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A NOTE ON λ-OPERATIONS IN ORTHOGONAL K-THEORY

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Abstract. In Comment. Math. Helv. **55** (1980), 233–254, Kratzer defined Lambda operations on classical algebraic K-theory by using exterior powers of representations and a splitting principle (R. G. Swan, Proc. Sympos. in Pure Math. **21** (1971), 155–159). Because hyperbolic forms are not stable under exterior powers, we instead use a larger class of symmetric bilinear forms to define the operation of exterior powers on the classifying space of the orthogonal K-theory.

1. Introduction

Let A be a commutative ring in which 2 is invertible, and let $\mathcal{C}_A$ be the category of symmetric bilinear A-modules. Denote by $\phi_{p,q}$ the diagonal bilinear form $\langle\underbrace{1,...,1}_{p},\underbrace{-1,...,-1}_{q}\rangle$ over A^{p+q} and let $\mathcal{Q}_A$ be the full subcategory of $\mathcal{C}_A$ with objects (M,ϕ) isomorphic to $(A^{p+q},\phi_{p,q})$ for a certain pair (p,q) (we were led to consider this category because the exterior powers of hyperbolic forms are not generally hyperbolic). The isometry group of $(A^{p+q},\phi_{p,q})$ will be denoted by $O_{p,q}(A)$. Let G be a group and let $R\mathcal{Q}_A(G)$ be the ring of isometry classes of representation of G in $\mathcal{Q}_A$. Recall [3, definition 3.1] that a λ-ring is a commutative ring with unit equipped with a sequence of operations $\{\lambda^n\}_{n\geq 0}$ satisfying the following properties: (a) $\lambda^0(x)=1$ and $\lambda^1(x)=x$, (b) $\lambda^m(x+y)=\sum_{k=0}^m \lambda^k(x)\lambda^{m-k}(y)$. The following observation is straightforward to check and can be found as proposition 1.4.3 on page 13 of [2].

Proposition 1.1. *The exterior powers induce a λ-ring structure on $R\mathcal{Q}_A(G)$.*

In fact this proposition comes from the known formula:

$$\Lambda^k[(M,\phi)\perp(N,\psi)]\cong\sum_{0\leq i\leq k}\Lambda^i(M,\phi)\otimes\Lambda^{k-i}(N,\psi)$$

where (M,ϕ) and (N,ψ) are symmetric bilinear modules (proposition 1.3.2 on page 11 in [2]). We denote by $O(A)$ the infinite orthogonal group, by $BO(A)^+$ the plus construction of Quillen applied to the classifying space $BO(A)$ [4], and by $[X,Y]$ the based homotopy class of maps from a based space X to another Y. The key to the result is the following construction of a morphism from $R\mathcal{Q}_A(G)$ to $[BG,BO(A)^+]$.

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2. A morphism from $RQ_A(G)$ to $[BG, BO(A)^+]$

Let ρ be a representation of the group G in an object (E, ϕ) of $\mathcal{Q}_A$. Then we have a commutative diagram:

$$
\begin{array}{ccc}
 & & (A^{p+q}, \phi_{p,q}) \\
 & \nearrow & \\
(E, \phi) & & \downarrow \Psi \\
 & \searrow & \\
 & & (A^{p'+q'}, \phi_{p',q'})
\end{array}
$$

The action of G on (E, ϕ) gives a mapping $\mathcal{R} : G \to Isom(E)$ and we obtain the commutative diagram:

$$
\begin{array}{ccc}
 & & O_{p,q}(A) \\
 & \nearrow & \\
G & & \downarrow \Upsilon \\
 & \searrow & \\
 & & O_{p',q'}(A)
\end{array}
$$

where the homomorphism Υ is the conjugate by Ψ. By using a theorem of Witt of extension of isometries in the case of a commutative ring in which 2 is invertible, which comes from [8, theorem 7.2], we can extend Ψ to an isometry $\overline{\Psi}$ on $(A^{2n}, \phi_{n,n})$ where $n = p + q = p' + q'$ and then obtain the commutative diagram:

$$
\begin{array}{ccccccc}
 & & O_{p,q}(A) & \longrightarrow & O_{n,n}(A) & \longrightarrow & O(A) \\
 & \nearrow & & & & & \\
G & & \downarrow & & \downarrow & & \downarrow \\
 & \searrow & & & & & \\
 & & O_{p',q'}(A) & \longrightarrow & O_{n,n}(A) & \longrightarrow & O(A)
\end{array}
$$

The composite mappings $u : G \to O_{p,q}(A) \to O_{n,n}(A)$ and $v : G \to O_{p',q'}(A) \to O_{n,n}(A)$ are conjugate by the element $\overline{\Psi}$ of the orthogonal group $O_{n,n}(A)$. By including $O_{n,n}(A)$ in $O_{2n,2n}(A)$ and replacing $\overline{\Psi}$ by $\overline{\Psi} \oplus \overline{\Psi}^{-1}$, we can suppose that $\overline{\Psi}$ is an element of the elementary group $E_{2n,2n}(A) \subset O_{2n,2n}(A)$. By Proposition 1.1.9 of [7], we conclude that u and v induce homotopic maps from BG to $BO(A)^+$. The homotopy class of these maps will be denoted by $r(E)$. We define the mapping from $RQ_A(G)$ to $[BG, BO(A)^+]$ on generators by:

$$
\begin{array}{rll}
r : & RQ_A(G) & \to [BG, BO(A)^+] \\
 & [\rho] & \mapsto r(E).
\end{array}
$$

Theorem 2.1. *There exist mappings λ^k, up to weak homotopy, from $BO(A)^+$ to $BO(A)^+$, inducing λ-operations on orthogonal K-theory.*

Proof. This was inspired by the first paragraph in section 5 of [6]. Let G be the group $O_{n,n}(A)$ and $[A_{id}^{n,n}]$ the class of the identity representation of G in the object $(A^{2n}, \phi_{n,n})$ of $\mathcal{Q}_A$. Denote by $2n.1\ : O_{n,n}(A) \to O_{n,n}(A)$ the trivial representation which sends each element to the identity matrix. The difference $[A_{id}^{n,n}] - [2n.1]$ will be denoted by $[A_{id}^{n,n}]^{\sim}$. Since the structure of a λ-ring in $R\mathcal{Q}_A$ is given by exterior powers, by considering the exterior powers of $[A_{id}^{n,n}]$ and $[2n.1]$ we see that the following diagram commutes:

$$\begin{array}{ccc} R\mathcal{Q}_A(O_{n+1,n+1}(A)) & \xrightarrow{i_n^*} & R\mathcal{Q}_A(O_{n,n}(A)) \\ {\scriptstyle \lambda^k}\downarrow & & {\scriptstyle \lambda^k}\downarrow \\ R\mathcal{Q}_A(O_{n+1,n+1}(A)) & \xrightarrow{i_n^*} & R\mathcal{Q}_A(O_{n,n}(A)) \end{array}$$

Let us consider the following diagram:

$$\begin{array}{ccc} R\mathcal{Q}_A(O_{n+1,n+1}(A)) & \xrightarrow{r} & [BO_{n+1,n+1}(A), BO(A)^+] \\ {\scriptstyle i_n^*}\downarrow & & \downarrow \\ R\mathcal{Q}_A(O_{n,n}(A)) & \xrightarrow{r} & [BO_{n,n}(A), BO(A)^+] \end{array}$$

From the construction of r and the fact that the restriction morphism sends $[A_{id}^{n+1,n+1}]^{\sim}$ to $[A_{id}^{n,n}]^{\sim}$, the restriction of $r(\lambda^k[A_{id}^{n+1,n+1}]^{\sim})$ to $BO_{n,n}(A)$ is $r(\lambda^k[A_{id}^{n,n}]^{\sim})$. Let $\lambda_{n,n}^k$ be a continuous map in the homotopy class $r(\lambda^k[A_{id}^{n,n}]^{\sim})$; then the following triangle is homotopy commutative:

$$\begin{array}{ccccc} BO_{n,n}(A) & & \xrightarrow{\quad \lambda_{n,n}^k \quad} & & BO(A)^+ \\ & \searrow & & \nearrow {\scriptstyle \lambda_{n+1,n+1}^k} & \\ & & BO_{n+1,n+1}(A) & & \end{array}$$

We also obtain a map $\lambda^k : BO(A) \longrightarrow BO(A)^+$ which passes to the Plus-construction and gives a map $\lambda^k : BO(A)^+ \longrightarrow BO(A)^+$. For any compact subset K of $BO(A)^+$, there exists a positive integer m such that $K \subset BO_{m,m}(A)^+$, and we deduce that the precedent map is defined up to weak homotopy. $\square$

We define the operations on orthogonal K-theory by:

$$\begin{array}{lll} \lambda^k : & \pi_n(BO(A)^+) = [S^n, BO(A)^+] & \to [S^n, BO(A)^+] \\ & \qquad [f] & \mapsto [\lambda \circ f]. \end{array}$$

These operations produce a λ-ring structure on orthogonal K-theory ([2], page 21) analogous to the λ-ring structure in classical algebraic K-theory [6, theorem 5.1].

Remark. To get a special λ-ring, one needs a splitting principle [10]. We do not know if there exists an analogue for the study of rank two modules in this case.

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