DE FORMATIONE
FRACTIONVM CONTINVARVM.

Auctore
L. EVLERO.

§. 1.

Principium vniuerſale ad fractiones continuas perducens reperitur in ſerie infinita quantitatum A, B, C etc.; quarum ternae ſibi ſuccedentes ſecundum certam legem, ſiue conſtantem ſiue vtcunque variabilem ita a ſe inuicem pendent, vt ſit

$$f\mathrm{A} = g\mathrm{B} + h\mathrm{C},\ f'\mathrm{B} = g'\mathrm{C} + h'\mathrm{D},\ f''\mathrm{C} = g'' + \mathrm{D} + h''\mathrm{E},$$
$$f'''\mathrm{D} = g'''\mathrm{E} + h'''\mathrm{F} \text{ etc.}$$

Hinc enim deducuntur ſequentes aequalitates:

$$\frac{f\mathrm{A}}{\mathrm{B}} = g + \frac{h\mathrm{C}}{\mathrm{B}} = g + \frac{f'h}{f'\mathrm{B:C}}$$
$$\frac{f'\mathrm{B}}{\mathrm{C}} = g' + \frac{h'\mathrm{D}}{\mathrm{C}} = g' + \frac{f''h'}{f''\mathrm{C:D}}$$
$$\frac{f''\mathrm{C}}{\mathrm{D}} = g'' + \frac{h''\mathrm{E}}{\mathrm{D}} = g'' + \frac{f'''h''}{f'''\mathrm{D:E}}$$
$$\frac{f'''\mathrm{D}}{\mathrm{E}} = g''' + \frac{h'''\mathrm{F}}{\mathrm{E}} = g''' + \frac{f''''h'''}{f'''\mathrm{E:F}}$$

etc. etc.

Quod si iam posteriores valores in prioribus continuo substituantur, sponte emerget sequens fractio continua:

$$\frac{f\,A}{B} = g + \cfrac{f'\,h}{g' + \cfrac{f''\,h'}{g'' + \cfrac{f'''\,h''}{g''' + \cfrac{f''''\,h'''}{g'''' + \text{etc.}}}}}$$

cuius ergo valor per solos duos primos terminos A & B seriei determinatur.

§. 2. Quoties igitur talis progressio quantitatum A, B, C, D, E etc. habetur, cuius lex ita fuerit comparata, vt terni quique eius termini sibi succedentes secundum legem quamcunque a se inuicem pendeant, toties inde deducitur fractio continua, cuius valor assignari potest. Quamobrem si formula quaecunque ita fuerit comparata, vt eius euolutio perducat ad huiusmodi seriem quantitatum A, B, C, D, E, etc. quarum quisque terminus per duos praecedentes determinatur, inde fractiones continuae deriuari poterunt, quod quomodo fiat, commodissime per aliquot exempla ostendi poterit.

I. Euolutio formulae.

$$s = x^n (\alpha - \beta x - \gamma x x).$$

§. 8. In hac formula exponens n indefinitus spectatur, successiue recipiens omnes valores 1, 2, 3, 4, 5, 6 etc., vnde, dummodo fuerit $n > 0$, haec formula euanescit, posito $x = 0$, tum vero etiam euanescit, sumto

$x = -$

$$x = -\frac{\beta \pm \surd(\beta\beta + 4\alpha\gamma)}{2\gamma}.$$

His notatis differentietur ista formula, vt fiat

$$ds = n\alpha x^{n-1} dx - (n+1)\beta x^n dx - (n+2)\gamma x^{n+1} dx,$$

vnde per partes integrando et integrationem tantum indicando fiet

$$n\alpha \int x^{n-1} dx = (n+1)\beta \int x^n dx + (n+2)\gamma \int x^{n+1} dx + s,$$

Hinc, si post quamque integrationem, ita peractam, vt integrale euanescat posito $x = 0$, statuatur

$$x = -\frac{\beta \pm \surd(\beta\beta + 4\alpha\gamma)}{2\gamma},$$

quippe quo casu fit $s = 0$, erit

$$n\alpha \int x^{n-1} dx = (n+1)\beta \int x^n dx + (n+2)\gamma \int x^{n+1} dx,$$

quae est eiusmodi relatio inter ternas formulas integrales sibi succedentes, qualem desideramus pro formatione fractionis continuae; quandoquidem hae formulae integrales, si loco n successiue scribantur numeri 1, 2, 3, 4, 5, 6 etc. nobis suppeditant quantitates A, B, C, D etc.

§. 4. Scribamus igitur loco n ordine numeros naturales 1, 2, 3, 4, etc. vt prodeant istae relationes:

$$\alpha \int dx = 2\beta \int x dx + 3\gamma \int x x dx$$
$$2\alpha \int x dx = 3\beta \int x x dx + 4\gamma \int x^3 dx$$
$$3\alpha \int x x dx = 4\beta \int x^3 dx + 3\gamma \int x^4 dx$$
$$4\alpha \int x^3 dx = 5\beta \int x^4 dx + 6\gamma \int x^5 dx$$

etc. etc.

Hinc igitur habebimus

$$A = \int dx = x = -\frac{\beta \pm \surd(\beta\beta + 4\alpha\gamma)}{2\gamma},$$
$$B = \int x dx = \tfrac{1}{2} x x = \tfrac{1}{2}\left(\frac{-\beta \pm \surd(\beta\beta + 4\alpha\gamma)}{2\gamma}\right)^2,$$
$$C = \int x x dx = \tfrac{1}{3} x^3, \quad D = \int x^3 dx = \tfrac{1}{4} x^4$$

etc. etc.

 Tunc

Tunc vero pro literis f, g, h habebuntur isti valores:

$$f = \alpha,\; f' = 2\alpha,\; f'' = 3\alpha,\; f''' = 4\alpha \text{ etc.}$$
$$g = 2\beta,\; g' = 3\beta,\; g'' = 4\beta,\; g''' = 5\beta \text{ etc.}$$
$$h = 3\gamma,\; h' = 4\gamma,\; h'' = 5\gamma,\; h''' = 6\gamma \text{ etc.}$$

ex quibus valoribus resultat sequens fractio continua:

$$\frac{\alpha A}{B} = 2\beta + \cfrac{6\alpha\gamma}{3\beta + \cfrac{12\alpha\gamma}{4\beta + \cfrac{20\alpha\gamma}{5\beta + \cfrac{30\alpha\gamma}{6\beta + \text{etc.}}}}}$$

cuius ergo valor est

$$\frac{4\alpha\gamma}{-\beta + \sqrt{(\beta\beta + 4\alpha\gamma)}} = \beta + \sqrt{(\beta\beta + 4\alpha\gamma)}.$$

§. 5. Quo haec fractio continua concinnior reddatur, loco $\alpha\gamma$ scribamus $\frac{1}{2}\delta$, et prodibit

$$\beta + \sqrt{(\beta\beta + 2\delta)} = 2\beta + \cfrac{3\delta}{3\beta + \cfrac{6\delta}{4\beta + \cfrac{10\delta}{5\beta + \cfrac{15\delta}{6\beta + \text{etc.}}}}}$$

Quoniam autem haec expressio capite truncata videtur, adiecto hoc capite ponamus

$$s = \beta + \cfrac{\delta}{2\beta + \cfrac{3\delta}{3\beta + \cfrac{6\delta}{4\beta + \cfrac{10\delta}{5\beta + \text{etc.}}}}},$$ eritque

$s = \beta$

$$s = \beta + \frac{\delta}{\beta + \sqrt{(\beta\beta + 2\delta)}}$$

quae expressio reducitur ad hanc:

$$s = \tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 2\delta)}.$$

§. 6. Haec autem fractio continua adhuc ad maiorem simplicitatem reduci potest, si loco δ scribamus 2ε, vt sit

$$\tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 4\varepsilon)} = \beta + \cfrac{2\varepsilon}{2\beta + \cfrac{6\varepsilon}{3\beta + \cfrac{12\varepsilon}{4\beta + \cfrac{20\varepsilon}{5\beta + 20 \text{ etc.}}}}}$$

Quod si iam prima fractio deprimatur per 2, secunda per 3, tertia per 4, quarta per 5 etc. prodibit sequens forma:

$$\tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 4\varepsilon)} = \beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \text{etc.}}}}}$$

quae est simplicissima, cuius summa si tanquam incognita spectetur, ac vocetur $= z$, erit vtique $z = \beta + \frac{\varepsilon}{z}$, ideoque $zz = \beta z + \varepsilon$, vnde fit $z = \frac{\beta + \sqrt{(\beta\beta + 4\varepsilon)}}{2}$, quae est eadem.

§. 7. Verum ista summa simplicissima immediate deduci potest ex ipsa formula initio assumta

$$s = x^n(\alpha - \beta x - \gamma x x),$$

quam

quam quoniam nihilo aequalem posuimus, erit vtique $\alpha = \beta x + \gamma x x$, eodemque modo

$$\alpha x = \beta x x + \gamma x^3,\ \alpha x x = \beta x^3 + \gamma x^4, \text{ etc.}$$

ita vt pro serie A, B, C, D, etc. habeamus hanc simplicem seriem potestatum: 1, x, x^2, x^3, x^4 etc., tum vero omnes literae, f, g, h etc. fiunt α, β, γ etc. vnde oritur ista fractio continua:

$$\frac{\alpha}{x} = \beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \text{etc.}}}}$$

vbi est $\frac{1}{x} = \frac{\beta + \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\alpha}$. Huius ergo fractionis valor est $\frac{1}{2}\beta + \frac{1}{2}\sqrt{(\beta\beta + 4\alpha\gamma)}$, vt ante, ob $\alpha\gamma = \varepsilon$.

II. Euolutio formulae.

$$s = x^n (a - x).$$

§. 8. Haec igitur formula euanescit, ponendo $x = a$; hinc autem fit $d s = n a x^{n-1} d x - (n+1) x^n d x$, quae expressio cum duobus tantum constet terminis, reducatur ad fractionem, cuius denominator sit $\alpha + \beta x$, ita vt fiat

$$d s = \frac{n a \alpha x^{n-1} d x + (\beta n a - \alpha(n+1)) x^n d x - \beta(n+1) x^{n+1} d x}{\alpha + \beta x}.$$

His igitur membris seorsim integratis fiet

$$s = n a \alpha \int \frac{x^{n-1} d x}{\alpha + \beta x} + (n \beta a - (n+1)\alpha) \int \frac{x^n d x}{\alpha + \beta x} - \beta(n+1) \int \frac{x^{n+1} d x}{\alpha + \beta x}$$

quae

quare si post singulas integrationes statuamus $x = a$, vt fiat $s = 0$, habebimus hanc reductionem:

$$n a \alpha \int \frac{x^{n-1} d x}{\alpha + \beta x} = ((n+1)\alpha - n\beta a) \int \frac{x^n d x}{\alpha + \beta x} + (n+1)\beta \int \frac{x^{n+1} d x}{\alpha + \beta x}.$$

§. 9. Loco n substituamus nunc successiue numeros 1, 2, 3, 4 etc. atque comparatione cum formulis generalibus instituta habebimus

$$A = \int \frac{d x}{\alpha + \beta x}, \quad B = \int \frac{x d x}{\alpha + \beta x}, \quad C = \int \frac{x x d x}{\alpha + \beta x} \text{ etc.}$$

vbi quidem post integrationem fieri debet $x = a$. Praeterea vero habebimus

$$f = a\alpha, \; f' = 2 a\alpha, \; f'' \, 3 a\alpha, \; f''' \, 4 a\alpha, \text{ etc.}$$

$$g = 2\alpha - \beta a, \; g' = 3\alpha - 2\beta a, \; g'' = 4\alpha - 3\beta a, \text{ etc.}$$

$$h = 2\beta, \; h' = 3\beta, \; h'' = 4\beta, \; h''' = 5\beta, \text{ etc.}$$

atque ex his oritur sequens fractio continua:

$$\frac{\alpha a A}{B} = (2\alpha - \beta a) + \cfrac{4 a\alpha\beta}{(3\alpha - 2\beta a) + \cfrac{9 a\alpha\beta}{(4\alpha - 3\beta a) + \cfrac{16 a\alpha\beta}{(5\alpha - 4\beta a) + \text{etc.}}}}$$

§. 10. Integratione autem instituta fit

$$\int \frac{d x}{\alpha + \beta x} = \frac{1}{\beta} l \frac{\alpha + \beta x}{\alpha},$$

quandoquidem integralia euanescere debent facto $x = 0$. Nunc igitur fiat $x = a$, eritque $A = \frac{1}{\beta} l \frac{\alpha + \beta a}{\alpha}$. Porro

$$\int \frac{x d x}{\alpha + \beta x} = \frac{1}{\beta}\left(x - \frac{\alpha}{\beta} l \frac{\alpha + \beta x}{\alpha}\right),$$ factoque $x = a$ fiet

$$B = \frac{a}{\beta} - \frac{\alpha}{\beta\beta} l \frac{\alpha + \beta a}{\alpha},$$

quamobrem valor noſtrae fractionis continuae erit

$$\frac{\alpha a \beta l \frac{\alpha + \beta a}{\alpha}}{a \beta - \alpha l \frac{\alpha + \beta a}{\alpha}}$$

euidens autem eſt, nihil de vniuerſalitate perire, etiamſi ſumatur $a = 1$; tum enim erit

$$\frac{\alpha \beta l \frac{\alpha + \beta}{\alpha}}{\beta - \alpha l \frac{\alpha + \beta}{\alpha}} = (2\alpha - \beta) + \cfrac{4 \alpha \beta}{(3\alpha - 2\beta) + \cfrac{9 \alpha \beta}{(4\alpha - 3\beta) + \text{etc.}}}$$

§. 11. Tota autem haec expreſſio manifeſto vnice pendet a ratione numerorum α et β; vnde ſumamus $\alpha = 1$ et $\beta = n$, atque orietur haec fractio continua:

$$\frac{n l (1 + n)}{n - l (1 + n)} = (2 - n) + \cfrac{4 n}{(3 - 2n) + \cfrac{9 n}{(4 - 3n) + \cfrac{16 n}{(5 - 4n) + \text{etc.}}}}$$

cui ſi praefigamus ſecundum ordinis legem $1 + n$ et ſummam ſtatuamus $= s$, vt ſit

$$s = 1 + \cfrac{n}{(2 - n) + \cfrac{4 n}{(3 - 2n) + \cfrac{9 n}{(4 - 3n) + \cfrac{16 n}{(5 - 4n) + \text{etc.}}}}}$$

erit

$$s = \frac{1 + n (n - l(1 + n))}{n l (1 + n)} = \frac{1 + n - l(1 + n)}{l(1 + n)} = \frac{n}{l(1 + n)}$$

§. 12.

§. 12. Exempla aliquot percurramus, sitque primo $n = 1$, erit

$$\frac{1}{l\,2} = 1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}$$

Posito autem $n = 2$ erit

$$\frac{2}{l\,3} = 1 + \cfrac{2}{0 + \cfrac{8}{-1 + \cfrac{18}{-2 + \cfrac{32}{-3 + \cfrac{50}{-4 + \text{etc.}}}}}}$$

quae autem expressio, ob quantitates negatiuas, non satis est commoda; quod cum eueniat quando $n > 1$, operae pretium erit eos casus euoluere, quibus n vnitate minor accipitur.

§. 13. Quo hoc facilius fieri possit, reuertamur ad expressionem literas α et β continentes, atque capite, quod deerat suppleto, prodit ista forma:

$$\frac{\beta}{l\,\frac{\alpha+\beta}{\alpha}} = \alpha + \cfrac{\alpha\beta}{(2\alpha-\beta) + \cfrac{4\alpha\beta}{(3\alpha-2\beta) + \cfrac{9\alpha\beta}{(4\alpha-3\beta) + \text{etc.}}}}$$

Ponamus nunc $n = n - m$ et $\beta = 2m$, vt obtineamus se-

 quen-

quentem formam:

$$\frac{2m}{l\frac{n+m}{n-m}} = n-m+\cfrac{2m(n-m)}{2n-4m+\cfrac{8m(n-m)}{3n-7m+\cfrac{18m(n-m)}{4n-10m+\text{etc.}}}}$$

vnde sequentes casus speciales deducuntur.

Si $m=1$ et $n=3$ erit

$$\frac{2}{l\,2} = 2+\cfrac{4}{2+\cfrac{16}{2+\cfrac{36}{2+\cfrac{64}{2+\text{etc.}}}}}$$

quae fractio per 2 diuisa et reducta praebet istam:

$$\frac{1}{l\,2} = 1+\cfrac{1}{1+\cfrac{4}{1+\cfrac{9}{1+\cfrac{16}{1+\text{etc.}}}}}$$

quae iam supra est inuenta.

Sit $m=1$ et $n=4$ erit

$$\frac{2}{l\frac{5}{3}} = 3+\cfrac{6}{4+\cfrac{24}{5+\cfrac{54}{6+\cfrac{96}{7+\text{etc.}}}}}$$

$$= 3+\cfrac{6.1}{4+\cfrac{6.4}{5+\cfrac{6.9}{6+\cfrac{6.16}{7+\text{etc.}}}}}$$

Sit

Sit $m = 1$ et $n = 5$, erit

$$\frac{2}{l\frac{3}{2}} = 4 + \cfrac{8}{6 + \cfrac{32}{8 + \cfrac{72}{10 + \cfrac{128}{12 + \text{etc.}}}}}$$

siue

$$\frac{1}{l\frac{3}{2}} = 2 + \cfrac{2}{3 + \cfrac{8}{4 + \cfrac{18}{5 + \cfrac{32}{6 + \text{etc.}}}}}$$

$$= 2 + \cfrac{2.1}{3 + \cfrac{2.4}{4 + \cfrac{2.9}{5 + \cfrac{2.16}{6 + \text{etc.}}}}}$$

III. Euolutio formulae.

$$s = x^n (1 - x^2)$$

§. 14. Haec ergo formula euanescit casibus $x = 0$ et $x = 1$. Quoniam vero hinc fit

$$ds = n x^{n-1} dx - (n + 2) x^{n+1} dx,$$

reducatur hoc differentiale ad denominatorem $\alpha + \beta x x$, fietque

$$ds = \frac{n \alpha x^{n-1} dx + (n\beta - (n+2)\alpha) x^{n+1} dx - (n+2) \beta x^{n+3} dx}{\alpha + \beta x x}.$$

 Hinc

Hinc iam iterum integrando fit

$$s = n\alpha\int\frac{x^{n-1}dx}{\alpha+\beta xx} + (n\beta-(n+2)\alpha)\int\frac{x^{n+1}dx}{\alpha+\beta xx} - (n+2)\beta\int\frac{x^{n+3}dx}{\alpha+\beta xx}.$$

Quod si iam post integrationes statuatur $x = 1$, prodibit haec integralium reductio:

$$n\alpha\int\frac{x^{n-1}dx}{\alpha+\beta xx} = ((n+2)\alpha - n\beta)\int\frac{x^{n+1}dx}{\alpha+\beta xx} + (n+2)\beta\int\frac{x^{n+3}dx}{\alpha+\beta xx}.$$

§. 15. Quoniam hic potestates ipsius x binario augentur, exponenti n successiue tribuamus valores 1, 3, 5, 7, 9 etc. ac statuatur:

$$A = \int\frac{dx}{\alpha+\beta xx},\quad B = \int\frac{xx\,dx}{\alpha+\beta xx},\quad C = \int\frac{x^4\,dx}{\alpha+\beta xx}\ \text{etc.}$$

Deinde vero literae f, g, h cum suis deriuatis erunt:

$$f = \alpha,\ f' = 3\alpha,\ f'' = 5\alpha,\ f''' = 7\alpha,\ \text{etc.}$$
$$g = 3\alpha-\beta,\ g' = 5\alpha-3\beta,\ g'' = 7\alpha-5\beta,\ \text{etc.}$$
$$h = 3\beta,\ h' = 5\beta,\ h'' = 7\beta,\ h''' = 9\beta,\ \text{etc.}$$

vnde nascitur sequens fractio continua:

$$\frac{\alpha A}{B} = 3\alpha-\beta+\cfrac{9\alpha\beta}{5\alpha-3\beta+\cfrac{25\alpha\beta}{7\alpha-5\beta+\cfrac{49\alpha\beta}{9\alpha-7\beta+\text{etc.}}}}$$

§. 16. Quia est $B = \int\frac{xx\,dx}{\alpha+\beta xx}$, erit

$B = \frac{1}{\beta}\int dx - \frac{\alpha}{\beta}\int\frac{dx}{\alpha+\beta xx}$, ideoque $B = \frac{1}{\beta} - \frac{\alpha}{\beta}A$,

quo valore substituto habebimus

$$\frac{\alpha\beta A}{1-\alpha A} = 3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \text{etc.}}}$$

cui, quia caput deeſt, praefigamus $\alpha + \beta + \alpha\beta$; tum autem erit ſumma $\beta + \frac{1}{A}$, ita vt habeamus

$$\beta + \frac{1}{A} = \alpha + \beta + \cfrac{\alpha\beta}{3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \text{etc.}}}}$$

exiſtente $A = \int \frac{dx}{\alpha + \beta xx}$, integrali ita ſumto, vt euaneſcat poſito $x = 0$, tum vero facto $x = 1$.

§. 17. Euoluamus primo caſum ſimpliciſſimum, quo $\alpha = 1$ et $\beta = 1$, vbi erit $A = \frac{\pi}{4}$, vnde habebimus

$$1 + \frac{4}{\pi} = 2 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{etc.}}}}}$$

ſiue erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \text{etc.}}}}$$

quae eſt ipſa fractio continua olim a *Brounkero* primum producta, cuius inueſtigatio, cum a *Walliſio* per calculos valde taedioſos ſit eruta, hic quaſi ſponte ex noſtra formula ſeſe prodidit.

§. 18.

§. 18. Noſtra autem forma generalis infinitas alias ſimiles expreſſiones ſuppeditat, prouti literae α et β vario modo accipiuntur. Ac primo quidem, ſi α et β fuerint numeri poſitiui, valor literae A ſemper per arcum circularem exprimetur, contra vero per logarithmos. Sit igitur primo $\beta = 1$, eritque

$$A = \int \frac{dx}{\alpha + xx} = \frac{1}{\sqrt{\alpha}} \text{ A tang. } \frac{x}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}} \text{ A tang. } \frac{1}{\sqrt{\alpha}},$$

vnde naſcitur haec fractio continua:

$$1 + \frac{\sqrt{\alpha}}{\text{A tang. } \frac{1}{\sqrt{\alpha}}} = \alpha + 1 + \cfrac{\alpha}{3\alpha - 1 + \cfrac{9\alpha}{5\alpha - 3 + \cfrac{25\alpha}{7\alpha - 5 + \text{etc.}}}}$$

Hinc igitur ſi ſumatur $\alpha = 3$, quia A tang. $\frac{1}{\sqrt{3}} = \frac{\pi}{6}$, habebimus

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3}{8 + \cfrac{27}{12 + \cfrac{75}{16 + \cfrac{147}{20 + \text{etc.}}}}}$$

ſiue

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3 \cdot 1}{8 + \cfrac{3 \cdot 9}{12 + \cfrac{3 \cdot 25}{16 + \cfrac{3 \cdot 49}{20 + \text{etc.}}}}}$$

§. 19.

§. 19. Sit nunc β numerus positiuus quicunque, et quia est

$$A = \int \frac{dx}{\alpha + \beta x x} = \frac{1}{\beta} \int \frac{dx}{\frac{\alpha}{\beta} + x x},$$

integrando fit $A = \frac{1}{\sqrt{\alpha\beta}} \text{ A tang. } \sqrt{\frac{\beta}{\alpha}}$. Hinc igitur habebimus

$$\beta + \frac{\sqrt{\alpha\beta}}{\text{A tang. } \sqrt{\frac{\beta}{\alpha}}} = \alpha + \beta + \cfrac{\alpha\beta}{3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \text{etc.}}}$$

Faciamus igitur $\alpha + \beta = 2n$ et $\alpha - \beta = 2m$, vt fit $\alpha = m + n$ et $\beta = n - m$, quibus valoribus positis erit

$$n - m + \frac{\sqrt{(nn - mm)}}{\text{A tang. } \sqrt{\frac{n-m}{n+m}}} = 2n + \cfrac{nn - mm}{2n + 4m + \cfrac{9(nn - mm)}{2n + 8m + \text{etc.}}}$$

§. 20. Consideremus etiam casum, quo β est numerus negatiuus, et ponendo $\beta = -\gamma$, erit

$$A = \int \frac{dx}{\alpha - \gamma x x} = \frac{1}{\gamma} \int \frac{dx}{\frac{\alpha}{\gamma} - x x},$$

cuius integrale est

$$A = \frac{1}{2\sqrt{\alpha\gamma}}\, l \frac{\sqrt{\frac{\alpha}{\gamma}} + x}{\sqrt{\frac{\alpha}{\gamma}} - x};$$

facto ergo $x = 1$ erit

$$A = \frac{1}{2\sqrt{\alpha\gamma}}\, l \frac{\sqrt{\alpha} + \sqrt{\gamma}}{\sqrt{\alpha} - \sqrt{\gamma}},$$

 vnde

vnde nafcitur ifta fractio continua:

$$-\gamma+\frac{2\sqrt{\alpha\gamma}}{l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}}=\alpha-\gamma-\cfrac{\alpha\gamma}{3\alpha+\gamma-\cfrac{9\alpha\gamma}{5\alpha+3\gamma-\cfrac{25\alpha\gamma}{7\alpha+5\gamma-\text{etc.}}}}$$

hocque modo nacti fumus nouas fractiones continuas, quarum valores etiam per logarithmos exhibere licet, et quae prorfus difcrepant ab illis, quas ante inuenimus.

§. 21. Hic cafus prae reliquis notatu dignus fe offert, quando $\gamma=\alpha$. Siue, quod eodem redit, $\alpha=1$ et $\gamma=1$; quia enim tum eft $l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}=l\infty=\infty$, habebimus

$$-1=0-\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}}$$

fiue mutatis fignis

$$1=\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}}$$

hinc primus denominator

$$4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}\quad\text{debet effe}=1.$$

Erit ergo $0=3-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}$

fiue

ſiue $1 = \cfrac{3}{8 - \cfrac{25}{12 - \text{etc.}}}$

vbi denominator debet eſſe $= 3$, vnde fit

$$0 = 5 - \cfrac{25}{12 - \text{etc.}}$$

cuius denominator debet eſſe $= 5$, vnde fit

$$0 = 7 - \cfrac{49}{16 - \cfrac{81}{20 - \text{etc.}}}$$

ex quo ordine facile veritas perſpicitur.

§. 22. Sumamus $\alpha = 4$ et $\gamma = 1$ et nanciſcemur hanc fractionem:

$$-1 + \frac{4}{l\,3} = 3 - \cfrac{4.1}{13 - \cfrac{4.9}{23 - \cfrac{4.25}{33 - \cfrac{4.49}{43 - \text{etc.}}}}}$$

Sin autem accipiamus $\alpha = 9$ et $\gamma = 1$ erit

$$-1 + \frac{6}{l\,2} = 8 - \cfrac{9.1}{28 - \cfrac{9.9}{48 - \cfrac{9.25}{68 - \cfrac{9.49}{88 - \text{etc.}}}}}$$

IV. Euolutio formulae.

$$s = x^n e^{\alpha x}(1 - x)$$

§. 22. Hic e denotat numerum cuius logarithmus hyperbolicus est vnitas, ita vt $d.\,e^{\alpha x} = \alpha\, dx\, e^{\alpha x}$. Hinc ergo erit

$$ds = n x^{n-1}\, dx\, e^{\alpha x} + (\alpha - (n+1))\, x^n\, dx\, e^{\alpha x} - \alpha x^{n+1}\, dx\, e^{\alpha x},$$

vnde viciſſim integrando fit

$$s = n\int x^{n-1}\, dx\, e^{\alpha x} + (\alpha - (n+1))\int x^n\, dx\, e^{\alpha x} - \alpha\int x^{n+1}\, dx\, e^{\alpha x}.$$

Quod ſi ergo poſt integrationem ſtatuatur $x = 1$, erit

$$n\int x^{n-1}\, dx\, e^{\alpha x} = (n+1-\alpha)\int x^n\, dx\, e^{\alpha x} + \alpha\int x^{n+1}\, dx\, e^{\alpha x}.$$

§. 23. Quodſi iam loco n ſucceſſiue ſcribamus numeros 1, 2, 3, 4, ac faciamus

$$A \int e^{\alpha x} dx = \tfrac{1}{\alpha}(e^{\alpha} - 1) \text{ et } B = \int x\, dx\, e^{\alpha x} = \tfrac{\alpha - 1}{\alpha\alpha} e^{\alpha} + \tfrac{1}{\alpha\alpha}$$

$$f = 1,\ f' = 2,\ f'' = 3,\ f''' = 4, \text{ etc.}$$

$$g = 2 - \alpha,\ g' = 3 - \alpha,\ g'' = 4 - \alpha, \text{ etc.}$$

$$h = \alpha,\ h' = \alpha,\ h'' = \alpha,\ h''' = \alpha, \text{ etc.}$$

prodibit iſta fractio continua:

$$\frac{A}{B} = 2 - \alpha + \cfrac{2\alpha}{3 - \alpha + \cfrac{3\alpha}{4 - \alpha + \cfrac{4\alpha}{5 - \alpha + \text{etc.}}}}$$

Adiungamus adhuc ſuperne $1 - \alpha + \alpha$, erit eius valor

$$1 - \alpha + \frac{(\alpha - 1)\, e^{\alpha} + 1}{e^{\alpha} - 1} = \frac{\alpha}{e^{\alpha} - 1},$$

vnde

vnde habebitur haec fractio continua satis concinna:

$$\frac{\alpha}{e^{\alpha}-1} = 1 - \alpha + \cfrac{\alpha}{2 - \alpha + \cfrac{2\alpha}{3 - \alpha + \cfrac{3\alpha}{4 - \alpha + \text{etc.}}}}$$

vnde patet, si fuerit $\alpha = 0$, ob $e^{\alpha} - 1 = \alpha$, fore vtique $1 = 1$.

§. 24. Consideremus nonnullos casus speciales; ac primo, si sit $\alpha = 1$, erit

$$\frac{1}{e-1} = 0 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \text{etc.}}}}}$$

quae fractio facile transfunditur in hanc:

$$\frac{1}{e-1} = \cfrac{1}{1 + \cfrac{\frac{1}{1}}{1 + \cfrac{\frac{1}{2}}{1 + \cfrac{\frac{1}{3}}{1 + \cfrac{\frac{1}{4}}{1 + \text{etc.}}}}}}$$

vnde fit

$$e - 1 = 1 + \cfrac{\frac{1}{1}}{1 + \cfrac{\frac{1}{2}}{1 + \cfrac{\frac{1}{3}}{1 + \text{etc.}}}}$$

Haec autem porro a fractionibus partialibus liberata dat

$$e - 1 = + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \text{etc.}}}}}}$$

vnde sequitur

$$\frac{1}{e-2} = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{4 + \text{etc.}}}}}$$

quae formae ob simplicitatem maxime sunt notatu dignae. Ex penultima, qua fit

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \text{etc.}}}}}$$

sumendo successiue 1, 2, 3, pluraue membra, orientur sequentes approximationes:

$e = 2,0000$
$e = 3.0000$
$e = 2,6666$
$e = 2,7272$
$e = 2,7169$

qui valores, alternatim maiores et minores, satis prompte ad veritatem conuergunt.

§. 25.

§. 25. Sumamus $\alpha = 2$ erit

$$\frac{2}{ee-1} = -1 + \cfrac{2}{0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}}$$

Ex hac fractione porro deducitur ista:

$$\frac{2(ee-1)}{ee+1} = 0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}$$

similique modo, si pro α maiores numeri accipiantur, reductio fieri poterit.

§. 26. Possunt etiam pro α numeri negativi accipi. Ita si fuerit $\alpha = -1$ fiet

$$\frac{e}{e-1} = 2 - \cfrac{1}{3 - \cfrac{2}{4 - \cfrac{3}{5 - \cfrac{4}{6 - \text{etc.}}}}}$$

quae reducitur ad hanc formam:

$$\frac{e}{e-1} = 2 + \cfrac{1}{-3 + \cfrac{2}{4 + \cfrac{3}{-5 + \cfrac{4}{6 + \text{etc.}}}}}$$

similique modo maiores valores expediri possunt.

§. 27.

§. 27. Statuamus etiam $\alpha = \frac{1}{2}$, ac repeSictur ista expressio:

$$\frac{1}{2(\sqrt{e}-1)} = \frac{1}{2} + \cfrac{\frac{1}{2}}{\frac{3}{2} + \cfrac{1}{\frac{5}{2} + \cfrac{\frac{3}{2}}{\frac{7}{2} + \cfrac{\frac{5}{2}}{\frac{9}{2} + \text{etc.}}}}}$$

quae liberata a fractionibus partialibus euadit

$$\frac{1}{-1+\sqrt{e}} = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \cfrac{8}{9 + \text{etc.}}}}}$$

Simili modo si sumus $\alpha = \frac{1}{3}$ erit

$$\frac{1}{3(\sqrt[3]{e}-1)} = 2:3 + \cfrac{1:3}{5:3 + \cfrac{2:3}{8:3 + \cfrac{3:3}{11:3 + \cfrac{4:3}{14:3 + \text{etc.}}}}}$$

quae a fractionibus partialibus liberata dat

$$\frac{1}{-1+\sqrt[3]{e}} = 2 + \cfrac{3}{5 + \cfrac{6}{8 + \cfrac{9}{11 + \cfrac{12}{14 + \text{etc.}}}}}$$

At

At ſi ponatur $\alpha = \frac{2}{3}$, prodit haec fractio continua:

$$\frac{2}{3(\sqrt[3]{(ee-1)})} = 1:3 + \cfrac{2:3}{4:3 + \cfrac{4:3}{7:3 + \cfrac{6:3}{10:3 + \cfrac{8:3}{13:3 + \text{etc.}}}}}$$

quae a fractionibus partialibus liberata fit

$$\frac{2}{\sqrt[3]{(ee-1)}} = 1 + \cfrac{6}{4 + \cfrac{12}{7 + \cfrac{18}{10 + \cfrac{24}{13 + \text{etc.}}}}}$$

§. 28. His formulis tanquam principalibus ac ſimplicioribus euolutis, ſimili modo alias multo generaliores tractare licebit, quae ad fractiones continuas multo magis abſconditas perducent, vti ex caſibus qui ſequuntur patebit.

V. Euolutio formulae.

$$s = x^n (a - b x^\theta - c x^{2\theta})^\lambda.$$

§. 29. Hinc igitur erit

$$ds = (a - b x^\theta - c x^{2\theta})^{\lambda-1} (n a x^{n-1} dx - b(n+\lambda\theta) x^{n+\theta-1} dx \\ - c(n + 2\lambda\theta) x^{n+2\theta-1} dx),$$

vnde per partes integrando, tum vero ſtatuendo $a - b x^\theta - c x^{2\theta} = 0$, (quod fit ſi fuerit $x^\theta = -\frac{b + \sqrt{(bb+4ac)}}{2c}$) habebitur iſta reductio generalis:

$$n a \int x^{n-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1}$$
$$= (n + \lambda\theta) b \int x^{n+\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1}$$
$$+ (n + 2\lambda\theta) c \int x^{n+2\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1}.$$

§. 30. Quodfi iam hanc formam cum noftra generali initio tradita comparare velimus, valores pro litera n fucceffiue affumendi per differentiam θ augeri debent. Deinde non neceffe eft vt primus valor ipfius n, vt hactenus fecimus, fumatur $= 1$; ftatuamus igitur eius primum valorem $= \alpha$, et quaeramus valores binarum fequentium formularum integralium, fcilicet:

$$A = \int x^{\alpha-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1} \text{ et}$$
$$B = \int x^{\alpha+\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1},$$

quae integralia ita funt capienda, vt euanefcant pofito $x = 0$, quo facto ipfi x ille valor tribui debet, qui reddat formulam $a - b x^{\theta} - c x^{2\theta} = 0$. Quoniam autem hoc in genere exfequi non licet, iftos valores per literas A et B indicare contenti fimus, quos ergo tanquam cognitos fpectemus.

§. 31. Praeterea vero literae f, g, h, cum fuis deriuatis fequentes induent valores:

$$f = \alpha a,\quad f' = (\alpha + \theta) a,\quad f'' = (\alpha + 2\theta) a,\quad f''' = (\alpha + 3\theta) a, \text{ etc.}$$
$$g = (\alpha + \lambda\theta) b,\quad g' = (\alpha + \theta + \lambda\theta) b,\quad g'' = (\alpha + 2\theta + \lambda\theta) b, \text{ etc.}$$
$$h = (\alpha + 2\lambda\theta) c,\quad h' = (\alpha + \theta + 2\lambda\theta) c,\quad h'' = (\alpha + 2\theta + 2\lambda\theta) c; \text{ etc.}$$

Ex his igitur formabitur fequens fractio continua:

$$\frac{\alpha a A}{B} = (\alpha+\lambda\theta)b + \cfrac{(\alpha+\theta)(\alpha+\lambda\theta)ac}{(\alpha+\theta+\lambda\theta)b + \cfrac{(\alpha+2\theta)(\alpha+\theta+2\lambda\theta)ac}{(\alpha+2\theta+\lambda\theta)b + \cfrac{(\alpha+3\theta)(\alpha+2\theta+2\lambda\theta)ac}{(\alpha+3\theta+\lambda\theta)b \text{ etc.}}}}$$

quae

quae forma vtique eſt maxime generalis, cuius autem vlteriori euolutioni non immoramur.

VI. Euolutio formulae.

$$s = x^n (1 - x^\theta)^\lambda$$

§. 32. Hinc ergo fit

$$ds = n x^{n-1} dx (1-x^\theta)^\lambda - \lambda \theta x^{n+\theta-1} dx (1-x^\theta)^{\lambda-1},$$

vnde tantum duae formulae integrales orirentur; quamobrem huic differentiali denominatorem arbitrarium tribuamus $a + b x^\theta$, vt habeamus:

$$ds = \frac{(1-x^\theta)^{\lambda-1}}{a + b x^\theta} (n a x^{n-1} dx - (a(n+\lambda\theta) - bn) x^{n+\theta-1} dx$$
$$- b(n+\lambda\theta) x^{n+2\theta-1} dx.$$

Nunc igitur, ponendo poſt integrationem $x = 1$, deducimus hanc reductionem:

$$na\int \frac{x^{n-1} dx (1-x^\theta)^{\lambda-1}}{a + b x^\theta} = (a(n+\lambda\theta) - bn) \int \frac{x^{n+\theta-1} dx (1-x^\theta)^{\lambda-1}}{a + b x^\theta}$$
$$+ b(n+\lambda\theta) \int \frac{x^{n+2\theta-1} dx (1-x^\theta)^{\lambda-1}}{a + b x^\theta}.$$

§. 33. Hic iterum euidens eſt valores ipſius n per differentiam θ creſcere debere. Statuatur autem primus valor ipſius $n = \alpha$, et quaerantur pro quouis caſu oblato binae ſequentes formulae integrales:

$$A = \int \frac{x^{\alpha-1} dx (1-x^\theta)^{\lambda-1}}{a + b x^\theta} \quad \text{et} \quad B = \int \frac{x^{\alpha+\theta-1} dx (1-x^\theta)^{\lambda-1}}{a + b x^\theta},$$

vbi ſcilicet poſt integrationem poſitum ſit $x = 1$. Quibus

bus inuentis, cum hinc fiat

$$f = \alpha a,\ f' = (\alpha + \theta) a,\ f'' = (\alpha + 2\theta) a,\ f''' = (\alpha + 3\theta) a, \text{ etc.}$$

$$g = (\alpha + \lambda\theta) a - \alpha b,\ g' - (\alpha + \theta + \lambda\theta) a - (\alpha + \theta) b,$$

$$g'' = (\alpha + 2\theta + \lambda\theta) a - (\alpha + 2\theta) b, \text{ etc.}$$

$$h = (\alpha + \lambda\theta) b,\ h = (\alpha + \theta + \lambda\theta) b,\ h'' = (\alpha + \theta + 2\lambda\theta) b, \text{ etc.}$$

inde formabitur ſequens fractio continua:

$$\frac{\alpha a A}{B} = (\alpha + \lambda\theta) a - \alpha b + \cfrac{(\alpha + \theta)(\alpha + \lambda\theta) a b}{(\alpha + \theta + \lambda\theta) a - (\alpha + \theta) b + \cfrac{(\alpha + 2\theta)(\alpha + \theta + \lambda\theta) a b}{(\alpha + 2\theta + \lambda\theta) a - (\alpha + 2\theta) b + (\alpha + 3\theta)(\alpha + 2\theta + \lambda\theta) a b \text{ etc.}}}$$

cuius formae vberiore euolutione ſuperſedemus.

VII. Euolutio formulae.

$$s = x^n (e^{\alpha x} (1 - x)^{\lambda})$$

§. 34. Hinc ergo fit

$$d s = (1 - x)^{\lambda - 1} (n x^{n - 1} d x - (n + \lambda - \alpha) x^n d x - \alpha x^n d x),$$

hinc igitur ſi poſt integrationem vbique ſtatuatur $x = 1$, quippe quo caſu fit $s = 0$, habebimus hanc reductionem:

$$n \int x^{n-1} d x\, e^{\alpha x} (1 - x)^{\lambda - 1} = (n + \lambda - \alpha) \int x^n d x\, e^{\alpha x} (1 - x)^{\lambda - 1}$$

$$+ \alpha \int x^{n+1} d x\, e^{\alpha x} (1 - x)^{\lambda - 1}.$$

§. 35. In his ergo formulis exponenti n valores vnitate creſcentes tribui debebunt, tum vero hic minimum eius valorem ſumamus $n = \delta$, atque valores literarum A et B ex his formulis erui oportebit, ponendo poſt integrationem $x = 1$,

$$A = \int x^{\delta - 1} d x\, e^{\alpha x} (1 - x)^{\lambda - 1},\quad B = \int x^{\delta} d x\, e^{\alpha x} (1 - x)^{\lambda - 1}$$

deinde vero ob hos valores:

$$f = \delta,$$

$$f=\delta,\ f'=\delta+1,\ f''=\delta+2,\ f'''=\delta+3,\ \text{etc.}$$
$$g=\delta+\lambda-\alpha,\ g'=\delta+1+\lambda-\alpha,\ g''=\delta+2+\lambda-\alpha,\ \text{etc.}$$
$$h=\alpha,\ h'=\alpha,\ h''=\alpha,\ \text{etc.}$$

fequitur ifta fractio continua:

$$\frac{\delta A}{B}=\delta+\lambda-\alpha+\cfrac{(\delta+1)\alpha}{\delta+1+\lambda-\alpha+\cfrac{(\delta+2)\alpha}{\delta+2+\lambda-\alpha+\cfrac{(\delta+3)\alpha}{\delta+3+\lambda-\alpha+\text{etc.}}}}$$

Vbi imprimis notari oportet, exponentes λ et δ neceffario nihilo maiores accipi debere, quia alioquin formula principalis $x^n e^{\alpha x}(1-x)^\lambda$ cafibus $x=1$ non euanefceret.

§. 36. Si literis δ et λ tribuatur valor $=1$, prodibit cafus iam fupra tractatus; ac fi his literis numeri integri affignentur, eiusmodi fractiones continuae orientur, quas per certas operationes ad priores reducere licebit. Verum fi his literis δ et λ, vel alterutri, vel vtrique, fractiones affignemus, tum formae orientur ad priores prorfus irreductibiles, quarumque valor haud aliter quam per quantitates maxime tranfcendentes exprimere liceat. Velvti fi fuerit $\delta=\frac{1}{2}$ et $\lambda=\frac{1}{2}$, valor literae A quaeri debebit ex hac formula integrali: $A=\frac{e^{\alpha x}\,dx}{\sqrt{(x-xx)}}$, cuius integratio ad quantitates maxime tranfcendentes perducit, ita vt valor talium fractionum continuarum prodeat maxime abftrufus.

On the formation of continued fractions*

Leonhard Euler†

1. A universal principle for unfolding continued fractions is found in the infinite series of quantities A, B, C, etc., of which each third succeeds from the preceding two by a certain law, either constant or variable inasmuch as they depend in turn upon each other, so that it will be

$$fA = gB + hC, f'B = g'C + h'D, f''C = g''D + h''E, f'''D = g'''E + h'''F, \text{ etc.}$$

From here indeed are deduced the following equalities:

$$\begin{aligned}
\frac{fA}{B} &= g + \frac{hC}{B} = g + \frac{f'h}{f'B : C} \\
\frac{f'B}{C} &= g' + \frac{h'D}{C} = g' + \frac{f''h'}{f''C : D} \\
\frac{f''C}{D} &= g'' + \frac{h''E}{D} = g'' + \frac{f'''h''}{f'''D : E} \\
\frac{f'''D}{E} &= g''' + \frac{h'''F}{E} = g''' + \frac{f''''h'''}{f''''E : F} \\
&\text{etc.} \qquad \text{etc.}
\end{aligned}$$

But if now the latter values are substituted successively in place of the prior ones, the following continued fraction spontaneously emerges:

$$\frac{fA}{B} = g + \cfrac{f'h}{g' + \cfrac{f''h'}{g'' + \cfrac{f'''h''}{g''' + \cfrac{f''''h'''}{g'''' + \text{etc.}}}}}$$

whose value therefore is determined by the two first terms A & B of the series alone.

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†Date of translation: August 12, 2005. Translated from the Latin by Jordan Bell, 3rd year undergraduate in Honours Mathematics, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada. Email: jbell3@connect.carleton.ca. This translation was written during an NSERC USRA supervised by Dr. B. Stevens.

2. Therefore whenever such a progression of quantities A, B, C, D, E, etc. is had, of which such a law is disposed, that each of the terms depends in turn on the succeeding two by a certain law, then from this a continued fraction may be deduced, whose value is able to be assigned. Therefore if such a particular formula were disposed, such that the expansion of it should lead to such a series of quantities A, B, C, D, E, etc. of which each term is determined by the preceding two, from this continued fractions will be able to be derived, in what way it will be able to be revealed most conveniently by several examples.

I. The expansion of the formula $s = x^n(\alpha - \beta x - \gamma xx)$

3. In this formula the exponent n is considered an indefinite, successively receiving all the values $1, 2, 3, 4, 5, 6$, etc., from which, providing it will be $n > 0$, this formula vanishes by putting $x = 0$, and then indeed it also vanishes by taking

$$x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma}.$$

With this having been noted, this formula will be differentiated so that it will become

$$ds = n\alpha x^{n-1}dx - (n+1)\beta \int x^n dx - (n+2)\gamma \int x^{n+1}dx + s,$$

from which by integrating the parts, this integration may then be indicated as such

$$x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma}.$$

Then, if after this integration, having been completed such that the integral should vanish by putting $x = 0$, it should be set

$$n\alpha \int x^{n-1}dx = (n+1)\beta \int x^n dx + (n+2)\gamma \int x^{n+1}dx,$$

of course in which case it would be $s = 0$, it will be

$$n\alpha \int x^{n-1}dx = (n+1)\beta \int x^n dx + (n+2)\gamma \int x^{n+1}dx,$$

which is a relation of this type between three integral formulas following each other, which we desire for the formation of a continued fraction, seeing that these integral formulas, if in place of n is written successively the number $1, 2, 3, 4, 5, 6$, etc., the quantities A, B, C, D, etc. are given to us.

4. We write therefore in place of n the consecutive numbers $1, 2, 3, 4$, etc., so that these relations may be produced:

$$\begin{aligned}
\alpha \int dx &= 2\beta \int x dx + 3\gamma \int xx dx \\
2\alpha \int x dx &= 3\beta \int xx dx + 4\gamma \int x^3 dx \\
3\alpha \int xx dx &= 4\beta \int x^3 dx + 3\gamma[\text{sic}] \int x^4 dx \\
4\alpha \int x^3 dx &= 5\beta \int x^4 dx + 6\gamma \int x^5 dx \\
&\qquad\qquad\quad \text{etc.} \qquad \text{etc.}
\end{aligned}$$

Then we will therefore have

$$A = \int dx = x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma},$$

$$B = \int x dx = \frac{1}{2} xx = \frac{1}{2}\Big(\frac{-\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma}\Big)^2,$$

$$C = \int xx dx = \frac{1}{3} x^3, \quad D = \int x^3 dx = \frac{1}{4} x^4$$

$$\text{etc.} \qquad \text{etc.}$$

Thereupon indeed, for the letters f, g, h will be had these values:

$$\begin{gathered}
f = \alpha,\ f' = 2\alpha,\ f'' = 3\alpha,\ f''' = 4\alpha \text{ etc.} \\
g = 2\beta,\ g' = 3\beta,\ g'' = 4\beta,\ g''' = 5\beta \text{ etc.} \\
h = 3\gamma,\ h' = 4\gamma,\ h'' = 5\gamma,\ h''' = 6\gamma \text{ etc.,}
\end{gathered}$$

from which values the following continued fraction results:

$$\frac{\alpha A}{B} = 2\beta + \cfrac{6\alpha\gamma}{3\beta + \cfrac{12\alpha\gamma}{4\beta + \cfrac{20\alpha\gamma}{5\beta + \cfrac{30\alpha\gamma}{6\beta + \text{etc.}}}}},$$

whose value therefore is

$$\frac{4\alpha\gamma}{-\beta + \sqrt{(\beta\beta + 4\alpha\gamma)}} = \beta + \sqrt{(\beta\beta + 4\alpha\gamma)}.$$

5. So that this continued fraction should be rendered more elegant, in place of $\alpha\gamma$ we write $\frac{1}{2}\delta$, and it will proceed

$$\beta + \sqrt{(\beta\beta + 2\delta)} = 2\beta + \cfrac{3\delta}{3\beta + \cfrac{6\delta}{4\beta + \cfrac{10\delta}{5\beta + \cfrac{15\delta}{6\beta + \text{etc.}}}}}.$$

Since moreover the head of this expression is seen to have been truncated, with this head having been added we may put

$$s = \beta + \cfrac{\delta}{2\beta + \cfrac{3\delta}{3\beta + \cfrac{6\delta}{4\beta + \cfrac{10\delta}{5\beta + \text{ etc.}}}}},$$

and it will be

$$s = \beta + \frac{\delta}{\beta + \surd(\beta\beta + 2\delta)},$$

which expression is reduced to this:

$$s = \frac{1}{2}\beta + \frac{1}{2}\surd(\beta\beta + 2\delta).$$

6. This continued fraction is in fact able to be reduced to even greater simplicity, if in place of δ we write 2ε, so that it would be

$$\frac{1}{2}\beta + \frac{1}{2}\surd(\beta\beta + 4\varepsilon) = \beta + \cfrac{2\varepsilon}{2\beta + \cfrac{6\varepsilon}{3\beta + \cfrac{12\varepsilon}{4\beta + \cfrac{20\varepsilon}{5\beta + 20[\text{sic}] \text{ etc.}}}}}.$$

And if now the first fraction is depressed by 2, the second by 3, the third by 4, the fourth by 5, etc., the following form will be produced:

$$\frac{1}{2}\beta + \frac{1}{2}\surd(\beta\beta + 4\varepsilon) = \beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \text{ etc.}}}}},$$

which is of the greatest simplicity, and if its sum is considered an unknown, and it is denoted as equal to z, it will be in turn be $z = \beta + \frac{\varepsilon}{z}$, and thus $zz = \beta z + \varepsilon$, from which it would be $z = \frac{\beta + \surd(\beta\beta + 4\varepsilon)}{2}$, which is the same.

7. Indeed this most simple sum is able to be immediately deduced from the formula taken initially

$$s = x^n(\alpha - \eta x - \gamma xx),$$

and since we have put this equal to nothing, it will certainly be $\alpha = \beta x + \gamma xx$, and in the very same way

$$\alpha x = \beta xx + \gamma x^3,\ \alpha xx = \beta x^3 + \gamma x^4, \text{ etc.},$$

thus so that for the series A, B, C, D, etc. we may have this simple series of powers: $1, x, x^3, x^3, x^4$, etc., then indeed each of the letters f, g, h, etc. becomes α, β, γ, etc., from which arises this continued fraction:

$$\frac{\alpha}{x} = \beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \text{ etc.}}}},$$

where it is $\frac{1}{x} = \frac{\beta + \surd(\beta\beta + 4\alpha\gamma)}{2\alpha}$. Therefore the value of this fraction is $\frac{1}{2}\beta + \frac{1}{2}\surd(\beta\beta + 4\alpha\gamma)$, such as before, since $\alpha\gamma = \varepsilon$.

II. The expansion of the formula $s = x^n(a - x)$

8. This formula will therefore vanish by setting $x = a$. Moreover it is $ds = nax^{n-1}dx - (n+1)x^n dx$, which expression is comprised by two terms; it may be reduced to a fraction, whose denominator is $\alpha + \beta x$, so that it will become

$$ds = \frac{na\alpha x^{n-1}dx + (\beta na - \alpha(n+1))x^n dx - \beta(n+1)x^{n+1}dx}{\alpha + \beta x}.$$

Therefore by integrating each of the members, it will be

$$s = na\alpha \int \frac{x^{n-1}dx}{\alpha + \beta x} + (n\beta a - (n+1)\alpha) \int \frac{x^n dx}{\alpha + \beta x} - \beta(n+1) \int \frac{x^{n+1}dx}{\alpha + \beta x},$$

which if after each are integrated we set $x = a$, so that it may become $s = 0$, we will have this reduction:

$$na\alpha \int \frac{x^{n-1}dx}{\alpha + \beta x} = ((n+1)\alpha - n\beta a) \int \frac{x^n dx}{\alpha + \beta x} + (n+1)\beta \int \frac{x^{n+1}dx}{\alpha + \beta x}).$$

9. In place of n we shall now successively substitute the numbers $1, 2, 3, 4$, etc., and then by comparison with the general formula that has been established we will have

$$A = \int \frac{dx}{\alpha + \beta x}, \qquad B = \int \frac{xdx}{\alpha + \beta x}, \qquad C = \int \frac{xxdx}{\alpha + \beta x}, \text{ etc.}$$

where indeed after integration it ought to be made $x = a$. Thereafter truly we will have

$$f = a\alpha,\ f' = 2a\alpha,\ f'' = 3a\alpha,\ f''' = 4a\alpha, \text{ etc.}$$
$$g = 2\alpha - \beta a,\ g' = 3\alpha - 2\beta a,\ g'' = 4\alpha - 3\beta a, \text{ etc.}$$
$$h = 2\beta, h' = 3\beta,\ h'' = 4\beta,\ h''' = 5\beta, \text{ etc.}$$

and thus from this arises the following continued fraction:

$$\frac{\alpha a A}{B} = (2\alpha - \beta a) + \cfrac{4a\alpha\beta}{(3\alpha - 2\beta a) + \cfrac{ga\alpha\beta}{(4\alpha - 3\beta a) + \cfrac{16a\alpha\beta}{(5\alpha - 4\beta a) + \text{ etc.}}}}.$$

10. By integrating, it may moreover be established

$$\int \frac{dx}{\alpha + \beta x} = \frac{1}{\beta} l \frac{\alpha + \beta x}{\alpha},$$

seeing that the integral ought to vanish by making it $x = 0$. Now therefore it may be $x = a$, and it will hence be $A = \frac{1}{\beta} l \frac{\alpha+\beta a}{\alpha}$. In turn

$$\int \frac{xdx}{\alpha+\beta x} = \frac{1}{\beta}\Big(x - \frac{\alpha}{\beta} l \frac{\alpha+\beta x}{\alpha}\Big),$$

and by it being made $x = a$ it shall become

$$B = \frac{a}{\beta} - \frac{\alpha}{\beta\beta} l \frac{\alpha+\beta a}{\alpha},$$

wherefore the value of our continued fraction will be

$$\frac{\alpha a \beta l \frac{\alpha+\beta a}{\alpha}}{a\beta - \alpha l \frac{\alpha+\beta a}{\alpha}};$$

moreover it is evident for nothing of universality to perish, even if it is set $a = 1$; then in fact it will be

$$\frac{\alpha\beta l \frac{\alpha+\beta}{\alpha}}{\beta - \alpha l \frac{\alpha+\beta}{\alpha}} = (2\alpha - \beta) + \frac{4\alpha\beta}{(3\alpha - 2\beta) + \frac{9\alpha\beta}{(4\alpha - 3\beta) + \text{ etc.}}}.$$

11. Moreover, the whole of this expression manifestly depends singularly on the ratio of the numbers α and β; from this, we may take $\alpha = 1$ and $\beta = n$, and then this continued fraction will be seen:

$$\frac{nl(1+n)}{n - l(1+n)} = (2-n) + \frac{4n}{(3-2n) + \frac{9n}{(4-3n) + \frac{16n}{(5-4n) + \text{ etc.}}}},$$

for which if we set this series after the law $1+n$ and set the sum equal to s, so that it will be

$$s = 1 + \frac{n}{(2-n) + \frac{4n}{(3-2n) + \frac{9n}{(4-3n) + \frac{16n}{(5-4n) + \text{ etc.}}}}},$$

it will be

$$s = 1 + \frac{n(n - l(1+n))}{nl(1+n)} = 1 + \frac{n - l(1+n)}{l(1+n)} = \frac{n}{l(1+n)}.$$

12. We shall run through several examples, and the first shall be $n = 1$, where it will be

$$\frac{1}{l2} = 1 + \frac{1}{1 + \frac{4}{1 + \frac{9}{1 + \frac{16}{1 + \text{ etc.}}}}}.$$

On the other hand by putting $n = 2$ it will be

$$\frac{2}{l3} = 1 + \frac{2}{0 + \frac{8}{-1 + \frac{18}{-2 + \frac{32}{-3 + \frac{50}{-4 + \text{ etc.}}}}}},$$

which expression however, on account of the negative quantities, is not very pleasant; insofar as it will turn out that whenever $n > 1$, it will be more worth the work to unfold these cases, than when n is taken as less than unity.

13. Therefore it is able to be made easily; it may be reverted to an expression containing the letters α and β, and then by supplying the head because it is missing, this form will be produced:

$$\frac{\beta}{l\frac{\alpha+\beta}{\alpha}} = \alpha + \cfrac{\alpha\beta}{(2\alpha-\beta) + \cfrac{4\alpha\beta}{(3\alpha-2\beta) + \cfrac{9\alpha\beta}{(4\alpha-3\beta) + \text{ etc.}}}}.$$

We now put $\alpha = n - m$ and $\beta = 2m$,[1] so that we may obtain the following form:

$$\frac{2m}{l\frac{n+m}{n-m}} = n - m + \cfrac{2m(n-m)}{2n-4m + \cfrac{8m(n-m)}{3n-7m + \cfrac{18m(n-m)}{4n-10m + \text{ etc.}}}},$$

from which the following special cases are deduced.

If $m = 1$ and $n = 3$ it will be

$$\frac{2}{l2} = 2 + \cfrac{4}{2 + \cfrac{16}{2 + \cfrac{36}{2 + \cfrac{64}{2 + \text{ etc.}}}}},$$

which fraction divided by 2 and reduced is rendered as such:

$$\frac{1}{l2} = 1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{ etc.}}}}},$$

which here is arrived at like before.

Were it $m = 1$ and $n = 4$ it will be

$$\frac{2}{l\frac{5}{2}} = 3 + \cfrac{6}{4 + \cfrac{24}{5 + \cfrac{54}{6 + \cfrac{96}{7 + \text{ etc.}}}}}$$

$$= 3 + \cfrac{6 \cdot 1}{4 + \cfrac{6 \cdot 4}{5 + \cfrac{6 \cdot 9}{6 + \cfrac{6 \cdot 16}{7 + \text{ etc.}}}}}.$$

Were it $m = 1$ and $n = 5$, it will be

$$\frac{2}{l\frac{3}{2}} = 4 + \cfrac{8}{6 + \cfrac{32}{8 + \cfrac{72}{10 + \cfrac{128}{12 + \text{ etc.}}}}},$$

[1] Translator: Original reads "$n = n - m$".

or

$$\frac{1}{l\frac{3}{2}} = 2 + \cfrac{2}{3 + \cfrac{8}{4 + \cfrac{18}{5 + \cfrac{32}{6 + \text{etc.}}}}}$$

$$= 2 + \cfrac{2\cdot 1}{3 + \cfrac{2\cdot 4}{4 + \cfrac{2\cdot 9}{5 + \cfrac{2\cdot 16}{6 + \text{etc.}}}}}.$$

III. The expansion of the formula $s = x^n(1 - x^2)$

14. This formula therefore vanishes in the cases $x = 0$ and $x = 1$. Then indeed, it shall be

$$ds = nx^{n-1}dx - (n+2)x^{n+1}dx,$$

which differential uplifted with the denominator $\alpha + \beta xx$ would be

$$ds = \frac{n\alpha x^{n-1}dx + (n\beta - (n+2)\alpha)x^{n+2}dx - (n+2)\beta x^{n+3}dx}{\alpha + \beta xx};$$

here, by now integrating this in turn it will be

$$s = n\alpha \int \frac{x^{n-1}dx}{\alpha + \beta xx} + (n\beta - (n+2)\alpha) \int \frac{x^{n+1}dx}{\alpha + \beta xx} - (n+2)\beta \int \frac{x^{n+3}dx}{\alpha + \beta xx}.$$

And if now after these integrations it is set $x = 1$, this reduced integral will be produced:

$$n\alpha \int \frac{x^{n-1}dx}{\alpha + \beta xx} = ((n+2)\alpha - n\beta) \int \frac{x^{n+1}dx}{\alpha + \beta xx} + (n+2)\beta \int \frac{x^{n+3}dx}{\alpha + \beta xx}.$$

15. Because these powers of x are augmented two by two, for the exponent n we shall assign successively the values $1, 3, 5, 7, 9$, etc., and it shall be set:

$$A = \int \frac{dx}{\alpha + \beta xx}, \qquad B = \int \frac{xxdx}{\alpha + \beta xx}, \qquad C = \int \frac{x^4 dx}{\alpha + \beta xx}, \text{ etc.}$$

Then indeed from these, the letters f, g, h will be derived:

$$f = \alpha,\ f' = 3\alpha,\ f'' = 5\alpha,\ f''' = 7\alpha, \text{ etc.}$$
$$g = 3\alpha - \beta,\ g' = 5\alpha - 3\beta,\ g'' = 7\alpha - 5\beta, \text{ etc.}$$
$$h = 3\beta, h' = 5\beta,\ h'' = 7\beta,\ h''' = 9\beta, \text{ etc.,}$$

from which is born the following continued fraction:

$$\frac{\alpha A}{B} = 3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \cfrac{49\alpha\beta}{9\alpha - 7\beta + \text{etc.}}}}.$$

16. Because it is $B = \int \frac{xxdx}{\alpha+\beta xx}$, it will be $B = \frac{1}{\beta}\int dx - \frac{\alpha}{}$ [sic] $\int \frac{dx}{\alpha+\beta xx}$, and thus $B = \frac{1}{\beta} - \frac{\alpha}{\beta}A$, by whose value being substituted in we will have

$$\frac{\alpha\beta A}{1-\alpha A} = 3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \text{ etc.}}},$$

which, because it is missing its head, we will set in front $\alpha + \beta + \alpha\beta$; then moreover the sum will be $\beta + \frac{1}{A}$, so that we will thus have

$$\beta + \frac{1}{A} = \alpha + \beta + \cfrac{\alpha\beta}{3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \text{ etc.}}}}.$$

with it arising that $A = \int \frac{dx}{\alpha+\beta xx}$, with this integral being taken so that it vanishes by putting $x = 0$, and indeed also by making it $x = 1$.

17. First we shall expand the simplest case, in which $\alpha = 1$ and $\beta = 1$, where it will be $A = \frac{\pi}{4}$, for which we will have

$$1 + \frac{4}{\pi} = 2 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{ etc.}}}}},$$

that is it will be

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \text{ etc.}}}},$$

which is the same continued fraction produced first before by Brouncker, the investigation of which, which was elicited by Wallis through greatly tedious calculations, here proceeds immediately from our formula by itself.

18. Our general form as well provides infinite other similar expressions, precisely as the letters α and β are taken in varied ways. And firstly indeed, if α and β were positive numbers, the value of the letter A is always able to be expressed by circular arcs, and conversely indeed by logarithms. Were it therefore $\beta = 1$, it will be

$$A = \int \frac{dx}{\alpha + xx} = \frac{1}{\sqrt{\alpha}} \text{ Atang. } \frac{x}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}} \text{ Atang. } \frac{1}{\sqrt{\alpha}},$$

from which is born this continued fraction:

$$1 + \frac{\sqrt{\alpha}}{\text{Atang. } \frac{1}{\sqrt{\alpha}}} = \alpha + 1 + \cfrac{\alpha}{3\alpha - 1 + \cfrac{9\alpha}{5\alpha - 3 + \cfrac{25\alpha}{7\alpha - 5 + \text{ etc.}}}}.$$

Here therefore it should be taken $\alpha = 3$, and because Atang. $\frac{1}{\sqrt{3}} = \frac{\pi}{3}$ we will have

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3}{8 + \cfrac{27}{12 + \cfrac{75}{16 + \cfrac{147}{20 + \text{ etc.}}}}},$$

or

$$1+\frac{6\sqrt{3}}{\pi}=4+\cfrac{3\cdot 1}{8+\cfrac{3\cdot 9}{12+\cfrac{3\cdot 25}{16+\cfrac{3\cdot 49}{20+\text{ etc.}}}}}.$$

19. Now B shall be a particular positive number, and indeed it is

$$A=\int\frac{dx}{\alpha+\beta xx}=\frac{1}{\beta}\int\frac{dx}{\frac{\alpha}{\beta}+xx},$$

which having been integrated will be $A=\frac{1}{\sqrt{\alpha\beta}}\,\text{Atang.}\sqrt{\frac{\beta}{\alpha}}$. Then therefore we will have

$$\beta+\frac{\sqrt{\alpha\beta}}{\text{Atang.}\sqrt{\frac{\beta}{\alpha}}}=\alpha+\beta+\cfrac{\alpha\beta}{3\alpha-\beta+\cfrac{9\alpha\beta}{5\alpha-3\beta+\text{ etc.}}}.$$

We shall now therefore set $\alpha+\beta=2n$ and $\alpha-\beta=2m$, so that it shall be $\alpha=m+n$ and $\beta=n-m$, with whose values having been put in it will be

$$n-m+\frac{\sqrt{(nn-mm)}}{\text{Atang.}\sqrt{\frac{n-m}{n+m}}}=2n+\cfrac{nn-mm}{2n+4m+\cfrac{9(nn-mm)}{2n+8m+\text{ etc.}}}.$$

20. We may consider as well the case, in which β is a negative number, and by putting $\beta=-\gamma$ it will be

$$A=\int\frac{dx}{\alpha-\gamma xx}=\frac{1}{\sqrt{}}\int\frac{dx}{\frac{\alpha}{\sqrt{}}-xx}[\text{sic}],$$

whose integral is

$$A=\frac{1}{2\sqrt{\alpha\gamma}}l\frac{\sqrt{\frac{\alpha}{\gamma}}+x}{\sqrt{\frac{\alpha}{\gamma}}-x};$$

with it having been made therefore $x=1$ it will be

$$A=\frac{1}{2\sqrt{\alpha\gamma}}l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}},$$

from which is born this continued fraction:

$$-\gamma+\frac{2\sqrt{\alpha\gamma}}{l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}}=\alpha-\gamma-\cfrac{\alpha\gamma}{3\alpha+\gamma-\cfrac{9\alpha\gamma}{5\alpha+3\gamma-\cfrac{25\alpha\gamma}{7\alpha+5\gamma-\text{ etc.}}}},$$

and thus we have made in this way new continued fractions, of which the values may moreover be exhibited by logarithms, and which are altogether different from that which we came upon before.

21. This case presents itself being more worthy of notice than the others, when $\gamma=\alpha$. Or, because it turns out the same, $\alpha=1$ and $\gamma=1$; because then it is $l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}=l\infty=\infty$, we will have

$$-1=0-\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{ etc.}}}},$$

or by changing the sign

$$1 = \cfrac{1}{4 - \cfrac{9}{8 - \cfrac{25}{12 - \text{ etc.}}}},$$

whence the first denominator

$$4 - \cfrac{9}{8 - \cfrac{25}{12 - \text{ etc.}}}$$

ought to be equal to 1. Therefore it will be

$$0 = 3 - \cfrac{9}{8 - \cfrac{25}{12 - \text{ etc.}}},$$

or

$$1 = \cfrac{3}{8 - \cfrac{25}{12 - \text{ etc.}}},$$

where the denominator ought to be equal to 3, from which it shall be

$$0 = 5 - \frac{25}{12 - \text{ etc.}},$$

whose denominator ought to be equal to 5, from which it shall be

$$0 = 7 - \cfrac{49}{16 - \cfrac{81}{20 - \text{ etc.}}},$$

from which the true order is easily perceived.

22. We take $\alpha = 4$ and $\gamma = 1$, and this fraction will be born

$$-1 + \frac{4}{l3} = 3 - \cfrac{4 \cdot 1}{13 - \cfrac{4 \cdot 9}{23 - \cfrac{4 \cdot 25}{33 - \cfrac{4 \cdot 49}{43 - \text{ etc.}}}}}$$

But if on the other hand we let $\alpha = 9$ and $\gamma = 1$ it will be

$$-1 + \frac{6}{l2} = 8 - \cfrac{9 \cdot 1}{28 - \cfrac{9 \cdot 9}{48 - \cfrac{9 \cdot 25}{68 - \cfrac{9 \cdot 49}{88 - \text{ etc.}}}}}.$$

IV. The expansion of the formula $s = x^n e^{\alpha x}(1 - x)$

22.[sic] This e denotes the number whose hyperbolic logarithm is unity, thus so that $d.e^{\alpha x} = \alpha dx e^{\alpha x}$. Then it will therefore be

$$ds = nx^{n-1}dxe^{\alpha x} + (\alpha - (n+1))x^n dxe^{\alpha x} - \alpha x^{n+1}dxe^{\alpha x},$$

from which in turn by integrating it will be

$$s = n\int x^{n-1}dxe^{\alpha x} + (\alpha - (n+1))\int x^n dxe^{\alpha x} - \alpha\int x^{n+1}dxe^{\alpha x}.$$

But if now after integration it is put $x = 1$, it will be

$$n\int x^{n-1}dxe^{\alpha x} = (n+1-\alpha)\int x^n dxe^{\alpha x} + \alpha\int x^{n+1}dxe^{\alpha x}.$$

23. But if now in place of n we successively write the numbers $1, 2, 3, 4$, and so on, we may make

$$A\int e^{\alpha x}dx = \frac{1}{\alpha}(e^\alpha - 1) \text{ and } B = \int xdxe^{\alpha x} = \frac{\alpha-1}{\alpha\alpha}e^\alpha + \frac{1}{\alpha\alpha}$$

$$f = 1,\ f' = 2,\ f'' = 3,\ f''' = 4, \text{ etc.}$$

$$g = 2-\alpha,\ g' = 3-\alpha,\ g'' = 4-\alpha, \text{ etc.}$$

$$h = \alpha,\ h' = \alpha,\ h'' = \alpha,\ h''' = \alpha, \text{ etc.}$$

and this continued fraction will be produced:

$$\frac{A}{B} = 2 - \alpha + \cfrac{2\alpha}{3 - \alpha + \cfrac{3\alpha}{4-\alpha+\cfrac{4\alpha}{5-\alpha+\text{ etc.}}}}.$$

We now adjoin at the head $1 - \alpha + \alpha$, whose value will be

$$1 - \alpha + \frac{(\alpha-1)e^\alpha + 1}{e^\alpha - 1} = \frac{\alpha}{e^\alpha - 1},$$

from which is obtained this quite elegant continued fraction:

$$\frac{\alpha}{e^\alpha - 1} = 1 - \alpha + \cfrac{\alpha}{2 - \alpha + \cfrac{2\alpha}{3-\alpha+\cfrac{3\alpha}{4-\alpha+\text{ etc.}}}},$$

from which it is apparent that if it were $\alpha = 0$, because $e^\alpha - 1 = \alpha$, for it to be certainly $1 = 1$.

24. We may consider several particular cases; and first, if it were $\alpha = 1$, it will be

$$\frac{1}{e-1} = 0 + \cfrac{1}{1 + \cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\text{ etc.}}}}},$$

which continued fraction is easily transformed into this:

$$\frac{1}{e-1} = \cfrac{1}{1 + \cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{4}}{1+\text{ etc.}}}}}},$$

from which it is

$$e - 1 = 1 + \cfrac{\frac{1}{1}}{1 + \cfrac{\frac{1}{2}}{1 + \cfrac{\frac{1}{3}}{1 + \text{etc.}}}}.$$

Moreover, this in turn having been resolved by partial fractions gives

$$e - 1 = + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \text{etc.}}}}}},$$

from which follows

$$\frac{1}{e-2} = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{4 \text{ [sic] } + \text{etc.}}}}},$$

which forms because of their great simplicity are noteworthy. From the second last, it may be

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \text{etc.}}}}};$$

by taking successively $1, 2, 3$, and so on for the members, the following approximations arise:

$$e = 2,0000$$
$$e = 3,0000$$
$$e = 2,6666$$
$$e = 2,7272$$
$$e = 2,7169$$

which values, alternately greater and smaller, converge to the truth readily enough.

25. We may take $\alpha = 2$ and it will be

$$\frac{2}{ee-1} = -1 + \cfrac{2}{0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}}.$$

From this fraction in turn is deduced this:

$$\frac{2(ee-1)}{ee+1} = 0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}},$$

and in a similar way, if larger numbers are taken for α, a reduction will be able to be made.

26. As well, negative numbers are able to be taken for α. In this way, if it were $\alpha = -1$, it will become

$$\frac{e}{e-1} = 2 - \cfrac{1}{3 - \cfrac{2}{4 - \cfrac{3}{5 - \cfrac{4}{6 - \text{etc.}}}}},$$

which is reduced to this form:

$$\frac{e}{e-1} = 2 + \cfrac{1}{-3 + \cfrac{2}{4 + \cfrac{3}{-5 + \cfrac{4}{6 + \text{etc.}}}}},$$

and in a similar way larger values are able to be dealt with.

27. Now too we set $\alpha = \frac{1}{2}$, and this expression will be found:

$$\frac{1}{2(\sqrt{e}-1)} = \frac{1}{2} + \cfrac{\frac{1}{2}}{\frac{3}{2} + \cfrac{1}{\frac{5}{2} + \cfrac{\frac{3}{2}}{\frac{7}{2} + \cfrac{\frac{5}{2}}{\frac{9}{2} + \text{etc.}}}}}$$

which reduced by partial fractions comes out as

$$\frac{1}{-1+\sqrt{e}} = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \cfrac{8}{9 + \text{etc.}}}}}.$$

In a similar way if we take $\alpha = \frac{1}{3}$, it will be

$$\frac{1}{3(\sqrt[3]{e}-1)} = 2:3 + \cfrac{1:3}{5:3 + \cfrac{2:3}{8:3 + \cfrac{3:3}{11:3 + \cfrac{4:3}{14:3 + \text{etc.}}}}},$$

which reduced by partial fractions gives

$$\frac{1}{-1+\sqrt[3]{e}} = 2 + \cfrac{3}{5 + \cfrac{6}{8 + \cfrac{9}{11 + \cfrac{12}{14 + \text{etc.}}}}}.$$

And if it is put $\alpha = \frac{2}{3}$, this continued fraction will be produced:

$$\frac{2}{3(\sqrt[3]{(ee}-1)} = 1:3 + \cfrac{2:3}{4:3 + \cfrac{4:3}{7:3 + \cfrac{6:3}{10:3 + \cfrac{8:3}{13:3 + \text{etc.}}}}},$$

which reduced by partial fractions will be

$$\frac{2}{\sqrt[3]{(ee-1)}} = 1 + \cfrac{6}{4 + \cfrac{12}{7 + \cfrac{18}{10 + \cfrac{24}{13 + \text{etc.}}}}}.$$

28. With these fairly simple principles disclosed, in a similar way it will be permitted to treat other more general ones, which will be led to continued fractions much more abstrusely, such that this will be accessible from the cases which follow.

V. The expansion of the formula $s = x^n(a - bx^\theta - cx^{2\theta})^\lambda$

29. Here therefore it will be

$$ds = (a-bx^\theta - cx^{2\theta})^{\lambda-1}(nax^{n-1}dx - b(n+\lambda\theta)x^{n+\theta-1}dx - c(n+2\lambda\theta)x^{n+2\theta-1}dx,$$

from which by its parts being integrated, then indeed by putting $a-bx^\theta - cx^{2\theta} = 0$, (insofar as if it were it will be $x^\theta = -\frac{b+\sqrt{(bb+4ac)}}{2c}$) this reduction will be had in general:

$$na\int x^{n-1}dx(a-bx^\theta - cx^{2\theta})^{\lambda-1}$$
$$= (n+\lambda\theta)b\int x^{n+\theta-1}dx(a-bx^\theta - cx^{2\theta})^{\lambda-1}$$
$$+(n+2\lambda\theta)c\int x^{n+2\theta-1}dx(a-bx^\theta - cx^{2\theta})^{\lambda-1}.$$

30. But if we want to compare this form with our general one related earlier, we must discover the values for the letter n which are to be successively assumed for different θ. Then it is not necessary that the first value for n, as we have so far made it, be taken as equal to 1; we shall set therefore the first value of it as equal to α, and we shall search for the values of the two following integral formulas, namely:

$$A = \int x^{\alpha-1}dx(a-bx^\theta - cx^{2\theta})^{\lambda-1} \text{ and}$$
$$B = \int x^{\alpha+\theta-1}dx(a-bx^\theta - cx^{2\theta})^{\lambda-1},$$

which integrals are taken such that they vanish by putting $x = 0$, which having been made the former value of x ought to be taken, which returns the formula $a - bx^\theta - c\alpha^{2\theta} = 0$. Since however it is not permitted for this to be carried out in general, we shall be content to indicate these values by the letters A and B, which therefore consider as known.

31. Thereafter indeed the letters f, g, h, with this having been derived will take on the following values:

$$f = \alpha a,\ f' = (\alpha+\theta)a,\ f'' = (\alpha+2\theta)a,\ f''' = (\alpha+3\theta)a, \text{ etc.}$$
$$g = (\alpha+\lambda\theta)b,\ g' = (\alpha+\theta+\lambda\theta)b,\ g'' = (\alpha+2\theta+\lambda\theta)b, \text{ etc.}$$
$$h = (\alpha+2\lambda\theta)c,\ h' = (\alpha+\theta+2\lambda\theta)c,\ h'' = (\alpha+2\theta+2\lambda\theta)c, \text{ etc.}$$

From this therefore will be formed the following continued fraction:

$$\frac{\alpha a A}{B} = (\alpha+\lambda\theta)b + \cfrac{(\alpha+\theta)(\alpha+\lambda\theta)ac}{(\alpha+\theta+\lambda\theta)b + \cfrac{(\alpha+2\theta)(\alpha+\theta+2\lambda\theta)ac}{(\alpha+2\theta+\lambda\theta)b + \cfrac{(\alpha+3\theta)(\alpha+2\theta+2\lambda\theta)ac}{(\alpha+3\theta+\lambda\theta)b \text{ etc.}}}},$$

which form is certainly most general, of which however we will not exclude another derivation.

VI. The expansion of the formula $s = x^n(1 - x^\theta)^\lambda$

32. Here therefore it would be

$$ds = nx^{n-1}dx(1 - x^\theta)^\lambda - \lambda\theta x^{n+\theta-1}dx(1 - x^\theta)^{\lambda-1},$$

from which will emerge two integral formulas; wherefore, for this, we may take an arbitrary denominator $a + bx^\theta$ of this differential, so that we may have:

$$ds = \frac{(1 - x^\theta)^{\lambda-1}}{a + bx^\theta}(nax^{n-1}dx - (a(n+\lambda\theta) - bn)x^{n+\theta-1}dx - b(n+\lambda\theta)x^{n+2\theta-1}dx).$$

Now therefore, by putting after integration $x = 1$, we deduce this reduction:

$$na\int \frac{x^{n-1}dx(1 - x^\theta)^{\lambda-1}}{a + bx^\theta} = (a(n + \lambda\theta) - bn)\int \frac{x^{n+\theta-1}dx(1 - x^\theta)^{\lambda-1}}{a + bx^\theta} + b(n + \lambda\theta)\int \frac{x^{n+2\theta-1}dx(1 - x^\theta)^{\lambda-1}}{a + bx^\theta}.$$

33. Here again it is clear what values should arise for n by differentiating by θ. Moreover, first the value of it should be set $n = \alpha$, and

$$A = \int \frac{x^{\alpha-1}dx(1 - x^\theta)^{\lambda-1}}{a + bx^\theta} \text{ and } B = \int \frac{x^{\alpha+\theta-1}dx(1 - x^\theta)^{\lambda-1}}{a + bx^\theta},$$

where namely after integration it shall be set $x = 1$. With this having been determined, it may then be made

$$f = \alpha a,\ f' = (\alpha + \theta)a,\ f'' = (\alpha + 2\theta)a,\ f''' = (\alpha + 3\theta)a, \text{ etc.}$$
$$g = (\alpha + \lambda\theta)a - \alpha b,\ g' = (\alpha + \theta + \lambda\theta)a - (\alpha + \theta)b,$$
$$g'' = (\alpha + 2\theta + \lambda\theta)a - (\alpha + 2\theta)b, \text{ etc.}$$
$$h = (\alpha + \lambda\theta)b,\ h' = (\alpha + \theta + \lambda\theta)b,\ h'' = (\alpha + \theta + 2\lambda\theta)b, \text{ etc. [sic]}$$

from which will be formed the following continued fraction:

$$\frac{\alpha aA}{B} = (\alpha+\lambda\theta)a - \alpha b + \cfrac{(\alpha + \theta)(\alpha + \lambda\theta)ab}{(\alpha + \theta + \lambda\theta)a - (\alpha + \theta)b + \cfrac{(\alpha+2\theta)(\alpha+\theta+\lambda\theta)ab}{(\alpha+2\theta+\lambda\theta)a - (\alpha+2\theta)b + (\alpha+3\theta)(\alpha+2\theta+\lambda\theta)ab \text{ etc.}}},$$

the copious expansion of which form we refrain from.

VII. The expansion of the formula $s = x^n(e^{\alpha x}(1-x)^\lambda)$

34. Here therefore it would be

$$ds = (1-x)^{\lambda-1}(nx^{n-1}dx - (n+\lambda-\alpha)x^n dx - \alpha x^n dx);$$

then consequently if after integration it is set everywhere $x = 1$, of course in which case it would be $s = 1$, we will have this reduction:

$$n\int x^{n-1}dxe^{\alpha x}(1-x)^{\lambda-1} = (n+\lambda-\alpha)\int x^n dxe^{\alpha x}(1-x)^{\lambda-1} + \alpha\int x^{n+1}dxe^{\alpha x}(1-x)^{\lambda-1}.$$

35. In this formula therefore, for the exponent n all the values ascending from unity ought to be taken, where indeed for the minimum value of this we shall take $n = \delta$, and then the values of the letters A and B will be able to be elicited from this formula, by it being put $x = 1$ after integration,

$$A = \int x^{\delta-1}dxe^{\alpha x}(1-x)^{\lambda-1}, \qquad B = \int x^\delta dxe^{\alpha x}(1-x)^{\lambda-1},$$

then indeed from these values:

$$f = \delta,\ f' = \delta+1,\ f'' = \delta+2,\ f''' = \delta+3, \text{ etc.}$$
$$g = \delta+\lambda-\alpha,\ g' = \delta+1+\lambda-\alpha,\ g'' = \delta+2+\lambda-\alpha, \text{ etc.}$$
$$h = \alpha,\ h' = \alpha,\ h'' = \alpha, \text{ etc.}$$

such a continued fraction follows:

$$\frac{\delta A}{B} = \delta+\lambda-\alpha + \cfrac{(\delta+1)\alpha}{\delta+1+\lambda-\alpha+\cfrac{(\delta+2)\alpha}{\delta+2+\lambda-\alpha+\cfrac{(\delta+3)\alpha}{\delta+3+\lambda-\alpha+\text{ etc.}}}}$$

Here it should be noted in particular that the exponents λ and δ should from necessity be taken as no greater than nothing, since otherwise the principal formula $x^n e^{\alpha x}(1-x)^\lambda$ will not vanish in those cases when $x = 1$.

36. If a value equal to 1 were taken for the letters δ and λ, the case treated above will come forth; and if integral numbers are assigned to these letters, in the same way continued fractions will arise, which by certain operations will be permitted to be reduced to the prior ones. Truly if either for either one or both of these letters δ and *lambda* fractions are assigned, then forms shall arise which are irreducible to the prior ones, of which the value would be able to be expressed in no other way than altogether transcendental quantities. For if it were $\delta = \frac{1}{2}$ and $\lambda = \frac{1}{2}$, the value of the letter A will be bound to be obtained from this integral formula: $A = \frac{e^{\alpha x}dx}{\sqrt{(x-xx)}}$ [sic], the integration of which leads to altogether transcendental quantities, thus so that the value of such continued fractions comes forth as most abstruse.