

From Quaternions to Cosmology: Spaces of Constant Curvature, ca. 1873–1925

Moritz Epple*

Abstract

After mathematicians and physicists had learned that the structure of physical space was not necessarily Euclidean, it became conceivable that the global topological structure of space was non-trivial. In the context of the late 19th century debates on physical space this speculation gave rise to the problem of classifying spaces of constant curvature from a topological point of view. William Kingdon Clifford, Felix Klein and Wilhelm Killing, the latter of whom devoted a substantial amount of work to the topic in the early 1890s, clearly perceived this problem as relevant for both mathematics and natural philosophy (i.e., physics or cosmology). To some extent, a cosmological interest may even be found among those authors who restated the space form problem in more modern terms in the early 20th century, such as Heinz Hopf.

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1. Scientific contexts of topology

The broader aim of the present paper is to contribute to a better understanding of the emergence of modern topology. From its very beginnings, *analysis situs* or *Topologie*, as Johann Benedikt Listing proposed to call the new field in the 1840s, was perceived as one of the most basic subfields of mathematics. Conceptually independent of many other branches of mathematics, it deserved thorough research in its own right. During the 20th century, this perception became even more pronounced with the gradual growth of structural thinking in mathematics. As is well known, topology — axiomatized in set-theoretical terms following the lead of Felix Hausdorff — became one of three “mother structures” in Bourbaki’s architecture of mathematics, making topology into a paradigm field of pure mathematics. Only

*Universität Stuttgart, Germany. E-mail: epple@math.uni-bonn.de

in recent decades have the immediate connections of topology with science, and physics in particular, been emphasized in many lines of research.

The purist view of topology also has dominated historical research on the emergence of topology for a long time. However, historians have begun to move beyond a history of topology focusing exclusively or at least predominantly on conceptual developments within pure mathematics. For instance, the importance of Poincaré's interest in celestial mechanics for the development of his qualitative theory of differential equations and of a number of crucial topological ideas (such as the notion of homoclinical points or his "last geometric theorem") has been underlined in several historical studies (see, e.g., [1]). Another area where the interaction of topological research and physics has been investigated thoroughly is the emergence of the theory of Lie groups [2]. Finally, the relations between 19th century studies of vortex motion in ideal fluids by Helmholtz and Thomson, the latter's influential theory of vortex atoms, and the early attempts at a classification of knots and links by Tait and his followers have been studied in detail [3], [4].

This and similar research has made it clear that the gradual formation of topology in the latter half of the 19th century and the first decades of the 20th century involved more than just pure mathematics. In addition to the growing need for topological notions in fields such as (algebraic) function theory or differential geometry, a need for topology was clearly felt in several domains of physics (and maybe even in chemistry). In the following, another major case will be discussed in which mathematical and physical thinking jointly contributed to the emergence of new topological ideas.¹

2. The topological space problem

In a well-known series of events ranging from the first mathematical discussions of non-Euclidean geometries to heated public debates in the late 19th century, mathematicians and physicists learned that the most adequate mathematical description of physical space was not necessarily Euclidean. This insight had a wide range of consequences both for the body and for the image of geometric, and indeed mathematical, knowledge (to use a distinction proposed by Yehuda Elkana). One of these consequences was to challenge not only the metric properties of Euclidean space (as a model of physical space) but to question its other properties as well. If there was no *a priori* reason for accepting the axiom of parallels, why should there be *a priori* reasons for accepting, e.g., the topological features of Euclidean space? What were the topological types of the best mathematical descriptions of physical space?

To phrase such a question in modern terms and within our understanding of the relations between geometry and topology sounds anachronistic. Nevertheless, a corresponding problem *was* raised in the terms available to 19th century scientists. Before the establishment of a coherent framework of topological notions, such terms were, in particular, the dimension, the "Zusammenhang" or connectivity, and the continuity of space. While it is well known that the issue of the dimension of space

¹ A more detailed treatment of this episode, including full references, will appear in [5].

was in the focus of several 19th century debates, it has less often been emphasized that the properties of connectivity and continuity of space came into question as well. Here I will concentrate on the problem of the connectivity of space.

The notion of “Zusammenhang” was originally introduced by Bernhard Riemann as a tool for distinguishing different types of (Riemann) surfaces in the context of function theory. This notion does not figure prominently in his famous talk *Über die Hypothesen, welche der Geometrie zu Grunde liegen*. There, Riemann introduced the crucial distinction between the “Ausdehnungsverhältnisse” and “Maßverhältnisse” (roughly: topological properties vs. metric properties) of a manifold, but the global topological aspects of manifolds received no special emphasis. This holds in particular for the final sections of his talk which were devoted to the geometry of physical space. While acknowledging that there exists a “discrete manifold” of possible “Ausdehnungsverhältnisse” of space, Riemann expressed scepticism about pursuing the global properties of space beyond the issue of dimension: “Questions about the immeasurably large are idle questions for the explanation of nature.”

When Riemann’s talk reached the scientific public in 1868, another contribution that shaped the later debates on the space problem was on its way. The physicist Hermann v. Helmholtz argued that for epistemological reasons, a crucial assumption in any mathematical description of physical space should be the “free mobility of rigid bodies” of arbitrary size. According to Helmholtz, the existence of freely movable rigid bodies was a precondition for measuring lengths. In mathematical terms, it implied that the classical non-Euclidean geometries were the only possible models of space. Although Helmholtz’s argument was soon criticised for technical reasons, his main assumption (not easily stated in precise mathematical terms) was accepted during the 19th century even by many proponents of liberal approaches to the geometry of physical space. With one exception and one crucial modification, this holds for all authors that will be treated below.

The exception is William Kingdon Clifford, the most imaginative follower of Riemann’s geometric speculations in Britain. His remarks on a “space theory of matter”, according to which all material phenomena might be explained by a time-dependent, wave-like variation of space curvature, are well known. In addition, several of Clifford’s writings show a marked interest in different *global* possibilities for manifolds or spaces. In [6], Clifford hinted at a large variety of “algebraic spaces”, higher dimensional analogues of Riemann’s surfaces. In the same paper, he presented his example of a closed surface embedded in elliptical 3-space, the inherited geometry of which is locally Euclidean. As this example came to play an essential role in the following, let me recall the main line of Clifford’s construction.

Identifying points in elliptic space with one-dimensional subspaces of the quaternions, any given quaternion different from zero induces two isometries of elliptic space by left and right multiplication. Such isometries Clifford called left and right “twists”, respectively. (Felix Klein would later term them “Schiebungen”, translations.) Every twist possessed a space-filling family of invariant lines, i.e. it moved points along these invariant lines by a constant distance. Any two members of one and the same such family were called “parallels” by Clifford. Next, given

any two intersecting lines l and l' , Clifford considered the ruled surface generated by all those Clifford (left) parallels to l which met l' . Equivalently, this surface could be described as being generated by all (right) Clifford parallels to l' meeting l . Moreover, there were two commuting one-parameter families of left and right twists inducing isometries of the surface, which had l and l' as invariant lines, respectively. Consequently, the surface had constant curvature zero. In topological terms, the surface was a torus as may be seen from Clifford's description of it as "a finite parallelogram whose opposite sides [given by the lines l and l'] are regarded as identical" [6, p. 193]. Closer inspection shows that the surface is indeed orientable. It is important to keep in mind that Clifford's example was not constructed by endowing the 2-torus with a geometrical structure, but rather as a particular surface embedded in elliptic 3-space arising from the consideration of a particular set of isometries, Clifford's twists or translations.

Several remarks in Clifford's philosophical articles indicate that he was aware of the implications this example had for the problem of giving an adequate mathematical description of physical space: The same local geometry might be tied to spaces that are globally different. Even for spaces of constant curvature one could make different "assumptions [...] about the *Zusammenhang* of space", as he wrote in 1873 [7, p. 387]. Clifford also saw that these differences were of a topological nature. A remark of 1875 may even be read as advocating a more radical kind of 'topologism': "There are many lines of mathematical thought which indicate that distance or quantity may come to be expressed in terms of *position* in the wide sense of the *analysis situs*. And the theory of space-curvature hints at a possibility of describing matter and motion in terms of extension only." [7, p. 289.]

In the 1870s, Clifford's critical remarks about the possibilities of globally different spaces with the same local geometry seem not to have generated resonances within the scientific communities either of physicists or of mathematicians. This changed during the 1880s for reasons that originally had nothing to do with Clifford's ideas. In 1877, the American astronomer Simon Newcomb published a paper on a geometry of space with constant positive curvature (in Kleinian terms: elliptic geometry). In a reaction to this paper, Wilhelm Killing, a student of Weierstrass and mathematics teacher, argued that Newcomb had overlooked the fact that there were actually *two* possible geometries with constant positive curvature that should be discussed: elliptic and spherical space. This prompted Felix Klein, whose earlier contributions on non-Euclidean geometry also had focused on elliptic rather than spherical geometry, to enter into a correspondence with Killing.² While Klein pointed out that Killing's remark was fairly obvious from the perspective that Klein had developed, Killing repeatedly emphasized the importance of a theorem (along Helmholtz's line of argument) specifying the full range of geometric spaces compatible with the idea of the free mobility of rigid bodies. According to Killing, there were exactly four such spaces: 3-dimensional Euclidean, hyperbolic, elliptic, and spherical space. Killing was clearly interested in what might be called the foundations of physical geometry as opposed to the framework of projective geometry that

²Killing's letters to Klein may be found in the Niedersächsische Staats- und Universitätsbibliothek (NSUB) Göttingen, Handschriftenabteilung, Cod. MS Klein 10.

guided Klein. One of Klein's reactions now was to refer to Clifford's flat surface in elliptic space. In his eyes, this example showed that there were many more manifolds satisfying the assumptions that Killing wanted to hold. Killing protested: Clifford's surface did not admit free mobility in the full sense (it did not allow global rotations) and thus was not a "space form satisfying our experience" (Killing to Klein, cf. note 2, 5 October 1880).

It took Killing and Klein several years to sort out their differences. In the end it became clear (not least because of Sophus Lie's additional work on Helmholtz's approach) that the conditions of constant curvature and free mobility in the Helmholtzian sense had to be distinguished. The former was a local, the latter both a local *and* a global property of space. However, Klein pointed out that there was no clear empirical sense which could be given to this latter property — contrary to both Helmholtz's and Killing's intentions. What *might* make sense as an empirical requirement was the free mobility of bodies of finite size, indeed of globally bounded finite size. Of course this restricted condition of free mobility still implied a constant curvature of space. Hence Klein felt justified in posing the following problem, first in a lecture course on non-Euclidean geometry in 1889/1890, then in print: "to enumerate all species of connectivity which may at all occur in closed manifolds of some constant measure of curvature" [8, p. 554]. Obviously, Klein was interested in the global topological differences of such manifolds, not in a finer classification up to isometry. In his paper, he gave a (not quite complete) discussion of the two-dimensional case, emphasizing again Clifford's work. Then he pointed out the general connection between regular tessellations of the standard non-Euclidean spaces of dimension 3 and manifolds of constant curvature. From his own and from Poincaré's work on automorphic functions he knew that this connection lead to quite involved problems. The corresponding section of his paper included an invitation "that the question would be taken up elsewhere". He underlined that the problem was "fundamental for the doctrine of space, inasmuch as we want to start the latter from the condition of free mobility of rigid bodies" [8, p. 564].

Note that Klein here referred to the *restricted* condition of free mobility. By now, Killing accepted Klein's argument that only this latter version of the condition had empirical content, and in the following years he took up the task that Klein had set. One may group his work on what he now called the problem of "Clifford-Klein space forms" (in the following: CK space forms) under three headings: a reformulation of the problem in group-theoretical terms, the construction of new classes of examples, and a discussion of the scientific relevance of spaces of constant curvature. I will return to the two more mathematical aspects in the next section. Here I want to comment on the third.

In both of his relevant publications, Killing included long sections defending a study of CK space forms in the context of the foundations of physical science [9], [10, part 4]. Repeating Klein's argument, Killing advocated an understanding of free mobility in the restricted sense and emphasized that nothing in experience excluded the possibility of space being different from the standard non-Euclidean spaces. In fact, he considered only one possible criticism as requiring a more careful discussion: As yet, neither mechanics nor any other physical theory existed for

CK space forms. But this was just the usual course of science. For the standard non-Euclidean spaces as well, mechanics was just in the process of being developed (Killing himself had made important contributions). In consequence, the primary task was to develop physical theories for CK space forms as well. Only then it would be possible to judge their scientific merits. As a particular phenomenon that mechanics in multiply connected CK space forms might bring up, Killing mentioned anisotropies of the gravitational force between two bodies [10, p. 347]. One may well read this as a hint at a possible local empirical phenomenon that might help in finding out global ‘connectivity properties’ of physical space.

Killing was not alone in the 1890’s in discussing the physical relevance of spaces of constant curvature. In 1899, Klein came into contact with the young and aspiring astronomer Karl Schwarzschild when the latter gave a talk at a large meeting of astronomers discussing “the admissible measure of curvature of space”. In this talk, Schwarzschild gave bounds on the radii of curvature of either an elliptic or a hyperbolic universe consistent with astronomical observations of star parallaxes. After the talk and in ensuing correspondence³, Klein made Schwarzschild aware of the fact that in such a discussion, CK space forms should also be taken into account. Schwarzschild agreed. In the printed version of his talk, he added an appendix in which he briefly discussed whether or not space might actually be a non-standard space of constant curvature. In very intuitive terms, he explained to his readers (the paper was published in an astronomical journal) how one could conceive of astronomical observations suggesting such kinds of spaces: by observing “identical, apparent repetitions of the same world-whole, be it in a Euclidean, elliptic, or hyperbolic space”. However, the time was not yet ripe for a full discussion of this possibility: “We may treat the other Clifford-Klein space forms very briefly, the more so since they have not yet been investigated completely even from a mathematical point of view. [...] experience only imposes, in all cases, the condition that their volume has to be larger than that of the visible star system.” [11, appendix.]

Killing’s mathematical work on CK space forms was reformulated in modern mathematical terms and substantially extended by Heinz Hopf who devoted one of two parts of his dissertation to the problem in 1925. Again I defer a discussion of the mathematical parts of Hopf’s work to the next section. However, it must be pointed out that Hopf also shared a cosmological interest in CK space forms with Killing and Klein. When, in 1928, Klein’s lectures on non-Euclidean geometry were edited posthumously in a completely rewritten form by Walter Rosemann, Hopf took over the task of writing a new section on spaces of constant curvature (at the time called “homogenous spaces” by him). He closed this section with discussing “the application of geometry to the external world”. Here, only “the possibility of homogenous space forms [had] to be taken into consideration”, as no empirical data were known that would force one to consider spaces of variable curvature. Of particular value was the “possibility of ascribing to the universe a finite volume, independently of its geometrical structure [...] since the idea of an infinite extent [...] causes various difficulties, for instance in the problem of the distribution of mass.” [12, p. 270.] One should note that this was written after the advent of Einstein’s

³See Schwarzschild’s letters in NSUB Göttingen, Cod. MS Teubner 44 and Cod. MS Klein 11.

theory of general relativity, and after the development of relativistic cosmology had seriously begun with contributions by Einstein, Schwarzschild, de Sitter, Weyl and others. In this context, constant curvature was no longer a pre-condition of measurement, but rather a consequence of the assumption of a homogenous average distribution of mass throughout the universe.

3. Killing's and Hopf's mathematical contributions

Killing's main mathematical contribution to the problem of CK space forms was its reduction to group theoretical terms. Killing tried to show that in all dimensions n , Klein's problem (see above) was equivalent to finding all finitely or at most countably generated subgroups G of $SL(\mathbb{R}, n+1)$ which for some real parameter $1/k^2$ leave invariant the bilinear form

$$a(x, y) = k^2 x_0 y_0 + x_1 y_1 + \dots + x_n y_n, \quad x, y \in \mathbb{R}^{n+1},$$

and satisfy a discontinuity condition that will become clear as we go along [10, p. 322]. Even if it is difficult to follow all details of Killing's argument, its main line is clear. (In the following, modern abbreviations are used to condense Killing's verbal style.) To begin with, if M was a manifold of dimension n with constant curvature $1/k^2$, Killing required the existence of some $r > 0$ such that for all points $P \in M$ there existed a ball $B_r(P) \subset M$ of radius r isometric with a similar ball in Euclidean, hyperbolic or spherical space of the same curvature. This was Killing's way of stating the restricted condition of free mobility. It implied both a kind of completeness of the manifold and the discontinuity condition just mentioned. Using a technique he had learned in a seminar of Weierstrass, Killing translated this into local "coordinates", by which he understood isometric mappings

$$B_r(P) \longrightarrow X_k := \{ x \in \mathbb{R}^{n+1} \mid a(x, x) = k^2 \}$$

mapping P to $\bar{P} := (1, 0, \dots, 0)$. (For negative curvature, the condition $x_0 > 0$ was added in the definition of X_k ; in the flat case, Killing just considered the hyperplane in \mathbb{R}^{n+1} defined by $x_0 = 1$.) Endowing X_k with the metric d given by

$$k^2 \cos \frac{d(x, y)}{k} = a(x, y), \quad x, y \in X_k,$$

made X_k into a model of the standard Euclidean and non-Euclidean spaces that had been used by Killing in most of his earlier work on these geometries.

Choosing a particular point P as the origin, these "Weierstrassian coordinates" defined a 1-1 correspondence of the bundles of geodesics through P and \bar{P} . Using this intuition, Killing extended a local coordinate system around P to a kind of global coordinate system, i.e., a 'mapping' $M \rightarrow X_k$, associating a point $Q \in M$ on some geodesic through P with a point $\bar{Q} \in X_k$ on the corresponding geodesic such that the distances between P and Q and between \bar{P} and \bar{Q} were equal. Partly without further argument and partly based on intuitive explanations, Killing assumed that this 'mapping' was in general multi-valued (since in M there might

exist closed geodesics), surjective, and locally isometric. If $\bar{Q}_1, \bar{Q}_2 \in X_k$ were two “coordinates” of the same $Q \in M$, then by construction there existed a (local) isometry $K_r(\bar{Q}_1) \rightarrow K_r(\bar{Q}_2)$. Killing assumed that this mapping could be uniquely extended to a global isometry $\psi : X_k \rightarrow X_k$. He knew that isometries of X_k were induced by elements of the group we denote by $SL(\mathbb{R}, n+1)$, leaving the bilinear form a invariant. Again on intuitive grounds Killing argued that any such ψ was in fact what we would call a covering transformation of M . The collection of all ψ arising in this way formed a discrete subgroup Γ of the isometry group of X_k which had the property that every $\psi \in \Gamma$ moved points by a distance of at least r . M itself was then equivalent to what later was called the quotient space X_k/Γ .

The gaps and intuitive turns in Killing’s argument give a striking illustration of the growing need for precise topological arguments in some areas of mathematics at this time. Notions relating to covering spaces or the fundamental group (in his intuitive explanations, Killing repeatedly relied on the consideration of “motions of bodies” along closed geodesics in M) would have helped Killing significantly in securing the vaguer parts of his considerations.

Killing was quite clear that the new problem in group theory was difficult. Accordingly, he was satisfied with describing a few simple Euclidean and spherical space forms. In the flat case, his main example was the analogue of Clifford’s surface, the manifold given by $\mathbb{R}^3/\mathbb{Z}^3$. In the case of positive curvature, Killing noticed that in even dimensions, only S^n and $\mathbb{R}P^n$ with their canonical metrics could occur. In dimension 3 he mentioned other possibilities, e.g. $\mathbb{R}P^3/\Gamma$, where Γ is a cyclic group of Clifford’s translations.

It was Heinz Hopf who reworked Killing’s arguments in a modern framework. In his dissertation of 1925, he presented a completely revised treatment of the problem of CK space forms that made it superfluous to look into the older literature any more [13]. His version of the problem was to classify all geodetically complete Riemannian manifolds of constant curvature in either of two possible senses: One could try to classify the resulting “geometries” (i.e., look for a classification up to isometry) or one might wish to classify just the manifolds carrying these geometries (i.e., look for a classification up to diffeomorphism). In Killing’s work, this distinction had never been clearly made.

After a preliminary clarification of the relation between Killing’s earlier completeness condition and the weaker condition of geodesic completeness, Hopf gave a new proof of Killing’s basic result. In the new setting, this theorem took the form that every geodetically complete Riemannian manifold of constant curvature (again called CK space form by Hopf) was a quotient of Euclidean, hyperbolic or spherical space by a discontinuous group Γ of isometries without fixed points and such that no orbit of Γ had a limit point. The geometric content of Hopf’s proof was very similar to Killing’s argument – the difference being that Hopf had conceptual tools at his disposal that Killing had missed. Hopf showed that if M was a CK space form in his sense, then every point $P \in M$ still had a neighbourhood that could be mapped isometrically onto a neighbourhood of some point \bar{P} in one of the standard spaces, say X . Using again the resulting 1-1 correspondence of the bundles of geodesics through P and \bar{P} , Hopf defined a mapping $X \rightarrow M$ (!) of

which he showed that it was an isometric covering. As X was simply connected, it was the universal covering space of M . Moreover, the fundamental group $\pi_1(M)$ acted freely and discontinuously in the sense explained above by isometries on X . Therefore, M was isometric to a quotient manifold of the required form.

Instead of multi-valued “coordinates”, Hopf could speak of coverings, inverting the direction of the crucial mapping. Relying on the notions of universal covering spaces, covering transformations, local isometries etc., Hopf was able to formulate several steps in his proof as simple arguments by contradiction. For this step into mathematical modernity, Hopf had an important model: Hermann Weyl. The conceptual framework used by Hopf was mainly an adaptation of that outlined in the topological sections of Weyl’s monograph *Die Idee der Riemannschen Fläche* of 1913. On the other hand, Weyl himself had pointed out that this framework could be used in discussing manifolds of constant curvature. Using the language of coverings, Weyl showed in an appendix to [14] that every “closed Euclidean space” (i.e., every closed manifold of constant curvature zero) was isometric to a “crystal”, i.e., a quotient of Euclidean space by a suitable discrete group of isometries. Weyl’s proof only worked in the flat case, and apparently he did not pursue analogous questions for curved manifolds.

Hopf not only used modern topological tools for reformulating the space form problem. He also showed how topology could profit from the geometric ideas involved in this problem. Looked at in this perspective, one point was fairly obvious: CK space forms, constructed as quotient spaces, furnished new examples of manifolds with known fundamental groups. At the time, this was particularly interesting for finite groups; very few examples of manifolds with finite fundamental groups had been known beyond those with cyclic groups. Based on Klein’s analysis of the isometry group of elliptic 3-space (which in turn relied on Clifford’s ideas), Hopf discussed a series of new 3-dimensional, spherical space forms with finite fundamental groups as well as infinitely many spaces with infinite groups.⁴ Moreover, Hopf’s research on spaces of constant positive curvature proved to be of decisive importance at a later point in his career: Clifford’s parallels provided him with the crucial example of a mapping from S^3 to S^2 that was not homotopic to a constant mapping (lifting a fibration of elliptic 3-space by Clifford parallels to the 2-sheeted covering of elliptic 3-space produced what became known as the Hopf fibration of S^3). Even Hopf’s basic invariant for classifying such maps, the linking number of fibres, was derived from an intuitive understanding that the linking behaviour of Clifford’s parallels was an obstacle for deforming the Hopf fibration into a constant [16].

4. Conclusions

In these ways, an important strand in the formation of modern manifold topology profited from geometric ideas that had their origin in a 19th century context in which mathematical and cosmological thinking were closely related. If I may use a topological metaphor: Once more it turns out that the fibres of historical develop-

⁴Spherical, 3-dimensional space forms with polyhedral fundamental groups had already been added to Killing’s examples by the American geometer Woods in 1905.

ments in different scientific fields are intertwined rather than grouped in a locally trivial bundle, the base of which would be some eternal architecture of concepts or structures that only need straightforward elaboration.

From cases like the one I have sketched one might learn that in periods of innovative research, boundaries between different mathematical fields and even between mathematical and physical thinking may tend to blur. In other periods, this may be different. In the present case, it seems (at least at first sight) that the later *solutions* of the Euclidean and the spherical space form problems were found in episodes of autonomous, purely mathematical research. Moreover, the history of the space form problem between Clifford and Hopf reveals a complicated relation between tradition and modernization within mathematics. The analysis of Killing's and Hopf's ways of approaching the space form problem shows that despite the crucial differences due to the non-availability or availability of precise topological notions, Killing's more traditional geometric ideas were taken up in the modern formulations. Another such core of geometric ideas that was handed down to modern topology were Clifford's geometric ideas: in fact *all* later authors discussed here made some use or other of these ideas.

In a period in which the global topological properties of the universe receive new interest among cosmologists, it seems fitting to recall that a century ago, such an interest also motivated some of the topological problems and ideas mathematicians have since become acquainted with. At least in this indirect way, questions about the immeasurably large have *not* been idle questions for the explanation of nature.

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