb{R}^\infty$ be a continuous injective map. Again taking ES_r to be the space $\mathcal{F}(\mathbb{R}^\infty;r)$, we define a map $\mathcal{S}_r(\beta)\colon \Delta_r(\alpha) \to D_rY$ by setting

$$\mathcal{S}_r(\beta)[k_1 \ldots, k_r] = [(\rho(k_1), \ldots, \rho(k_r)), (\beta(k_1), \ldots, \beta(k_r))].$$

Note that $\mathcal{S}_r(\beta)$ is independent of the choice of ρ up to homotopy, since any two such ρ are homotopic through continuous injective maps. One may then verify that the triple $(\Delta_r(\alpha), \Psi_r(\alpha), \mathcal{S}_r(\beta))$ is a spreading of type D_rY in X, and we obtain a well-defined function on bordism classes giving the following analogue of Definition 3.4.

Definition 4.5. *There are operations*

$$\Psi_r\colon \mathcal{J}(-;Y) \to \mathcal{J}(-;D_rY),$$

defined for each $r \geq 1$ by setting

$$\Psi_r[(K, \alpha, \beta)] = [(\Delta_r(\alpha), \Psi_r(\alpha), \mathcal{S}_r(\beta))].$$

These self-intersection operations *are natural transformations of set-valued co-functors.*

5 The Isomorphism Θ

Suppose we are given a triple (M, f, v) representing the class $[f] \in \mathcal{I}(N;\zeta)$. Consider the unit normal disc bundle $D\nu(f)$ of f. We can find an immersion $F\colon D\nu(f) \looparrowright N$ which extends f and is injective on the fibres. Such an immersion F will be called a *spreading of the immersion f*.

Now suppose the bundle map $v\colon \nu(f) \to \zeta$ was chosen to be isometric on fibres, so that we have a composition

$$\widetilde{v}\colon D\nu(f) \to D\zeta \to T\zeta.$$

Then the triple $(D\nu(f), F, \widetilde{v})$ is a spreading of type $T\zeta$ in N_+, justifying the above terminology. In fact, this gives a well defined function

$$\Theta\colon \mathcal{I}(N;\zeta) \to \mathcal{J}(N_+;T\zeta).$$

Theorem 5.1. *Θ is an isomorphism of monoids, and is natural in the following sense. If $g\colon Q \looparrowright N$ is a proper immersion and $\eta\colon \zeta \to \xi$ is a bundle map, then the following diagrams commute.*

$$\begin{array}{ccc} \mathcal{I}(N;\zeta) & \xrightarrow{\Theta} & \mathcal{J}(N_+;T\zeta) \\ \downarrow g^* & & \downarrow (g_+)^* \\ \mathcal{I}(Q;\zeta) & \xrightarrow{\Theta} & \mathcal{J}(Q_+;T\zeta) \end{array} \qquad \begin{array}{ccc} \mathcal{I}(N;\zeta) & \xrightarrow{\Theta} & \mathcal{J}(N_+;T\zeta) \\ \downarrow \eta_* & & \downarrow (T\eta)_* \\ \mathcal{I}(N;\xi) & \xrightarrow{\Theta} & \mathcal{J}(N_+;T\xi) \end{array}$$

Proof. The map Θ is evidently a monoid homomorphism. Vogel constructs an inverse $\Upsilon\colon \mathcal{J}(N_+;T\zeta) \to \mathcal{I}(N;\zeta)$ for Θ, using a slight modification of the proof of Thom's theorem ([12], chapter IV) on the bordism of embedded submanifolds.

Firstly, we may assume that ζ is a smooth k-vector bundle over a manifold X, whose total space $E(\zeta)$ is therefore a manifold. This is because $\mathcal{I}(N;-)$ and $\mathcal{J}(N_+;-)$ are homotopy functors and X has the homotopy type of a manifold.

Now consider a spreading (K, α, β) of type $(T\zeta, *)$ in N_+. By the definition of a spreading,

$$\alpha|_{K-\beta^{-1}(*)}\colon K - \beta^{-1}(*) \to N$$

is a local homeomorphism, hence $K - \beta^{-1}(*)$ has the structure of an n-manifold without boundary. Using arguments similar to those found in

Thom's seminal work [12], β is homotopic rel $\beta^{-1}(*)$ to a map $\beta'\colon K \to T\zeta$ with the following properties:

(i) $\beta'|_{K-(\beta')^{-1}(*)}\colon K-(\beta')^{-1}(*) \to E(\zeta)$ is smooth and transverse to the zero section $X \hookrightarrow E(\zeta)$;

(ii) $M := (\beta')^{-1}(X)$ is a codimension k closed submanifold of $K - \beta^{-1}(*)$, whose normal bundle ν admits a bundle map $v\colon \nu \to \zeta$.

Now restriction of α to M gives an immersion $f\colon M \to N$ which again has normal bundle ν. Hence we may set

$$\Upsilon(K,\alpha,\beta) = (M,f,v).$$

This passes to a well-defined function $\Upsilon\colon \mathcal{J}(N_+;T\zeta) \to \mathcal{I}(N;\zeta)$ on bordism classes. It is also clear that $\Upsilon\circ\Theta$ is the identity on $\mathcal{I}(N;\zeta)$. To prove that $\Theta\circ\Upsilon$ is the identity on $\mathcal{J}(N_+;T\zeta)$, one may use the following lemma, which says that the bordism class of a spreading (K,α,β) of type Y in X is not affected by 'throwing away' almost all of $\beta^{-1}(y)$.

Lemma 5.2. *Let (K,α,β) be a spreading of type (Y,y) in X. Let $C \subseteq K$ be the closure of $K-\beta^{-1}(y)$. Then*

$$[(C,\alpha|_C,\beta|_C)] = [(K,\alpha,\beta)] \in \mathcal{J}(X;Y)$$

Proof. Let $i\colon C \hookrightarrow K$ be the inclusion. The mapping cylinder

$$M_i = \frac{C\times I \sqcup K}{(c,1)\sim i(c)},$$

along with the obvious maps $M_i \to X\times I$ and $M_i \to Y$, gives the required bordism of spreadings. □

Suppose we have constructed $\Upsilon(K,\alpha,\beta)$ as above, and let U be an open tubular neighbourhood of the submanifold $M \subseteq K-\beta^{-1}(*)$. Again by arguments found in [12], β' is further homotopic to a map $\beta''\colon K \to T\zeta$ in *standard form*, that is to say, β'' agrees with the vector bundle morphism $v\colon \nu(f) \to \zeta$ on U and maps $K-U$ to the base point $*\in T\zeta$. Since $\mathcal{J}(N_+;-)$ is a homotopy functor, (K,α,β) is bordant to (K,α,β''). Then by Lemma 5.2, (K,α,β'') is bordant to $(D\nu(f),\alpha|_{D\nu(f)},\beta''|_{D\nu(f)})$, which is a representative of $\Theta\Upsilon[(K,\alpha,\beta)]$.

With the aid of this inverse to Θ, the statements about naturality can be easily verified. □

The next result says that the isomorphism Θ behaves well with respect to products and the self-intersection operations.

Theorem 5.3. *The following are commutative diagrams.*

$$\begin{tikzcd} \mathcal{I}(N;\zeta) \arrow[r, "\Theta"] \arrow[d, "\psi_r"'] & \mathcal{J}(N_+;T\zeta) \arrow[d, "\Psi_r"] \\ \mathcal{I}(N;\mathcal{S}_r\zeta) \arrow[r, "\Theta"] & \mathcal{J}(N_+;D_rT\zeta) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{I}(N;\zeta)\times\mathcal{I}(N';\zeta') \arrow[r, "\Theta\times\Theta"] \arrow[d, "\times"'] & \mathcal{J}(N_+;T\zeta)\times\mathcal{J}(N'_+;T\zeta') \arrow[d, "\wedge"] \\ \mathcal{I}(N\times N';\zeta\times\zeta') \arrow[r, "\Theta"] & \mathcal{J}((N\times N')_+;T\zeta\wedge T\zeta') \end{tikzcd}$$

Proof. We shall prove commutativity only of the first diagram, the proof for the second diagram being a simplification of this.

Note that an embedding $\rho\colon\ \nu(f)\to\mathbb{R}^\infty$ restricts to an embedding of $D\nu(f)$, which we shall also denote by ρ. Thus the same embedding can be use to define both $\mathcal{S}_r(v)$ and $\mathcal{S}_r(\widetilde{v})$ (see §3.4 and §4.4).

We begin with a class $[M,f,v]\in\mathcal{I}(N;\zeta)$ with $f\colon\ M\looparrowright N$ self-transverse, and apply Θ to obtain a spreading $(D\nu(f),F,\widetilde{v})$ of f. The spreading $(\Delta_r(F),\Psi_r(F),\mathcal{S}_r(\widetilde{v}))$ represents $\Psi_r\circ\Theta[M,f,v]$, where

$$\Delta_r(F)=\{[x_1,\dots,x_r]\in\mathcal{F}(D\nu(f);r)/S_r\mid F(x_1)=\ldots=F(x_r)\}$$

and $\Psi_r(F)([x_1,\dots,x_r])=F(x_1)$. We shall apply Υ to this spreading.

The map $\mathcal{S}_r(\widetilde{v})\colon\ \Delta_r(F)\to D_rT\zeta$ is already transverse to the zero section

$$\mathcal{F}(\mathbb{R}^\infty;r)\times_{S_r}X^{(r)}\hookrightarrow\mathcal{F}(\mathbb{R}^\infty;r)\times_{S_r}E(\zeta)^{(r)}$$

by virtue of being constructed from a product of bundle maps $v\colon\ \nu(f)\to E(\zeta)$. We then see that

$$\mathcal{S}_r(\widetilde{v})^{-1}(\mathcal{F}(\mathbb{R}^\infty;r)\times_{S_r}X^{(r)})=\Delta_r(f)\hookrightarrow\Delta_r(F).$$

Since the immersion $\psi_r(f)$ factorises as

$$\Delta_r(f)\hookrightarrow\Delta_r(F)\xrightarrow{\Psi_r(F)}N,$$

its normal bundle is isomorphic to the normal bundle of $\Delta_r(f)$ in $\Delta_r(F)$. Thus

$$\Upsilon\circ\Psi_r\circ\Theta[M,f,v]=[\Delta_r(f),\psi_r(f),\mathcal{S}_r(v)]=\psi_r[M,f,v]$$

as claimed. $\square$

6 The Main Result

In this section, $f\colon M^{n-k} \looparrowright N^n$ will be a self-transverse immersion of closed manifolds whose normal bundle has a ζ-structure given by a bundle map $v\colon \nu(f) \to \zeta$. For every $l \in \mathbb{N}$ the standard double cover $c_l\colon S^l \to \mathbb{R}P^l$ is a local homeomorphism, and hence an immersion. Therefore c_l represents an element $[c_l] \in \mathcal{I}(\mathbb{R}P^l;\star)$, where $\star$ denotes the trivial point bundle over a point. The immersion of closed manifolds $c_l \times f\colon S^l \times M \looparrowright \mathbb{R}P^l \times N$ represents a class

$$[c_l \times f] \in \mathcal{I}(\mathbb{R}P^l \times N; \star \times \zeta) = \mathcal{I}(\mathbb{R}P^l \times N; \zeta)$$

We wish to examine the bordism class of the double point immersion of $c_l \times f$, that is, the class

$$\psi_2[c_l \times f] \in \mathcal{I}(\mathbb{R}P^l \times N; \mathcal{S}_2\zeta).$$

Since $c_l \times f$ is *not self-transverse*, we shall employ the technical results of §5, which allow us to look instead at the double points of its spreading.

First we must recall some notation from §2.2, where the internal Steenrod operation $\mathcal{P}^k$ in MO-cobordism was constructed as a limit over l of transformations

$$\mathcal{P}^k_l\colon\ \mathsf{MO}^k(X) \to \mathsf{MO}^{2k}(\mathbb{R}P^l \times X).$$

Given our immersion $f\colon M \looparrowright N$ we have defined an immersion

$$\lambda_l(f)\colon\ S^l \times_{\mathbb{Z}_2} M^{(2)} \to S^l \times_{\mathbb{Z}_2} N^{(2)}$$

extending $f \times f$, and an immersion $\xi_l(f)\colon \Sigma(f) \to \mathbb{R}P^l \times N$ which was the pullback of $\lambda_l(f)$ along a transverse representative of the generalised diagonal embedding

$$\triangle^l_{\mathbb{Z}_2}\colon\ \mathbb{R}P^l \times N \to S^l \times_{\mathbb{Z}_2} N^{(2)}.$$

Lemma 6.1. *The immersions $\lambda_l(f)$ and $\xi_l(f)$ have a $\mathcal{S}_2\zeta$-structure on their normal bundles.*

Proof. Since $\xi_l(f)$ is a pullback of $\lambda_l(f)$, by §3.2 it suffices to prove the statement for $\lambda_l(f)$. Let $p\colon \nu(f) \to M$ be the projection of the normal bundle of f. Then the normal bundle $\nu(\lambda_l(f))$ of $\lambda_l(f)$ has projection

$$1 \times_{\mathbb{Z}_2} p^{(2)}\colon\ S^l \times_{\mathbb{Z}_2} \nu(f)^{(2)} \to S^l \times_{\mathbb{Z}_2} M^{(2)}.$$

We fix, for the remainder of this paper, an embedding $\rho\colon S^l \times \nu(f) \hookrightarrow \mathbb{R}^\infty$. Then define a bundle map $\lambda_l(v)\colon \nu(\lambda_l(f)) \to \mathcal{S}_2\zeta$ by setting

$$\lambda_l(v)[w, x_1, x_2] = [\rho(w, x_1), \rho(-w, x_2), v(x_1), v(x_2)]$$

where $w \in S^l$ and $x_1, x_2 \in \nu(f)$. $\square$

Theorem 6.2. $\psi_2[c_l \times f] = [\xi_l(f)] + c_l \times \psi_2[f] \in \mathcal{I}(\mathbb{R}P^l \times N; \mathcal{S}_2\zeta)$.

Proof. Since $[\xi_l(f)] = (\triangle^l_{\mathbb{Z}_2})^*[\lambda_l(f)]$, we may apply Θ and use the results of §5 to reduce the statement of the Theorem to the equivalent statement

$$\Psi_2\Theta[c_l \times f] = (\triangle^l_{\mathbb{Z}_2})^*\Theta[\lambda_l(f)] + \Theta[c_l] \wedge \Psi_2\Theta[f] \in \mathcal{J}((\mathbb{R}P^l \times N)_+; D_2T\zeta).$$

To prove this version, we need to find a spreading of $c_l \times f$. The normal bundle $\nu(c_l \times f)$ has projection $1 \times p\colon S^l \times \nu(f) \to S^l \times M$, and a ζ-structure $v_0\colon \nu(c_l \times f) \to \zeta$ given by $v_0(w, x) = v(x)$. Hence $\Theta[c_l \times f] \in \mathcal{J}((\mathbb{R}P^l \times N)_+; T\zeta)$ is represented by the triple $(S^l \times D\nu(f), c_l \times F, \widetilde{v_0})$, where $F\colon D\nu(f) \looparrowright N$ is a spreading of f.

We now apply the operation Ψ_2 to this spreading. We obtain the class $\Psi_2\Theta[c_l \times f]$, represented by the triple

$$(\Delta_2(c_l \times F), \Psi_2(c_l \times F), \mathcal{S}_2(\widetilde{v_0}))$$

where the map $\mathcal{S}_2(\widetilde{v_0})$ is defined as in §4.4, using the embedding $\rho\colon S^l \times D\nu(f) \hookrightarrow \mathbb{R}^\infty$ from the proof of Lemma 6.1. Now $\Delta_2(c_l \times F)$ is the compact space defined as

$$\{[(w_1, x_1), (w_2, x_2)] \in \mathcal{F}(S^l \times D\nu(f); 2)/\mathbb{Z}_2 \mid (c_l \times F)(w_1, x_1) = (c_l \times F)(w_2, x_2)\},$$

which splits as a disjoint union of compact spaces $\Delta_2(c_l \times F) = \Sigma_1 \sqcup \Sigma_2$, where

$$\Sigma_1 = \{[(w, x_1), (-w, x_2)] \in \Delta_2(c_l \times F)\},$$

$$\Sigma_2 = \{[(w, x_1), (w, x_2)] \in \Delta_2(c_l \times F) \mid x_1 \neq x_2\}.$$

Hence $\Psi_2\Theta[c_l \times f]$ splits as a sum of bordism classes.

We claim that the triple

$$(\Sigma_1, \Psi_2(c_l \times F)|_{\Sigma_1}, \mathcal{S}_2(\widetilde{v_0})|_{\Sigma_1})$$

represents $(\triangle^l_{\mathbb{Z}_2})^*\Theta[\lambda_l(f)] \in \mathcal{J}(\mathbb{R}P^l \times N; D_2T\zeta)$. From Lemma 6.1, a spreading $\Theta[\lambda_l(f)]$ of $\lambda_l(f)$ is given by the triple

$$(S^l \times_{\mathbb{Z}_2} D\nu(f)^{(2)}, 1 \times_{\mathbb{Z}_2} F^{(2)}, \widetilde{\lambda_l(v)}).$$

To obtain $(\triangle^l_{\mathbb{Z}_2})^*\Theta[\lambda_l(f)]$ we must form the pullback

$$\begin{tikzcd} \Sigma(F) \arrow[r, "i"] \arrow[d, "\xi_l(F)"'] & S^l \times_{\mathbb{Z}_2} D\nu(f)^{(2)} \arrow[d, "1 \times_{\mathbb{Z}_2} F^{(2)}"] \\ \mathbb{R}P^l \times N \arrow[r, "\triangle^l_{\mathbb{Z}_2}"] & S^l \times_{\mathbb{Z}_2} N^{(2)} \end{tikzcd}$$

Here i is the inclusion of

$$\Sigma(F) = \{[w, x_1, x_2] \in S^l \times_{\mathbb{Z}_2} D\nu(f)^{(2)} \mid F(x_1) = F(x_2)\}$$

and $\xi_l(F)[w, x_1, x_2] = ([w], F(x_1))$. Hence $(\triangle^l_{\mathbb{Z}_2})^*\Theta[\lambda_l(f)]$ is the class of

$$(\Sigma(F), \xi_l(f), \widetilde{\lambda_l(v)} \circ i).$$

There is a diffeomorphism $\Sigma_1 \approx \Sigma(F)$ given by $[(w, x_1), (-w, x_2)] \leftrightarrow [w, x_1, x_2]$ under which $\xi_l(F)$ agrees with $\Psi_2(c_l \times F)|_{\Sigma_1}$ and $\widetilde{\lambda_l(v)} \circ i$ agrees (by construction) with $\mathcal{S}_2(\widetilde{v_0})|_{\Sigma_1}$. This proves the claim.

It remains only to show that the triple

$$(\Sigma_2, \Psi_2(c_l \times F)|_{\Sigma_2}, \mathcal{S}_2(\widetilde{v_0})|_{\Sigma_2})$$

represents the product spreading $\Theta(c_l) \wedge \Psi_2\Theta[f]$. This product may be represented by the triple

$$(S^l \times \Delta_2(F), c_l \times \Psi_2(F), (\widetilde{\mathcal{S}_2(v)})_0),$$

where for $\mathbf{1} = (1, 0, \ldots 0) \in S^l$ and $(w, [x_1, x_2]) \in S^l \times \Delta_2(F)$,

$$(\widetilde{\mathcal{S}_2(v)})_0(w, [x_1, x_2]) = [\rho(\mathbf{1}, x_1), \rho(\mathbf{1}, x_2), \widetilde{v}(x_1), \widetilde{v}(x_2)]$$

Under the diffeomorphism $\Sigma_2 \approx S^l \times \Delta_2(F)$ given by $[(w, x_1), (w, x_2)] \leftrightarrow (w, [x_1, x_2])$, the maps $\Psi_2(c_l \times F)|_{\Sigma_2}$ and $c_l \times \Psi_2(F)$ agree. The map $\mathcal{S}_2(\widetilde{v_0})|_{\Sigma_2}$ corresponds to the map

$$(w, [x_1, x_2]) \mapsto (\rho(w, x_1), \rho(w, x_2), \widetilde{v}(x_1), \widetilde{v}(x_2)).$$

Assuming that ρ extends to an embedding of $D^l \times D\nu(f)$, where D^l is the unit disc, this map is homotopic to $(\widetilde{\mathcal{S}_2(v)})_0$. This proves the claim and the theorem. $\square$

Having obtained this result for every $l \in \mathbb{N}$, we would like to pass to the limit and say something about the double point immersion of $c \times f\colon S^\infty \times M \looparrowright \mathbb{R}P^\infty \times N$, where c is the universal principal $\mathbb{Z}_2$-bundle. Care must be taken, since S^∞ is not closed and we have not yet defined $\mathcal{I}(-;\zeta)$ on infinite dimensional manifolds.

Definition 6.3. *Let X be an infinite dimensional manifold without boundary, filtered by finite dimensional submanifolds X_l such that $dim(X_{l+1}) > dim(X_l)$ for $l \geq 1$ and $\bigcup_l X_l = X$, forming a direct system of embeddings*

$$X_1 \overset{j_1}{\hookrightarrow} X_2 \overset{j_2}{\hookrightarrow} \dots \overset{j_{l-1}}{\hookrightarrow} X_l \overset{j_l}{\hookrightarrow} \dots \hookrightarrow X.$$

Then the induced homomorphisms $(j_l)^*\colon \mathcal{I}(X_{l+1};\zeta) \to \mathcal{I}(X_l;\zeta)$ form a direct system, and we may define

$$\mathcal{I}(X;\zeta) := \varprojlim \mathcal{I}(X_l;\zeta).$$

Proposition 6.4. *For each $r \geq 1$ the operations $\psi_r\colon \mathcal{I}(X_l;\zeta) \to \mathcal{I}(X_l;\mathcal{S}_r\zeta)$ give a morphism of direct systems over l, and so induce an operation*

$$\psi_r\colon \mathcal{I}(X;\zeta) \to \mathcal{I}(X;\mathcal{S}_r\zeta)$$

by passage to the limit.

Proof. The operations ψ_r are natural. □

As an example of the above we have the infinite dimensional manifolds $\mathbb{R}P^\infty$ and $\mathbb{R}P^\infty \times N$, filtered by the inclusions $\iota_l^m\colon \mathbb{R}P^l \hookrightarrow \mathbb{R}P^m$ and $\iota_l^m \times 1\colon \mathbb{R}P^l \times N \to \mathbb{R}P^m \times N$ for $l \leq m$. We make the following definitions:

$$[c] = \varprojlim [c_l] \in \mathcal{I}(\mathbb{R}P^\infty;\star),$$

$$[c \times f] = \varprojlim [c_l \times f] \in \mathcal{I}(\mathbb{R}P^\infty \times N;\zeta),$$
$$[\xi(f)] = \varprojlim [\xi_l(f)] \in \mathcal{I}(\mathbb{R}P^\infty \times N;\mathcal{S}_2\zeta).$$

Corollary 6.5.

$$\psi_2[c \times f] = [\xi(f)] + c \times \psi_2[f] \in \mathcal{I}(\mathbb{R}P^\infty \times N;\mathcal{S}_2\zeta).$$

This result relates to the Steenrod operations of §2 in the following manner. An immersion of closed manifolds $f\colon M^{n-k} \looparrowright N^n$ is proper and so also represents a class $[f] \in \mathsf{MO}^k(N)$. In fact this describes, for any k-dimensional bundle ζ, a natural transformation from $\mathcal{I}(N;\zeta)$ to $\mathsf{MO}^k(N)$ which preserves sums and products. A similar transformation exists when N is infinite dimensional and filtered as in Definition 6.3, by passage to inverse limits.

Now recall from §2 that there exist Steenrod operations of type $(\mathbb{Z}_2, 1)$ in the theories MO^* and $\mathsf{H}\mathbb{Z}_2^*$. Denoting both corresponding internal operations by $\mathcal{P}$, Proposition 2.14 implies that $\mu \circ \mathcal{P} = \mathcal{P} \circ \mu$, where $\mu\colon \mathsf{MO}^* \to \mathsf{H}\mathbb{Z}_2^*$ is the multiplicative natural transformation given by the universal Thom class. We shall say that $[f] \in \mathsf{MO}^k(N)$ and $\mu[f] \in H^k(N;\mathbb{Z}_2)$ (which is dual to the homology class $f_*[M] \in H_{n-k}(N;\mathbb{Z}_2)$) are both *represented by the immersion* $f\colon M \looparrowright N$. Note that $\xi(f)$ represents $\mathcal{P}^k[f]$, and in fact we have the following.

Corollary 6.6.

$$\psi_2[c \times f] = \mathcal{P}^k[f] + c \times \psi_2[f] \in \mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N), \text{ and}$$

$$\mu\psi_2[c \times f] = \mathcal{P}^k\mu[f] + \mu[c] \times \mu\psi_2[f] \in H^{2k}(\mathbb{R}P^\infty \times N;\mathbb{Z}_2).$$

Now $\mu[c]$ is defined as the limit over l of elements $\mu[c_l] \in H^0(\mathbb{R}P^l;\mathbb{Z}_2)$, which are the Poincaré duals of elements $(c_l)_*[S^l] \in H_l(\mathbb{R}P^l;\mathbb{Z}_2)$. Since the double cover c_l has degree ± 2 these elements are all zero, and so is $\mu[c]$. This leads us to the following result, which gives a geometric construction of the internal Steenrod operation on cohomology classes represented by immersions.

Corollary 6.7. *Let $\alpha \in H^k(N;\mathbb{Z}_2)$ be represented by $f\colon M^{n-k} \looparrowright N^n$. Then $\mathcal{P}^k\alpha \in H^{2k}(\mathbb{R}P^\infty \times N;\mathbb{Z}_2)$ is represented by the double point immersion of $c \times f$, where $c\colon S^\infty \to \mathbb{R}P^\infty$ is the universal double cover.*

Remark. The analogue of 6.6 holds true in $\mathsf{M}\Gamma^*$ when $\Gamma = U$ or Sp and the immersion f has a complex or symplectic structure on its normal bundle. The operations of type $(\mathbb{Z}_2, 1)$ in these theories are constructed geometrically using finite manifolds as in §2.2. Of course, the conditions of Lemma 2.11 are not satisfied by the spectra MU, MSp; however it is not always necessary to pass to the limit and many authors prefer not to do so.

The analogous theorem is false when $\Gamma = SO$ or SU, since $\mathbb{Z}_2$ consists of an odd permutation.

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