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Meyer's signature cocycle and hyperelliptic fibrations

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Abstract. We show that the cohomology class represented by Meyer's signature cocycle is of order $2g+1$ in the 2-dimensional cohomology group of the hyperelliptic mapping class group of genus g. By using the 1-cochain cobounding the signature cocycle, we extend the local signature for singular fibers of genus 2 fibrations due to Y. Matsumoto [18] to that for singular fibers of hyperelliptic fibrations of arbitrary genus g and calculate its values on Lefschetz singular fibers. Finally, we compare our local signature with another local signature which arises from algebraic geometry.

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1. Introduction

Let Σ_g be a closed oriented surface of genus g and $\mathcal{M}_g$ the mapping class group of Σ_g, namely the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g.

Meyer [21][22] introduced a cocycle $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \longrightarrow \mathbb{Z}$ (Meyer's *signature cocycle*) and gave a signature formula for surface bundles over surfaces. He showed that the signature $Sign(E)$ of the total space E of an oriented Σ_g-bundle over an oriented surface is always zero when g is equal to $0, 1$ or 2. It is easily seen that the signature $Sign(E)$ of any E is divisible by 4 because E has an almost complex structure. Meyer also showed that every multiple of 4 is equal to the signature $Sign(E)$ of some E when $g \geq 3$. These results are related to the order of the cohomology class of the signature cocycle τ_g in the cohomology group $H^2(\mathcal{M}_g, \mathbb{Z})$: namely the orders for τ_1 and τ_2 are finite but that for $\tau_g (g \geq 3)$ is infinite (see also Hoster's paper [13] for an elementary purely topological proof of the fact that Meyer's signature cocycle represents 4 times a generator of the cohomology group $H^2(\mathcal{M}_g, \mathbb{Z})$ and the author's paper [7] for an improvement of the proof of Meyer's theorem using Wajnryb's presentation [30] of $\mathcal{M}_g$).

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Matsumoto [16][18] defined a ‘local signature’, which he called the σ-number or the fractional signature, for singular fibers of genus 1 and 2 fibrations by using the finiteness of the order of the cohomology class of the signature cocycle $\tau_g (g = 1, 2)$. He calculated the values of the local signature for singular fibers of good torus fibrations, elliptic surfaces and Lefschetz fibrations of genus 2. The signature of the total space of such a fibration of genus 1 or 2 is equal to the sum of local signatures of all the singular fibers in the fibration. He also gave some remarkable examples of Lefschetz fibrations of genus 2 including some ‘exotic’ phenomena (see [8]) and reproved theorems of Persson [28] and Xiao-Ueno [31][29] with topological methods.

For $g \geq 3$, this kind of concentration of the signature were not expected because the order of the cohomology class of the signature cocycle τ_g in $H^2(\mathcal{M}_g, \mathbb{Z})$ is not finite. But on the hyperelliptic mapping class group $\mathcal{H}_g$, a certain subgroup of $\mathcal{M}_g$ which we explain in Sect. 2, the order of the cohomology class of the signature cocycle τ_g is actually finite.

In this paper we first show that the order of the cohomology class of the signature cocycle τ_g^H restricted to the hyperelliptic mapping class group $\mathcal{H}_g$ of genus g is equal to $2g + 1$ and we next define the local signature for singular fibers of hyperelliptic fibrations of arbitrary genus g and calculate some values of it. It turns out that the finiteness of the order of the cohomology class of the signature cocycle $\tau_g (g = 1, 2)$ comes from that of the order of the cohomology class of the signature cocycle τ_g^H restricted to the hyperelliptic mapping class group $\mathcal{H}_g$.

In Sect. 2 we determine the order of the cohomology class of the signature cocycle τ_g^H restricted to the hyperelliptic mapping class group $\mathcal{H}_g$ in the cohomology group $H^2(\mathcal{H}_g, \mathbb{Z})$ using Birman-Hilden’s presentation of $\mathcal{H}_g$. We calculate some values of the 1-cochain ϕ_g cobounding the signature cocycle τ_g^H restricted to $\mathcal{H}_g$ in Sect. 3. Then we define the local signature of a singular fiber of a hyperelliptic locally analytic fibration of arbitrary genus g and determine its values on Lefschetz singular fibers in Sect. 4. We also give some examples of (hyperelliptic) Lefschetz fibrations including Lefschetz fibrations with positive signature. And in the last section, Sect. 5, we compare our local signature with another local signature which arises from a study about degeneration of hyperelliptic curves in algebraic geometry and present Terasoma’s theorem which asserts that the values of these two kinds of local signature always coincide.

Morifuji [24][25] gives a certain formula for the 1-cochain ϕ_g cobounding the signature cocycle τ_g^H restricted to $\mathcal{H}_g$ and studies a relation between the 1-cochain ϕ_g and the Atiyah-Patodi-Singer η-invariant.

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2. Meyer's signature cocycle restricted to the hyperelliptic mapping class group

In this section we first recall the definition of Meyer's signature cocycle τ_g [22] and Birman-Hilden's presentation [5] of the hyperelliptic mapping class group $\mathcal{H}_g$ of genus g and we next determine the order of the cohomology class of the signature cocycle τ_g^H restricted to $\mathcal{H}_g$ in the cohomology group $H^2(\mathcal{H}_g, \mathbb{Z})$.

(1) *Meyer's signature cocycle* τ_g

For a pair (A, B) of symplectic matricies $A, B \in Sp(2g, \mathbb{Z})$, the vector space $V_{A,B}$ is defined by:

$$V_{A,B} := \{(x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \mid (A^{-1} - I)x + (B - I)y = 0\},$$

where I is the identity matrix. Consider the (possibly degenerate) symmetric bilinear form

$$< \ , \ >_{A,B} : V_{A,B} \times V_{A,B} \longrightarrow \mathbb{R}$$

on $V_{A,B}$ defined by:

$$< (x_1, y_1), (x_2, y_2) >_{A,B} := < x_1 + y_1, (I - B)y_2 >,$$

$$(x_i, y_i) \in V_{A,B} \quad (i = 1, 2),$$

where $< \ , \ >$ is the standard symplectic form on $\mathbb{R}^{2g}$ given by:

$$< x, y > = {}^t x J y \quad (x, y \in \mathbb{R}^{2g}),$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in M_{2g}(\mathbb{R}).$$

Meyer's *signature cocycle* [21][22]

$$\tau_g : Sp(2g, \mathbb{Z}) \times Sp(2g, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is defined by:

$$\tau_g(A, B) := sign(V_{A,B}, < \ , \ >_{A,B})$$

$$(A, B \in Sp(2g, \mathbb{Z})).$$

From the Novikov additivity, τ_g is a 2-cocycle of $Sp(2g, \mathbb{Z})$ and represents the cohomology class $4c_1 = [\tau_g] \in H^2(Sp(2g, \mathbb{Z}), \mathbb{Z})$ (cf. [26]).

Lemma 2.1 (Meyer [21][22]**).** *The signature cocycle τ_g satisfies the following properties:*

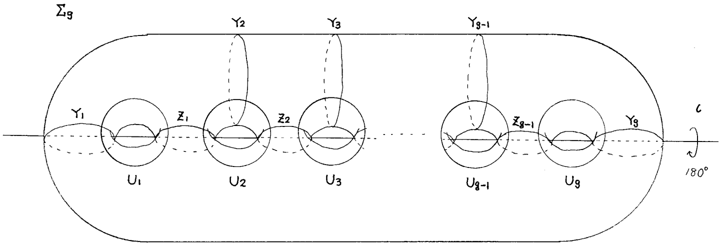

Fig. 1.

(1) $\tau_g(A, B) + \tau_g(AB, C) = \tau_g(A, BC) + \tau_g(B, C)$;
(2) if $ABC = I$, then $\tau_g(A, B) = \tau_g(B, C) = \tau_g(C, A)$;
(3) $\tau_g(A, I) = \tau_g(A, A^{-1}) = 0$;
(4) $\tau_g(B, A) = \tau_g(A, B)$;
(5) $\tau_g(A^{-1}, B^{-1}) = -\tau_g(A, B)$;
(6) $\tau_g(CAC^{-1}, CBC^{-1}) = \tau_g(A, B)$,

where $A, B, C \in Sp(2g, \mathbb{Z})$.

(2) *The hyperelliptic mapping class group* $\mathcal{H}_g$

Let Σ_g be a closed oriented surface of genus g embedded in $\mathbb{R}^3$ and $Y_1, \dots, Y_g, U_1, \dots U_g, Z_1, \dots, Z_{g-1}$ simple closed curves embedded in Σ_g in the manner illustrated in Fig. 1 (see [5] Sect. 2). We can define the *hyperelliptic involution* $\iota : \Sigma_g \longrightarrow \Sigma_g$ as in Fig. 1.

Let $\mathcal{M}_g$ be the mapping class group of the surface Σ_g of genus g, namely the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g and $\mathcal{H}_g$ the *hyperelliptic mapping class group* of genus g, namely the subgroup of $\mathcal{M}_g$ which consists of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g commuting with the isotopy class of ι.

Theorem 2.2 ((1) Lickorish [15]**, (2) Birman-Hilden** [5]**).**

(1) *The group* $\mathcal{M}_g$ *is generated by negative Dehn twists* $y_1, \dots, y_g, u_1, \dots, u_g, z_1, \dots, z_{g-1}$ *along simple closed curves* $Y_1, \dots Y_g$, $U_1, \dots U_g, Z_1, \dots, Z_{g-1}$ *illustrated in Fig. 1.*
(2) *The group* $\mathcal{H}_g$ *is the subgroup of* $\mathcal{M}_g$ *generated by* $y_1, u_1, z_1, u_2, \dots, z_{g-1}, u_g, y_g$ *and a complete set of defining relators for* $\mathcal{H}_g$ *are given as follows:*

$$\begin{aligned}
\alpha^1 &:= [y_1, y_g],\\
\alpha^2_{i,j} &:= [y_i, u_j] \quad (i = 1, g, 1 \le j \le g, i \ne j),\\
\alpha^3_{i,j} &:= [y_i, z_j] \quad (i = 1, g, 1 \le j \le g-1),\\
\alpha^4_{i,j} &:= [u_i, u_j] \quad (1 \le i < j \le g),\\
\alpha^5_{i,j} &:= [u_i, z_j] \quad (1 \le i \le g, 1 \le j \le g-1, j \ne i, i+1),\\
\alpha^6_{i,j} &:= [z_i, z_j] \quad (1 \le i < j \le g-1),\\
\beta^1_i &:= y_i u_i y_i u_i^{-1} y_i^{-1} u_i^{-1} \quad (i = 1, g),\\
\beta^2_i &:= u_i z_i u_i z_i^{-1} u_i^{-1} z_i^{-1} \quad (1 \le i \le g-1),\\
\beta^3_i &:= z_i u_{i+1} z_i u_{i+1}^{-1} z_i^{-1} u_{i+1}^{-1} \quad (1 \le i \le g-1),\\
\gamma^1 &:= (y_1 u_1 z_1 u_2 \cdots z_{g-1} u_g y_g)^{2g+2},\\
\delta^1 &:= (y_1 u_1 z_1 u_2 \cdots z_{g-1} u_g y_g^2 u_g z_{g-1} \cdots u_2 z_1 u_1 y_1)^2,\\
\epsilon^1 &:= [y_1 u_1 z_1 u_2 \cdots z_{g-1} u_g y_g^2 u_g z_{g-1} \cdots u_2 z_1 u_1 y_1, y_1].
\end{aligned}$$

Particularly, the group $\mathcal{H}_g$ is equal to the group $\mathcal{M}_g$ if $g = 1$ or 2.

By choosing a suitable symplectic basis of $H_1(\Sigma_g; \mathbb{Z})$, we fix an explicit representation $\sigma : \mathcal{M}_g \longrightarrow Sp(2g; \mathbb{Z})$ by:

$$\begin{aligned}
\sigma &: y_i \longmapsto \begin{pmatrix} I & 0 \\ -E_{ii} & I \end{pmatrix} \quad (1 \le i \le g),\\
\sigma &: u_i \longmapsto \begin{pmatrix} I & E_{ii} \\ 0 & I \end{pmatrix} \quad (1 \le i \le g),\\
\sigma &: z_i \longmapsto \begin{pmatrix} I & 0 \\ -E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} & I \end{pmatrix} \quad (1 \le i \le g-1),
\end{aligned}$$

where $E_{ij} \in M_g(\mathbb{Z})$ is the (i, j)-matrix unit (see [22]). For simplicity, we often denote $\tau_g \circ (\sigma \times \sigma)$ by τ_g.

(3) *The order of the cohomology class of τ_g in $H^2(\mathcal{H}_g, \mathbb{Z})$*

Let G be a finitely presentable group and $z : G \times G \longrightarrow \mathbb{Z}$ a 2-cocycle of G (i.e. $z(x, y) + z(xy, w) = z(x, yw) + z(y, w)$ for every $x, y, w \in G$) which satisfies

$$z(x, 1) = z(1, x) = z(x, x^{-1}) = 0$$

for arbitrary element x of G (cf. [6]). The order of the cohomology class $k := [z]$ of z in $H^2(G, \mathbb{Z})$ can be calculated as follows.

There exists an exact sequence:

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1,$$

where F is the free group of finite rank r generated by a free basis $E = \{e_1, \dots, e_r\}$. For each generator $e_i (i = 1, \dots, r)$, the homomorphism $e_i^* : F \longrightarrow \mathbb{Z}$ is defined by:

$$e_i^*(e_j) := \delta_{ij} \quad (i, j = 1, \dots, r).$$

The map $c : F \longrightarrow \mathbb{Z}$ is defined by:

$$c(x) := \sum_{j=1}^{m} z(\pi(\widetilde{x}_{j-1}), \pi(x_j))$$

$$\left(x = \prod_{i=1}^{m} x_i,\ x_i \in E \cup E^{-1},\ \widetilde{x}_j = \prod_{i=1}^{j} x_i \right).$$

It can be checked that the restriction $c \mid_R : R \longrightarrow \mathbb{Z}$ of c to R is actually a homomorphism.

Proposition 2.3 (Meyer [22] **p.249).** *A positive integer n is divided by the order $ord(k)$ of the cohomology class $k = [z]$ in $H^2(G, \mathbb{Z})$ if and only if there exist integers $n_1, \dots, n_r$ such that the following holds:*

$$nc \mid_R \equiv \sum_{i=1}^{r} n_i e_i^* \mid_R .$$

Furthermore, if the abelianization $G/[G, G]$ of G is finite, then such integers $n, n_1, \dots, n_r$ are uniquely determined under the condition $g.c.d.(n, n_1, \dots, n_r) = 1$ and the order $ord(k)$ of k is exactly equal to n.

Now we use Proposition 2.3 in the case that $G = \mathcal{H}_g$, $z = \tau_g^H := \tau_g \circ (\sigma \times \sigma) \mid_{\mathcal{H}_g \times \mathcal{H}_g} \in Z^2(\mathcal{H}_g, \mathbb{Z})$, $F = < y_1, u_1, z_1, u_2, \dots, z_{g-1}, u_g, y_g >$ and R is the normal closure of the set of the Birman-Hilden relators $\{\alpha_{ij}^l, \beta_i^l, \gamma^1, \delta^1, \epsilon^1\}$ in F.

Lemma 2.4. *The values of the homomorphism $c \mid_R$: $R \longrightarrow \mathbb{Z}$ on the relators of Birman-Hilden's presentation of $\mathcal{H}_g (g \geq 1)$ are calculated as follows:*

(1) $c(\alpha_{ij}^l) = 0$ (*for every* l, i, j);
(2) $c(\beta_i^l) = 0$ (*for every* l, i);
(3) $c(\gamma^1) = 2(g + 1)^2$;
(4) $c(\delta^1) = 4(g + 1)$;
(5) $c(\epsilon^1) = 0$.

Proof. (1),(2) and (5) follow from a formula of Meyer ([22] p.254-l.2). (3) and (4) are formulae (30) and (31) in [22]. □

Proposition 2.5. *The order $ord([\tau_g^H])$ of the cohomology class $[\tau_g^H]$ of the signature cocycle τ_g^H restricted to the hyperelliptic mapping class group $\mathcal{H}_g$ of genus g in the cohomology group $H^2(\mathcal{H}_g, \mathbb{Z})$ is exactly equal to $2g+1$.*

Proof. Lemma 2.4 implies

$$(2g+1)c\mid_R \equiv (g+1)(y_1^* + u_1^* + z_1^* + u_2^* + \cdots + z_{g-1}^* + u_g^* + y_g^*)\mid_R .$$

It can be easily checked that $\mathcal{H}_g/[\mathcal{H}_g, \mathcal{H}_g]$ is a finite cyclic group by using Birman-Hilden's presentation of $\mathcal{H}_g$. Therefore $ord([\tau_g^H]) = 2g+1$ follows from Proposition 2.3. □

Corollary 2.6 (Meyer [22]**).** *The order of $[\tau_1]$ in $H^2(\mathcal{M}_1, \mathbb{Z})$ is equal to 3 and the order of $[\tau_2]$ in $H^2(\mathcal{M}_2, \mathbb{Z})$ is equal to 5.*

3. The 1-cochain cobounding the signature cocycle

In this section we calculate some values of the 1-cochain ϕ_g cobounding the signature cocycle τ_g^H restricted to the hyperelliptic mapping class group $\mathcal{H}_g$.

(1) *The class function ϕ_g on $\mathcal{H}_g$*

Proposition 3.1. *There exists a unique function*

$$\phi_g : \mathcal{H}_g \longrightarrow \frac{1}{2g+1}\mathbb{Z} \quad (g \geq 1)$$

with the following properties:

(1) $\tau_g^H(x, y) = \phi_g(x) + \phi_g(y) - \phi_g(xy)$;
(2) $\phi_g(1) = 0$;
(3) $\phi_g(x^{-1}) = -\phi_g(x)$;
(4) $\phi_g(yxy^{-1}) = \phi_g(x)$,

where $x, y \in \mathcal{H}_g$.

Proof. The existence of ϕ_g and property (1) are assured by Proposition 2.5 and its uniqueness follows from the finiteness of $\mathcal{H}_g/[\mathcal{H}_g, \mathcal{H}_g]$. Properties (2)(3)(4) are easily shown by using property (1) and Lemma 2.1(3)(4). □

Lemma 3.2. *If an element x of $\mathcal{H}_g$ satisfies $x^2 = 1$, then the value $\phi_g(x)$ is equal to 0.*

Proof. The statement immediately follows from $x = x^{-1}$ and Lemma 2.1(3). □

(2) *The values of ϕ_g on the Lickorish generators*

Lemma 3.3. *The values of the function ϕ_g on the Lickorish generators $y_1, u_1, z_1, u_2, \ldots, z_{g-1}, u_g, y_g$ of the hyperelliptic mapping class group $\mathcal{H}_g$ are calculated as follows:*

$$\begin{aligned}\phi_g(y_1) &= \phi_g(y_g) = \phi_g(u_i) = \phi_g(z_j) \\ &= \frac{g+1}{2g+1} \quad (i = 1, \ldots, g,\ j = 1, \ldots, g-1).\end{aligned}$$

Proof. By using Proposition 3.1 for $\phi_g(\beta_i^1)(i = 1, g)$, we have

$$\begin{aligned}0 = \phi_g(1) = \phi_g(\beta_i^1) &= \phi_g(y_i u_i y_i u_i^{-1} y_i^{-1} u_i^{-1}) \\ &= \phi_g(y_i u_i y_i u_i^{-1} y_i^{-1}) + \phi_g(u_i^{-1}) - \tau_g^H(y_i u_i y_i u_i^{-1} y_i^{-1}, u_i^{-1}) \\ &= \phi_g(y_i) - \phi_g(u_i) - \tau_g^H(u_i y_i u_i^{-1}, u_i y_i^{-1} u_i^{-1}) \\ &= \phi_g(y_i) - \phi_g(u_i) \quad (i = 1, g).\end{aligned}$$

By combining with similar calculation for $\phi_g(\beta_i^2)(i = 1, \ldots, g)$ and $\phi_g(\beta_j^3)(j = 1, \ldots, g-1)$, we obtain

$$\phi_g(y_1) = \phi_g(y_g) = \phi_g(u_i) = \phi_g(z_j) \quad (i = 1, \ldots, g,\ j = 1, \ldots, g-1)$$

and put $\lambda := \phi_g(y_1)$. Next we use Proposition 3.1 for $\phi_g(\gamma^1)$. (We put $p := y_1 u_1 z_1 u_2 \cdots z_{g-1} u_g y_g$.)

$$\begin{aligned}0 = \phi_g(1) = \phi_g(\gamma^1) = \phi_g(p^{2g+2}) &= (g+1)\phi_g(p^2) - \sum_{i=1}^{g} \tau_g^H(p^2, p^{2i}) \\ &= (g+1)(2\phi_g(p) - \tau_g^H(p, p)) = 2(g+1)(\phi_g(p) - g),\end{aligned}$$

where $\tau_g^H(p^2, p^{2i}) = 0\,(i = 1, \ldots, g)$ and $\tau_g^H(p, p) = 2g$ ([22] p.255). Thus we have $\phi_g(p) = g$. From direct computation of the signature cocycle τ_g^H, it follows that $\tau_g^H(y_1 u_1 z_1 \cdots u_i z_i, u_{i+1}) = 0$, $\tau_g^H(y_1 u_1 z_1 \cdots z_{i-1} u_i, z_i) = 0 (i = 1, \ldots, g-1)$, $\tau_g^H(y_1 u_1 z_1 \cdots z_{g-1} u_g, y_g) = 1$ and $\tau_g^H(y_1, u_1) = 0$. Therefore we get

$$\begin{aligned}\phi_g(p) &= \phi_g(y_1 u_1 z_1 u_2 \cdots z_{g-1} u_g y_g) \\ &= (2g+1)\lambda - \sum_{i=1}^{g-1} (\tau_g^H(y_1 u_1 z_1 \cdots u_i z_i, u_{i+1}) + \tau_g^H(y_1 u_1 z_1 \cdots z_{i-1} u_i, z_i)) \\ &\qquad - \tau_g^H(y_1 u_1 z_1 \cdots z_{g-1} u_g, y_g) - \tau_g^H(y_1, u_1) \\ &= (2g+1)\lambda - 1.\end{aligned}$$

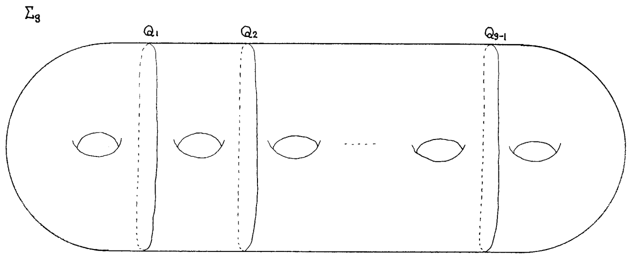

Fig. 2.

Hence we obtain

$$\lambda = \frac{g+1}{2g+1}$$

and this completes the proof. □

Remark 3.4. T. Morifuji spoke about a purely algebraic proof of Corollary 3.7 of [18] and the author about Proposition 2.5 of this paper at Hokkaido University in September 1997. They arrived at the same conclusion (Lemma 3.3 above and corresponding theorems in [24][25]) independently by slightly different calculations.

(3) *The values of ϕ_g on the BSCC maps*

On the surface Σ_g in Fig. 1 we draw simple closed curves $Q_1, \dots, Q_{g-1}$ as in Fig. 2 and denote by q_h the negative Dehn twist along $Q_h (h = 1, \dots, g-1)$. Each $q_h (h = 1, \dots, g-1)$ belongs to $\mathcal{H}_g$ and q_{g-h} is conjugate to q_h in $\mathcal{H}_g$. This q_h is often called a *BSCC map* of genus h, where BSCC means 'bounding simple closed curve' (see [14]).

Lemma 3.5. *The value of the function ϕ_g on the BSCC map q_h of genus h are calculated as follows:*

$$\phi_g(q_h) = -\frac{4h(g-h)}{2g+1} \quad (h = 1, \dots, g-1).$$

Proof. As an element of $\mathcal{H}_g$, q_h is equal to $p_h^{2(2h+1)}$, where $p_h := y_1 u_1 z_1 u_2 \cdots z_{h-1} u_h$ $(h = 1, \dots, g-1)$(cf. Lemma 4.13). From direct computation of τ_g^H, it follows that $\tau_g^H(p_h, p_h) = 2h$, $\tau_g^H(p_h^2, p_h^{2h}) = 2h$ and $\tau_g^H(p_h^2, p_h^{4h}) = 0$. Then we have

$$
\begin{aligned}
\phi_g(p_h) &= \phi_g(y_1u_1z_1u_2\cdots z_{h-1}u_h)\\
&= \frac{g+1}{2g+1}\times 2h - \sum_{i=1}^{h-1}(\tau_g^H(y_1u_1z_1\cdots z_{i-1}u_i, z_i)\\
&\qquad\qquad + \tau_g^H(y_1u_1z_1\cdots u_iz_i, u_{i+1})) - \tau_g^H(y_1, u_1)\\
&= \frac{g+1}{2g+1}\times 2h,\\
\phi_g(q_h) &= \phi_g(p_h^{2(2h+1)})\\
&= (2h+1)\phi_g(p_h^2) - \sum_{i=1}^{h-1}\tau_g^H(p_h^2, p_h^{2i}) - \tau_g^H(p_h^2, p_h^{2h})\\
&\qquad - \sum_{i=h+1}^{2h-1}\tau_g^H(p_h^2, p_h^{2i}) - \tau_g^H(p_h^2, p_h^{4h})\\
&= (2h+1)(2\phi_g(p_h) - \tau_g^H(p_h, p_h)) - 2h - 2\lambda(g,h)\\
&= -\frac{4h(g-h)}{2g+1} - 2\lambda(g,h),
\end{aligned}
$$

where we put

$$
\begin{aligned}
&\lambda(g,h) := \sum_{i=1}^{h-1}\tau_g^H(p_h^2, p_h^{2i}) = \sum_{i=h+1}^{2h-1}\tau_g^H(p_h^2, p_h^{2i}) \quad (g > h > 1),\\
&\lambda(g,1) := 0 \quad (g \geq 2).
\end{aligned}
$$

By virtue of Proposition 3.1(4), we have $\phi_g(q_{g-h}) = \phi_g(q_h)(h = 1,\dots, g-1)$ because q_{g-h} is conjugate to q_h in $\mathcal{H}_g$. It immediately follows that $\lambda(g, g-h) = \lambda(g,h)(g \geq 2, h = 1, \dots g-1)$. On the other hand, it can be observed that $\lambda(g+1,h) = \lambda(g,h)(g \geq 2, h = 1, \dots, g-1)$ by using formula (13) in [22]. Therefore we conclude that $\lambda(g,h) = 0(g \geq 2, h = 1, \dots, g-1)$ and this completes the proof. □

4. The local signature for hyperelliptic fibrations

In this section we define the local signature σ_g for singular fibers of hyperelliptic fibrations and calculate its values on Lefschetz singular fibers.

We follow the notation and terminology of [18].

(1) *The local signature* σ_g

Definition 4.1([18]**).** Let M and B be compact oriented (not necessarily closed) smooth manifolds of dimension 4 and 2, respectively. A smooth map $f : M \longrightarrow$

B is called a *locally analytic fibration* of genus g if it satisfies the following conditions:

(1) $\partial M = f^{-1}(\partial B)$;
(2) there is a finite set of points $b_1, \ldots, b_n$ (called the *critical values* of f) in $Int\, B$ $(= B - \partial B)$ such that $f|f^{-1}(B - \{b_1, \ldots, b_n\}) : f^{-1}(B - \{b_1, \ldots, b_n\}) \longrightarrow B - \{b_1, \ldots, b_n\}$ is a smooth fiber bundle with the fiber diffeomorphic to Σ_g;
(3) for each i $(1 \leq i \leq n)$ and at each point $p \in f^{-1}(b_i)$, the germ (f, p) is conjugate via (not necessarilly orientation-preserving) diffeomorphism to the germ at $\mathbf{0}$ of a holomorphic function $\mathbb{C}^2 \longrightarrow \mathbb{C}$. Moreover, there exists at least one critical point of f on the fiber $f^{-1}(b_i)$.

We call a fiber $f^{-1}(b)$ a *singular fiber* if $b \in \{b_1, \ldots b_n\}$ or else a *general fiber*. Also we call M the *total space, B* the *base space* and f the *projection*.

Let $f : M \longrightarrow B$ be a locally analytic fibration of genus g and $b_1, \ldots, b_n$ the critical values of f. Take a base point $b_0 \in Int\, B - \{b_1, \ldots, b_n\}$, and identify the general fiber $F_0 := f^{-1}(b_0)$ with Σ_g by an orientation-preserving diffeomorphism $\Phi : \Sigma_g \longrightarrow F_0$. Then we obtain the monodromy representation

$$\rho : \pi_1(B - \{b_1, \ldots, b_n\}, b_0) \longrightarrow \mathcal{M}_g$$

of $f : M \longrightarrow B$ associated with $\Phi : \Sigma_g \longrightarrow F_0$.

Definition 4.2. A locally analytic fibration $f : M \longrightarrow B$ of genus g is said to be *hyperelliptic* if there exists an orientation-preserving diffeomorphism $\Phi : \Sigma_g \longrightarrow F_0$ such that the image $Im\,\rho$ of the monodromy representation ρ of the fibration f associated with Φ is included in the hyperelliptic mapping class group $\mathcal{H}_g$.

Definition 4.3([18]**).** Let F and F' be (singular or non-singular) fibers in locally analytic fibrations $f : M \longrightarrow B$ and $f' : M' \longrightarrow B'$, respectively. F and F' are said to be *topologically equivalent* if there exist neighborhoods $N(\subset B)$ and $N'(\subset B')$ of points $b = f(F)$ and $b' = f'(F')$, respectively, and orientation-preserving homeomorphisms $H : (f)^{-1}(N) \longrightarrow (f')^{-1}(N')$ and $h : N \longrightarrow N'$ so that $h(b) = b'$ and $f' \circ H = h \circ f$.

We denote by $\mathcal{S}_g$ the set of topological equivalence classes of all fibers which appear in locally analytic fibrations of genus g and by $\mathcal{S}_g^H$ the subset of $\mathcal{S}_g$ which consists of all equivalence classes of hyperelliptic fibers.

Theorem 4.4. *There exists a unique function*

$$\sigma_g : \mathcal{S}_g^H \longrightarrow \frac{1}{2g+1}\mathbb{Z} \quad (g \geq 1)$$

with the following properties:

(1) *if* $F \in \mathcal{S}_g^H$ *is non-singular, then* $\sigma_g(F) = 0$;
(2) *if* $f : M \longrightarrow B$ *is a hyperelliptic locally analytic fibration of genus* g *over a closed surface* B, *and if* $F_1, \ldots, F_n$ *are all singular fibers in this fibration, then*

$$Sign(M) = \sum_{i=1}^{n} \sigma_g(F_i),$$

where $Sign(M)(\in \mathbb{Z})$ *is the signature of* M. *We call* $\sigma_g(F_i)$ *the* local signature *of the singular fiber* F_i.

Proof. Let $\{b_1, \ldots, b_n\}$ be the set of critical values of f, D_i a small 2-disk on B centered at b_i and $F_i = f^{-1}(b_i)$. We put M_0 for $M - \bigcup_{i=1}^{n} f^{-1}(Int\ D_i)$. We give an orientation to ∂D_i from the inside of D_i and let $\alpha_i \in \mathcal{H}_g$ be the monodromy of the bundle $f^{-1}(\partial D_i) \longrightarrow \partial D_i$ in this direction of ∂D_i. By virtue of Meyer's signature formula ([22] Satz 1) and Proposition 3.1, we obtain

$$Sign(M_0) = -\sum_{i=1}^{n} \phi_g(\alpha_i).$$

If we define $\sigma_g(F_i)$ by

$$\sigma_g(F_i) := -\phi_g(\alpha_i) + Sign(f^{-1}(D_i)),$$

then we get the formula above from Novikov's additivity.

Let σ_g' be another function satisfying properties (1) and (2). If $\alpha \in \mathcal{H}_g$ is the monodromy of a singular fiber $F \in \mathcal{S}_g^H$, then $\alpha^{4(2g+1)}$ is included in $[\mathcal{H}_g, \mathcal{H}_g]$ because the order of $\mathcal{H}_g/[\mathcal{H}_g, \mathcal{H}_g]$ divides $4(2g+1)$. Thus there exists a hyperelliptic locally analytic fibration $f : M \longrightarrow B$ of genus g over a closed surface B which has $4(2g+1)$ times F as the singular fibers. Then we have $4(2g+1)\sigma_g'(F) = 4(2g+1)\sigma_g(F)$ from property (2). Hence σ_g' must be equal to σ_g. □

Remark 4.5. The value of σ_g is independent of a choice of Φ because ϕ_g is a class function on $\mathcal{H}_g$ (Proposition 3.1(4)).

We recall the Euler contribution of a fiber of locally analytic fibrations (see [18] Definition 3.8).

For $F \in \mathcal{S}_g (g \geq 1)$, $\epsilon(F) := e(F) - e(\Sigma_g)$ is called the *Euler contribution* of F. If $f : M \longrightarrow B$ be a locally analytic fibration of genus g over a closed surface B, and if $F_1, \ldots, F_n$ are all singular fibers in this fibration f, then

$$e(M) = e(B)e(\Sigma_g) + \sum_{i=1}^{n} \epsilon(F_i).$$

Remark 4.6. The local signature $\sigma_g(F)$ and the Euler contribution $\epsilon(F)$ of a singular fiber $F \in \mathcal{S}_g^H$ are conserved at splitting of F.

(2) *The local signature for Lefschetz singular fibers*

Definition 4.7([18]**).** A locally analytic fibration $f : M \longrightarrow B$ of genus g is called a *Lefschetz fibration* of genus g if the following conditions are satisfied:

(1) for each critical value b_i $(1 \le i \le n)$, there exists a single point $p_i \in f^{-1}(b_i)$ such that (i) $(df)_p : T_p(M) \longrightarrow T_{f(p)}(B)$ is onto for any $p \in f^{-1}(b_i) - \{p_i\}$, (ii) about p_i (resp. b_i), there exist local complex coordinates z_1, z_2 with $z_1(p_i) = z_2(p_i) = 0$ (resp. local complex coordinate ξ with $\xi(b_i) = 0$), so that f is locally written as $\xi = f(z_1, z_2) = z_1 z_2$;
(2) no fiber contains a (-1)-sphere.

In Condition (1)(ii), we require the orientation of M (resp. B) to coincide with the canonical orientation determined by the local complex coordinates z_1, z_2 (resp. local canonical coordinate ξ) about the critical point p_i (resp. b_i).

Y. Matsumoto proved that the global monodromy of a Lefschetz fibration determines the fibering structure up to isomorphism ([18] Theorem 2.4 and Theorem 2.6).

There are exactly $[\frac{g}{2}] + 1$ topological types of Lefschetz singular fibers of genus g, which we call singular fibers of type I and type $II_h (h = 1, \ldots, [\frac{g}{2}])$. All of these singular fibers are hyperelliptic: $I, II_1, \ldots, II_{[\frac{g}{2}]} \in \mathcal{S}_g^H$. The monodromy of a singular fiber of type I is the negative Dehn twist along a non-separating simple closed curve on Σ_g and that of a singular fiber of type II_h is the negative Dehn twist along a separating (or bounding) simple closed curve of genus h on Σ_g. We can easily see that $\epsilon(I) = \epsilon(II_h) = 1 (h = 1, \ldots, [\frac{g}{2}])$.

Theorem 4.8. *The local signature σ_g for Lefschetz singular fibers are calculated as follows:*

$$\sigma_g(I) = -\frac{g+1}{2g+1}, \quad \sigma_g(II_h) = \frac{4h(g-h)}{2g+1} - 1 \quad \left(h = 1, \ldots, \left[\frac{g}{2}\right]\right).$$

Proof. These values are easily computed from the definition of σ_g, Lemma 3.3 and Lemma 3.5. □

Remark 4.9. The value $\sigma_g(II_h)(h = 1, \ldots, [\frac{g}{2}])$ is always positive for $g \ge 3$.

We are interested in the number of singular fibers in a hyperelliptic Lefschetz fibration.

Proposition 4.10. *Let $f : M \longrightarrow B$ be a hyperelliptic Lefschetz fibration of genus g over a closed surface B and $a, b_1, \ldots, b_{[\frac{g}{2}]}$ numbers of Lefschetz singular fibers of type $I, II_1, \ldots, II_{[\frac{g}{2}]}$, respectively. Then the following congruences* *hold:*

$$(g+1)a - 4\sum_{h=1}^{[\frac{g}{2}]} h(g-h)b_h \equiv 0 \quad (mod\ 2g+1) \tag{1}$$

$$a + 4\sum_{h=1}^{[\frac{g}{2}]} h(2h+1)b_h \equiv 0 \quad \left(mod \begin{Bmatrix} 2(2g+1)\ (g: even) \\ 4(2g+1)\ (g: odd) \end{Bmatrix} \right). \tag{2}$$

Proof. (1) This congruence follows from Theorem 4.4, Theorem 4.8 and the integrality of the signature of M:

$$Sign(M) = \frac{g+1}{2g+1}a - \sum_{h=1}^{[\frac{g}{2}]} \frac{4h(g-h)}{2g+1}b_h \in \mathbb{Z}.$$

(2) The abelianization $\mathcal{H}_g^{ab} = \mathcal{H}/[\mathcal{H}_g, \mathcal{H}_g]$ of $\mathcal{H}_g$ is isomorphic to $\mathbb{Z}/2(2g+1)$ (resp. $\mathbb{Z}/4(2g+1)$) if g is even (resp. odd)(see [5]). Under the identification $\mathcal{H}_g^{ab} = \mathbb{Z}/2(2g+1)$ (resp. $\mathbb{Z}/4(2g+1)$), each of Lickorish generators $y_1, u_1, z_1, u_2, \dots, z_{g-1}, u_g, y_g$ corresponds to 1 and

$$q_h = (y_1 u_1 z_1 u_2 \cdots z_{g-1} u_g y_g)^{2(2g+1)}$$

to $4h(2h+1)$ $(h = 1, \dots, [\frac{g}{2}])$. Thus the total monodromy of $f : M \longrightarrow B$ corresponds to

$$a + 4\sum_{h=1}^{[\frac{g}{2}]} h(2h+1)b_h.$$

Since the total monodromy of $f : M \longrightarrow B$ must be freely equal to a commutator of $\mathcal{H}_g$, the abelianized total monodromy is equal to 0. Thus we obtain the congruence above. □

The next corollary corresponds to Noether's condition $c_1^2 + c_2 \equiv 0 (mod\ 12)$ on complex manifolds (cf. [4]).

Corollary 4.11. *If $f : M \longrightarrow B$ be a hyperelliptic Lefschetz fibration of genus g over a closed surface B, then the integer $Sign(M) + e(M)$ is divisible by* 4, *where $e(M)$ is the Euler characteristic of M.*

Proof. If $a, b_1, \dots, b_{[\frac{g}{2}]}$ are numbers of Lefschetz singular fibers of type $I, II_1, \dots, II_{[\frac{g}{2}]}$ in this fibration f, then we have

$$Sign(M) + e(M) = \frac{g}{2g+1}a + \sum_{h=1}^{[\frac{g}{2}]} \frac{4h(g-h)}{2g+1}b_h + e(B)(2-2g).$$

Both $4h(g-h)b_h$ and $e(B)(2-2g)$ are clearly divisible by 4. By using Proposition 4.10(2), we can show that ga is also divisible by 4. □

Remark 4.12. The congruence $Sign(M)+e(M) \equiv 0\ (mod\ 4)$ holds for any (not necessarily hyperelliptic) Lefschetz fibration $f : M \longrightarrow B$ of genus g over a closed surface B because M has an almost complex structure (see [10] Chapter 8).

(3) *Examples of Lefschetz fibrations*

We need the following lemma.

Lemma 4.13. *In the hyperelliptic mapping class group $\mathcal{H}_g$, the next relation holds:*

$$(y_1u_1z_1u_2\cdots z_{g-1}u_g)^{2g+1} = \iota\ ,$$

where ι is the hyperelliptic involution.

Proof. We denote $y_1, u_1, z_1, u_2, \ldots, z_{g-1}, u_g, y_g$ by $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \ldots, \zeta_{2g-1}, \zeta_{2g}, \zeta_{2g+1}$ and set $a = \zeta_1\zeta_2\zeta_3\cdots\zeta_{2g}\zeta_{2g+1}$ and $b = \zeta_1^2\zeta_2\zeta_3\cdots\zeta_{2g}\zeta_{2g+1}$. Then by Theorem 2.2(2), $a^{2g+2} = 1$ and $\zeta_1 = ba^{-1} = ba^{2g+1}$. We can verify $\zeta_i = a^{i-1}\zeta_1 a^{1-i} (i = 2, \ldots, 2g+1)$. We obtain

$$\zeta_i = a^{i-1}ba^{2g-i+2} \quad (i = 2, \ldots, 2g+1)$$

by substituting ba^{2g+1} for ζ_1. We rewrite the known relation

$$\zeta_1\zeta_2\zeta_3\cdots\zeta_{2g}\zeta_{2g+1}^2\zeta_{2g}\cdots\zeta_3\zeta_2\zeta_1 = \iota$$

(cf. [5]) using the above expression of ζ_i in terms of a and b, then we obtain

$$a^{-1}(ba^{2g})^{2g+1}a = \iota.$$

Since ι belongs to the center of $\mathcal{H}_g$, we have $(ba^{2g})^{2g+1} = \iota$. We will further rewrite ba^{2g} as follows:

$$\begin{aligned} ba^{2g} &= b\cdot a^{2g}ba\cdot a^{-1}b^{-1} = b\zeta_{2g+1}a^{-1}b^{-1} \\ &= b(\zeta_{2g+1}\cdot\zeta_{2g+1}^{-1}\zeta_{2g}^{-1}\cdots\zeta_3^{-1}\zeta_2^{-1}\zeta_1^{-1})b^{-1} \\ &= b(\zeta_1\zeta_2\zeta_3\cdots\zeta_{2g-1}\zeta_{2g})^{-1}b^{-1}. \end{aligned}$$

Then we have $b(\zeta_1\zeta_2\zeta_3\cdots\zeta_{2g-1}\zeta_{2g})^{-(2g+1)}b^{-1} = \iota$. Using the fact that ι belongs to the center of $\mathcal{H}_g$ and that $\iota^2 = 1$, we finally obtain

$$(\zeta_1\zeta_2\zeta_3\cdots\zeta_{2g-1}\zeta_{2g})^{2g+1} = \iota$$

as asserted. □

We generalize Example D of [18] to examples in higher genara.

Example 4.14. (Homeomorphic but non-isomorphic Lefschetz fibrations)

From Theorem 2.2(2) and Lemma 4.13, we have the following relations in $\mathcal{H}_g$:

$$(y_1 u_1 z_1 u_2 \cdots z_{g-1} u_g y_g^2 u_g z_{g-1} \cdots u_2 z_1 u_1 y_1)^{2g} = 1,$$

$$(y_1 u_1 z_1 u_2 \cdots z_{g-1} u_g)^{4g+2} = 1,$$

which produce two hyperelliptic Lefschetz fibrations $M_1 \longrightarrow S^2$ and $M_2 \longrightarrow S^2$ of type $4g(2g+1)I$ ([18] Theorem 2.6). By computing the local signature σ_g and the Euler contribution ϵ, both of M_1 and M_2 have the same signature $-4g(g+1)$ and the same Euler characteristic $4(2g^2+1)$. If $g \equiv 1$ or $2 (mod\ 4)$, then M_1 and M_2 are homeomorphic to $(2g^2-2g+1)\mathbb{C}P^2 \sharp (6g^2+2g+1)\overline{\mathbb{C}P}^2$ from Freedman's theorem. However, they are not isomorphic as Lefschetz fibrations because the monodromy representation of M_1 is surjective onto the whole mapping class group $\mathcal{M}_g$ while that of M_2 is not so. In the case $g=2$, it is shown that M_1 and M_2 are not diffeomorphic to each other by using Kirby calculus ([8]).

We can also construct Lefschetz fibrations of genus $g (\geq 3)$ with positive signature.

Example 4.15.(Lefschetz fibrations with positive signature)

We assume $g \geq 3$. The total monodromy $(y_1 u_1)^{6(2g+1)}$ of $2g+1$ singular fibers of type II_1 with monodromy $(y_1 u_1)^6 (= q_1)$ is included in $[\mathcal{H}_g, \mathcal{H}_g]$ because the order of $\mathcal{H}_g/[\mathcal{H}_g, \mathcal{H}_g]$ divides $4(2g+1)$. Thus $(y_1 u_1)^{6(2g+1)}$ is equal to a product of k commutators of $\mathcal{H}_g$ for some $k (> 0)$ and we obtain a Lefschetz fibration $f : M \longrightarrow B$ of genus g of type $(2g+1)II_1$ over a closed surface B of genus k ([18] Theorem 2.6). Then we have

$$\begin{aligned} Sign(M) &= (2g+1)\sigma_g(II_1) = (2g+1) \times \left(\frac{4(g-1)}{2g+1} - 1 \right) \\ &= 2g - 5. \end{aligned}$$

We compute a concrete value of k.

Let denote by W_g the product of $(y_1 u_1)^{6(2g+1)}$ with

$$(\beta_1^1)^{-6(2g+1)} (\beta_g^1)^{12} \prod_{i=1}^{g-1} (\beta_i^2)^{-12(2g-2i+1)} \prod_{i=1}^{g-1} (\beta_i^3)^{-12(2g-2i)} (\delta^1)^{-3} \quad (= 1).$$

The word W_g, which is actually equal to $(y_1 u_1)^{6(2g+1)}$ as an element of $\mathcal{H}_g$, is of length $12(2g+1)(6g-1)$ and the algebraic number of each generator included in W_g is equal to zero. Such a word W_g is decomposed to a product of at most $3(2g+1)(6g-1)$ commutators ([9] Sect. 3 Corollary 2) and we can set $k = 3(2g+1)(6g-1)$.

We can construct a non-hyperelliptic Lefschetz fibration of genus $g (\geq 3)$ with positive signature (cf. [3] Introduction).

Let $f : M \longrightarrow B$ be a Lefschetz fibration of genus g obtained above and $f' : M' \longrightarrow B'$ a Σ_g-bundle over a closed surface B' with zero signature whose monodromy representation is surjective onto $\mathcal{M}_g$. (Such a Σ_g-bundle is easily constructed in the smooth category.) The desired Lefschetz fibration is the fiber connected sum of $f : M \longrightarrow B$ with $f' : M' \longrightarrow B'$.

Y. Matsumoto remarked about the construction of Example 4.15 as follows.

Remark 4.16. (See [17] Sect. 5 Remark(1) for genus 1 case)

A similar construction to Example 4.15 in genus 2 case yields a Lefschetz fibration $f : M \longrightarrow B$ of genus 2 of type $5II_1$ over a closed surface B. The total monodromy of all singular fibers of f is equal to $(y_1u_1)^{30}(= (q_1)^5)$ and the signature $Sign(M)$ of the total space M is equal to -1. This Lefschetz fibration is not isomorphic to any fiber connected sum of a Lefschetz fibration over S^2 with a Σ_g-bundle over B because such a Lefschetz fibration of genus 2 over S^2 does not exist. If such a Lefschetz fibration over S^2 exists, the total monodromy $(q_1)^5$ of it must be equal to 1. But this contradicts a result of Mess [20].

5. Another local signature

In this section we work in the complex category.

For (holomorphic) hyperelliptic fibrations, another local signature can be defined with a quite different method from ours.

The next theorem is known.

Theorem 5.1 (Horikawa [12], Persson [27] and Arakawa-Ashikaga [2]). *Let $f : S \longrightarrow C$ be a relatively minimal (holomorphic) hyperelliptic fibration of genus g over a compact Riemann surface C of genus π and $F_1, \dots, F_n$ all the singular fibers of this fibration. There exists a non-negative rational number $Ind(F_i) \in \frac{1}{g}\mathbb{Z}$ called the Horikawa index of the singular fiber $F_i (i = 1, \dots, n)$ such that the next equality holds:*

$$K_S^2 = \frac{4(g-1)}{g}\{\chi + (g+1)(\pi - 1)\} + \sum_{i=1}^{n} Ind(F_i),$$

where K_S is the canonical bundle and χ is the holomorphic Euler-Poincaré characteristic of the compact complex surface S.

Arakawa and Ashikaga [2] (and the author) noticed the following proposition.

Proposition 5.2 (Arakawa-Ashikaga [2]). *If $f : S \longrightarrow C$ is a relatively minimal (holomorphic) hyperelliptic fibration of genus g over a compact Riemann surface B of genus π and $F_1, \dots, F_n$ all the singular fibers of it, then*

$$Sign(S) = \sum_{i=1}^{n} \hat{\sigma}_g(F_i),$$

where

$$\hat{\sigma}_g(F_i) = \frac{gInd(F_i) - (g+1)\epsilon(F_i)}{2g+1} \in \frac{1}{2g+1}\mathbb{Z}$$

is called the local signature *of the singular fiber* $F_i (i = 1, \dots, n)$ *and* $\epsilon(F_i)$ *is the Euler contribution of* F_i.

Proof. The statement follows from Theorem 5.1 and relations among invariants of S, namely $K_S^2 = c_1^2$, $12\chi = c_1^2 + c_2$, $c_1^2 = 2e(S) + 3Sign(S)$ and $c_2 = e(S)$. □

Let us denote our local signature σ_g by σ_g^{top} and Arakawa-Ashikaga's local signature $\hat{\sigma}_g$ by σ_g^{hol}. We call σ_g^{top} (resp. σ_g^{hol}) the *topological* (resp. *holomorphic*) *local signature*.

When the author spoke about Proposition 5.2 above and Example 5.4 below at Hokkaido University in September 1998, T. Terasoma told him the next theorem.

Theorem 5.3 (Terasoma). *Let F be a singular fiber of a relatively minimal (holomorphic) hyperelliptic fibration $f : S \longrightarrow C$ of genus g over a Riemann surface C. Then the topological local signature $\sigma_g^{top}(F)$ of F is always equal to the holomorphic local signature $\sigma_g^{hol}(F)$ of F: namely*

$$\sigma_g^{top} = \sigma_g^{hol}.$$

The proof of Theorem 5.3 due to Terasoma is given in Appendix below. We observe the coincidence of these two kinds of local signature σ_g^{top} and σ_g^{hol} in several cases.

Example 5.4. (The local signature for ‘atomic fibers’)

Theorem 5.3 can be easily checked when $g \leq 4$. Any singular fiber of relatively minimal (holomorphic) hyperelliptic fibrations of genus 1 (resp. 2, 3 and 4) splits into singular fibers of 1 type (resp. 2 types, 5 types and 5 types), which are called ‘atomic fibers’, under the conservation of local signatures σ_g^{top} and σ_g^{hol} ([1][2]) and it is easily seen that $\sigma_g^{top}(F) = \sigma_g^{hol}(F)$ if F is an ‘atomic fiber’ (see [2] Example 4.8).

(1) $g = 1$: $\sigma_1(\text{type } 0_0) = -2/3$;
(2) $g = 2$: $\sigma_2(\text{type } 0_0) = -3/5$, $\sigma_2(\text{class II-(i)}) = -1/5$;
(3) $g = 3$: $\sigma_3(\text{type } 0_0) = -4/7$, $\sigma_3(\text{class I } (g' = 1)) = -6/7$,
$\sigma_3(\text{class II-(i)}) = 1/7$, $\sigma_3(\text{class II-(ii)}) = -6/7$, $\sigma_3(\text{class II-(iii)}) = 0$;
(4) $g = 4$: $\sigma_4(\text{type } 0_0) = -5/9$, $\sigma_4(\text{class I } (g' = 1)) = -2/3$,
$\sigma_4(\text{class II-(i)}) = 1/3$, $\sigma_4(\text{class II-(ii)}) = 7/9$, $\sigma_4(\text{class II-(iii)}) = -1$,

where type 0_0, class I and class II are classes of singular fibers defined in [2]. A singular fiber of type 0_0 corresponds to a Lefschetz singular fiber of type I: type 0_0=Lefschetz I. It can be observed that $\sigma_g^{top}(\text{Lefschetz } I) = \sigma_g^{hol}(\text{type } 0_0) =$

$-(g+1)/(2g+1)$ for every $g(\geq 1)$ (see Theorem 4.8 and [2] Example 4.8). If $g \leq 4$, all Lefschetz singular fibers of type $II_h(h = 1, \ldots, [\frac{g}{2}])$ are included in class II: class II-(i)=Lefschetz II_1 $(g = 2, 3, 4)$; class II-(ii)=Lefschetz II_2 $(g = 4)$.

Because the coincidence of two local signatures σ_g^{top} and σ_g^{hol} (Theorem 5.3) is proven by using the uniqueness of σ_g^{top} and that of σ_g^{hol}, conceptual reason of this coincidence is still not clear. If one wants to clarify more of the relation between these two local signatures, it might be important to study the correspondence of singular fibers to their monodromies (cf. [19]).

Appendix. Proof of Theorem 5.3 (by T. Terasoma)

Let $\Delta = \{z \in \mathbf{C} \mid\mid z \mid< \epsilon\}$ and $f : \mathcal{C} \to \Delta$ be a family of hyperelliptic curves over Δ. Let σ^{top} and σ^{hol} be the topological and holomorphic invariant. Let $\eta \in \mathbf{C}$ such that $0 <\mid \eta \mid< \epsilon$. If the geomeric monodromy of $\mathcal{C}$ on $\pi_1(f^{-1}(\eta))$ is equal to the Dehn twist by a simple closed curve which does not separate the surface $f^{-1}(\eta)$ into two components, then we have $\sigma^{top} = \sigma^{hol}$ (see Example 5.4).

Definition. Such a family $\mathcal{C} \to \Delta$ is called *irreducible Lefschetz type*.

Theorem. *For any family of hyperelliptic cirve $\mathcal{C} \to \Delta$, we have $\sigma^{top} = \sigma^{hol}$.*

Proof. We denote $\mathcal{C}_n$ by the fiber product of $\mathcal{C}$ and $Spec(\mathbf{C}[t]/t^{n+1})$ over Δ. First we note that there exists a sufficiently large n such that for any $\mathcal{C}' \to \Delta$ such that $\mathcal{C}'_n \simeq \mathcal{C}_n$ over $Spec(\mathbf{C}[t]/t^{n+1})$, then (1)$\sigma(\mathcal{C}')^{top} = \sigma(\mathcal{C})^{top}$ and (2)$\sigma(\mathcal{C}')^{hol} = \sigma(\mathcal{C})^{hol}$. For the statement (1), the conjugacy class of the geometric monodromy representation depends only on the data for a stable reduction and the action of the Galois group for the covering of Δ corresponding to the stabilization on the stable graph and skrew numbers defined in [19]. This impiles the existence of a sufficiently large number n with the property (1). For (2), one can prove this statement using formal function theorem (cf. [11] III,Sect. 11) and the definiton of $Ind(\mathcal{C})$.

Lemma. *We can construct a relatively minimal hyperelliptic fibration $\phi : S \to \mathbf{P}^1$ such that*

(1) $S \times_{\mathbf{P}^1} Spec(\mathbf{C}[t]/t^{n+1})$ *is isomorphic to* $\mathcal{C}_n$, *where* $Spec(\mathbf{C}[t]/t^{n+1}) \to \mathbf{P}^1$ *is the* n*-th infinitesimal neibourhood of* $0 \in \mathbf{P}^1$.
(2) *The restriciton* $S \mid_{f^{-1}(\mathbf{P}^1 - \{0\})} \to \mathbf{P}^1 - \{0\}$ *of* ϕ *has at most singularities of irreducible Lefschetz type.*

Proof. Since $\mathcal{C}$ is a relatively minimal hyperellipic fibration, we can take $2:1$ covering $\mathcal{C} - f^{-1}(0) \to \mathbf{P}^1 \times (\Delta - \{0\})$ branching at $(2g+2)$-multisection Σ^0. The closure of Σ^0 in $\mathbf{P}^1 \times \Delta$ is denoted by Σ. Then $\mathcal{C}$ is bimeromorphically equivalent to the double covering branching at Σ or $\Sigma \cup (\mathbf{P}^1 \times \{0\})$. Let $\tilde{\Sigma}$ be the branching divisor of $\mathbf{P}^1 \times \Delta$. Then for any n there exists a divisor $\tilde{\Sigma}_{gl}$ in $\mathbf{P}^1 \times \mathbf{P}^1$ such that the image of $\tilde{\Sigma}_{gl}$ in $\mathbf{P}^1 \times_{\mathbf{P}^1} Spec(\mathbf{C}[t]/t^{n+1})$ corresponds that of $\tilde{\Sigma}$ under the isomorphism between $\mathbf{P}^1 \times_{\mathbf{P}^1} Spec(\mathbf{C}[t]/t^{n+1})$ and $\mathbf{P}^1 \times_{\Delta} Spec(\mathbf{C}[t]/t^{n+1})$.

Next we claim that there exists a divisor $\tilde{\Sigma}_{gl,Lef} \subset \mathbf{P}^1 \times \mathbf{P}^1$ such that

(1) $\tilde{\Sigma}_{gl,Lef} \mid_{\mathbf{P}^1 \times Spec(\mathbf{C}[t]/t^{n+1})} = \tilde{\Sigma}_{gl} \mid_{\mathbf{P}^1 \times Spec(\mathbf{C}[t]/t^{n+1})}$.
(2) $\tilde{\Sigma}_{gl,Lef}$ has at most nodes and each nodes does not lay on the same fiber for $\mathbf{P}^1 \times (\mathbf{P}^1 - \{0\}) \to \mathbf{P}^1 - \{0\}$.
(3) The divisor class of $\tilde{\Sigma}_{gl,Lef}$ is divisible by 2.

The type of $\tilde{\Sigma}_{gl}$ is denoted by $(2g+2, m)$. Let m' be an integer such that $m + m' > n + 1$ and $m + m'$ is divisible by 2. Let F be the defining equation of $\tilde{\Sigma}_{gl,Lef} + m'(\mathbf{P}^1 \times \{\infty\})$. Fix a section $t \in \mathcal{O}_{\mathbf{P}^1}(1)$ such that $t(0) = 0$ and G be a generic member of $\mathcal{O}(2g+2+m+m'-n-1)$. Then for a generic k, the zero locus of $F + k \cdot pr_2^*(t)^{n+1} G$ satisfies the required properties. Note that the condition (2) is open condition for k which is satisfied for $k = \infty$.

To show the existence of S in Lemma, we take S as the relatively minimal model of the double covering of $\mathbf{P}^1 \times \mathbf{P}^1$ branching at $\tilde{\Sigma}_{gl,Lef}$. □

We can finish the proof of Theorem by using Theorem 4.4 and Proposition 5.2. Actually,

$$\begin{aligned} \operatorname{sign}(S) =& \sigma^{top}(\mathcal{C} \mid_{\Delta_0}) + \sum_{p \neq 0, \mathcal{C} \to \mathbf{P}^1 \text{is singular at } p} \sigma^{top}(\mathcal{C} \mid_{\Delta_p}) \\ =& \sigma^{hol}(\mathcal{C} \mid_{\Delta_0}) + \sum_{p \neq 0, \mathcal{C} \to \mathbf{P}^1 \text{is singular at } p} \sigma^{hol}(\mathcal{C} \mid_{\Delta_p}), \end{aligned}$$

where Δ_p is a sufficiently small neigbourhood of p in $\mathbf{P}^1$. Since $\sigma^{top}(\mathcal{C} \mid_{\Delta_p}) = \sigma^{hol}(\mathcal{C} \mid_{\Delta_p})$ for $p \neq 0$, we have $\sigma^{top}(\mathcal{C} \mid_{\Delta_0}) = \sigma^{hol}(\mathcal{C} \mid_{\Delta_0})$. □

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