A NOTE ON λ -OPERATIONS IN ORTHOGONAL K-THEORY

MOHAMED ELHAMDADI

(Communicated by Ralph Cohen)

ABSTRACT. In Comment. Math. Helv. **55** (1980), 233–254, Kratzer defined Lambda operations on classical algebraic K-theory by using exterior powers of representations and a splitting principle (R. G. Swan, Proc. Sympos. in Pure Math. **21** (1971), 155–159). Because hyperbolic forms are not stable under exterior powers, we instead use a larger class of symmetric bilinear forms to define the operation of exterior powers on the classifying space of the orthogonal K-theory.

1. Introduction

Let A be a commutative ring in which 2 is invertible, and let \mathcal{C}_A be the category of symmetric bilinear A-modules. Denote by $\phi_{p,q}$ the diagonal bilinear form $(\underbrace{1,...,1}_p,\underbrace{-1,...,-1}_q)$ over A^{p+q} and let \mathcal{Q}_A be the full subcategory of \mathcal{C}_A with objects

 (M,ϕ) isomorphic to $(A^{p+q},\phi_{p,q})$ for a certain pair (p,q) (we were led to consider this category because the exterior powers of hyperbolic forms are not generally hyperbolic). The isometry group of $(A^{p+q},\phi_{p,q})$ will be denoted by $O_{p,q}(A)$. Let G be a group and let $RQ_A(G)$ be the ring of isometry classes of representation of G in Q_A . Recall [3, definition 3.1] that a λ -ring is a commutative ring with unit equipped with a sequence of operations $\{\lambda^n\}_{n\geq 0}$ satisfying the following properties: (a) $\lambda^0(x)=1$ and $\lambda^1(x)=x$, (b) $\lambda^m(x+y)=\sum_{k=0}^m \lambda^k(x)\lambda^{m-k}(y)$. The following observation is straightforward to check and can be found as proposition 1.4.3 on page 13 of [2].

Proposition 1.1. The exterior powers induce a λ -ring structure on $RQ_A(G)$.

In fact this proposition comes from the known formula:

$$\Lambda^{k}[(M,\phi) \perp (N,\psi)] \cong \sum_{0 \le i \le k} \Lambda^{i}(M,\phi) \otimes \Lambda^{k-i}(N,\psi)$$

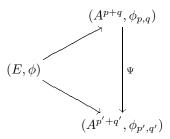
where (M, ϕ) and (N, ψ) are symmetric bilinear modules (proposition 1.3.2 on page 11 in [2]). We denote by O(A) the infinite orthogonal group, by $BO(A)^+$ the plus construction of Quillen applied to the classifying space BO(A) [4], and by [X, Y] the based homotopy class of maps from a based space X to another Y. The key to the result is the following construction of a morphism from $RQ_A(G)$ to $[BG, BO(A)^+]$.

Received by the editors January 23, 1998.

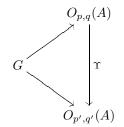
¹⁹⁹¹ Mathematics Subject Classification. Primary 19G38, 11E57.

2. A MORPHISM FROM $RQ_A(G)$ TO $[BG, BO(A)^+]$

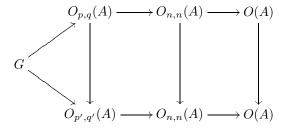
Let ρ be a representation of the group G in an object (E, ϕ) of \mathcal{Q}_A . Then we have a commutative diagram:



The action of G on (E, ϕ) gives a mapping $\mathcal{R}: G \to Isom(E)$ and we obtain the commutative diagram:



where the homomorphism Υ is the conjugate by Ψ . By using a theorem of Witt of extension of isometries in the case of a commutative ring in which 2 is invertible, which comes from [8, theorem 7.2], we can extend Ψ to an isometry $\overline{\Psi}$ on $(A^{2n}, \phi_{n,n})$ where n = p + q = p' + q' and then obtain the commutative diagram:



The composite mappings $u: G \to O_{p,q}(A) \to O_{n,n}(A)$ and $v: G \to O_{p',q'}(A) \to O_{n,n}(A)$ are conjugate by the element $\overline{\Psi}$ of the orthogonal group $O_{n,n}(A)$. By including $O_{n,n}(A)$ in $O_{2n,2n}(A)$ and replacing $\overline{\Psi}$ by $\overline{\Psi} \oplus \overline{\Psi}^{-1}$, we can suppose that $\overline{\Psi}$ is an element of the elementary group $E_{2n,2n}(A) \subset O_{2n,2n}(A)$. By Proposition 1.1.9 of [7], we conclude that u and v induce homotopic maps from BG to $BO(A)^+$. The homotopy class of these maps will be denoted by r(E). We define the mapping from $R\mathcal{Q}_A(G)$ to $[BG, BO(A)^+]$ on generators by:

$$r: R\mathcal{Q}_A(G) \to [BG, BO(A)^+]$$

 $[\rho] \mapsto r(E).$

Theorem 2.1. There exist mappings λ^k , up to weak homotopy, from $BO(A)^+$ to $BO(A)^+$, inducing λ -operations on orthogonal K-theory.

Proof. This was inspired by the first paragraph in section 5 of [6]. Let G be the group $O_{n,n}(A)$ and $[A_{id}^{n,n}]$ the class of the identity representation of G in the object $(A^{2n}, \phi_{n,n})$ of \mathcal{Q}_A . Denote by $2n.1: O_{n,n}(A) \to O_{n,n}(A)$ the trivial representation which sends each element to the identity matrix. The difference $[A_{id}^{n,n}] - [2n.1]$ will be denoted by $[A_{id}^{n,n}]^{\sim}$. Since the structure of a λ -ring in $R\mathcal{Q}_A$ is given by exterior powers, by considering the exterior powers of $[A_{id}^{n,n}]$ and [2n.1] we see that the following diagram commutes:

$$R\mathcal{Q}_{A}(O_{n+1,n+1}(A)) \xrightarrow{i_{n}^{*}} R\mathcal{Q}_{A}(O_{n,n}(A))$$

$$\downarrow^{\lambda^{k}} \qquad \qquad \downarrow^{\lambda^{k}} \downarrow$$

$$R\mathcal{Q}_{A}(O_{n+1,n+1}(A)) \xrightarrow{i_{n}^{*}} R\mathcal{Q}_{A}(O_{n,n}(A))$$

Let us consider the following diagram:

$$RQ_{A}(O_{n+1,n+1}(A)) \xrightarrow{r} [BO_{n+1,n+1}(A), BO(A)^{+}]$$

$$\downarrow i_{n}^{*} \downarrow \qquad \qquad \downarrow$$

$$RQ_{A}(O_{n,n}(A)) \xrightarrow{r} [BO_{n,n}(A), BO(A)^{+}]$$

From the construction of r and the fact that the restriction morphism sends $[A_{id}^{n+1,n+1}]^{\sim}$ to $[A_{id}^{n,n}]^{\sim}$, the restriction of $r(\lambda^k[A_{id}^{n+1,n+1}]^{\sim})$ to $BO_{n,n}(A)$ is $r(\lambda^k[A_{id}^{n,n}]^{\sim})$. Let $\lambda^k_{n,n}$ be a continuous map in the homotopy class $r(\lambda^k[A_{id}^{n,n}]^{\sim})$; then the following triangle is homotopy commutative:

$$BO_{n,n}(A) \xrightarrow{\lambda_{n,n}^k} BO(A)^+$$

$$BO_{n+1,n+1}(A)$$

We also obtain a map $\lambda^k : BO(A) \longrightarrow BO(A)^+$ which passes to the Plus-construction and gives a map $\lambda^k : BO(A)^+ \longrightarrow BO(A)^+$. For any compact subset K of $BO(A)^+$, there exists a positive integer m such that $K \subset BO_{m,m}(A)^+$, and we deduce that the precedent map is defined up to weak homotopy.

We define the operations on orthogonal K-theory by:

$$\lambda^k: \pi_n(BO(A)^+) = [S^n, BO(A)^+] \rightarrow [S^n, BO(A)^+]$$

$$[f] \mapsto [\lambda \circ f].$$

These operations produce a λ -ring structure on orthogonal K-theory ([2], page 21) analogous to the λ -ring structure in classical algebraic K-theory [6, theorem 5.1].

Remark. To get a special λ -ring, one needs a splitting principle [10]. We do not know if there exists an analogue for the study of rank two modules in this case.

Acknowledgements

I would like to thank Professor Cathelineau for his assistance, Professor Karoubi for his valuable discussions, and the referee for his helpful and insightful comments. Thanks are also due to the University of South Florida Department of Mathematics.

References

- [1] A. Bak, K-theory of forms. Annals of Math. Studies, Princ. Univ. Press 1980. MR 84m:10012
- [2] M. Elhamdadi, Sur les Lambda-opérations et la L-théorie, Thèse de Doctorat de l'université de Nice-Sophia Antipolis (1996).
- [3] H. Hiller, λ -rings and algebraic K-theory, J. Pure Appl. Algebra **20** (1981), 241-266. MR **82e**:18016
- [4] M. Karoubi, Théorie de Quillen et homologie du groupe orthogonal, Ann. of Math. 112 (1980), 207-257. MR 82h:18011
- [5] M. A. Knus, Quadratic and Hermitian Forms over Rings, Grundlehren der math. Wissenschaften 294. Springer, Berlin 1991. MR 92i:11039
- [6] Ch. Kratzer, Lambda-structures en K-théorie algébrique, Comment. Math. Helv. 55 (1980), 233-254. MR 81m:18011
- [7] J. L. Loday, K-théorie algébrique et représentations de groupes, Ann. Sci. Ecole Norm. Sup. 9 (1976), 309-377. MR 56:5686
- [8] A. Roy, Cancellation of quadratic forms over commutative rings, J. Algebra 10 (1968), 286-298. MR 38:172
- [9] C. Soulé, Opérations en K-théorie algébrique, Can. J. Math. vol. XXXVII, N 3 (1985), 488-550. MR 87b:18013
- [10] R. G. Swan, A Splitting Principle in Algebraic K-theory, Proc. Symp. in Pure Math. 21 (1971), 155-159. MR 47:5079

Department of Mathematics, University of South Florida, 4202 East Fowler Ave., PHY 114, Tampa, Florida 33620-5700

E-mail address: emohamed@math.usf.edu