Double Point Manifolds of Immersions of Spheres in Euclidean Space

Peter J. Eccles

Abstract

Anyone who has been intrigued by the relationship between homotopy theory and differential topology will have been inspired by the work of Bill Browder. This note contains an example of the power of these interconnections.

We prove that, in the metastable range, the double point manifold of a self-transverse immersion $S^n \looparrowright \mathbb{R}^{n+k}$ is either a boundary or bordant to the real projective space $\mathbb{R}P^{n-k}$. The values of n and k for which non-trivial double point manifolds arise are determined.

1 Introduction

Given a self-transverse immersion $f: S^n \looparrowright \mathbb{R}^{n+k}$, the *r-fold intersection set* $I_r(f)$ is defined as follows:

$$I_r(f) = \{ f(x_1) = f(x_2) = \ldots = f(x_r) \mid x_i \in S^n,\ i \neq j \Rightarrow x_i \neq x_j \}.$$

The self-transversality of f implies that this subset of $\mathbb{R}^{n+k}$ is itself the image of an immersion (not necessarily self-transverse)

$$\theta_r(f): L^{n-k(r-1)} \looparrowright \mathbb{R}^{n+k}$$

of a manifold L of dimension $n - k(r-1)$ called the *r-fold intersection manifold of f*. It is natural to ask for a given manifold L whether it can arise as an intersection manifold of an immersed sphere and if so for which dimensions. Alternatively we can consider simply the bordism class of L.

In the stable range $n < k$ the map f is necessarily an embedding as $I_r(f)$ is empty for $r \geqslant 2$. In this note we consider immersions in the metastable range, $n < 2k$. In these cases the intersection manifolds are empty for $r \geqslant 3$ and so the double point manifold is embedded by $\theta_2(f)$.

Theorem 1.1 *In the metastable range, $k \leqslant n < 2k$,*

(a) *if $n-k$ is odd then the double point manifold of a self-transverse immersion $S^n \looparrowright \mathbb{R}^{n+k}$ is a boundary;*

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(b) *if $n-k$ is even then the double point manifold of a self-transverse immersion $S^n \looparrowright \mathbb{R}^{n+k}$ is either a boundary or bordant to the real projective space $\mathbb{R}P^{n-k}$.*

It should be recalled that odd dimensional real projective spaces are boundaries.

Theorem 1.2 *In the metastable range, $0 \leqslant 4p < n$, there exists an immersion $S^n \looparrowright \mathbb{R}^{2n-2p}$ with double point manifold bordant to $\mathbb{R}P^{2p}$ if and only if $n \equiv 2^q - 1 \bmod 2^q$, where $q = p$ if $p \equiv 0$ or $3 \bmod 4$ and $q = p+1$ if $p \equiv 1$ or $2 \bmod 4$.*

In fact, it is known ([17]) that for $p \geqslant 5$ there is an immersion with double point manifold homeomeorphic to $\mathbb{R}P^{2p}$ in the dimensions given by this theorem. I am grateful to András Szűcs for drawing this reference to my attention. In addition, it was my efforts to understand his result that there exists an immersion $M^n \looparrowright \mathbb{R}^{2n-2}$ with double point manifold bordant to the projective plane if and only if $n \equiv 3 \bmod 4$ ([14]) which led to the present work. Theorem 1.2 shows that in this result the manifold M can always be taken to be a sphere.

The proof of these theorems is an application of the general method introduced in [3]. It may be summarized as follows.

Step 1. Write $\mathbb{R}P_k^\infty$ for the truncated real projective space $\mathbb{R}P^\infty/\mathbb{R}P^{k-1}$. Then to an element $\alpha \in \pi_n \mathbb{R}P_k^\infty$ we can associate an immersion $i_\alpha: S^n \looparrowright \mathbb{R}^{n+k}$ well-defined up to regular homotopy. In the metastable range every regular homotopy class of such immersions arises in this way.

Step 2. Let $MO(k)$ be the Thom space of the universal $O(k)$-bundle so that each element of $\pi_n MO(k)$ represents a bordism class of $(n-k)$-dimensional manifolds. Then there is a map

$$Mk\eta: \mathbb{R}P_k^\infty \to MO(k)$$

of Thom complexes, induced by the bundle $k\eta$ over $\mathbb{R}P^\infty$, the Whitney sum of k copies of the Hopf line bundle. The element $(Mk\eta)_*\alpha \in \pi_n MO(k)$ represents the bordism class of the double point manifold of the immersion i_α.

Step 3. Since the bordism classes of a manifold is determined by its Stiefel-Whitney numbers we apply the Hurewicz map in $\mathbb{Z}/2$-homology to the class of the double point manifold. The following diagram commutes by naturality.

$$\begin{array}{ccccc} & \pi_n \mathbb{R}P_k^\infty & \xrightarrow{(Mk\eta)_*} & \pi_n MO(k) \\ & \Big\downarrow h & & \Big\downarrow h \\ \mathbb{Z}/2 \cong & H_n(\mathbb{R}P_k^\infty; \mathbb{Z}/2) & \xrightarrow{(Mk\eta)_*} & H_n(MO(k); \mathbb{Z}/2) \end{array}$$

Results about the S-reducability of truncated projective spaces ([1]) imply that, for $k \leqslant n < 2k$, the left hand Hurewicz homomorphism is zero if and only if $k < n + 1 - \rho(n+1)$. Here $\rho(n+1)$ is the Hurwitz-Radon-Eckmann number which has the property that $\rho(n+1) - 1$ is the maximum number of linearly independent tangent vector fields on S^n. Thus, in this case, the double point manifold is a boundary. The complete solution is obtained by evaluating the map $(Mk\eta)_*$ in homology.

This paper is organized as follows. In §2 we discuss the generalized J-homomorphism which leads to the first step of the above proof. In §3 we consider the relationship between this homomorphism and the Hopf invariant. The relationship between the Hopf invariant and the double point manifold described in §4 then leads to the second step. The calculations required for the third step are described in §5.

2 The generalized J-homomorphism

Let $G(m)$ be a closed subgroup of the orthogonal group $O(m)$ with inclusion map $i: G(m) \to O(m)$. Then, if ξ is an m-dimensional vector bundle on S^n represented by $\xi \in \pi_n BO(m)$, a $G(m)$-structure on ξ is a choice of element $\bar{\xi} \in \pi_n BG(m)$ such that $i_* \bar{\xi} = \xi$.

The normal bundle ν of the standard embedding $S^n \hookrightarrow \mathbb{R}^{n+m}$ is trivial and so each element of $\pi_n O(m)/G(m)$ determines a $G(m)$-structure on ν. With this structure, the embedded sphere represents an element of $\pi_{n+m} MG(m)$ by the Pontrjagin-Thom construction. This process defines the *generalized J-homomorphism*

$$J: \pi_n O(m)/G(m) \to \pi_{n+m} MG(m)$$

introduced by Bruno Harris ([4]).

The image of this map J consists of those elements which may be represented by the standard embedding of S^n in $\mathbb{R}^{n+m}$ with some $G(m)$-structure. When $G(m)$ is the trivial group, this reduces to the classical J-homomorphism $\pi_n O(m) \to \pi_{n+m} S^m$.

We now consider the case of $G(m) = O(k)$, for $k \leqslant m$, embedded in the standard way. In this case $O(m)/O(k)$ is the real Stiefel manifold $V_{m-k}(\mathbb{R}^m)$ and $MG(m)$ is the suspension $\Sigma^{m-k} MO(k)$ so that we have the following map:

$$J: \pi_n V_{m-k}(\mathbb{R}^m) \to \pi_{n+m} \Sigma^{m-k} MO(k).$$

As an aside, we now describe a map of spaces inducing this J-homomorphism which will be needed later. Recall that, as the classifying space $BO(k)$, we can take the infinite Grassmannian $G_k(\mathbb{R}^\infty)$ of k-dimensional linear subspaces of $\mathbb{R}^\infty$ (see for example [9]). In this case, the total space $EO(k)$ of the universal bundle is given by

$$EO(k) = \{ (u, U) \mid u \in U,\ U \in G_k(\mathbb{R}^\infty \}.$$

Now, let $v = (v_1, \ldots, v_{m-k}) \in V_{m-k}(\mathbb{R}^m)$ be an orthogonal $(m-k)$-frame. Write $U = \langle v_1, \ldots, v_{m-k}\rangle^\perp \subseteq \mathbb{R}^m \subseteq \mathbb{R}^\infty$, the subspace of $\mathbb{R}^m$ orthogonal to each v_i. Then a point of $\mathbb{R}^m$ may be written uniquely as $u + t_1 v_1 + \ldots + t_{m-k} v_{m-k}$ where $t_i \in \mathbb{R}$ and $u \in U$. So we may define a continuous map $J(v)\colon \mathbb{R}^m \to EO(k) \times \mathbb{R}^{m-k}$ by

$$J(v)(u + t_1 v_1 + \ldots + t_{m-k} v_{m-k}) = \big((u, U), t_1, \ldots, t_{m-k}\big).$$

This induces a map $J(v)\colon S^m \to MO(k) \wedge S^{m-k}$, i.e. $J(v) \in \Omega^m \Sigma^{m-k} MO(k)$. Checking the definitions carefully gives the following result.

Proposition 2.1 *The continuous map* $J\colon V_{m-k}(\mathbb{R}^m) \to \Omega^m \Sigma^{m-k} MO(k)$ *induces the generalized* J*-homomorphism*

$$J\colon \pi_n V_{m-k}(\mathbb{R}^m) \to \pi_n \Omega^m \Sigma^{m-k} MO(k) \cong \pi_{n+m} \Sigma^{m-k} MO(k).$$

Returning to the differential topology, recall ([16]) that the bordism group of immersed manifolds $M^n \looparrowright \mathbb{R}^{n+k}$ is isomorphic to the stable homotopy group $\pi_n^S MO(k)$. An n-manifold embedded in $\mathbb{R}^{n+m}$ with an $O(k)$-structure, as above, has $(m-k)$ linearly independent normal vector fields and by [5] these lead to a regular homotopy of the embedding to an immersion in $\mathbb{R}^{n+k} \subseteq \mathbb{R}^{n+m}$. This process corresponds to the stabilization map

$$\pi_{n+m} \Sigma^{m-k} MO(k) \to \pi_{n+k}^S MO(k).$$

Composing this with J gives a stable J-homomorphism

$$J^S\colon \pi_n V_{m-k}(\mathbb{R}^m) \to \pi_{n+k}^S MO(k)$$

whose image consists of classes represented by immersed spheres, and all such classes if $m > n$ since then the stabilization map is an epimorphism. Recall the classical result of Stephen Smale ([10]) that for $m > n+1$ the homotopy group $\pi_n V_{m-k}(\mathbb{R}^m)$ represents regular homotopy classes of immersed spheres $S^n \looparrowright \mathbb{R}^{n+k}$; the stable J-homomorphism maps regular homotopy classes to bordism classes.

Finally, recall that hyperplane reflection defines a $(2k-1)$-equivalence $\lambda\colon \mathbb{R}P_k^{m-1} \to V_{m-k}(\mathbb{R}^m)$. The composition

$$J^S \circ \lambda_*\colon \pi_n \mathbb{R}P_k^{m-1} \to \pi_{n+k}^S MO(k)$$

provides the map $\alpha \mapsto [i_\alpha\colon S^n \looparrowright \mathbb{R}^{n+k}]$ claimed in Step 1 of the introduction. From the above discussion, all bordism classes represented by immersed spheres lie in the image of this map if $m > n$ and $2k > n$.

3 The Hopf invariant and the J-homomorphism

In this section we demonstrate that the multiple suspension Hopf invariant is closely related to the generalized J-homomorphism described in the previous section.

To describe this invariant we need a preliminary definition. The *quadratic construction* on a pointed space X is defined to be

$$D_2X = X \wedge X \rtimes_{\mathbb{Z}/2} S^\infty = X \wedge X \times_{\mathbb{Z}/2} S^\infty / \{*\} \times_{\mathbb{Z}/2} S^\infty,$$

where the non-trivial element of the group $\mathbb{Z}/2$ acts on $X \wedge X$ by permuting the coordinates and on the infinite sphere S^∞ by the antipodal action. This space has a natural filtration given by $D_2^iX = X \wedge X \rtimes_{\mathbb{Z}/2} S^i$. The following homeomorphisms may be obtained directly from this definition.

Proposition 3.1 *For a pointed space X,*

(a) $D_2^0X \cong X \wedge X$;

(b) *for* $i \geqslant 1$, $D_2^iX/D_2^{i-1}X \cong \Sigma^i(X \wedge X)$;

(c) *for* $i, j \geqslant 1$, $D_2^i\Sigma^jX \cong \Sigma^j(D_2^{i+j}X/D_2^{j-1}X)$;

(d) *in particular, for* $i, j \geqslant 1$, $D_2^iS^j \cong \Sigma^j\mathbb{R}P_j^{i+j}$.

We can now describe the basic properties of the multiple suspension *James-Hopf invariants* h_2^i. Write QX for the direct limit $\Omega^\infty\Sigma^\infty X = \lim \Omega^n\Sigma^nX$. For a connected space X and $i \geqslant 0$, there is a natural transformation (see [11])

$$h_2^i = h_2^i(X)\colon \Omega^{i+1}\Sigma^{i+1}X \to QD_2^iX.$$

In the case $i = 0$ this is simply the stabilization of the classical James-Hopf invariant:

$$\Omega\Sigma X \to \Omega\Sigma(X \wedge X) \to Q(X \wedge X).$$

The main property of these maps is a generalization of the classical EHP-sequence.

Theorem 3.2 ([8]) *Let X be a $(k-1)$-connected space where $k \geqslant 1$. Then, in the metastable range $j < 3k$, there is an exact sequence as follows:*

$$\pi_jX \xrightarrow{i_*} \pi_j\Omega^{i+1}\Sigma^{i+1}X \xrightarrow{h_2^i} \pi_jQD_2^iX \longrightarrow \pi_{j-1}X.$$

Using adjointness and stability we can rewrite this as follows:

$$\pi_jX \xrightarrow{\Sigma^{i+1}} \pi_{i+j+1}\Sigma^{i+1}X \xrightarrow{h_2^i} \pi_jD_2^iX \longrightarrow \pi_{j-1}X.$$

We also need to record the relationships between these invariants for different values of i. These follow directly from their definitions.

Proposition 3.3 (a) *For each $i \geqslant 0$ the following diagram is commutative.*

$$\begin{array}{ccc} \Omega^{i+1}\Sigma^{i+1}X & \xrightarrow{\;h_2^i\;} & QD_2^iX \\ \big\downarrow & & \big\downarrow \\ \Omega^{i+2}\Sigma^{i+2}X & \xrightarrow{\;h_2^{i+1}\;} & QD_2^{i+1}X \end{array}$$

(b) *For each $i, j \geqslant 0$, the following diagram is commutative.*

$$\begin{array}{ccccc} \Omega^{i+j+1}\Sigma^{i+j+1}X & & \xrightarrow{\quad h_2^{i+j}(X)\quad} & & QD_2^{i+j}X \\ \big\downarrow{\scriptstyle 1} & & & & \big\downarrow \\ \Omega^j\Omega^{i+1}\Sigma^{i+1}\Sigma^jX & \xrightarrow{\;\Omega^j h_2^i(\Sigma^jX)\;} & \Omega^jQD_2^i\Sigma^jX & \cong & Q(D_2^{i+j}X/D_2^{j-1}X) \end{array}$$

The first part of this proposition means that the invariants h_2^i combine to form the *stable James-Hopf invariant* $h_2^S\colon QX \to QD_2X$.

We can now state the main result relating the Hopf invariant and the J-homomorphism.

Theorem 3.4 *For $1 \leqslant k < m$, the following diagram is commutative.*

$$\begin{array}{ccccc} \pi_n\mathbb{R}P_k^{m-1} & \xrightarrow{\;\lambda_*\;} & \pi_nV_{m-k}(\mathbb{R}^m) & \xrightarrow{\;J\;} & \pi_{n+m}\Sigma^{m-k}MO(k) \\ & & & & \big\downarrow{\scriptstyle\cong} \\ \big\downarrow{\scriptstyle\Sigma^k} & & & & \pi_{n+k}\Omega^{m-k}\Sigma^{m-k}MO(k) \\ & & & & \big\downarrow{\scriptstyle h_2^{m-k-1}} \\ \pi_{n+k}^S\Sigma^k\mathbb{R}P_k^{m-1} & \xrightarrow{\;\cong\;} & \pi_{n+k}^SD_2^{m-k-1}S^k & \xrightarrow{\;i_*\;} & \pi_{n+k}^SD_2^{m-k-1}MO(k) \end{array}$$

Here the isomorphism in the bottom row comes from the homeomorphism of Proposition 3.1(d) and the map i is induced by the map $S^k \to MO(k)$ arising from the inclusion of a fibre in the universal bundle.

This result may be proved by making use of the map inducing the J-homomorphism introduced in the previous section beginning with a lemma which corresponds to the $k = 0$ version of the theorem.

Lemma 3.5 *For $m \geqslant 2$, the following diagram is homotopy commutative.*

$$
\begin{array}{ccc}
\mathbb{R}P^{m-1} & \xrightarrow{\quad\lambda\quad} & O(m) \\
\Big\downarrow i & & \Big\downarrow J \\
Q\mathbb{R}P^{m-1} \cong \Omega Q D_2^{m-2} S^1 & \xleftarrow{\;\Omega h_2^{m-2}(S^1)\;} & \Omega^m S^m
\end{array}
$$

Proof. The proof is by induction on m. For $m = 2$, the result is a formulation of the statement that the Hopf map $S^3 \to S^2$ has Hopf invariant 1.

The inductive step is based on the following commutative diagram.

$$
\begin{array}{ccccc}
\mathbb{R}P^{m-1} & \longrightarrow & \mathbb{R}P^{m} & \longrightarrow & S^m \\
\lambda \Big\downarrow & & \lambda \Big\downarrow & & \Big\downarrow 1 \\
O(m) & \longrightarrow & O(m+1) & \longrightarrow & S^m \\
J \Big\downarrow & & J \Big\downarrow & \mathbf{C} & \Big| \\
\Omega^m S^m & \longrightarrow & \Omega^{m+1} S^{m+1} & & \Big| \, i \\
\Omega h_2^{m-2}(S^1) \Big\downarrow & \mathbf{A} & \Omega h_2^{m-1}(S^1) \Big\downarrow & \mathbf{B} \;\; \searrow^{\Omega^m h_2^0(S^m)} & \Big\downarrow \\
Q\mathbb{R}P^{m-1} & \longrightarrow & Q\mathbb{R}P^{m} & \longrightarrow & QS^m
\end{array}
$$

In this diagram, **A** is commutative by Proposition 3.3(a), **B** is commutative by Proposition 3.3(b) and **C** is commutative by [6]. Adjointing the vertical maps in the diagram gives the following diagram of stable maps.

$$
\begin{array}{ccccc}
\mathbb{R}P^{m-1} & \longrightarrow & \mathbb{R}P^{m} & \longrightarrow & S^{m} \\
\downarrow{\scriptstyle \phi_{m-1}} & & \downarrow{\scriptstyle \phi_{m}} & & \downarrow{\scriptstyle 1} \\
\mathbb{R}P^{m-1} & \longrightarrow & \mathbb{R}P^{m} & \longrightarrow & S^{m}
\end{array}
$$

The inductive hypothesis is that $\phi_{m-1} \simeq 1$. Since the rows are cofibre sequences this implies that the stable map $\phi_m \simeq 1$ as required to complete the inductive step. □

Proof of Theorem 3.4. Consider the following commutative diagram.

$$
\begin{array}{ccccccc}
\mathbb{R}P^{k-1} & \longrightarrow & \mathbb{R}P^{m-1} & \longrightarrow & & \longrightarrow & \mathbb{R}P_k^{m-1} \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} & & & & \downarrow{\scriptstyle \lambda} \\
O(k) & \longrightarrow & O(m) & & & & V_{m-k}(\mathbb{R}^m) \\
\downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J} & & & & \downarrow{\scriptstyle J} \\
\Omega^k S^k & \longrightarrow & \Omega^m S^m & \longrightarrow & \Omega^k \Omega^{m-k} \Sigma^{m-k} S^k & \longrightarrow & \Omega^m \Sigma^{m-k} MO(k) \\
\downarrow{\scriptstyle \Omega h_2^{k-2}(S^1)} & & \downarrow{\scriptstyle \Omega h_2^{m-2}(S^1)} & & \downarrow{\scriptstyle \Omega^k h_2^{m-k-1}(S^k)} & & \downarrow{\scriptstyle \Omega^k h_2^{m-k-1}(MO(k))} \\
Q\mathbb{R}P^{k-1} & \longrightarrow & Q\mathbb{R}P^{m-1} & \longrightarrow & \Omega^k Q D_2^{m-k-1} S^k & \longrightarrow & \Omega^k Q D_2^{m-k-1} MO(k)
\end{array}
$$

In this case, adjointing the vertical maps and using $D_2^{m-k-1}S^k \cong \Sigma^k \mathbb{R}P_k^{m-1}$ we obtain the following commutative diagram of stable maps.

$$
\begin{array}{ccccccc}
\Sigma^k\mathbb{R}P^{k-1} & \longrightarrow & \Sigma^k\mathbb{R}P^{m-1} & \longrightarrow & & \longrightarrow & \Sigma^k\mathbb{R}P_k^{m-1} \\
\downarrow{\scriptstyle \phi_{k-1}} & & \downarrow{\scriptstyle \phi_{m-1}} & & & \swarrow{\scriptstyle \psi} & \downarrow \\
\Sigma^k\mathbb{R}P^{k-1} & \longrightarrow & \Sigma^k\mathbb{R}P^{m-1} & \longrightarrow & \Sigma^k\mathbb{R}P_k^{m-1} & \longrightarrow & D_2^{m-k-1}MO(k)
\end{array}
$$

Here the stable map ψ exists by the basic properties of cofibre sequences. By the lemma, $\phi_{k-1} \simeq 1$ and $\phi_{m-1} \simeq 1$. It follows that the stable map $\psi \simeq 1$. Now applying π_n to the unstable adjoint of the right hand triangle gives the theorem. □

4 The Hopf invariant and the double point manifold

The final ingredient required to enable us to read off the bordism class of the double point manifold of an immersed sphere is the observation that this is determined by the stable Hopf invariant $h_2^S\colon \pi_{n+k}^S MO(k) \to \pi_{n+k}^S D_2MO(k)$ in the following sense. If the self-transverse immersion $f\colon M^n \looparrowright \mathbb{R}^{n+k}$ represents the element $\alpha \in \pi_{n+k}^S MO(k)$, then the double point manifold $\theta_2(f)\colon L^{n-k} \looparrowright \mathbb{R}^{n+k}$ respresents the element $h_2^S(\alpha)$. This has been proved independently by Pierre Vogel ([15]), by András Szűcs ([12], [13]), and by Ulrich Koschorke and Brian Sanderson ([7]).

Notice that $\theta_2(f)$ represents an element in the stable homotopy of $D_2MO(k)$ because the immersion of the double point manifold automatically acquires some additional structure on its normal bundle, namely that at each point $f(x_1) = f(x_2)$, the normal $2k$-dimensional space may be decomposed as the direct sum of the two (unordered) k-dimensional normal spaces of f at the points x_1 and x_2. The universal bundle for this structure is

$$EO(k) \times EO(k) \times_{\mathbb{Z}/2} S^\infty \to BO(k) \times BO(k) \times_{\mathbb{Z}/2} S^\infty$$

which has the Thom complex $D_2MO(k)$.

In the case of immersed spheres in the metastable range, Theorem 3.4 shows that the double point manifold has a more refined structure corresponding to the restriction of the above universal bundle to $\{*\} \times \{*\} \times_{\mathbb{Z}/2} S^\infty$:

$$\mathbb{R}^k \times \mathbb{R}^k \times_{Z/2} S^\infty \to \mathbb{R}P^\infty.$$

As before, the involution on $\mathbb{R}^k \times \mathbb{R}^k$ is obtained by interchanging coordinates, but in this case by a change of basis it can be written as $(x, y) \mapsto (-x, y)$. This demonstrates that this bundle is simply the Whitney sum $k\eta \oplus k$ where η is the Hopf bundle. This is another way of proving that the Thom complex D_2S^k is homeomorphic to the suspended truncated real projective space $\Sigma^k\mathbb{R}P_k^\infty$ (using [2]).

In fact we are here not interested in this additional structure but merely in the unoriented bordism class of the double point manifold. Forgetting the additional structure induces a map of Thom complexes $\zeta\colon D_2MO(k) \to MO(2k)$. The bordism class of the double point manifold is then given by the composition

$$\pi_{n+k}^S D_2MO(k) \xrightarrow{\zeta_*} \pi_{n+k}^S MO(2k) \to \pi_{n-k}MO$$

where the final stabilization map to the homotopy of the MO-spectrum corresponds to forgetting the immersion. The above identification of the restricted universal bundle imples that the composition $\zeta_* i_*\colon \pi_{n+k}^S \Sigma^k\mathbb{R}P_k^\infty \cong \pi_{n+k}^S D_2S^k \to \pi_{n+k}^S D_2MO(k) \to \pi_{n+k}^S MO(2k)$ is given by $\Sigma^k(Mk\eta)_*$.

We can sum this up in the following result.

Theorem 4.1 *Let $\theta_2\colon \pi^S_{n+k}MO(k) \to \pi_{n-k}MO$ be the map defined geometrically by the double point manifold of a self-transverse immersion. Then the following diagram is commutative.*

$$
\begin{array}{ccccccc}
\pi^S_{n+k}MO(k) & \xrightarrow{h_2^S} & \pi^S_{n+k}D_2MO(k) & \longleftarrow & \pi^S_{n+k}D_2S^k & \xleftarrow{\cong} & \pi_{n+k}\Sigma^k\mathbb{R}P^\infty_k \\
\downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \zeta_*} & & & & \downarrow{\scriptstyle \cong} \\
\pi_{n-k}MO & \longleftarrow & \pi^S_{n+k}MO(2k) & \longleftarrow & \pi^S_n MO(k) & \xleftarrow{(Mk\eta)_*} & \pi^S_n\mathbb{R}P^\infty_k
\end{array}
$$

Some geometric comments

It is easy to see directly that the double point manifold of an immersed sphere has the refined structure discussed above.

Let $\tilde{L}$ be the double cover of the double point manifold L so that there is the following commutative diagram.

$$
\begin{array}{ccccc}
\tilde{L}^{n-k} & \longrightarrow & S^n & \xrightarrow{\nu} & BO(n+k) \\
\downarrow & & \downarrow{\scriptstyle f} & & \\
L^{n-k} & \xrightarrow{\theta_2(f)} & \mathbb{R}^{n+k} & &
\end{array}
$$

Then $\nu|\tilde{L}^{n-k}$ is trivial since $n-k<n$. A choice of trivialization induces a $\mathbb{Z}/2$-structure on the normal bundle of L^{n-k} corresponding to the universal bundle discussed above with Thom complex $D_2S^k \cong \Sigma^k\mathbb{R}P^\infty_k$.

Notice that this construction applies to any immersion of a sphere and suggests that a stable retraction of $V_{m-k}(\mathbb{R}^m)$ onto $\mathbb{R}P^{m-1}_k$ can be included in the diagram of Theorem 3.4. This ought to lead to an extension of the results of this paper beyond the metastable range to the general case.

5 Determining the double point manifold

In this section we prove Theorems 1.1 and 1.2. The calculation is based upon the following immediate corollary of Theorems 3.4 and 4.1.

Proposition 5.1 *For $k \leqslant n < 2k$, the following diagram is commutative.*

$$
\begin{array}{ccccc}
\pi_n\mathbb{R}P_k^{m-1} & \xrightarrow{i_*} & \pi_n\mathbb{R}P_k^{\infty} & \xrightarrow{(Mk\eta)_*} & \pi_n MO(k) \\
\Big\downarrow{\scriptstyle J^S\circ\lambda_*} & & & & \Big\downarrow{\scriptstyle\sigma} \\
\pi^S_{n+k}MO(k) & & \xrightarrow{\qquad\theta_2\qquad} & & \pi_{n-k}MO
\end{array}
$$

Thus, given $\alpha \in \pi_n\mathbb{R}P_k^{m-1}$, if $i_\alpha: S^n \looparrowright \mathbb{R}^{n+k}$ is a self-transverse immersion representing $J^S \circ \lambda_*(\alpha)$ then the bordism class of the double point manifold of i_α is represented by $(Mk\eta)_* i_*(\alpha) \in \pi_n MO(k)$.

For $m > n+1$ the above map i_* is an isomorphism. We may complete the proof by applying the Hurewicz homomorphism in $\mathbb{Z}/2$-homology to $Mk\eta$ because this Hurewicz homomorphism for MO is a monomorphism ([9]).

$$
\begin{array}{ccccc}
\pi_n\mathbb{R}P_k^{\infty} & \xrightarrow{(Mk\eta)_*} & \pi_n MO(k) & \xrightarrow{\sigma} & \pi_{n-k}MO \\
\Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h} \\
\mathbb{Z}/2 \cong H_n\mathbb{R}P_k^{\infty} & \xrightarrow{(Mk\eta)_*} & H_n MO(k) & \xrightarrow{\sigma} & H_{n-k}MO
\end{array}
$$

Here and from now on we write H_nX for $H_n(X;\mathbb{Z}/2)$.

It is well-known ([1]) that, for $k \leqslant n < 2k$, the left hand Hurewicz homomorphism is zero if and only if $k < n+1-\rho(n+1)$. So in this case $\sigma\circ(Mk\eta)_* = 0: \pi_n\mathbb{R}P_k^{\infty} \to \pi_{n-k}MO$ and the double point manifold is a boundary.

Proposition 5.2 *For $k \leqslant n < 2k$, if $k \geqslant n+1-\rho(n+1)$, let $\alpha \in \pi_n\mathbb{R}P_k^{\infty}$ be such that $h(\alpha) \neq 0$. Let L^{n-k} be the double point manifold of a corresponding immersion $i_\alpha: S^n \looparrowright \mathbb{R}^{n+k}$. Then the (normal) Stiefel-Whitney numbers of L are given by*

$$w_I[L^{n-k}] = \binom{k}{i_1}\binom{k}{i_2}\cdots\binom{k}{i_t}$$

for partitions $I = (i_1, i_2, \ldots, i_t)$ such that $i_1 + i_2 + \ldots + i_t = n-k$.

Proof. The Thom complex $MO(k)$ is homotopy equivalent to the quotient space
$BO(k)/BO(k-1)$ so that $H^*MO(k) \cong w_k\mathbb{Z}/2[w_1, \ldots, w_k]$ with the Thom isomorphism $\tilde{H}^{k+i}MO(k) \cong H^iBO(k)$ given by $w_Iw_k \leftrightarrow w_I$. Hence the Stiefel-Whitney number

$$
\begin{aligned}
w_I[L] &= \langle h(Mk\eta)_*(\alpha), w_Iw_k\rangle && \text{by the Thom isomorphism} \\
&= \langle h(\alpha), w_I(k\eta)a^k\rangle && \text{by naturality}
\end{aligned}
$$

where $a \in H^1\mathbb{R}P^\infty$ is the generator so that $a^k \in H^k\mathbb{R}P_k^\infty$ is the Thom class.

Now the total Stiefel-Whitney class $w(k\eta) = (1+a)^k \in H^*\mathbb{R}P^\infty$ so that $w_i(k\eta) = \binom{k}{i}a^i$. The result follows. □

To complete the calculation we confirm that these are the normal Stiefel-Whitney numbers of $\mathbb{R}P^{n-k}$.

Proposition 5.3 *If $k \geqslant n+1-\rho(n+1)$, then the (normal) Stiefel-Whitney numbers of $\mathbb{R}P^{n-k}$ are given by*

$$w_I[\mathbb{R}P^{n-k}] = \binom{k}{i_1}\binom{k}{i_2}\cdots\binom{k}{i_t}.$$

Proof. Put $n+1 = (2a+1)2^b$. It is clear from the definition of ρ that $\rho(n+1) \leqslant 2^b$ so that $k \geqslant n+1-\rho(n+1) \Rightarrow n-k \leqslant \rho(n+1)-1 < 2^b$.

The total tangent Stiefel-Whitney class of $\mathbb{R}P^{n-k}$ is given by $(1+a)^{n-k+1}$ since $\tau\oplus 1 \cong (n-k+1)\eta$. Hence the total normal class is given by $w(\mathbb{R}P^{n-k}) = (1+a)^{-n-1+k} \in H^*\mathbb{R}P^{n-k}$. Hence, for $i \leqslant n-k$, $w_i(\mathbb{R}P^{n-k}) = \binom{-n-1+k}{i}a^i = \binom{k}{i}a^i$ since, by the above, $i \leqslant n-k \Rightarrow i < 2^b$ and $n+1$ is a multiple of 2^b. The result follows. □

This completes the proof of Theorem 1.1.

To complete the proof of Theorem 1.2 we simply examine the definition of the function ρ. We have proved that in the case of even $n-k$, say $2p$, there exists an immersion $S^n \looparrowright \mathbb{R}^{n+k}$ with double point manifold bordant to $\mathbb{R}P^{n-k}$ if and only if $k \geqslant n+1-\rho(n+1)$, i.e. $\rho(n+1) \geqslant 2p+1$.

Recall that, for $n+1 = (2a+1)2^b$ where $b = c+4d$ for $0 \leqslant c \leqslant 3$ and $d \geqslant 0$, the value of ρ is given by $\rho(n+1) = 2^c + 8d$. It follows that the least value of b for which $\rho(n+1) \geqslant 2p+1$ is given by

for $p \equiv 0 \bmod 4$: $c = 0$ and $8d = 2p$, i.e. $b = p$;
for $p \equiv 1 \bmod 4$: $c = 2$ and $8d = 2(p-1)$, i.e. $b = p+1$;
for $p \equiv 2 \bmod 4$: $c = 3$ and $8d = 2(p-2)$, i.e. $b = p+1$;
for $p \equiv 3 \bmod 4$: $c = 3$ and $8d = 2(p-3)$, i.e. $b = p$.

This completes the proof of Theorem 1.2.

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