Is algebraic topology a respectable field?

Preliminary remark

This is the text of a lecture delivered shortly before the 40 years's celebration of FIM, as the last lecture of the *Zürich Graduate Colloquium 2003/04*, which took place in the Hermann Weyl Zimmer of the FIM. I had been asked to recall some memories of my long life in mathematics. Without revealing the topic, I suggested the title "Some Old Time Mathematics: 40 Years and Beyond". The topic was only formulated after I had mentioned my personal contacts with Hermann Weyl.

Is Algebraic Topology a respectable field? Of course it is. Even more than that: it is commonplace that today Algebraic Topology is a general name for various more or less different branches, like differential topology, manifold theory, combinatorial methods, ℓ_2-cohomology, general homology and K-theory, homological algebra – each of them interesting in itself but also for applications in many other fields of mathematics. But this was not always the case. After the discovery – or invention? – of Algebraic Topology (called Analysis Situs) by Poincaré in 1895 it took many decades for this field to be recognized generally as a "respectable" field of mathematics.

What follows is not meant to be a historical survey of that long development. There exist many very detailed writings about it, and comparing them closely one realizes that the history was quite complicated indeed. I just want to describe, mostly from my own personal experience in that field, some of the facts which support the claims formulated above, tell how gradually the field became respectable and fully accepted in the family of mathematicians. Thus there is no claim of completeness; to the contrary, what follows is just a number of specific items chosen from a personal viewpoint.

1. Hermann Weyl, 1923/24

"Why did you publish your two 1923/1924 papers on Algebraic Topology ("Analysis Situs Combinatorio") in Spanish in the Revista Matematica Hispano-Americana, a periodical which was not well-known and not easily accessible at that time?"

After his retirement from the Institute for Advanced Study, Hermann Weyl spent most of his time in Zurich. I had known him before in Princeton and our contacts continued in Zurich. I asked him the above question in 1954 when he was just preparing the laudatio for the Fields Medals to be awarded to J-P. Serre and to K. Kodaira at the International Congress in Amsterdam. Hermann Weyl answered that he simply did not want to draw attention to those two publications [484]*, the colleagues should not read them! The field was not considered to be serious mathematics like the classical fields of Analysis, Algebra, Geometry. In the spirit of the modern term political correctness it was at that early time not "mathematically correct" to work in such a field. But one has to recall that the medal was awarded to Serre for his famous thesis work in Algebraic Topology (homotopy groups of spheres) [429]. So in the meanwhile things must have changed considerably.

The two articles by Weyl give an elegant, very detailed and largely algebraic presentation of Combinatorial Topology as described by Poincaré in the *Compléments* (see below).

Before going further into the development of "mathematical correctness" of Algebraic Topology one has to take a short look at the early history from the very beginning. This of course took place long before I was involved in mathematics and topology. I say what I can find in the original papers.

2. Poincaré, 1895–1904

The birth of Algebraic Topology can be fixed historically in a very precise way: the papers of Henri Poincaré from 1895 to 1904 [369] began with "Analysis Situs" and were continued in a series of "Compléments". They clearly do not look like Algebraic Topology in a modern book. But everything connected with homology of spaces and homological algebra can be traced back to these old papers. This applies in particular to the multiple applications in Complex Analysis, in Algebraic Geometry, in Algebra and Group Theory, and in Theoretical Physics.

Thus not only the vast fields of the various aspects of modern topology, but many concepts used in mathematics today go back to one person, Henri Poincaré. His Analysis Situs was inspired by earlier ideas of Riemann and Betti, but these could not really be called a theory.

In Poincaré we find the concepts of cell complex, the cells being portions of bounded manifolds; incidence numbers describing the boundary of a cell, i.e. the way boundary cells of the next-lower dimension lie on a cell; cycles and homology; Betti numbers β_i and Euler characteristic $\chi = \sum(-1)^i\alpha_i$ where α_i is the number of cells of dimension i; the Euler–Poincaré formula

$$\chi = \sum(-1)^i\alpha_i = \sum(-1)^i\beta_i.$$

* Our references in [ ] refer to the bibliography of the monumental work by Dieudonné "A History of Algebraic and Differential Topology 1900–1960"

and Poincaré duality for a closed manifold of dimension n

$$\beta_i = \beta_{n-i}.$$

In the beginning everything was topologically invariant, at least in the differentiable sense, not really rigorous by today's standards. Then Poincaré turned to the rigorous concept of simplicial complex with invariance of homology under subdivision. But there topological invariance got lost – this is something we all know from our own work: you gain something, but you have to pay for it! The idea of simplicial approximation was already in the air; it later became one of the most important tools.

3. Hilbert, 1900

Many of us have reread, in 2000, Hilbert's famous address at the 1900 International Congress of Mathematicians, when the Millennium mathematical problems of the Clay Institute were formulated. Hilbert had established a program for the development of mathematics in the century to come (from letters addressed to his friends one knows that the original title was "the future of mathematics"). Partly he formulated explicit problems and partly he asked, in a more general way, for certain fields to be investigated and developed. Everywhere he insisted on rigor in the sense of axioms and proofs. One knows to what extent that lecture influenced mathematical research at least for the first half of the century, and in certain fields up to now.

But – not a word about Analysis Situs, not a word of the tremendous effort of Poincaré to establish this entirely new field! Was it on purpose, or a Freudian slip? One must admit that Hilbert simply did not realize that here was something to become more and more important throughout the century. This is in strong contrast to his remarkable anticipation of things to come in practically all other fields.

It is interesting to note that the papers by Hermann Weyl mentioned above are presented in a rigorous axiomatic way, in contrast to Poincaré's highly intuitive approach. Maybe this would have been more to Hilbert's taste.

4. After Poincaré

So it is a fact, mentioned explicitly by Hadamard in [217], that at the beginning of the twentieth century only a few mathematicians were interested in Analysis Situs. On the other hand those who were made very remarkable contributions; we mention some of them. Brouwer [89] proved in 1911 topological invariance of the dimension of $\mathbb{R}^n$; he solved a problem which had intrigued analysts since Cantor's (not continuous) bijective map of the real interval onto higher dimensional cubes, and Peano's continuous not bijective map of the interval onto the square. Very important for the future development was Brouwer's method of simplicial approximation and the concept of degree for mappings of manifolds. It is not clear whether even in the small family of topologists all this was really known.

In 1915 the topological invariance of the Betti numbers, and thus of the Euler characteristic, of a cell complex, was proved by Alexander [9]. In 1922 Alexander [11] found another interesting result: his duality theorem generalizing the classical Jordan curve theorem to all higher dimensions.

As for the topological invariance proof simplicial maps and simplicial approximation played an important role, combined with the concept of homotopy (making precise the earlier rather vague idea of deformation). Much later the invariance proofs became very simple thanks to the concept of homotopy equivalence and its algebraic counterpart.

All such results were considered as ingenious but somewhat exotic achievements, and it seems that not many mathematicians really knew exactly about them.

5. Heinz Hopf

With the appearance of Heinz Hopf's thesis and with his papers and lectures immediately afterwards [238] things seem to have changed considerably. Topology - that was now the standard name – was somehow accepted, though still considered a strange field. This change, what was the reason? Was it the fact that Hopf's work was intimately linked to easily accessible problems in differential geometry (Clifford-Klein problem, Curvatura Integra)? Was it his style, clear and rigorous, his inventing methods and solving "concrete" problems at the same time? Or his wonderful personality? Or his collaboration with Paul Alexandroff, beginning in Göttingen 1926 and lasting for many years? Hopf used to say later that his main merit was to have read, understood and made accessible the difficult work of Brouwer. According to Alexandroff and Hopf they both had largely been inspired by wonderful lectures of Erhardt Schmidt, Hopf's thesis adviser, on some of Brouwer's papers. In any case, certain papers of Hopf had a decisive influence on the later place of topology within mathematics, and we list them in more detail.

5.1 Hopf, 1925

In close connection with his work relating topological arguments to global differential geometry Hopf [240] proved for arbitrary dimension the famous theorem on tangent non-zero vector fields on a closed manifold (extending Poincaré's result for surfaces): if the field has isolated singularities (or zeros) then the sum of their indices is equal to the Euler characteristic of the manifold – whence a topological invariant. The index is an integer, defined as a mapping degree, which is zero if and only if the field can be modified in the neighborhood of the singularity so that the singularity disappears.

It follows, in particular, that a sphere of even dimension cannot admit tangent vector fields without singularities, while on an odd-dimensional sphere such fields exist (and can easily be described).

5.2 Hopf, 1928

On the other hand the influence of Emmy Noether on Hopf must have played a decisive role. In a 1928 paper by Hopf [241] algebraic concepts such as groups and homomorphisms were used for the first time to describe "combinatorial" aspects of (finite) cell complexes and homology. Instead of the matrices of incidence numbers, the free Abelian groups C_i generated by the i-dimensional cells of a complex were considered. The boundary ∂ becomes a homomorphism $C_i \longrightarrow C_{i-1}$, its kernel is the cycle group Z_i and $Z_i/\partial C_{i-1}$ is the Homology group H_i of the cell complex; the Betti number β_i is its $\mathbb{Q}$-rank. The sequence

$$C_n \longrightarrow ...C_{i+1} \longrightarrow C_i \longrightarrow C_{i-1} \longrightarrow ...C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

was later called the chain complex of the cellular space: the boundary of a 0-cell, a vertex, is by definition $= 1 \in \mathbb{Z}$. That chain complex is exact (kernel=image) if and only if all homology groups with $i \geq 1$ are 0.

Very soon algebraization took over; this may also be one of the reasons why, after the first papers of Hopf, some more people got interested in what could now truly be called Algebraic Topology. The term Analysis Situs disappeared, the name Topology seems to be old – after Poincaré both terms had been used for some time. In the thirties the field was pretty well established. Several books appeared and special meetings were organized.

5.3 Hopf, 1931 and 1935

In 1931 Hopf [243] showed that there are (infinitely many) maps $S^3 \longrightarrow S^2$ which are contractible i.e. not homotopic to the constant map. This fact, quite unexpected from the viewpoint of homology, was not recognized as being important – for example topologists like Lefschetz did not find it interesting. It turned out later to be the starting point of a new branch of topology, homotopy theory.

In 1935 Hopf [245] extended that result to maps $S^{4k-1} \longrightarrow S^{2k}$ for all $k \geq 1$. In an appendix special such maps are constructed with the help of a simple geometrical idea, namely "fibrations". Later these again turned out to be the root of a very vast and important theory.

The fibrations considered were essentially the following

(1) $S^{2k+1} \longrightarrow \mathbb{C}P^k$, with fiber $S1$, $k \geq 1$
(2) $S^{4k+3} \longrightarrow \mathbb{H}P^k$, with fiber $S3$, $k \geq 1$
(3) $S^{8k+7} \longrightarrow \mathbb{O}P^k$, with fiber $S8$, $k = 1$ only

The spheres on the left are the unit spheres in complex (or quaternionic, or octonionic respectively) number space of dimension $k+1$. The arrows denote the passage to homogeneous coordinates and thus are (continuous) maps onto the respective projective spaces. Since the octonions are not associative, the procedure is possible in (3) for $k = 1$ only. The fibers, the inverse images of the points of these projective spaces, are easily seen to be the respective spheres.

Since the projective lines ($k = 1$) are the spheres $S2$, $S4$, and $S8$ respectively one gets maps

(1') $S3 \longrightarrow S2$
(2') $S7 \longrightarrow S4$
(3') $S^{15} \longrightarrow S8$

which according to Hopf's method are non-contractible.

Before telling about the generalization of the Hopf fiberings (fiber spaces) and further results of Hopf we turn to another important event in Algebraic Topology:

6. Hurewicz, 1935/36

The four Dutch Academy Notes by Witold Hurewicz [256] on the "Theory of Deformations" had a great impact on the whole further development, although in the beginning they remained almost unnoticed. There are two aspects:

6.1 Homotopy groups

A few words about the definition of the homotopy groups $\pi_i(X)$ of a path-connected space X with base-point, $i \geq 1$. Its elements are the homotopy classes of based maps $S^i \longrightarrow X$, thus for $i = 1$ the homotopy classes of loops, and the group operation is a natural generalisation of the composition of loops. The structure of the group $\pi_i(X)$ is independent of the base-points. For $i \geq 2$ these groups are Abelian. They had been proposed, in 1932 already, by Cech: but then topologists did not consider them as important because of the commutativity – Hurewicz however put them to work. For any covering $\overline{X}$ of X the homotopy groups $\pi_i(\overline{X})$ and $\pi_i(X)$ are isomorphic for $i \geq 2$. X is called aspherical if all $\pi_i(X)$, $i \geq 2$ are 0.

6.2 Homotopy equivalence

A most important concept introduced by Hurewicz is homotopy equivalence, generalizing homeomorphism. A map $f : X \longrightarrow Y$ is called a homotopy equivalence if there is a map $g : Y \longrightarrow X$ such that the two compositions gf and fg are homotopic to the respective identities. The spaces X and Y are then called homotopy equivalent. Their homotopy groups and their homology groups are isomorphic.

Hurewicz proved, in particular, that two aspherical spaces X and Y with isomorphic fundamental groups are homotopy equivalent; any isomorphism between their fundamental groups is induced by a homotopy equivalence. Thus, in particular, an aspherical space with vanishing fundamental group is homotopy equivalent to the trivial space consisting of a single point (contractible).

7.

We approach the time when my own research began [148, 149, 151]. In 1939 Hopf asked me to study the papers of Hurewicz mentioned above. Some of my other Professors said that with Hopf I could certainly not go wrong, although Topology was not a well-known field. But something exotic like homotopy groups? Who might be interested?

Well, I was impressed by what I read and very soon noticed two extraordinary things – miracles.

7.1 Miracle one.

The degree of a map $S^n \longrightarrow S^n$ could easily be seen to be a homomorphism $\pi_n(S^n) \longrightarrow \mathbb{Z}$, and by simplicial approximation one realized that $\pi_n(S^n)$ is generated by the identity (degree $= 1$). Thus

$$\pi_n(S^n) = \mathbb{Z},$$

i.e. one recovers by this simple argument Hopf's Theorem that the homotopy classes of maps $S^n \longrightarrow S^n$ are characterized by the degree.

7.2 A concept which proved to be very suitable in connection with homotopy groups was that of fiber spaces (or fibrations) generalizing the Hopf fibrations (see 5.2). A fiber space is in the simplest case a map of spaces $p : E \longrightarrow B$ such that the fibers F, i.e. the inverse images of the points of B are all homeomorphic among themselves and constitute locally a topological product. The map p is called projection, the space B the base space of the fibration. In the context of homotopy groups, E and B are path-connected and have base-points (respected by maps and homotopies), and F is the inverse image of the base-point of B.

I noticed that a fibration gives rise to an exact sequence

$$\ldots \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \pi_{i-1}(F) \longrightarrow \ldots$$

(The lowest dimensions require some changes which we do not mention here.) The first homomorphism is induced by the imbedding of F into E, the second by the map p. To define the third homomorphism and to prove exactness an additional property is required, the *homotopy lifting*. It tells that if f is a map $f = pg : X \longrightarrow B$ via E then any homotopy of f is also obtained via E by a homotopy of g. This "axiom" for fibrations (there were later many variants of it) was easily verified in all geometrical examples I was dealing with. Then the third map in the sequence is constructed as follows: one represents an element of $\pi_i(B)$ by a map of the i-ball into B with boundary sphere S^{i-1} mapped to the base-point and lifts it up to a map into E with S^{i-1} mapped into F.

7.3 Miracle two.

We apply the sequence to the Hopf fibration $S^3 \longrightarrow S^2$ above and get

$$\ldots \pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) \longrightarrow \ldots$$

But $\pi_i(S^1) = 0$ for $i \geq 2$ since the universal covering is contractible. Thus

$$\pi_3(S^2) = \pi_3(S^3) = \mathbb{Z}$$

and we get (even in a more precise way) Hopf's result about non-contractible maps $S^3 \longrightarrow S^2$.

7.4 Using homotopy groups, the homotopy lifting, and exactness, various problems of geometrical nature could be solved but many questions remained open. We mention here only the vector field problem.

On a sphere S^n of odd dimension n there exist tangent unit vector fields without singularities. Do there exist two or more (or even the maximum possible number n) of such fields which are linearly independent at each point of S^n? I proved that for $n = 4k+1$ there cannot exist two independent such fields. Later, with the development of algebraic topology, more and more results of this kind were obtained: Kervaire [272] and Milnor showed that only the spheres S^n with $n = 1,\ 3,\ 7$ admit the maximum number n of independent fields (parallelizability). This problem is related to (actually a special case of) the existence of a continuous multiplication in $\mathbb{R}^{n+1}$ with two-sided unit and with norm-product rule. Adams [2] showed in 1960 that this is possible for $n + 1 = 1,\ 2,\ 4,\ 8$ only; in these cases bilinear multiplications of the required type were known already before 1900.

8. Hopf, 1944

According to Hurewicz (see **6.2**) aspherical spaces X and Y with isomorphic fundamental group G are homotopy equivalent and thus have isomorphic homology groups. Thus these homology groups are determined by G. A natural problem came up: to express them in a purely algebraic way from the group G.

Hopf [249] solved this problem by constructing a free resolution of a module M over the group algebra $\mathbb{Z}G$ of G (actually over any ring). This was a fundamental concept in the development of the algebraic field which later was called Homological Algebra. A free resolution of M is an exact sequence

$$\ldots \longrightarrow C_i \longrightarrow C_{i-1} \longrightarrow \ldots C_1 \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

where all C_i are free $\mathbb{Z}G$-modules. It can easily be constructed since any module is the quotient of a free module.

This was, of course, patterned after the methods of Hurewicz. If X is an aspherical (cellular) space then its universal covering $\widetilde{X}$ is contractible and has vanishing integral homology groups $H_i(\widetilde{X})$ for $i \geq 1$ and $\mathbb{Z}$ for $i = 0$. The fundamental group G acts freely on $\widetilde{X}$ and the chain groups are free $\mathbb{Z}G$-modules. Thus the chain complex of $\widetilde{X}$ is precisely a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The homotopy equivalence of all aspherical spaces with the same G was imitated in an algebraic way by Hopf; thus all free resolutions of $\mathbb{Z}$ yield the same homology groups with various coefficients, in particular those with coefficients $\mathbb{Z}$ (trivial action of $\mathbb{Z}G$); these yield in the case of $\widetilde{X}$ the homology of X.

9. The exact sequence

Here comes a correction: All the sequences, exact or not, mentioned in our text so far were NOT at all expressed with arrows. The arrow notation for maps $A \longrightarrow B$ with domain A and range B did not yet exist. Maps were just described by words. Arrows occurred together with a certain sequence for the first time in 1941 in a short announcement by Hurewicz [257] which seems to have remained unnoticed. Even in a note by Hurewicz and Steenrod (1941) [260] where the exact homotopy sequence appears implicitly no arrows nor sequences occur. As late as 1947 the importance of arrows and sequences was emphasized by Kelley and Pitcher [271]; they invented the name "exact" and showed that exact sequences play an important role in Algebraic Topology. Immediately this was taken up by topologists and algebraists. The simplification in notation and in concepts was so evident that Henri Cartan said in an Oberwolfach-meeting 1952:

S'il est vrai que la mathématique est la reine des sciences, qui est la reine de la mathématique? La suite exacte!

This plaisanterie was not meant too seriously. But it showed that here was a real improvement, in notation, concept and intuition. Not only sequences, but large diagrams of sequences were used very soon (Eilenberg-Steenrod, Foundations of Algebraic Topology). To express more complicated statements (and to prove them!) without that new notation was almost impossible.

In the *pre*-arrow and *pre*-exact sequences time we (Hopf, the author, and everybody else) used lengthy descriptions of the maps and of the fact that an image was equal to the kernel of another map – or not. It is today, for the authors themselves, but even more so for younger mathematicians, difficult to read the "old" papers.

10. After World War II

During the War a great deal of work was done independently on both sides of the Atlantic. Communication was almost impossible. After the War people got together and were happy to compare results. In the meanwhile Algebraic Topology had become a respectable field, recognized world-wide.

Not only that; the interest in this field seemed to grow every day. People learned about various applications and wanted to understand the techniques, which were more and more simplified and elegant, and useful here and there.

Most famous was certainly Hopf's Theorem [246] on the Betti numbers of compact Lie groups, as follows.

10.1 Hopf algebras

This had occurred in 1939 already. The paper was submitted to Compositio, but that periodical stopped publication. The manuscript found its way to the U.S. and was published in 1941 in Annals of Mathematics [246]. It became really known after the war only. It was a real surprise: the results of Elie Cartan (1936) on the topology of certain compact Lie groups turned out to be a corollary of a topological theorem. It was about closed manifolds provided with a multiplication with unit; the results were valid for all compact Lie groups without using their deep Lie structure. This was exactly what Elie Cartan had asked for, namely to find a general reason for the special topology of compact Lie groups.

The multiplication was used by Hopf to give the cohomology ring of the manifold (modern terminology) a second structure, a co multiplication. Such a superposition was called later a Hopf Algebra; it turned out to be one of the most important concepts, until today, in many fields beyond topology (e.g. theoretical physics).

11. A list of highlights

There was, in the years following 1946, a real explosion of interesting applications of Algebraic Topology to various fields, due to a continuous development of the techniques. We mention only some spectacular ones, with very few explanations.

11.1 Serre 1953

In his Ph.D thesis [429], Serre obtained a wealth of results on the homotopy groups of spheres; before, only very little was known. Serre used the Hopf algebra structure of the cohomology of loop spaces and other recent techniques.

11.2 Cartan-Serre

In the 1953 paper "Variétés analytiques complexes et et cohomologie" [105] cohomology with sheaf coefficients was applied to the Cousin problem in the theory of functions of several complex variables. They consider a complex manifold X and the sheaves Ω and $\mathcal{M}$ of germs of local holomorphic, and meromorphic respectively, functions. Since Ω is contained in $\mathcal{M}$ one has an exact coefficient cohomology sequence

$$\ldots \longrightarrow H^i(X;\Omega) \longrightarrow H^i(X;\mathcal{M}) \longrightarrow H^i(X;\mathcal{M}/\Omega) \longrightarrow H^{i+1}(X;\Omega) \longrightarrow \ldots$$

where the quotient sheaf is the sheaf of germs of locally given principal parts. $H0(\mathcal{M})$ is the group of global meromorphic functions, and $H0(X;\mathcal{M}/\Omega)$ of global principal parts on X. The existence of a meromorphic function on X with given principal part (additive Cousin problem) is thus guaranteed if $H1(X;\Omega) = 0$. This is proved for Stein manifolds X (complex manifolds with enough holomorphic functions).

11.3 Hirzebruch, 1953/54

The Hirzebruch–Riemann–Roch Theorem for algebraic manifolds [234, 235] expressed, in its simplest form the holomorphic Euler–Poincar'e characteristic in terms of topological invariants (Chern classes). It was based on many topological theories established before (Thom cobordism theory, Steenrod operations, sheaf theory etc). There were later many generalizations, in particular Atiyah-Hirzebruch, "Differentiable Riemann-Roch and K-Theory".

11.3 Bott, 1956

It was known in the thesis of the author already (1942) [148] that the homotopy groups $\pi_i(U(n))$ of the unitary groups $U(n)$ are constant for $n \geq 1/2(i+2)$ for even i and $n \geq 1/2(i+1)$ for odd i: these "stable" groups were known to be $= 0$ for $i = 0, 2, 4$ and $= \mathbb{Z}$ for $i = 1, 3, 5$. Bott [77] proved by very elaborate combination of Morse theory and differential geometry that the stable group is $= 0$ for all even i and $= \mathbb{Z}$ for all odd i (periodicity modulo 2; similar result for the orthogonal groups with periodicity modulo 8). There were later many different and more transparent proofs. Bott's theorem stimulated other developments: topological K-theory, general cohomological functors.

11.4 Adams, 1960 and 1962

In 1960 appeared Adams' theorem [2] about continuous multiplications in $\mathbb{R}^n$ with unit and norm product rule: they exist for $n = 1, 2, 4, 8$ only, with many interesting corollaries (parallelizability of spheres, bilinear division algebras etc). The proof was a real tour de force using the whole range of cohomological techniques developed before. Later the proof could be simplified thanks to topological K-theory and the Atiyah-Hirzebruch integrality results.

In 1962 Adams [Ann.of Math 75] solved completely the vector field problem for spheres: the maximum number of independent tangent vector fields on S^n is exactly the same as the corresponding number for vector fields which are linear with respect to the coordinates of S^n in $\mathbb{R}^{n+1}$ – known long ago. Here no simplification of the proof seems to be known.

12. The climax

12.1 ICM Stockholm, 1962

The International Congress Stockholm witnessed the triumph of Algebraic Topology (after that things calmed down). But there everything was topology even if the field was very different; some connections could always be established. The enthusiasm went very far. A joke went around, even quoted by the Congress president L. Garding at the official dinner: *All the different sections of the Congress should be named "Topology" with some attribute, Algebraic. Differential, Manifold-, Combinatorial, Geometrical, Analytical, Arithmetical, Nu-*

merical, Computational, etc etc, and finally there should even be a Section on Topological Topology!

12.2 Topology and Differential Geometry Zurich, 1960. FIM, 1964

The Swiss Mathematical Society organized in 1960 an international meeting devoted mainly to topology and global geometry. There was great general interest for this "new" field of mathematics. An article for a general public appeared in the Neue Zürcher Zeitung on the front page.

12.3. FIM, 1964

After Zurich 1960 and Stockholm 1962 I felt, and so did many others, that the rapid development in all fields of mathematics – algebraic topology was just a striking example – required much more and different contacts between mathematicians. The idea was that there should be at the Department of the ETH Zurich an institution for inviting people from all over the world, involved in newest research for extended stays in Zurich. Thus professors and students could learn from them and exchange views and problems, and collaboration would be stimulated. The system should be as flexible as possible and provide all necessary facilities for the visitors.

I approached President H. Pallmann of the ETH Zurich. I went to see him and explained the idea, really quite new at that time. After thinking for a few moments he said: "We have no funds, no rooms, no infrastructure for this, nothing. But we will get it. You have the idea, just go ahead".

Before any formal decision, we were allowed to start the Forschungsinstitut für Mathematik on January 1, 1964, with distinguished visitors, among them K. Chandrasekharan and L. Bers.