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COHOMOLOGY AND CONTINUOUS MAPPINGS

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I. INTRODUCTION[1]

Given a geometrical cell-complex K and a polyhedron Y, let us consider all continuous mappings $f(K) \subset Y$. These maps are divided into homotopy classes, the maps in any class being homotopic to each other. The problem of determining these classes by means of known invariants (for instance, homology properties) of K and Y is of extremely great importance in modern topology.

The discussion of the case when $K = S^i$ is an i-dimensional spherical manifold led Hurewicz[2] to the definition of the i^{th} homotopy group $\pi_i(Y)$ of Y. Although our knowledge of these groups is still very imperfect, they have proved to be a powerful instrument in all considerations connected with the general question.

Special interest has always been paid by topologists to the case when $Y = S^n$. In this direction there is the fundamental result of Hopf solving the question completely if $K = K^n$.[3] As newly shown by Whitney[4] this theorem may be stated as follows:

The classes of maps of K^n into S^n are in a $(1 - 1)$-correspondence with the elements of the n^{th} cohomology group ${}^n\boldsymbol{H}_I(K^n)$ of K^n with the group I of all integers as coefficient group.

The theorem holds even if we replace the condition $Y = S^n$ by the condition $\pi_i(Y) = 0$ for $i < n$ $(n > 1)$, provided the group I is replaced by $\pi_n(Y)$. This shows that the appearance of the group I in Hopf's theorem (as stated above) is due to the fact that I and $\pi_n(S^n)$ are isomorphic.

The use of cohomology in Hopf's theorem is natural, also, for the following reason. The theorem and the proofs hold for infinite, locally finite complexes as well as for finite, provided infinite chains and cocycles are admitted.[5] The statement of Hopf's theorem for infinite complexes in the language of homology seems to be much more complicated.[6]

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[1] Some of the results were published by the author without proofs, in C. R. Paris 208 (1939), p. 68. See also H. Freudenthal, Proc. Akad. Amsterdam 42(1939), p. 139.

[2] W. Hurewicz, Proc. Akad. Amsterdam 38(1935), pp. 112–115.

[3] K^n stands for the closed subcomplex of K consisting of all its cells of dimension $\leqq n$; $K^{-1} = 0$.

[4] H. Whitney, Duke Math. Jour. 3(1937), p. 51. References to Hopf, Hurewicz, and others will be found there.

[5] C. H. Dowker, Proc. Nat. Acad. U. S. A. 23(1937), p. 293.

[6] In a note published in C. R. Paris 206(1938), p. 1436, L. Pontrjagin signalizes some results concerning the case $K = K^{n+1}$, $Y = S^n$. $(n + 1)$-chains with coefficients from $\pi_{n+1}(S^n)$ are implicitly introduced.

In a recent paper[7] I have made systematic use of chains with elements of homotopy groups as coefficients. Let Q^r be a geometrical cell-complex which is an r-dimensional oriented combinatorial manifold, and let P^i be an i-dimensional closed subcomplex of Q^r. Given a map $f(Q^r - P^i) \subset S^n$, I define an i-chain $\gamma^i(f)$ in P^i with coefficients in $\pi_{r-i-1}(S^n)$ as follows: Let σ^i be an i-cell of P^i and s^{r-i-1} a "small" $(r - i - 1)$-sphere contained in $Q^r - P^i$ and "simply linked" with σ^i, then the element of $\pi_{r-i-1}(S^n)$ defined by the map $f(s^{r-i-1}) \subset S^n$ is the coefficient of σ^i in $\gamma^i(f)$. I proved that this chain is a cycle and that its homology properties are closely related with the homotopy properties of f.

The purpose of the present paper is to build up an analogous theory for arbitrary complexes. In parts III, IV, and V we develop such a theory which, applied to manifolds, will at once give us, in part VI, all previous results concerning $\gamma^i(f)$, even in a slightly stronger form.

The basic definition is introduced in part III as follows: Let K be an arbitrary geometrical cell-complex and $f(K^n) \subset Y$ a continuous mapping. Let σ^{n+1} be a $(n + 1)$-cell of K and $c(f, \sigma^{n+1})$ the element of $\pi_n(Y)$ defined by considering f on the boundary of σ^{n+1}. Taking $c(f, \sigma^{n+1})$ as the coefficient of σ^{n+1}, we obtain an $(n + 1)$-chain $c^{n+1}(f)$. It is proved that $c^{n+1}(f)$ is a cocycle. Extension theorem I, which is the main theorem of this paper, shows how closely the cohomology properties of this cocycle are connected with the extension-possibilities of f. In all of part III Y can be an arbitrary topological space which is simple in dimension n, a condition introduced in part II that is necessary in order to make the definition of $c(f, \sigma^{n+1})$ unique.

Part IV contains the application to the case when $\pi_i(Y) = 0$ for $i < n$. A generalization of Hopf's theorem is given which includes the generalization of Hurewicz-Whitney[4] and a generalization given by the author[8] arising from replacing the hypothesis $K = K^n$ by some hypothesis concerning cohomology groups for dimensions $> n$.

A homology interpretation of the results of part IV is given in part V, under some additional hypothesis on Y.

Two appendices discussing special topics are given at the end of the paper.

II. Preliminaries

1. Let K be a geometrical locally finite complex, with oriented convex cells σ_i^n of dimension $n = 0, 1, \cdots$. Their number may be infinite and their dimensions may form an unbounded sequence. The cells are open, and the closure of the n-cell σ_i^n will be denoted by $\bar{\sigma}_i^n$.

Let $\partial_{ji}^n = 1, -1$ or 0 according as σ_j^{n-1} is positively, negatively, or not at all, on the boundary $\bar{\sigma}_i^n - \sigma_i^n$ of σ_i^n. The *boundary* and *coboundary* of σ_i^n are defined by

$$\partial\sigma_i^n = \sum_j \partial_{ji}^n \sigma_j^{n-1}, \qquad \delta\sigma_i^n = \sum_j \partial_{ij}^{n+1} \sigma_j^{n+1}.$$

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[7] Fund. Math. 31(1938), pp. 179–200.

[8] Compositio Math. 6(1939), p. 429.

An n-chain is an infinite linear form $A^n = \sum_i \alpha_i \sigma_i^n$, in which α_i are elements of an abelian group G[9]. The *boundary* and *coboundary* of A^n are defined by

$$\partial A^n = \sum_i \alpha_i \partial \sigma_i^n, \qquad \delta A^n = \sum_i \alpha_i \delta \sigma_i^n.$$

2. Let K' be a closed subcomplex of K. An n-chain $A^n = \sum_i \alpha_i \sigma_i^n$ is contained in K' or in $K - K'$ (notation: $A^n \subset K'$ or $A^n \subset K - K'$) if $\alpha_i = 0$ for each n-cell σ_i^n in $K - K'$ or in K'. Clearly $A^n \subset K'$ implies $\partial A^n \subset K'$ and $A^n \subset K - K'$ implies $\delta A^n \subset K - K'$.

A^n is a *cycle* mod K' if $\partial A^n \subset K'$; A^n is a *cocycle in* $K - K'$ if $A^n \subset K - K'$ and $\delta A^n = 0$. Two cycles A_0^n and A_1^n mod K' are *homologous* mod K' (notation: $A_0^n \sim A_1^n$ mod K') if there is an $(n + 1)$-chain A^{n+1} such that $\partial A^{n+1} - A_1^n - A_0^n \subset K$. Two cocycles A_0^n and A_1^n in $K - K'$ are *cohomologous in* $K - K'$ (notation: $A_0^n \backsim A_1^n$ in $K - K'$) if there is an $(n - 1)$-chain $A^{n-1} \subset K - K'$ such that $\delta A^{n-1} = A_0^n - A_1^n$.

Using the relations $\partial\partial A^n = 0 = \delta\delta A^n$ we may define as usual the *homology* and *cohomology groups*

$$^n\boldsymbol{H}^G(K) \text{ mod } K', \qquad ^n\boldsymbol{H}_G(K - K'),$$

where G is an arbitrary abelian group whose elements are taken as coefficients in the chains.

If $K' = 0$ we write $A_0^n \sim A_1^n$, $A_0^n \backsim A_1^n$ and $^n\boldsymbol{H}^G(K)$ instead of $A_0^n \sim A_1^n$ mod K', $A_0^n \backsim A_1^n$ in $K - K'$ and $^n\boldsymbol{H}^G(K)$ mod K'.

Everything can be repeated starting from *finite* chains. We shall use similar notations, replacing $\sim$, $\backsim$ and $\boldsymbol{H}$ by $\sim^*$, $\backsim^*$ and $\boldsymbol{H}^*$.

3. LEMMA. *Given an n-chain* A^n, *an* $(n + 1)$-*cell* σ^{n+1} *and an n-cell* $\sigma^n \subset \bar{\sigma}^{n+1}$, *there is an* $(n - 1)$-*chain* A^{n-1} *such that*

$$A^n - \delta A^{n-1} = \alpha\sigma^n + A_1^n, \qquad A_1^n \subset K - \bar{\sigma}^{n+1}, \qquad \alpha \in G. \tag{3.1}$$

PROOF. Let $\sigma^n, \sigma_1^n, \sigma_2^n, \cdots, \sigma_r^n$ be all the n-cells of $\bar{\sigma}^{n+1}$ and let

$$A^n = \beta\sigma^n + \sum_{i=1}^{r} \beta_i \sigma_i^n + B^n, \qquad B^n \subset K - \bar{\sigma}^{n+1}.$$

The boundary $\bar{\sigma}^{n+1} - \sigma^{n+1}$ of σ^{n+1} being an n-dimensional manifold there is[10] an $(n - 1)$-chain A_i^{n-1} with integer coefficients such that

$$\delta A_i^{n-1} = \sigma_i^n - \epsilon_i \sigma^n + B_i^n, \qquad \epsilon_i = \pm 1, \qquad B_i^n \subset K - \bar{\sigma}^{n+1}$$

Writing $A^{n-1} = \sum_{i=1}^r \beta_i A_i^{n-1}$, $\alpha = \beta + \sum_{i=1}^r \epsilon_i \beta_i$ and $A_1^n = B^n + \sum_{i=1}^r \beta_i B_i^n$ we obtain (3.1).

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[9] More exactly, an n-chain is a function with n-cells as arguments and elements of G as values.

[10] H. Whitney, Duke Math. Jour. 3(1937), p. 44.

4. Let X and Y be two topological spaces and Y^X the family of all continuous transformations f such that $f(X) \subset Y$. Given $f \in Y^X$ and $A \subset X$, we denote by $f \mid A$ the "partial" function obtained considering f only on A. $Y^X(A, f)$ will be the subfamily of Y^X containing all the functions f' such that $f' \mid A = f \mid A$.

Let us fix a point $y_0 \in Y$. We shall denote by 0 the function mapping the whole of X into y_0, 0 will also stand for $0 \mid A$ and $Y^X(A, y_0)$ for $Y^X(A, 0)$.

We shall denote the closed interval (0, 1) by E and the Cartesian product of X and E by $X \times E$.

Two functions $f_0, f_1 \in Y^X(A, f)$ will be called *homotopic relative to* A (notation: $f_0 \simeq f_1$ rel. A) if there is a $g \in Y^{X \times E}$ such that

$$g(x, i) = f_i(x) \quad \text{for} \quad x \in X, i = 0, 1,$$
$$g(x, t) = f(x) \quad \text{for} \quad x \in A, t \in E.$$

If $A = 0$, the functions f_0 and f_1 will merely be called *homotopic* (notation: $f_0 \simeq f_1$) instead of homotopic rel. 0.

In this way the family Y^X is divided into *homotopy classes* and $Y^X(A, f)$ into *homotopy classes rel. A*.

5. It is well known[11] that the set $T = K \times 0 + K' \times E$ is a retract[12] of the product $K \times E$, where K' is a closed subcomplex of K, and therefore

(5.1) *Every* $f \in Y^T$ *has an extension* $f' \in Y^{K \times E}$.

Let $A \subset K'$ and $f \in Y^K$.

(5.2) *Given* $f_0 \in Y^K(A, f)$ *such that* $f_0 \mid K' \simeq f \mid K'$ *rel.* A, *there is an* $f_0' \in Y^K(K', f)$ *such that* $f_0' \simeq f_0$ *rel.* A.

PROOF. Let $g \in Y^{K' \times E}$ be such that

$$g(x, 0) = f_0(x),\ g(x, 1) = f(x) \text{ for } x \in K',$$
$$g(x, t) = f(x) \qquad \text{for } x \in A, t \in E.$$

Writing
$$g'(x, t) = g(x, t) \text{ for } (x, t) \in K' \times E,$$
$$g'(x, t) = f_0(x) \quad \text{for } (x, t) \in K \times 0$$

we have $g' \in Y^T$ and by (5.1) there is an extension $g'' \in Y^{K \times E}$ of g'. The function $f_0'(x) = g''(x, 1)$ satisfies the conditions of (5.2).

Taking $A = 0$ in (5.2) we have

(5.3) *Given* $f_0, f_1 \in Y^K$ *such that* $f_0 \mid K' \simeq f_1 \mid K'$, *there is an* $f_0' \in Y^K$ *such that* $f_0' \simeq f_0$ *and* $f_0' \mid K' = f_1 \mid K'$.

6. Let S^n be an oriented n-dimensional sphere, let $S^n = E^n_+ + E^n_-$ be a decomposition of S^n into two hemispheres (oriented as S^n), and let x_0 be a point of the equator $S^{n-1} = E^n_+ \cdot E^n_-$.

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[11] See P. Alexandroff and H. Hopf, *Topologie I*, Berlin 1935, p. 501; K. Borsuk, Ann. Soc. Polon. Math. 16(1937), p. 218.

[12] $A \subset X$ is a *retract* of X if there is an $r \in A^X$ such that $r(x) = x$ for $x \in A$.

We shall denote by $[f]$ the homotopy class rel. x_0 of a map $f \in Y^{S_n}(x_0, y_0)$. Given $f_0, f_1 \in Y^{S_n}(x_0, y_0)$, we obviously have

$$f_0 \mid E^n_- \simeq 0 \text{ rel. } x_0 \quad \text{and} \quad f_1 \mid E^n_+ \simeq 0 \text{ rel. } x_0 .$$

Therefore, by (5.2), there are two functions $f_0' \in [f_0]$ and $f_1' \in [f_1]$ such that

$$f_0' \mid E^n_- = 0 \quad \text{and} \quad f_1' \mid E^n_+ = 0.$$

Writing

$$f \mid E^n_+ = f_0' \mid E^n_+ \quad \text{and} \quad f \mid E^n_- = f_1' \mid E^n_-$$

we have $f \in Y^{S_n}(x_0, y_0)$. We define

$$[f] = [f_0] + [f_1].$$

It is easy to see that with this definition the homotopy classes rel. x_0 of $Y^{S_n}(x_0, y_0)$ form a group $\pi_n = \pi_n(Y)$ which is called[13] the n^{th} *homotopy group* of Y (with respect to y_0). The unit element of this group is obviously the class [0].

(6.1) *π_n is abelian for $n > 1$.*[14]

7. We shall call Y *simple in dimension n*,[15] or, more briefly, *n-simple* if every homotopy class of Y^{S_n} contains exactly one homotopy class rel. x_0 of $Y^{S_n}(x_0, y_0)$. In other words, Y is n-simple if

(a) *for each $f \in Y^{S_n}$ there is an $f' \in Y^{S_n}(x_0, y_0)$ such that $f' \simeq f$*

(b) *$f_0, f_1 \in Y^{S_n}(x_0, y_0)$ and $f_0 \simeq f_1$ imply $f_0 \simeq f_1$ rel. x_0.*

It can easily be shown, using (5.3), that (a) is equivalent with

(a′) *Y is arcwise connected.*

From now on we shall assume that Y is arcwise connected.

Obviously if Y is n-simple then every $f \in Y^{S_n}$ determines uniquely an element of π_n, and we may suppose that the elements of π_n are the homotopy classes of Y^{S_n}. This is the property that makes n-simple spaces useful for our further discussion.

(7.1) *π_1 is abelian if and only if Y is 1-simple.*[15]

(7.2) *If $\pi_n = 0$ then Y is n-simple.*[15]

(7.3) *If $\pi_1 = 0$ then Y is n-simple.*[15]

(7.4) *S^r is n-simple for $r = 1, 2, \cdots$.*

8. Let E^n be an oriented n-dimensional element bounded by S^{n-1}. Let us choose two homeomorphisms

$$h_+(E^n_+) = E^n \quad \text{and} \quad h_-(E^n_-) = E^n$$

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[13] W. Hurewicz, Proc. Akad. Amsterdam 38(1935), p. 113.

[14] Ibid., p. 114. Though π_1 is in general non-abelian, it is abelian in all the cases considered here, so that additive notations will be used throughout this paper.

[15] S. Eilenberg, Fund. Math. 32(1939), pp. 167–175.

such that $h_+ \mid S^{n-1} = h_- \mid S^{n-1}$, $S^{n-1} = E^n_+ \cdot E^n_-$ being the equator of S^n. Let us agree further that h_+ transforms E^n_+ into E^n positively. Then h_- transforms E^n_- into E^n negatively.

Given two functions $f_0, f_1 \in Y^{E^n}$ such that $f_0 \mid S^{n-1} = f_1 \mid S^{n-1}$, let $(f_0, f_1) \in Y^{S_n}$ be a function defined by

$$(f_0 , f_1) \mid E^n_+ = f_0 h_+, \qquad (f_0 , f_1) \mid E^n_- = f_1 h_-.$$

Assuming that Y is n-simple we denote by $d(f_0, f_1)$ the element of π_n corresponding to (f_0, f_1).

(8.1) *$d(f_0, f_1) = 0$ if and only if $f_0 \simeq f_1$ rel. S^{n-1}.*

(8.2) *If $f_0, f_1, f_2 \in Y^{E^n}$ and $f_0 \mid S^{n-1} = f_1 \mid S^{n-1} = f_2 \mid S^{n-1}$ then $d(f_0, f_1) + d(f_1, f_2) = d(f_0, f_2)$,*

(8.3) $d(f_0, f_1) = -d(f_1, f_0)$.

(8.4) *Given $f_0 \in Y^{E^n}$ and $\alpha \in \pi_n$ there is an $f_1 \in Y^{E^n}$ such that $f_0 \mid S^{n-1} = f_1 \mid S^{n-1}$ and that $d(f_0, f_1) = \alpha$.*

PROOFS. AD (8.1). $d(f_0, f_1) = 0$ is equivalent to $(f_0, f_1) \simeq 0$ and this obviously holds if and only if $f_0 \simeq f_1$ rel. S^{n-1}.

AD (8.2). Since $f_1 \simeq 0$ there is a map $g_1 \in Y^{E^n \times E}$ such that

$$g_1(x, 0) = f_1(x), \qquad g_1(x, 1) = y_0 \quad \text{for} \quad x \in E^n$$

using (5.1) we can find two maps $g_0, g_2 \in Y^{E^n \times E}$ such that

$$g_i(x, 0) = f_i(x) \quad \text{for} \quad x \in E^n, \qquad i = 0, 2,$$

$$g_i(x, t) = g_1(x, t) \quad \text{for} \quad (x, t) \in S^{n-1} \times E, \qquad i = 0, 2.$$

Writing $f_{i,t}(x) = g_i(x, t)$ for $t \in E$ and $i = 0, 1, 2$ we have for $i, j = 0, 1, 2$

$$f_{i,t} \mid S^{n-1} = f_{j,t} \mid S^{n-1} \quad \text{and} \quad (f_i , f_j) \simeq (f_{i,t} , f_{j,t}).$$

In particular we have $d(f_i, f_j) = d(f_{i,1}, f_{j,1})$ and since $f_{1,1} = 0$, (8.2) reduces to the formula

$$d(f_{0,1} , 0) + d(0, f_{2,1}) = d(f_{0,1} , f_{2,1}),$$

which is a direct consequence of the definition of π_n.

AD (8.3). Follows from (8.1) and (8.2) taking $f_2 = f_0$.

AD (8.4). Let $g \in Y^{S_n}$ be a map representing the element $\alpha \in \pi_n$. Since $g \mid E^n_+ \simeq 0$ and $f_0 \simeq 0$ we have

$$g \mid E^n_+ \simeq f_0 h_+ .$$

Using (5.3) we find a $g' \in Y^{S^n}$ such that

$$g' \simeq g \quad \text{and} \quad g' \mid E^n_+ = f_0 h_+ .$$

Writing $f_1 = g' h_-^{-1}$ we have $f_1 \in Y^{E^n}$, $f_0 \mid S^{n-1} = f_1 \mid S^{n-1}$ and $g' = (f_0, f_1)$. Therefore $d(f_0, f_1) = \alpha$.

III. The General Theory

9. We assume that Y is n-simple. All chains considered here will have coefficients taken from $\pi_n = \pi_n(Y)$.

Let $f \in Y^{K^n}$.[3] For each $(n + 1)$-cell σ^{n+1} the map $f \mid (\sigma^{n+1})^{\cdot}$, the n-sphere $(\sigma^{n+1})^{\cdot} = \bar{\sigma}^{n+1} - \sigma^{n+1}$ being oriented by the n-cycle $\partial\sigma^{n+1}$, defines uniquely an element of π_n which we denote by $c(f, \sigma^{n+1})$. We define an $(n + 1)$-chain $c^{n+1}(f)$ writing

(9.1) $$c^{n+1}(f) = \sum_i c(f, \sigma_i^{n+1})\sigma_i^{n+1}.$$[16]

Obviously $c(f, \sigma^{n+1}) = 0$ is equivalent with $f \mid (\sigma^{n+1})^{\cdot} \simeq 0$ and therefore

(9.2) *$c(f, \sigma^{n+1}) = 0$ if and only if there is an extension $f' \in Y^{K^n + \sigma^{n+1}}$ of f.*

(9.3) *If $f' \in Y^{K^n}$ and $f \simeq f'$ then $c^{n+1}(f) = c^{n+1}(f')$.*

Let $f_0, f_1 \in Y^{K^n}$, $f_0 \mid K^{n-1} = f_1 \mid K^{n-1}$. For each n-cell σ^n the maps $f_0 \mid \bar{\sigma}^n$ and $f_1 \mid \bar{\sigma}^n$ define according to **8** an element $d(f_0, f_1, \sigma^n)$ of π_n. We define[17] an n-chain $d^n(f_0, f_1)$ writing

(9.4) $$d^n(f_0, f_1) = \sum_i d(f_0, f_1, \sigma_i^n)\sigma_i^n.$$

It follows from (8.1) that

(9.5) $$d(f_0, f_1, \sigma^n) = 0 \textit{ if and only if } f_0 \mid \bar{\sigma}^n \simeq f_1 \mid \bar{\sigma}^n \textit{ rel. } \bar{\sigma}^n - \sigma^n.$$

10. We shall prove the following fundamental properties of $c^{n+1}(f)$ and $d^n(f_0, f_1)$:

(10.1) *$c^{n+1}(f)$ is a cocycle* (i.e. $\delta c^{n+1}(f) = 0$).

(10.2) $\delta d^n(f_0, f_1) = c^{n+1}(f_0) - c^{n+1}(f_1)$.

(10.3) *If $f_0, f_1, f_2 \in Y^{K^n}$ and $f_0 \mid K^{n-1} = f_1 \mid K^{n-1} = f_2 \mid K^{n-1}$ then $d^n(f_0, f_1) + d^n(f_1, f_2) = d^n(f_0, f_2)$.*

(10.4) $d^n(f_0, f_1) = -d^n(f_1, f_0)$

(10.5) *Given $f_0 \in Y^{K^n}$ and an n-chain d^n there is an $f_1 \in Y^{K^n}$ such that $f_0 \mid K^{n-1} = f_1 \mid K^{n-1}$ and that $d^n(f_0, f_1) = d^n$.*

Proofs. (10.3) follows from (8.2); (10.4) follows from (8.3) and (10.5) is an immediate consequence of (8.4).

Ad (10.2). Let σ_i^n be an n-cell. We suppose first that

(10.6) $$f_0 \mid K^n - \sigma_i^n = f_1 \mid K^n - \sigma_i^n.$$

We then have

$$d^n(f_0, f_1) = d(f_0, f_1, \sigma_i^n)\sigma_i^n,$$

and therefore

$$\delta d^n(f_0, f_1) = d(f_0, f_1, \sigma_i^n)\delta\sigma_i^n = d(f_0, f_1, \sigma_i^n) \sum_j \partial_{ij}^{n+1}\sigma_j^{n+1}$$

[16] According to footnote[9] we obtain $c^{n+1}(f)$ by considering $c(f, \sigma_i^{n+1})$ as a function of the argument σ_i^{n+1}.

[17] See the paper of L. Pontrjagin quoted in [6].

The formula (10.2) therefore, takes the form

$$(10.7) \qquad \partial_{ij}^{n+1} d(f_0, f_1, \sigma_i^n) = c(f_0, \sigma_j^{n+1}) - c(f_1, \sigma_j^{n+1}).$$

If $\partial_{ij}^{n+1} = 0$, then because of (10.6) we have $f_0 \mid (\sigma_j^{n+1})^{\cdot} = f_1 \mid (\sigma_j^{n+1})^{\cdot}$. It follows that $c(f_0, \sigma_j^{n+1}) = c(f_1, \sigma_j^{n+1})$, and (10.7) holds.

If $\partial_{ij}^{n+1} \neq 0$ we have $\sigma_i^n \subset \bar{\sigma}_j^{n+1}$. Let $h(\bar{\sigma}_i^n) = (\sigma_j^{n+1})^{\cdot} - \sigma_i^n$ be a homeomorphism such $h(x) = x$ for $x \in (\sigma_i^n)^{\cdot}$. Because of (10.6) we have

$$(10.8) \qquad f_0 h = f_1 h.$$

It follows from **8** that for $k = 0, 1$ we have

$$c(f_k, \sigma_j^{n+1}) = d(f_k, f_k h, \sigma_i^n) \quad \text{if} \quad \partial_{ij}^{n+1} = 1,$$

$$c(f_k, \sigma_j^{n+1}) = d(f_k h, f_k, \sigma_i^n) \quad \text{if} \quad \partial_{ij}^{n+1} = -1.$$

Using (10.8) and (8.3) we obtain

$$c(f_0, \sigma_j^{n+1}) - c(f_1, \sigma_j^{n+1}) = \partial_{ij}^{n+1}[d(f_0, f_0 h, \sigma_i^n) + d(f_0 h, f_1, \sigma_i^n)].$$

(10.7) follows, therefore, from (8.2).

Now, let $\sigma_1^n, \sigma_2^n, \cdots, \sigma_r^n$ be the n-cells of $\bar{\sigma}^{n+1}$, where σ_j^{n+1} is an arbitrary $(n+1)$-cell. We define the functions $g_0, g_1, \cdots, g_r \in Y^{K^n}$ as follows:

$$g_0 = f_0,$$

$$g_i \mid K^n - \sigma_i^n = g_{i-1} \mid K^n - \sigma_i^n, \; g_i \mid \bar{\sigma}_i^n = f_1 \mid \bar{\sigma}_i^n \text{ for } i > 0.$$

We then have for $i = 0, 1, \cdots, r-1$

$$\delta d^n(g_i, g_{i+1}) = c^{n+1}(g_i) - c^{n+1}(g_{i+1}),$$

and by (10.3)

$$(10.9) \qquad \delta d^n(f_0, g_r) = c^{n+1}(f_0) - c^{n+1}(g_r).$$

But since $g_r \mid (\sigma_j^{n+1})^{\cdot} = f_2 \mid (\sigma_j^{n+1})^{\cdot}$ we have

$$c(g_r, \sigma_j^{n+1}) = c(f_1, \sigma_j^{n+1}),$$

$$d(f_0, g_r, \sigma_i^n) = d(f_0, f_1, \sigma_i^n) \quad \text{for} \quad \sigma_i^n \subset \bar{\sigma}_j^{n+1}.$$

Consequently we deduce from (10.9) that $c(f_0, \sigma_j^{n+1}) - c(f_1, \sigma_j^{n+1})$ is the coefficient of σ_j^{n+1} in the $(n+1)$-chain $\delta d^n(f_0, f_1)$, and hence (10.2) is completely proved.

AD (10.1). Let σ^{n+2} be an $(n+2)$-cell and $\sigma^{n+1} \subset \bar{\sigma}^{n+2}$ an $(n+1)$-cell. By the lemma of **3** there is an n-chain d^n such that

$$(10.10) \qquad c^{n+1}(f) - \delta d^n = \alpha\sigma^{n+1} + A^{n+1}, \qquad A^{n+1} \subset K - \bar{\sigma}^{n+2}, \qquad \alpha \in \pi_n.$$

According to (10.5) there is an $f' \in Y^{K^n}$ such that $f \mid K^{n-1} = f' \mid K^{n-1}$ and $d^n(f, f') = d^n$. Therefore, by (10.2),

$$(10.11) \qquad c^{n+1}(f) - \delta d^n = c^{n+1}(f').$$

It follows from (10.10) and (10.11) that $c(f', \sigma^{n+1}) = \alpha$ and that $c(f', \sigma_i^{n+1}) = 0$ for each $(n + 1)$-cell $\sigma_i^{n+1} \subset \bar{\sigma}^{n+2} - \sigma^{n+1}$. By (9.2) we may therefore suppose that f' is extended on $(\sigma^{n+2})^{\cdot} - \sigma^{n+1}$. This complex being an $(n + 1)$-element it follows $f' \mid (\sigma^{n+1})^{\cdot} \simeq 0$ therefore $c(f', \sigma^{n+1}) = 0$ and therefore $\alpha = 0$. This and (10.10) imply

$$c^{n+1}(f) - \delta d^n \subset K - \bar{\sigma}^{n+2},$$

and therefore

$$\delta c^{n+1}(f) = \delta[c^{n+1}(f) - \delta d^n] \subset K - \bar{\sigma}^{n+2}.$$

σ^{n+2} being an arbitrary $(n + 2)$-cell it follows that $\delta c^{n+1}(f) = 0$.

11. Let $f \in Y^{K'+K^n}$, where K' is a fixed closed subcomplex of K.

(11.1) $c^{n+1}(f) \subset K - K'$.

(11.2) $c^{n+1}(f) = 0$ *if and only if there is an extension* $f' \in Y^{K'+K^{n+1}}$ *of* f.

In fact, for each $\sigma^{n+1} \subset K'$ we have $f \mid \bar{\sigma}^{n+1} \simeq 0$, whence $c(f, \sigma^{n+1}) = 0$. Therefore (11.1) holds. (11.2) is a consequence of (9.2).

Let $f_0, f_1 \in Y^{K'+K^n}$, $f_0 \mid K' + K^{n-1} = f_1 \mid K' + K^{n-1}$.

(11.3) $d^n(f_0, f_1) \subset K - K'$.

(11.4) $d^n(f_0, f_1) = 0$ *if and only if* $f_0 \simeq f_1$ *rel.* $K' + K^{n-1}$.

For each $\sigma^n \subset K'$ we have $f_0 \mid \bar{\sigma}^n = f_1 \mid \bar{\sigma}^n$ and from (9.5) we deduce $d(f_0, f_1, \sigma^n) = 0$. Therefore (11.3) holds. (11.4) follows from (9.5).

(11.5) *Given* $f_0 \in Y^{K'+K^n}$ *and an n-chain* $d^n \subset K - K'$, *there is an* $f_1 \in Y^{K'+K^n}$ *such that* $f_0 \mid K' + K^{n-1} = f_1 \mid K' + K^{n-1}$ *and that* $d^n(f_0, f_1) = d^n$.

(11.6) *Given* $f \in Y^{K'+K^n}$ *and a cocycle* $c^{n+1} \subset K - K'$ *such that* $c^{n+1} \backsim c^{n+1}(f)$ *in* $K - K'$, *there is an* $f' \in Y^{K'+K^n}$ *such that* $f \mid K' + K^{n-1} = f' \mid K + K^{n-1}$ *and that* $c^{n+1}(f') = c^{n+1}$.

Proofs. (11.5) is an immediate consequence of (8.4). In order to prove (11.6) let us consider an n-chain $d^n \subset K - K'$ such that

$$\delta d^n = c^{n+1}(f) - c^{n+1}.$$

By (11.5) there is an $f' \in Y^{K'+K^n}$ such that $f \mid K' + K^{n-1} = f' \mid K' + K^{n-1}$ and that $d^n(f, f') = d^n$. According to (10.2) we then have

$$\delta d^n = c^{n+1}(f) - c^{n+1}(f'),$$

and $c^{n+1}(f') = c^{n+1}$ follows.

12. Extension Theorem I. *Let* $f \in Y^{K'+K^n}$. *The* $(n + 1)$*-chain* $c^{n+1}(f)$ *defined by* (9.1) *is a cocycle in* $K - K'$. *Moreover,* $c^{n+1}(f) \backsim 0$ *in* $K - K'$ *if and only if there is an* $f' \in Y^{K'+K^{n+1}}$ *such that* $f \mid K' + K^{n-1} = f' \mid K' + K^{n-1}$.

Proof. The first part of the theorem follows from (10.1) and (11.1). If $c^{n+1}(f) \backsim 0$ in $K - K'$ then applying (11.6) for $c^{n+1} = 0$ we find an $f'' \in Y^{K'+K^n}$ such that $f \mid K' + K^{n-1} = f'' \mid K' + K^{n-1}$ and that $c^{n+1}(f'') = 0$. By (11.2) there is an extension $f' \in Y^{K'+K^{n+1}}$ of f''. We then have $f \mid K' + K^{n-1} = f' \mid K + K^{n-1}$. On the other hand, if such an f' exists we have: $\delta d^n(f, f') =$

$c^{n+1}(f) - c^{n+1}(f')$ because of (10.2), $c^{n+1}(f') = 0$ because of (11.2), and $d^n(f, f') \subset K - K'$ because of (11.3). It follows that $c^{n+1}(f) \backsim 0$ in $K - K'$.

HOMOTOPY THEOREM I. *Given two functions f_0, $f_1 \in Y^K$ such that $f_0 \mid K' + K^{n-1} = f_1 \mid K' + K^{n-1}$, the n-chain $d^n(f_0, f_1)$ defined by* (9.4) *is a cocycle in $K - K'$. Moreover, $d^n(f_0, f_1) \backsim 0$ in $K - K'$ if and only if*

(12.1) $$f_0 \mid K' + K^n \simeq f_1 \mid K' + K^n \textit{ rel. } K' + K^{n-2}.$$

PROOF. We shall consider the interval E as a complex containing two 0-cells 0 and 1 and one 1-cell ϵ oriented so that $\partial\epsilon = 0 - 1$. The product $L = K \times E$ is considered as a complex with cells of the form $\sigma^n \times 0$, $\sigma^n \times 1$ and $\sigma^n \times \epsilon$. The n-cell $\sigma^n \times 0$ is oriented as σ^n, $\sigma^n \times 1$ as $-\sigma^n$. The $(n + 1)$-cell $\sigma^n \times \epsilon$ is oriented so as to have $\partial(\sigma^n \times \epsilon) = \partial\sigma^n \times \epsilon + \sigma^n \times \partial\epsilon$.

Let $L' = K' \times E + K \times 0 + K \times 1$. Let $g \in Y^{L'+L^n}$ be the map defined by

$$g(x, t) = f_0(x) \quad \text{for} \quad x \in K' + K^{n-1}, \qquad t \in E,$$

$$g(x, i) = f_i(x) \quad \text{for} \quad x \in K, \qquad i = 0, 1.$$

By (9.2) we then have $c(g, \sigma^{n+1} \times 0) = 0$ and $c(g, \sigma^{n+1} \times 1) = 0$. It follows from **8** that $c(g, \sigma^n \times \epsilon) = d(f_0, f_1, \sigma^n)$. Therefore we have

$$c^{n+1}(g) = d^n(f_0, f_1) \times \epsilon.$$

The condition $d^n(f_0, f_1) \backsim 0$ in $K - K'$ is therefore equivalent with the condition $c^{n+1}(g) \backsim 0$ in $L - L'$. By Ext. th. I this is equivalent with the existence of a $g' \in Y^{L'+L^{n+1}}$ such that $g \mid L' + L^{n-1} = g' \mid L' + L^{n-1}$. This, however, means exactly (12.1).

HOMOTOPY THEOREM IA. *If ${}^nH_{\pi_n}(K - K') = 0$ and f_0, $f_1 \in Y^K$ then*

(12.2) $$f_0 \mid K' + K^{n-1} \simeq f_1 \mid K' + K^{n-1} \textit{ rel. } A$$

implies

(12.3) $$f_0 \mid K' + K^n \simeq f_1 \mid K' + K^n \textit{ rel. } A$$

for any subset A of K.

PROOF. By (5.2) and (12.2) there is an $f_0' \in Y^K$ such that $f_0' \mid K' + K^{n-1} = f_1 \mid K' + K^{n-1}$ and that $f_0 \simeq f_0'$ rel. A. Since $d^n(f_0', f_1) \backsim 0$ in $K - K'$ it follows from Hom. th. I that $f_0' \mid K' + K^n \simeq f_1 \mid K' + K^n$ rel. A. This implies (12.3).

13. In this section Y is i-simple for $i = n, n + 1, \cdots, \dim (K - K')$.

EXTENSION THEOREM II. *If ${}^{i+1}H_{\pi_i}(K - K') = 0$ for $i = n, n + 1, \cdots$, where $\pi_i = \pi_i(Y)$, then every $f \in Y^{K'}$ which has an extension $f' \in Y^{K'+K^n}$ has also an extension $f'' \in Y^K$.*

PROOF. We define a sequence $f_n = f'$, f_{n+1}, f_{n+2}, $\cdots$ of maps $f_i \in Y^{K'+K^i}$ such that $f_{i+1} \mid K' + K^{i-1} = f_i \mid K' + K^{i-1}$. The sequence exists by Ext. th. I. Writing $f''(x) = \lim f_i(x)$ we have $f'' \in Y^K$ and $f \mid K' = f' \mid K' = f'' \mid K$.

HOMOTOPY THEOREM II. *If ${}^iH_{\pi_i}(K - K') = 0$ for $i = n, n + 1, \cdots$, where $\pi_i = \pi_i(Y)$, then f_0, $f_1 \in Y^K$ and $f_0 \mid K' + K^{n-1} \simeq f_1 \mid K' + K^{n-1}$ rel. A imply $f_0 \simeq f_1$ rel. A for any subset A of K'.*

PROOF. Let $L = K \times E$, $L' = K' \times E + K \times 0 + K \times 1$. We then have ${}^{i+1}\boldsymbol{H}_{\pi_i}(L - L') = 0$ for $i = n, n+1, \cdots$. It follows by Ext. th. II that every $g \in Y^L$ which has an extension $g' \in Y^{L'+L^n}$ also has an extension $g'' \in Y^L$.

If $f_0 \mid K' + K^{n-1} \simeq f_1 \mid K' + K^{n-1}$ rel. A, there is a $g' \in Y^{L'+L^n}$ such that

$$g'(x, t) = f_0(x) \quad \text{for} \quad x \in A, \qquad t \in E,$$

$$g'(x, i) = f_i(x) \quad \text{for} \quad x \in K, \qquad i = 0, 1.$$

Writing $g' \mid L = g$, we see that there is an extension $g'' \in Y^L$ of g. This implies $f_0 \simeq f_1$ rel. A.

IV. THE CASE $\pi_i(Y) = 0$ FOR $i < n$.

14. In this part we assume that $\pi_i(Y) = 0$ for $i < n$. If $n > 1$ this implies $\pi_1 = 0$ and by (7.3) Y is i-simple for all i. If $n = 1$ we assume that Y is 1-simple, or (see (7.1)) that π_1 is abelian. In particular everything can be applied for $Y = S^n$.

(14.1) *Every* $f \in Y^{K'}$ *has an extension* $f' \in Y^{K'+K^n}$.

(14.2) $f_0, f_1 \in Y^K$ *and* $f_0 \mid K' \simeq f_1 \mid K'$ *rel.* A *imply* $f_0 \mid K' + K^{n-1} \simeq f_1 \mid K' + K^{n-1}$ *rel.* A *for any* $A \subset K'$.

PROOFS. Obviously there is an extension $f_0 \in Y^{K'+K^0}$ of f. Since Y is arcwise connected (see **7**) there is also an extension $f_1 \in Y^{K'+K^1}$ of f. Since $\pi_i = 0$ for $i < n$, we have ${}^{i+1}\boldsymbol{H}_{\pi_i}(K - K') = 0$ and applying Ext. th. I we obtain successively extensions $f_{i+1} \in Y^{K'+K^{i+1}}$ for $i = 1, 2, \cdots, n-1$.

Y being arcwise connected, it follows from $f_0 \mid K' \simeq f_1 \mid K'$ rel. A that $f_0 \mid K' + K^0 \simeq f_1 \mid K' + K^0$ rel. A. Since ${}^{i}\boldsymbol{H}_{\pi_i}(K - K') = 0$ for $i = 1, 2, \cdots, n-1$, we obtain $f_0 \mid K' + K^{n-1} \simeq f_1 \mid K' + K^{n-1}$ rel. A applying Hom. th. I A.

15.

(15.1) $f_0, f_1 \in Y^{K'+K^n}$ *and* $f_0 \mid K' \simeq f_1 \mid K'$ *imply* $c^{n+1}(f_0) \simeq c^{n+1}(f_i)$ *in* $K - K'$.

PROOF. By (14.2) we have $f_0 \mid K' + K^{n-1} \simeq f_1 \mid K' + K^{n-1}$, and according to (5.2) there is an $f_0' \in Y^{K'+K^n}$ such that $f_0 \simeq f_0'$ and $f_0' \mid K' + K^{n-1} = f_1 \mid K' + K^{n-1}$. From (9.3), (10.2), and (11.3) we then have: $c^{n+1}(f_0) = c^{n+1}(f_0')$, $\delta d^n(f_0, f_1) = c^{n+1}(f_0') - c^{n+1}(f_1)$, and $d^n(f_0, f_1) \subset K^n - K'$. It follows that $c^{n+1}(f_0) \backsim c^{n+1}(f_1)$ in $K - K'$.

By (14.1) every $f \in Y^{K'}$ has an extension $f' \in Y^{K'+K^n}$. Let $\boldsymbol{c}^{n+1}(f)$ be the element of ${}^{n+1}\boldsymbol{H}_{\pi_n}(K - K')$ determined by the cocycle $c^{n+1}(f')$. It follows from (15.1) that the choice of f' does not matter and that

(15.2) $$f_0, f_1 \in Y^{K'} \textit{ and } f_0 \simeq f_1 \textit{ imply } \boldsymbol{c}^{n+1}(f_0) = \boldsymbol{c}^{n+1}(f_1).$$

From Ext. th. I we obtain the following

EXTENSION THEOREM III. *Given* $f \in Y^{K'}$ *we have* $\boldsymbol{c}^{n+1}(f) = 0$ *if and only if there is an extension* $f' \in Y^{K'+K^{n+1}}$ *of* f.

16.

(16.1) *Given $f_0, f_1, f_0', f_1' \epsilon Y^K$ such that $f_0 \mid K' + K^{n-1} = f_1 \mid K' + K^{n-1}$, $f_0' \mid K' + K^{n-1} = f_1' \mid K' + K^{n-1}$, $f_0 \simeq f_0'$ rel. K' and $f_1 \simeq f_1'$ rel. K' we have $d^n(f_0, f_1) \backsim d^n(f_0', f_1')$ in $K - K'$.*

Proof. Let $L = K \times E$, $L' = K' \times E + K \times 0 + K \times 1$. Let $g \epsilon Y^{L'+L^n}$ be defined by

$$g(x, t) = f_0(x) \quad \text{for} \quad x \epsilon K' + K^{n-1}, \qquad t \epsilon E,$$

$$g(x, i) = f_i(x) \quad \text{for} \quad x \epsilon K, \qquad i = 0, 1.$$

In an analogous way we define $g' \epsilon Y^{L'+L^n}$ using f_0' and f_1' instead of f_0 and f_1. As in **12** we then have

(16.2) $$c^{n+1}(g) = d^n(f_0, f_1) \times \epsilon, \qquad c^{n+1}(g') = d^n(f_0', f_1') \times \epsilon.$$

Now, $f_0 \simeq f_0'$ rel. K' and $f_1 \simeq f_1'$ rel. K' obviously imply $g \mid L' \simeq g' \mid L'$, and therefore, by (15.1), $c^{n+1}(g) \backsim c^{n+1}(g')$ in $L - L'$. Using (16.2) we then obtain $d^n(f_0, f_1) \backsim d^n(f_0, f_1)$ in $K - K'$.

Given two functions $f_0, f_1, \epsilon Y^K$ such that $f_0 \mid K' = f_1 \mid K'$ we have, by (14.2), $f_0 \mid K' + K^{n-1} \simeq f_1 \mid K' + K^{n-1}$ rel. K', and by (5.2) there are two functions $f_0', f_1' \epsilon Y^K$, such that

(16.3) $$f_0 \simeq f_0' \text{ rel. } K', \; f_1 \simeq f_1' \text{ rel. } K', \qquad f_0' \mid K' + K^{n-1} = f_1' \mid K' + K^{n-1}.$$

Let $\boldsymbol{d}^n(f_0, f_1)$ be the element of ${}^n\boldsymbol{H}_{\pi_n}(K - K')$ determined by the cocycle $d^n(f_0', f_1')$. It follows from (16.1) that $\boldsymbol{d}^n(f_0, f_1)$ is independent of the particular choice of f_0' and f_1' such that (16.3) holds. By (10.3) and (10.4) we have

(16.4) *If $f_0, f_1, f_2 \epsilon Y^K$ and $f_0 \mid K' = f_1 \mid K' = f_2 \mid K'$ then* $\boldsymbol{d}^n(f_0, f_1) + \boldsymbol{d}^n(f_1, f_2) = \boldsymbol{d}^n(f_0, f_2)$,

(16.5) $$\boldsymbol{d}^n(f_0, f_1) = -\boldsymbol{d}^n(f_1, f_0).$$

Homotopy Theorem III. *Given $f_0, f_1 \epsilon Y^K$ such that $f_0 \mid K' = f_1 \mid K'$ we have $\boldsymbol{d}^n(f_0, f_1) = 0$ if and only if*

(16.6) $$f_0 \mid K' + K^n \simeq f_1 \mid K' + K^n \text{ rel. } K'.$$

Proof. If $\boldsymbol{d}^n(f_0, f_1) = 0$ then $d^n(f_0', f_1') \backsim 0$ in $K - K'$ and by Hom. th. I we have $f_0' \mid K' + K^n \simeq f_1' \mid K' + K^n$ rel. K'. Using (16.3) we obtain (16.6). On the other hand, by (5.2), (16.6) implies the existence of an $f_0' \epsilon Y^K$ such that $f_0 \simeq f_0'$ rel. K' and that $f_0' \mid K' + K^n = f_1 \mid K' + K^n$. It follows $d^n(f_0', f_1) = 0$ and therefore $\boldsymbol{d}^n(f_0, f_1) = 0$.

17. In this section (and in **18**) we assume as before that $\pi_i(Y) = 0$ for $i < n$. If $n = 1$ we assume that Y is i-simple for $i = 1, 2, \cdots, \dim (K - K')$. As before, everything can be applied for $Y = S^n$.

Combining Ext. th. II and III we have the

Extension Theorem IV. *Let ${}^{i+1}\boldsymbol{H}_{\pi_i}(K - K') = 0$ for $i = n + 1, n + 2, \cdots,$*

where $\pi_i = \pi_i(Y)$. *Given* $f \in Y^{K'}$ *we have* $\mathbf{c}^{n+1}(f) = 0$ *if and only if there is an extension* $f' \in Y^K$ *of* f.

Similarly Hom. th. II and III imply

HOMOTOPY THEOREM IV. *Let* ${}^i\boldsymbol{H}_{\pi_i}(K - K') = 0$ *for* $i = n + 1, n + 2, \cdots$, *where* $\pi_i = \pi_i(Y)$. *Given* $f_0, f_1 \in Y^K$ *such that* $f_0 \mid K' = f_1 \mid K'$ *we have* $\mathbf{d}^n(f_0, f_1) = 0$ *if and only if* $f_0 \simeq f_1$ *rel.* K'.

The hypotheses of Ext. th. IV and of Hom. th. IV are obviously satisfied if $K - K' \subset K^n$.

EXISTENCE THEOREM. *Let* ${}^{i+1}\boldsymbol{H}_{\pi_i}(K - K') = 0$ *for* $i = n + 1, n + 2, \cdots$, *where* $\pi_i = \pi_i(Y)$. *Given* $f_1 \in Y^K$ *and* $\mathbf{d}^n \in {}^n\boldsymbol{H}_{\pi_n}(K - K')$ *there is an* $f_0 \in Y^K$ *such that* $f_0 \mid K' = f_1 \mid K'$ *and that* $\mathbf{d}^n(f_0, f_1) = \mathbf{d}^n$.

PROOF. Let d^n be a cocycle of the cohomology class $\mathbf{d}^n$. By (11.5) and (10.4) there is a map $f_0 \in Y^{K'+K^n}$ such that $f_0 \mid K' + K^{n-1} = f_1 \mid K' + K^{n-1}$ and that $d^n(f_0, f_1) = d^n$. By (10.2) we have $\delta d^n = c^{n+1}(f_0) - c^{n+1}(f_1)$. Since $\delta d^n = 0$ and $c^{n+1}(f_1) = 0$ (see (11.2)) it follows $c^{n+1}(f_0) = 0$ and by (11.2) there is an extension $f_0' \in Y^{K'+K^{n+1}}$ of f_0.

Now, let $K'' = K' + K^n$. We then have ${}^{i+1}\boldsymbol{H}_{\pi_i}(K - K'') = 0$ for $i = n + 1, n + 2, \cdots$ and by Ext. th. II there is an extension $f_0'' \in Y^K$ of f_0. We then have $d^n(f_0'', f_1) = d^n$ and therefore $\mathbf{d}^n(f_0'', f_1) = \mathbf{d}^n$.

18. Let $f^* \in Y^K$. As in **4** $Y^K(K', f^*)$ will be the sub-family of Y^K containing all $f \in Y^K$ such that $f \mid K' = f^* \mid K'$. The family $Y^K(K', f^*)$ is divided into homotopy classes rel. K', two maps $f_0, f_1, \in Y^K(K, f^*)$ being in the same class if, and only if, $f_0 \simeq f_1$ rel. K'.

Given a homotopy class Φ rel. K' of $Y^K(K', f^*)$ we define $\mathbf{d}^n(\Phi) = \mathbf{d}^n(f, f^*)$ where $f \in \Phi$. It follows from (16.4), (16.5), and Hom. th. III, that the element $\mathbf{d}^n(\Phi)$ of ${}^n\boldsymbol{H}_{\pi_n}(K - K')$ is defined uniquely. Under the hypothesis of Hom. th. IV we have $\Phi_0 = \Phi_1$ if and only if $\mathbf{d}^n(\Phi_0) = \mathbf{d}^n(\Phi_1)$. Under the hypothesis of the Existence th. there is for each $\mathbf{d}^n \in {}^n\boldsymbol{H}_{\pi_n}(K - K')$ a homotopy class Φ rel. K' of $Y^K(K', f^*)$ such that $\mathbf{d}^n(\Phi) = \mathbf{d}^n$. We obtain, therefore,

CLASSIFICATION THEOREM I. *Let* ${}^i\boldsymbol{H}_{\pi_i}(K - K') = {}^{i+1}\boldsymbol{H}_{\pi_i}(K - K') = 0$ *for* $i = n + 1, n + 2, \cdots$, *where* $\pi_i = \pi_i(Y)$. *The elements of* ${}^n\boldsymbol{H}_{\pi_n}(K - K')$ *are in a* $(1 - 1)$*-correspondence with the homotopy classes rel.* K' *of* $Y^K(K', f^*)$. *The correspondence is determined by the operation* $\mathbf{d}^n(f, f^*)$.

Taking $K' = 0$ and $f^* = 0$ we obtain

CLASSIFICATION THEOREM II. *Let* ${}^i\boldsymbol{H}_{\pi_i}(K) = {}^{i+1}\boldsymbol{H}_{\pi_i}(K) = 0$ *for* $i = n + 1, n + 2, \cdots$ *where* $\pi_i = \pi_i(Y)$. *The elements of* ${}^n\boldsymbol{H}_{\pi_n}(K)$ *are in a* $(1 - 1)$*-correspondence with the homotopy classes of* Y^K. *The correspondence is determined by the operation* $\mathbf{d}^n(f)(=\mathbf{d}^n(f, 0))$.

Note that according to **16** $\mathbf{d}^n(f)$ is defined as follows: by (14.2) we have $f \mid K^{n-1} \simeq 0$, therefore by (5.2) there is an $f' \in Y^K$ such that $f \simeq f'$ and that $f' \mid K^{n-1} = 0$. $\mathbf{d}^n(f)$ is then the element of ${}^n\boldsymbol{H}_{\pi_n}(K)$ corresponding to the cocycle $d^n(f') = d^n(f', 0)$.

The hypothesis of Class. th. I are obviously satisfied if $K - K' \subset K^n$. Taking $K = K^n$ in Class. th. II we obtain the theorem of Whitney quoted in the introduction.

V. Application to Homology

19. Let G, H, Z be three abelian groups. If to each $\alpha \epsilon G$ and $\beta \epsilon H$ there corresponds and $\alpha \cdot \beta$ in Z, and both distributive laws are satisfied, we say[18] G and H form a *group-pair* with respect to Z. If I is the group of rational integers then clearly G and I form a group pair with respect to G.

Given a finite n-chain $A^n = \sum_i \alpha_i \sigma_i^n$ with $\alpha_i \epsilon G$ and an arbitrary n-chain $B^n = \sum_i \beta_i \sigma_i^n$ with $\beta_i \epsilon H$ we write

$$A^n \cdot B^n = \sum_i \alpha_i \cdot \beta_i .$$

It is easy to see that

$$\partial A^{n+1} \cdot B^n = A^{n+1} \cdot \delta B^n$$

for every finite $(n + 1)$-chain A^{n+1}. It follows that if A_0^n and A_1^n are finite cycles mod K' (coef. G) and B_0^n and B_1^n are cocycles in $K - K'$ (coef. H) then

$$A_0^n \sim^* A_1^n \quad \text{and} \quad B_0^n \backsim B_1^n$$

imply

$$A_0^n \cdot B_0^n = A_1^n \cdot B_1^n .$$

We see then that ${}^n\boldsymbol{H}^{*G}(K)$ mod K' and ${}^n\boldsymbol{H}_H(K - K')$ form a group pair with respect to Z. Similar relations hold for ${}^n\boldsymbol{H}^G(K)$ mod K' and ${}^n\boldsymbol{H}_H^*(K - K')$.

20. We assume that

1°) *Y is locally connected in dimensions $\leqq n$*[19]

2°) $\pi_i(Y) = 0$ *for* $i < n$

3°) $\pi_n(Y)$ *is isomorphic with* I.

It follows from (7.3) and (7.1) that Y is n-simple.

It follows from our hypothesis that $\pi_n(Y)$ can be considered as identical with the n^{th} homology group ${}^n\mathcal{H}^1(Y)$ with integer coefficients.[20] Further, there is in Y an n-dimensional cycle Γ_0^n (coef. I) such that for every n-dimensional cycle Γ^n (coef. G) in Y there is a unique $\alpha \epsilon G$ such that $\Gamma^n \sim \alpha\Gamma_0^n$.[21] Owing to this fact we may write ${}^n\mathcal{H}^G(Y) = G$ and in particular $\pi_n(Y) = {}^n\mathcal{H}^I(Y) = I$.

21. Let $f \epsilon Y^{K'}$. For every finite cycle A^n in K' with coefficients in G we have

$$f(A^n) \sim \alpha\Gamma_0^n \quad \text{where} \quad \alpha \epsilon G.$$

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[18] See E. Čech, Ann. of Math. 37(1936), p. 684.

[19] See e.g. C. Kuratowski, Fund. Math. 24(1935), p. 269.

[20] W. Hurewicz, Proc. Akad. Amsterdam 38(1935), pp. 521–2. Explanations concerning homology in Y will be found there.

[21] See N. E. Steenrod, Amer. Jour. of Math. 58(1936), pp. 661–701.

The element α will be called the *degree of f on A^n* and denoted by $g(f, A^n)$. It is obvious that $A_0^n \sim^* A_1^n$ in K' implies $g(f, A_0^n) = g(f, A_1^n)$ and therefore $g(f, \boldsymbol{a}^n)$ is defined for every $\boldsymbol{a}^n \in {}^n\boldsymbol{H}^{*G}(K')$ and is a homomorphic mapping of ${}^n\boldsymbol{H}^{*G}(K')$ into G. Obviously $f_0, f_1 \in Y^{K'}$ and $f_0 \simeq f_1$ imply $g(f_0, \boldsymbol{a}^n) = g(f_1, \boldsymbol{a}^n)$.

Let $f' \in Y^{K'+K^n}$ be an extension of f. For any $(n+1)$-cell σ_i^{n+1} we then have according to **9** and **20** $c(f', \sigma_i^{n+1}) = g(f', \partial\sigma_i^{n+1})$ which can be written

$$\sigma_i^{n+1}\cdot c^{n+1}(f') = g(f', \partial\sigma_i^{n+1}).$$

It follows that

$$A^{n+1}\cdot c^{n+1}(f') = g(f', \partial A^{n+1})$$

for any finite $(n+1)$-chain A^{n+1} with coefficients in G. If, in particular, A^{n+1} is a finite cycle mod K' then $\partial A^{n+1} \subset K'$ and

$$A^{n+1}\cdot c^{n+1}(f') = g(f, \partial A^{n+1}). \tag{21.1}$$

Therefore, according to **15** and **19**

$$\boldsymbol{a}^{n+1}\cdot \boldsymbol{c}^{n+1}(f) = g(f, \partial\boldsymbol{a}^{n+1}) \textit{ for every } \boldsymbol{a}^{n+1} \in {}^{n+1}\boldsymbol{H}^{*G}(K) \text{ mod } K'. \tag{21.2}$$

22. Let $f_0, f_1 \in Y^K$ and $f_0 \mid K' = f_1 \mid K'$. For every finite cycle A^n mod K' (coef. G) $f_0(A^n) - f_1(A^n)$ is an n-cycle in Y, and therefore $f_0(A^n) - f_1(A^n) \sim \alpha\Gamma_0^n$ for some $\alpha \in G$. We write $g(f_0, f_1, A^n) = \alpha$. As before we verify easily that $g(f_0, f_1, \boldsymbol{a}^n) \in G$ is a homomorphic map defined for every $\boldsymbol{a}^n \in {}^n\boldsymbol{H}^{*G}(K)$ mod K'. Clearly $f_0', f_1' \in Y^K, f_0 \simeq f_0'$ rel. K' and $f_1 \simeq f_1'$ rel. K' imply $g(f_0, f_1, \boldsymbol{a}^n) = (f_0', f_1', \boldsymbol{a}^n)$.

Now let us suppose that $f_0' \mid K' + K^{n-1} = f_1' \mid K' + K^{n-1}$. We then have according to **8**, **9**, and **20** $d(f_0', f_1', \sigma_i^n) = g(f_0', f_1', \sigma_i^n)$ and as before

$$A^n\cdot d^n(f_0', f_1') = g(f_0', f_1', A^n)$$

and therefore

$$A^n\cdot d^n(f_0', f_1') = g(f_0, f_1, A^n) \tag{22.1}$$

for every finite cycle A^n mod K'. According to **16** and **19** we then have

$$\boldsymbol{a}^n\cdot \boldsymbol{d}^n(f_0, f_1) = g(f_0, f_1, \boldsymbol{a}^n) \textit{ for every } \boldsymbol{a}^n \in {}^n\boldsymbol{H}^{*G}(K) \text{ mod } K'. \tag{22.2}$$

23. We assume now (besides the hypothesis on Y made in **20**) that

1°) *$K - K'$ is finite,*

2°) *$G = R$ is the group of real numbers reduced* mod 1.

In this case the groups ${}^n\boldsymbol{H}^R(K)$ mod K'[22] and ${}^n\boldsymbol{H}_I(K - K')$ are orthogonal[23] and therefore every element of ${}^n\boldsymbol{H}_I(K - K')$ can be considered as a character[24] of

[22] This group has to be considered as a topological compact group. See L. Pontrjagin, Ann. of Math. 35(1934), p. 908.

[23] L. Pontrjagin, Ann. of Math. 35(1934), pp. 361–388; H. Whitney, Duke Math. Jour. 3(1937), p. 40.

[24] L. Pontrjagin, Loc. cit.

${}^{n}\boldsymbol{H}^{R}(K)$ mod K'. By (21.2) and (22.2) we see that the cohomology classes $\boldsymbol{c}^{n+1}(f) \in {}^{n+1}\boldsymbol{H}_{I}(K - K')$ and $\boldsymbol{d}^{n}(f_0, f_1) \in {}^{n}\boldsymbol{H}_{I}(K - K')$, when considered as characters, are just equal to $g(f, \partial \boldsymbol{a}^{n+1})$ and $g(f_0, f_1, \boldsymbol{a}^{n})$.[25] Using Ext. th. III and Hom. th. III we obtain therefore

EXTENSION THEOREM III*. *Given $f \in Y^{K'}$ we have $g(f, \partial \boldsymbol{a}^{n+1}) = 0$ for every $\boldsymbol{a}^{n+1} \in {}^{n+1}\boldsymbol{H}^{R}(K)$ mod K' if and only if there is an extension $f' \in Y^{K'+K^{n+1}}$ of f.*

HOMOTOPY THEOREM III*. *Given $f_0, f_1 \in Y^{K}$ such that $f_0 \mid K' = f_1 \mid K'$ we have $g(f_0, f_1, \boldsymbol{a}^{n}) = 0$ for every $\boldsymbol{a}^{n} \in {}^{n}\boldsymbol{H}^{R}(K)$ mod K' if and only if $f_0 \mid K' + K^{n} \simeq f_1 \mid K' + K^{n}$ rel. K'.*

24. In order to obtain classification theorems in terms of homology we have to admit that

1°) *Y is locally connected in dimensions* $\leqq$ dim $(K - K')$;

2°) *$\pi_i(Y) = 0$ for $i < n$;*

3°) *$\pi_n(Y)$ is isomorphic with I.*

If $n > 1$ it follows from (7.3) that Y is i-simple for all i. If $n = 1$ we require Y to be i-simple for $i = 1, 2, \cdots$ dim $(K - K')$. In particular we may take $Y = S^{n}$.

The group $\pi_i = \pi_i(Y)$ being countable at most for $i \leqq$ dim $(K - K')$, there is[26] a topological compact group ρ_i orthogonal to π_i. In particular we may take $\rho_n = R$.

If we admit further that $K - K'$ is finite, then the groups ${}^{j}\boldsymbol{H}^{\rho_i}(K)$ mod K'[22] and ${}^{j}\boldsymbol{H}_{\pi_i}(K - K')$ are orthogonal[23] and therefore the formulas

$${}^{j}\boldsymbol{H}^{\rho_i}(K) \bmod K' = 0, \qquad {}^{j}\boldsymbol{H}_{\pi_i}(K - K') = 0$$

are equivalent.

Using the argument of **23** we may restate all the theorems of **17** and **18** replacing cohomology by homology. In particular we obtain

CLASSIFICATION THEOREM I*. *Let ${}^{i}\boldsymbol{H}^{\rho_i}(K)$ mod $K' = {}^{i+1}\boldsymbol{H}^{\rho_i}(K)$ mod $K' = 0$ for $i = n + 1, n + 2, \cdots$. The characters of the group ${}^{n}\boldsymbol{H}^{R}(K)$ mod K' are in a $(1 - 1)$-correspondence with the homotopy classes rel. K' of $Y^{K}(K', f^*)$. To each $f \in Y^{K}(K', f^*)$ there corresponds the character $g(f, f^*, \boldsymbol{a}^{n})$.*

CLASSIFICATION THEOREM II*.[27] *Let ${}^{i}\boldsymbol{H}^{\rho_i}(K) = {}^{i+1}\boldsymbol{H}^{\rho_i}(K) = 0$ for $i = n + 1, n + 2, \cdots$. The characters of the group ${}^{n}\boldsymbol{H}^{R}(K)$ are in a $(1 - 1)$-correspondence with the homotopy classes of Y^{K}. To each $f \in Y^{K}$ there corresponds the character $g(f, \boldsymbol{a}^{n})$.*

VI. MANIFOLDS

25. Let Q^{r} be a finite or infinite geometrical cell-complex which is an oriented r-dimensional combinatorial manifold.[28] The first barycentric subdivision

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[25] This conclusion (and consequently also the theorems which follow) can be obtained even when $K - K'$ is infinite provided the groups ${}^{i}\boldsymbol{H}^{R}(K - K')$ and ${}^{i}\boldsymbol{H}_{I}^{*}(K)$ mod K' are orthogonal for $i = 1, 2, \cdots$.

[26] L. Pontrjagin, Ann. of Math. 35(1934), pp. 361–388.

[27] See S. Eilenberg, Compositio Math. 6(1939), p. 429.

[28] See e.g. K. Reidemeister, *Topologie der Polyeder*, Leipzig 1938, p. 151.

Q_1^r of Q^r is a simplicial complex, and using the simplices of Q_1^r we may define as usual the (oriented) dual $(r - n)$-cell for each n-cell of Q^r. The dual cells form a cell-complex K^r which is also an r-manifold and has Q_1^r as a barycentric subdivision.

If σ_i^{r-n} is the $(r - n)$-cell of K^r dual to the n-cell τ_i^n of Q^r we write $\mathfrak{D}(\tau_i^n) = \sigma_i^{r-n}$ and $\mathfrak{D}^*(\sigma_i^{r-n}) = \tau_i^n$. More generally, given an n-chain.

$$A^n = \sum_i \alpha_i \tau_i^n$$

we write

$$\mathfrak{D}(A^n) = \sum_i \alpha_i \sigma_i^{r-n}$$

and $\mathfrak{D}^*[\mathfrak{D}(A^n)] = A^n$. It is well known that

$$\partial \mathfrak{D}(A^n) = (-1)^{n+1} \mathfrak{D}(\delta A^n).^{29}$$

Similar relations hold for $\mathfrak{D}^*$. We obtain thus $(1 - 1)$-isomorphisms $\mathfrak{D}[{}^n\boldsymbol{H}^G(Q^r)] = {}^{r-n}\boldsymbol{H}_G(K^r)$ and $\mathfrak{D}[{}^n\boldsymbol{H}_G(Q^r)] = {}^{r-n}\boldsymbol{H}^G(K^r)$. The inverse isomorphisms are given by $\mathfrak{D}^*$.

26. P will stand for an arbitrary closed subcomplex of Q^r. We shall denote by $\mathfrak{D}(P)$ the subcomplex of K^r consisting of all the cells $\mathfrak{D}(\tau_i^n)$ where $\tau_i^n \in P$.

(26.1) $P \subset \mathfrak{D}(P)$.

(26.2) $P_1 \subset P_2$ *implies* $\mathfrak{D}(P_1) \subset \mathfrak{D}(P_2)$.

(26.3) $K^r - \mathfrak{D}(P)$ *is a closed subcomplex of* K^r.

(26.4) $P = P^i$ *implies* $K^{r-i-1} \subset K^r - \mathfrak{D}(P)$.

(26.5) $\mathfrak{D}(A^n) \subset \mathfrak{D}(P)$ *for every n-chain* A^n *in* P. $\mathfrak{D}^*(A^n) \subset P$ *for every n-chain* A^n *in* $\mathfrak{D}(P)$.

(26.6) $K^r - \mathfrak{D}(P)$ *is a deformation retract*[30] *of* $Q^r - P$.[31]

(26.7) *Every* $f \in Y^{K^r - \mathfrak{D}(P)}$ *has an extension* $f' \in Y^{Q^r - P}$.

(26.8) $f_0, f_1 \in Y^{Q^r-P}$ *and* $f_0 \mid K^r - \mathfrak{D}(P) \simeq f_1 \mid K^r - \mathfrak{D}(P)$ *imply* $f_0 \simeq f_1$.

(26.1)–(26.5) follow directly from the definition. In order to prove (26.6), notice that $K^r - \mathfrak{D}(P)$ consists of all simplices of the barycentric subdivision Q_1^r of Q^r which have no vertex on P. (26.6) is therefore a consequence of the following quite general and elementary lemma:

Let Q_1 *be a geometric simplicial complex,* P_1 *a closed subcomplex of* Q_1 *and* $C(P_1)$ *the closed subcomplex of* Q_1 *consisting of all simplices of* Q_1 *which have no vertex on* P_1. *The complex* $C(P_1)$ *is then a deformation retract of* $Q_1 - P_1$.

(26.7) and (26.8) follow from (26.6).

27. We assume that Y is n-simple All chains will have coefficients from $\pi_n = \pi_n(Y)$.

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[29] See H. Seifert and W. Threfall, *Lehrbuch der Topologie*, Leipzig-Berlin 1934, p. 245.

[30] $A \subset X$ is a deformation retract of X if there is a map $g \in X^{X \times E}$ such that $g(x, 0) = x$, $g(x, 1) \in A$ for $x \in X$ and $g(x, 1) = x$ for $x \in A$.

[31] See S. Lefschetz, *Topology*, New York 1930, p. 141.

Let P be a closed subcomplex of Q^r and let $f \in Y^{Q^r - P^{r-n-1}}$. Taking $K' = K^r - \mathfrak{D}(P)$ we have according to (26.1) and (26.4)

$$K' + K^n \subset K^r - \mathfrak{D}(P^{r-n-1}) \subset Q^r - P^{r-n-1}.$$

Write

$$f_1 = f \mid K' + K^n, \qquad \gamma^{r-n-1}(f) = \mathfrak{D}^*[c^{n+1}(f_1)].^{32}$$

(27.1) *$\gamma^{r-n-1}(f)$ is a cycle in P^{r-n-1} (with coef. in $\pi_n(Y)$).*

(27.2) *$f' \in Y^{Q^r - P^{r-n-1}}$ and $f \simeq f'$ imply $\gamma^{r-n-1}(f) = \gamma^{r-n-1}(f')$.*

(27.3) *If $\gamma^{r-n-1}(f) \subset P_1 \subset P$ then there is an $f' \in Y^{Q^r - (P_1 + P^{r-n-2})}$ such that $f \simeq f' \mid Q^r - P^{r-n-1}$.*

(27.4) *$\gamma^{r-n-1}(f) = 0$ if and only if there is an $f' \in Y^{Q^r - P^{r-n-2}}$ such that $f \simeq f' \mid Q^r - P^{r-n-1}$.*

(27.5) *Given an $(r - n - 1)$-cycle $\gamma^{r-n-1} \subset P$ such that $\gamma^{r-n-1} \sim \gamma^{r-n-1}(f)$ in P, there is an $f' \in Y^{Q^r - P^{r-n-1}}$ such that $\gamma^{r-n-1}(f') = \gamma^{r-n-1}$ and that $f \mid Q^r - P^{r-n} \simeq f' \mid Q^r - P^{r-n}$.*

(27.6) *$\gamma^{r-n-1}(f) \sim 0$ in P if and only if there is an $f' \in Y^{Q^r - P^{r-n-2}}$ such that $f \mid Q^r - P^{r-n} \simeq f' \mid Q^r - P^{r-n}$.*

Proofs. Ad (27.1). By (10.1) and (11.1) $c^{n+1}(f_1)$ is a cocycle in $\mathfrak{D}(P)$. Therefore, by (26.5), $\gamma^{r-n-1}(f)$ is a cycle in P.

Ad (27.2). $f \simeq f'$ implies $f \mid K' + K^n \simeq f' \mid K + K^n$. Therefore $f_1 \simeq f_1'$, and by (9.3) $c^{n+1}(f_1) = c^{n+1}(f_1)$. It follows that $\gamma^{r-n-1}(f) = \gamma^{r-n-1}(f')$.

Ad (27.3). $\gamma^{r-n-1}(f) \subset P_1$ implies $c^{n+1}(f_1) \subset \mathfrak{D}(P_1)$, whence $c(f_1, \sigma_i^{n+1}) = 0$ for every $\sigma_i^{n+1} \subset \mathfrak{D}(P) - \mathfrak{D}(P_1)$. By (9.2) f_1 admits an extension on every such σ_i^{n+1} and therefore there is an extension $f_1' \in Y^{K^r - \mathfrak{D}(P_1 - P^{r-n-2})}$ of f_1. According to (26.7) there is an extension $f' \in Y^{Q^r - (P_1 - P^{r-n-2})}$ of f_1'. We then have $f \mid K^r - \mathfrak{D}(P^{r-n-1}) = f_1 = f' \mid K^r - \mathfrak{D}(P^{r-n-1})$ and by (26.8) $f \simeq f' \mid Q^r - P^{r-n-1}$.

Ad (27.4). If $\gamma^{r-n-1}(f) = 0$ then taking $P_1 = 0$ in (27.3) we obtain an $f' \in Y^{Q^r - P^{r-n-2}}$ such that $f \simeq f' \mid Q^r - P^{r-n-1}$. On the other hand if such an f' exists we have $\gamma^{r-n-1}(f) = \gamma^{r-n-1}(f')$ by (27.2) and therefore $\gamma^{r-n-1}(f) = 0$ by (27.1) since $\gamma^{r-n-1}(f') \subset P^{r-n-2}$.

Ad (27.5). Let $c^{n+1} = \mathfrak{D}(\gamma^{r-n-1})$. Then $\gamma^{r-n-1} \sim \gamma^{r-n-1}(f)$ in P implies $\mathfrak{D}(\gamma^{r-n-1}) \backsim \mathfrak{D}[\gamma^{r-n-1}(f)]$ in $\mathfrak{D}(P)$ and therefore $c^{n+1} \backsim c^{n+1}(f_1)$ in $K^r - K'$. By (11.6) there is an $f_1' \in Y^{K' + K^n}$ such that $f_1 \mid K' + K^{n-1} = f_1' \mid K' + K^{n-1}$ and that $c^{n+1}(f_1') = c^{n+1}$. Since $K' + K^n = K^r - \mathfrak{D}(P^{r-n-1})$ there is, by (26.7), an extension $f' \in Y^{Q^r - P^{r-n-1}}$ of f_1'. We then have $\gamma^{r-n-1}(f') = \mathfrak{D}[c^{n+1}(f_1')] = \gamma^{r-n-1}$. Since $K' + K^{n-1} = K^r - \mathfrak{D}(P^{r-n})$ and $f \mid K' + K^{n-1} = f_1 \mid K + K^{n-1} = f_1' \mid K' + K^{n-1} = f' \mid K + K^{n-1}$, we have $f \mid K^r - \mathfrak{D}(P^{r-n}) = f' \mid K^r - \mathfrak{D}(P^{r-n})$, and by (26.8) $f \mid Q^r - P^{r-n} \simeq f' \mid Q^r - P^{r-n}$.

Ad (27.6). If $\gamma^{r-n-1}(f) \sim 0$ in P, then, taking $\gamma^{r-n-1} = 0$ in (27.5), we obtain an $f'' \in Y^{Q^r - P^{r-n-1}}$ such that $\gamma^{r-n-1}(f'') = 0$ and that $f \mid Q^r - P^{r-n} \simeq$

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[32] This definition of $\gamma^{r-n-1}(f)$ is obviously equivalent with that given in the introduction. Cf. footnote [15].

$f'' \mid Q^r - P^{r-n}$. Applying (27.4) we obtain an $f' \in Y^{Q^r - P^{r-n-2}}$ such that $f \mid Q^r - P^{r-n} \simeq f' \mid Q^r - P^{r-n}$. On the other hand, if such an f' exists then $f \mid K' + K^{n-1} \simeq f' \mid K' + K^{n-1}$ since $K' + K^{n-1} \subset Q^r - P^{r-n}$. By (5.2) we may therefore suppose that $f \mid K' + K^{n-1} = f' \mid K' + K^{n-1}$. Since $K' + K^n \subset Q^r - P^{r-n-1}$ and $K' + K^{n+1} \subset Q^r - P^{r-n-2}$ it follows from Ext. th. I that $c^{n+1}(f \mid K' + K^n) \backsim 0$ in $K - K' = \mathfrak{D}(P)$ and therefore that $\gamma^{r-n-1}(f) \sim 0$ in P.

28. In this section we assume that $\pi_i(Y) = 0$ for $i < n$ if $n > 1$, and that Y is 1-simple (i.e. that $\pi_1(Y)$ is abelian) if $n = 1$.

(28.1) *Given* $f \in Y^{Q^r - P}$ *there is an* $f' \in Y^{Q^r - P^{r-n-1}}$ *such that* $f \simeq f' \mid Q^r - P$.

(28.2) $f_0, f_1 \in Y^{Q^r - P^{r-n}}$ *and* $f_0 \mid Q^r - P \simeq f_1 \mid Q^r - P$ *imply* $f_0 \simeq f_1$.

(28.3) $f_0, f_1 \in Y^{Q^r - P^{r-n-1}}$ *and* $f_0 \mid Q^r - P \simeq f_1 \mid Q^r - P$ *imply* $\gamma^{r-n-1}(f_0) \sim \gamma^{r-n-1}(f_1)$ *in* P.

Proofs. Ad (28.1). By (14.1) there is an $f_1 \in Y^{K'+K^n}$ such that $f \mid K' = f_1 \mid K'$, where as before we take $K' = K^r - \mathfrak{D}(P)$. Since $K' + K^n = K^r - \mathfrak{D}(P^{r-n-1})$ there is, by (26.7), an extension $f' \in Y^{Q^r - P^{r-n-1}}$ of f_1. We then have $f \mid K^r - \mathfrak{D}(P) = f' \mid K^r - \mathfrak{D}(P)$ and therefore $f \simeq f' \mid Q^r - P$ by (26.8).

Ad (28.2). $f_0 \mid Q^r - P \simeq f_1 \mid Q^r - P$ implies $f_0 \mid K' \simeq f_1 \mid K'$ and by (14.2) $f_0 \mid K' + K^{n-1} \simeq f_1 \mid K' + K^{n-1}$. Since $K' + K^{n-1} = K^r - \mathfrak{D}(P^{r-n})$ it follows from (26.8) that $f_0 \simeq f_1$.

Ad (28.3). $f_0 \mid Q^r - P \simeq f_1 \mid Q^r - P$ implies $f_0 \mid K' \simeq f_1 \mid K'$, and by (15.1) $c^{n+1}(f_0 \mid K' + K^n) \backsim c^{n+1}(f_1 \mid K' + K^n)$ in $K^r - K' = \mathfrak{D}(P)$. It follows that $\gamma^{r-n-1}(f_0) \sim \gamma^{r-n-1}(f_1)$ in P.

Let $f \in Y^{Q^r - P}$ and let f' be given by (28.1). Let $\boldsymbol{\gamma}^{r-n-1}(f)$ be the element of ${}^{r-n-1}\boldsymbol{H}^{\pi_n}(P)$ determined by the cycle $\gamma^{r-n-1}(f')$. It follows from (28.3) that $\boldsymbol{\gamma}^{r-n-1}(f)$ is independent of the choice of f'.

(28.4) $\boldsymbol{\gamma}^{r-n-1}(f) = \mathfrak{D}^*[\boldsymbol{c}^{n+1}(f \mid K')]$ *where* $K' = K^r - \mathfrak{D}(P)$.

(28.5) *Given* $f \in Y^{Q^r - P}$ *we have* $\boldsymbol{\gamma}^{r-n-1}(f) = 0$ *if and only if there is an* $f' \in Y^{Q^r - P^{r-n-2}}$ *such that* $f \simeq f' \mid Q^r - P$.

Proofs. (28.4) follows straight from the definition of $\boldsymbol{\gamma}^{r-n-1}$ and $\boldsymbol{c}^{n+1}$. If $\boldsymbol{\gamma}^{r-n-1}(f) = 0$ then for every $f'' \in Y^{Q^r - P^{r-n-1}}$ such that $f \simeq f'' \mid Q^r - P$ we have $\gamma^{r-n-1}(f'') \sim 0$ in P. It follows by (27.6) that there is an $f' \in Y^{Q^r - P^{r-n-2}}$ such that $f \simeq f'' \mid Q^r - P \simeq f' \mid Q^r - P$. On the other hand, if such an f' exists we have $\gamma^{r-n-1}(f') = 0$ by (27.4), and therefore $\boldsymbol{\gamma}^{r-n-1}(f) = 0$.

29. We assume now that $\pi_i(Y) = 0$ for $i < n$ if $n > 1$ and that Y is i-simple for $i = 1, 2, \cdots, r - 1$ if $n = 1$.

(29.1) *Let* ${}^{r-i-1}\boldsymbol{H}^{\pi_i}(P) = 0$ *for* $i = n + 1, n + 2, \cdots, r - 1$ *where* $\pi_i = \pi_i(Y)$. *Given* $f \in Y^{Q^r - P}$ *we have* $\boldsymbol{\gamma}^{r-n-1}(f) = 0$ *if and only if there is an* $f' \in Y^{Q^r}$ *such that* $f \simeq f' \mid Q^r - P$.

This follows from (28.5) and (27.6) applied successively for $n + 1, n + 2, \cdots, r - 1$.

30. Let G and H be two groups forming a group pair with respect to a group Z. Given a finite n-chain $A^n = \sum_i \alpha_i \sigma_i^n$ in K^r with coefficients in G and an arbitrary $(r - n)$-chain $B^{r-n} = \sum_i \beta_i \tau_i^{r-n}$ in Q^r with coefficients in H, we write

$$\chi(A^n, A^{r-n}) = \sum_i \alpha_i \cdot \beta_i .^{33}$$

We obviously have

$$(30.1) \qquad \chi(A^n, A^{r-n}) = A^n \cdot \mathfrak{D}(B^{r-n}) = \mathfrak{D}^*(A^n) \cdot B^{r-n}$$

It follows therefore from **19**, **25**, and **26** that ${}^n\boldsymbol{H}^{*G}(K^r) \bmod K^r - \mathfrak{D}(P)$ and ${}^{r-n}\boldsymbol{H}^H(P)$ form a group pair with respect to Z. Similar relations hold for ${}^n\boldsymbol{H}^G(K^r) \bmod K^r - \mathfrak{D}(P)$ and ${}^{r-n}\boldsymbol{H}^{*H}(P)$.

31. We make the same hypothesis about Y as in **20**. Let $f \in Y^{Q^r-P}$. By (21.2), (28.4) and (30.1) we have

(31.1) $\chi[\boldsymbol{a}^{n+1}, \gamma^{r-n-1}(f)] = g(f, \partial \boldsymbol{a}^{n+1})$ *for every* $\boldsymbol{a}^{n+1} \in {}^{n+1}\boldsymbol{H}^{*G}(K^r) \bmod K^r - \mathfrak{D}(P)$

Now, if P is finite and $G = R$ (see **23** and footnote[25]) the complex $\mathfrak{D}(P)$ is finite and ${}^{n+1}\boldsymbol{H}^R(K^r) \bmod K^r - \mathfrak{D}(P)$ is orthogonal to ${}^{r-n-1}\boldsymbol{H}^I(P)$, since it is orthogonal to ${}^{n-r-1}\boldsymbol{H}_I[\mathfrak{D}(P)]$ (see **23**). It follows therefore from (31.1) that $\gamma^{n-r-1}(f)$ considered as a character of ${}^{n+1}\boldsymbol{H}^R(K^r) \bmod K^r - \mathfrak{D}(P)$ is equal to $g(f, \partial \boldsymbol{a}^{n+1})$. Therefore it follows from (28.5) that

(31.2) *Given* $f \in Y^{Q^r-P}$ *we have* $g(f, \partial \boldsymbol{a}^{n+1}) = 0$ *for every* $\boldsymbol{a}^{n+1} \in {}^{n+1}\boldsymbol{H}^R(K^r)$ mod $K^r - \mathfrak{D}(P)$ *if and only if there is an* $f' \in Y^{Q^r-P^{r-n-2}}$ *such that* $f \simeq f' \mid Q^r - P$.

Appendix I. On Normal Mappings[33]

Let Y be an arbitrary topological space and let $y_0 \in Y$. A map $f \in Y^{K^n}$ will be called n-normal if $f \mid K^{n-1} = 0$ (i.e. if $f(x) = y_0$ for $x \in K^{n-1}$).

Given an oriented n-cell σ_i^n in K^n, the map $f \mid \bar{\sigma}_i^n$ defines (if f is n-normal) uniquely an element $d(f, \sigma_i^n)$ of $\pi_n(Y)$. If $n > 1$ then $\pi_n(Y)$ is abelian and we may define the n-chain $d^n(f)$ and the $(n + 1)$-cocycle $c^{n+1}(f)$ by

$$d^n(f) = \sum_i d(f, \sigma_i^n)\sigma_i^n , \qquad c^{n+1}(f) = \delta d^n(f).$$

$c(f, \sigma_i^{n+1})$ will be defined as the coefficient of the $(n + 1)$-cell σ_i^{n+1} in $c^{n+1}(f)$. Given two n-normal maps $f_0 , f_1 \in Y^K$ we take

$$d^n(f_0 , f_1) = d^n(f_0) - d^n(f_1).$$

Starting from the definition of $\pi_n(Y)$ we can prove that $c(f, \sigma_i^{n+1}) = 0$ if and only if f can be extended on σ_i^{n+1}. It follows that

$d^n(f)$ is a cocycle (i.e. $c^{n+1}(f) = 0$) *if and only if there is an extension* $f' \in Y^{K^{n+1}}$ *of* f.

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[33] The purpose of this appendix is to make clear the position of the results of H. Whitney (Duke Math. Jour. 3(1937), pp. 51–55) in the theory developed in this paper. The proofs are on the same lines as those of Whitney and may be left to the reader.

From this, statements analogous to those of **9–12** can easily be deduced. In particular we have

EXTENSION THEOREM. *Given an n-normal map* $f \in Y^{K'}$, *the n-chain* $d^n(f) \subset K'$ *is part of a cocycle in* K[34] *if and only if there is an n-normal extension* $f' \in Y^{K'+K^{n+1}}$ *of* f.

HOMOTOPY THEOREM. *Given two n-normal maps* $f_0, f_1 \in Y^K$ *such that* $f_0 \mid K' = f_1 \mid K'$, *the n-chain* $d^n(f_0, f_1)$ *is a cocycle in* $K - K'$. *Moreover* $d^n(f_0, f_1) \backsim 0$ *in* $K - K'$ *if and only if* $f_0 \mid K' + K^n \simeq f_1 \mid K' + K^n$ *rel.* $K' + K^{n-2}$.

APPENDIX II. MAPPINGS OF INFINITE $(n + 1)$-MANIFOLDS

We assume that $\pi_i(Y) = 0$ for $i < n$ if $n > 1$ and that Y is i-simple for $i = 1, 2$ if $n = 1$. In particular we may take $Y = S^n$.

Let Q^{n+1} be an *infinite* geometrical cell-complex which is an oriented $(n + 1)$-dimensional combinatorial manifold.

CLASSIFICATION THEOREM. *The homotopy classes of* $Y^{Q^{n+1}}$ *are in* (1-1)-*correspondence with the elements of the group* ${}^1\boldsymbol{H}^{\pi_n}(Q^{n+1})$ *where* $\pi_n = \pi_n(Y)$.

PROOF. Let K^{n+1} be the dual of Q^{n+1}. Since Q^{n+1} is a connected infinite complex, it is easy to see that ${}^0\boldsymbol{H}^G(Q^{n+1}) = 0$ and therefore, by **25**, that ${}^{n+1}\boldsymbol{H}_G(K^{n+1}) = 0$ for every abelian group G. The hypotheses of Class. th. II are thus satisfied and the homotopy classes of $Y^{K^{n+1}} = Y^{Q^{n+1}}$ are in a $(1 - 1)$-correspondence with the elements of ${}^n\boldsymbol{H}_{\pi_n}(K^{n+1})$. By **25** this group is isomorphic with ${}^1\boldsymbol{H}^{\pi_n}(Q^{n+1})$. This proves the theorem.

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[34] Whitney, loc. cit., p. 53. It is easy to verify that $d^n(f)$ is part of a cocycle in K if and only if $c^{n+1}(f) \backsim 0$ in $K - K'$; ibid., p. 54.