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EXACT HOMOMORPHISM SEQUENCES IN HOMOLOGY THEORY

By J. L. KELLEY AND EVERETT PITCHER

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The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2]) which form an extension of the Mayer-Vietoris formulas concerning coverings of a complex by two complexes. In this paper these relations are formalized and are seen to be consequences of the existence of an exact homomorphism sequence (see Definition 3.1). The concept of an exact sequence seems to be due to Hurewicz [WH]. Its principal property is used in the presentation of the Mayer-Vietoris formulas by Alexandroff and Hopf [A-H, pp. 297–299]. It has been used most notably by Eilenberg and Steenrod (reference [E-S]) as one of a very simple system of axioms for homology theory. (See [C] for another use.) In one sense, this paper might have been written from the point of view of exploiting this axiom. Actually we have found the most flexible approach to be the consideration of the exact sequence of homology groups on a Mayer complex as a fundamental algebraic identity. We exploit this identity in two directions. First we obtain a number of duality theorems and second, we investigate chain mappings and some closely related topics on coverings. A considerable part of the paper is methodological in character in that known results are deduced as part of a general line of reasoning.

In geometric applications involving Čech homology groups there are two procedures available. One might consider fundamental complexes for a compact metric space, in which case the fundamental algebraic identity could be used, or else one might set up a limiting process. We have used the latter method.

Section 1 is concerned with preliminary remarks on notation. In Section 2 we define the term *homomorphism sequence* and recall the definition of an abstract complex according to Mayer [WM 1, 2, 4] distinguishing chain and cochain complexes notationally and defining homology and cohomology groups. In Section 3 we define the term *exact sequence* and establish the fundamental construction of the paper, Theorem 3.3.

In Section 4, the limit of a direct system of homomorphism sequences is defined and direct limits of exact sequences are shown to be exact. Section 5 gives a summary of needed results on character theory. Compare with papers by Alexandroff [A] and Mayer [WM 4]. These results are used in Section 6 to establish algebraic duality in Mayer complexes. Compare with [WM 4]. The results of Section 5 are used again in Section 7, where limits of inverse systems of homomorphism sequences of compact groups are defined, to show that inverse limits of exact sequences of compact groups are compact.

In Section 8, the theory of inverse limits is applied to prove exactness of the homomorphism sequence of homology groups relative to a compact coefficient group of a space and a closed subspace (this was announced by Hurewicz [WH]) and to identify the dual sequence. For compact spaces a duality theorem of the Alexander type results. The exactness of the sequence of the singular homology groups of an arbitrary space and subspace is exhibited.

Poincaré duality is discussed in Section 9. If M is a manifold and N a closed subspace the form of statement here asserts the duality of the exact sequence of Čech homology groups of M and N and the exact sequence of singular homology groups of complementary dimension of M and M - N. Alexander-Pontrjagin duality is a consequence.

Chain mappings are discussed in Section 10. The homology groups of vanishing chains and the homology groups of image chains are introduced and the relevant exact sequences are set up. For singular homology theory and an arbitrary continuous mapping this introduces two sets of groups which are functions of the mapping.

A general construction relating the homology groups of a complex with the homology groups of the elements of a finite covering and of their intersections and of the nerve of the covering is described in Section 11. This extends the Mayer-Vietoris formulas to the case of coverings by more than two subcomplexes.

Some remarks on critical level theory are made in Section 12. When singular homology theory is used inequalities analogous to those of Morse [MM 1, 2, 3] are proved for an arbitrary bounded function with a finite number of critical levels.

References to classical results are made to Alexandroff and Hopf [A-H] and Lefschetz [L] when possible.

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1. Notation

The following notation is essentially that of Lefschetz, [L].

If A and B are sets $A \cap B$ is their intersection, $A \cup B$ their union. If A_i is a collection of sets $\bigcap A_i$ and $\bigcup A_i$ are respectively the intersection and union. The set of all elements of A which are not elements of B is written A - B.

If a is an element, $\{a\}$ is the set consisting of the single element a.

The set of elements satisfying a given condition is written $\{a \mid \text{condition}\}$. For example, for any set $A, A = \{a \mid a \in A\}$.

If f is a function whose domain is A and whose range is B we write $f: A \to B$ and read "f carries A into B." If C is a subset of A, fC is the set of images of C, that is, $fC = \{b \mid b \in B \text{ and for some } a \in C, fa = b\}$. The function whose domain is C and which has the same functional values as f is denoted by $f \mid C: C \to B$. If $f: A \to B$ and $C \subset B$, $f^{-1}C$ is the set of elements carried into C by f, that is $f^{-1}C = \{a \mid a \in A \text{ and } fa \in C\}$.

By group we shall mean additive, abelian, topological group. No separation axiom is assumed and the term *subgroup* is not restricted to closed sets. (Compare with "generalized topological groups," [E-M].)

If A and B are subsets of a group, A + B is the set of elements which can be written as the sum of an element of A and an element of B.

If A and B are groups $A \times B$ is the direct product, consisting of elements $(a, b), a \in A, b \in B$.

If B is a subgroup of a group A the collection of subsets of the form $\{a\} + B$, $a \in A$, form a group under the operation + defined above. This is the *factor* group, written A/B. A set is open in A/B if the union of its elements is open in A.

If A and B are groups, by a homomorphism $f:A \to B$, we mean a strongly continuous homomorphism. Explicitly, if V is an open set in A, we require that fV be open in fA. This requirement implies that a homomorphism of A onto B which is 1-1 is an isomorphism. We notice that the map of a group into its factor group is a (strongly continuous) homomorphism. For compact groups strong continuity is a consequence of continuity.

The kernel of a homomorphism $f: A \to B$ is denoted by K[f, A] and is $\{a \mid a \in A \text{ and } f a = 0\}$. For a subgroup $C \subset A$ we also define

$$K[f, C] = \{a \mid a \in C \text{ and } fa = 0\} = K[f, A] \cap C.$$

If $f:A \to B$ is a homomorphism and $A' \subset A$, $B' \subset B$ are subgroups such that $fA' \subset B'$ then f *induces* a homomorphism on A/A' to B/B' in which the image of $\{a\} + A'$ is $\{fa\} + B'$ for all $a \in A$.

The symbol \approx will be used for group isomorphism. "Isomorphism" will be used in the sense "isomorphism onto."

The following proposition is well-known.

NOETHER THEOREM 1.1. If A, B, C are groups with $A \supset B \supset C$, then B/C is a subgroup of A/C and $(A/C)/(B/C) \approx A/B$.

We shall frequently have to exhibit particular groups and homomorphisms. This will ordinarily be done by exhibiting homomorphisms on isomorphic copies. Formally, we shall say that the homomorphism $f: A \to B$ is equivalent to the homomorphism $g: C \to D$ under the isomorphisms h_1, h_2 if $h_1: A \to C, h_2: B \to D$ and $h_2f = h_1g$ holds as an identity on elements of A. The reader is referred to [E-M] where ideas of this sort are discussed in detail. Theorem 5.1e will provide an important construction for equivalences.

2. Mayer complexes

We begin with a preliminary definition. A sequence of groups and homomorphisms is termed a homomorphism sequence if it can be indexed from the integers so that, if G_r and g_r denote respectively the group and homomorphism with index r, then $g_r: G_r \to G_{r-1}$, $r = \cdots, -1, 0, 1, \cdots$.

Notations $\{G_r, g_r\}$ or $\{G_r, g\}$ and $G_{r+1} \xrightarrow{\rightarrow} G_r$, $G_{r+1} \rightarrow G_r$, with as many terms enumerated as needed for clarity, will be used for homomorphism sequences. Although the definition requires that the sequence permit indexing in a prescribed fashion it may in an application be indexed in the reverse direction or in some other fashion.

We shall have occasion to use homomorphisms of one homomorphism sequence into another. Homomorphisms

$$f_r: G_r \to G'_r$$

will be said to establish a homomorphism on $\{G_r, g_r\}$ to $\{G'_r, g'_r\}$, written $f_r: \{G_r, g_r\} \to \{G'_r, g'_r\}$, provided the homomorphisms f_r commute with the homomorphisms of the sequences in the sense that $g'_r f_r = f_{r-1}g_r$ is an identity on elements of G_r for all r.

In case the homomorphism of one homomorphism sequence into another is a group isomorphism for all values of the index, the two sequences are termed *equivalent*. A problem of finding a homomorphism sequence with certain properties may ordinarily be regarded as solved when an equivalent sequence is exhibited and we shall use the concept of equivalent sequence in this way.

The homomorphism sequences of greatest interest for the purposes of this paper are Mayer chain and cochain complexes, the first of which is defined as follows.

DEFINITION 2.1. A Mayer chain complex M is a homomorphism sequence $M = \{C_r, \beta\}, \beta: C_r \to C_{r-1}$, such that $\beta\beta = 0$ for all groups C_r . The groups C_r are called *chain groups*, their elements are *chains*, and β is called the *boundary homomorphism*.

For notational convenience if M is a Mayer chain complex then $C_r(M)$ may be used as notation for its r^{th} chain group.

A chain complex N is termed a subcomplex of M if $C_r(N) \subset C_r(M)$ and if the boundary homomorphism of M agrees on $C_r(N)$ with that of N. The same symbol will be used for the boundary homomorphism on a complex and on a subcomplex.

If N is a subcomplex of M a quotient complex M/N may be constructed. Its chains groups $C_r(M/N)$ are defined to be $C_r(M)/C_r(N)$ and its boundary homomorphism is induced by that of M. Explicitly, the boundary of $\{a\} + C_r(N)$ is $\{\beta a\} + C_{r-1}(N)$.

We define the homology groups of a Mayer complex in the conventional manner.

Definition 2.2.

$$\begin{split} Z_r(M) &= group \text{ of } r\text{-dimensional cycles of } M \\ &= K[\beta, C_r(M)] \\ B_r(M) &= group \text{ of } r\text{-dimensional bounding cycles of } M \\ &= \beta C_{r+1}(M) \\ H_r(M) &= r\text{-dimensional homology group of } M \\ &= Z_r(M)/B_r(M). \end{split}$$

The chain groups of a cellular complex (explicitly, finite chains in a closure finite complex or infinite chains on a star-finite complex; see [L]) with the usual

boundary homomorphism form a Mayer chain complex whose cycles, boundaries and homology groups are the usual ones. A subcomplex generates a Mayer subcomplex. The homology theory of the quotient Mayer complex corresponds to the usual *relative homology* theory. To show this we make the following definition.

DEFINITION 2.3. If N is a subcomplex of the Mayer chain complex M,

$$Z_r(M \mod N) = group \text{ of } r\text{-dimensional cycles } \mod N$$
$$= \beta^{-1}C_{r-1}(N)$$
$$B_r(M \mod N) = group \text{ of } r\text{-dimensional bounding cycles } \mod N$$
$$= \beta C_{r+1}(M) + C_r(N)$$
$$H_r(M \mod N) = r\text{-dimensional homology group of } M \mod N$$

 $= Z_r(M \mod N)/B_r(M \mod N).$

We easily establish the following lemma.

LEMMA 2.4. The groups $H_r(M \mod N)$ are isomorphic to the groups $H_r(M/N)$ under the isomorphism of the Noether theorem.

For following definition (2.2) and the definition of a quotient complex

$$Z_{r}(M/N) = \beta^{-1}C_{r-1}(N)/C_{r}(N)$$

$$B_{r}(M/N) = [\beta C_{r+1}(M) + C_{r}(N)]/C_{r}(N)$$

When this is compared with definition (2.3) the truth of the lemma is seen.

On account of this isomorphism we can and do identify the groups $H_r(M \mod N)$ and $H_r(M/N)$, and we use either of the definitions as is convenient. The group $H_r(M/N)$ is useful since it is the group of a Mayer complex, but has the disadvantage that its elements are sets of sets of chains.

We shall also use Mayer cochain complexes, which differ from chain complexes essentially only in notation.

DEFINITION 2.5. A homomorphism sequence $M = \{C^r, \beta'\}, \beta': C^r \to C^{r+1}$, is a Mayer cochain complex if $\beta'\beta' = 0$ for all C^r .

Superscript notation will always be used for cochain complexes and subscript notation for chain complexes. The groups C' = C'(M) will be termed cochain groups and β' the coboundary homomorphism. If one defines $C_r = C^{-r}$ the cochain complex corresponds to a chain complex. Cocycles, coboundaries and cohomology groups are defined so as to correspond in this way to cycles, boundaries and homology groups. Also subcomplex and quotient complex of a cochain complex correspond to the concepts already defined for a chain complex and the relative theory and quotient theories are again to be identified.

Cochain groups of a cellular complex in the usual sense (finite cochains for a star-finite complex or infinite cochains for a closure finite complex) form a Mayer cochain complex. A Mayer subcomplex corresponds to cochains of an open cellular subcomplex and the quotient complex to cochains relative to an open cellular subcomplex.

Any results on chain complexes can be immediately transferred to cochain

complexes by associating with each cochain complex $\{C^r, \beta'\}$ the chain complex $\{C_r, \beta\}$, where $C^r = C_{-r}$ and $\beta = \beta'$.

3. Exact homomorphism sequences

The terms Mayer chain and cochain complex have been defined by restricting the concept of homomorphism sequence. The term exact sequence is defined by imposing additional restrictions.

DEFINITION 3.1. A homomorphism sequence is said to be *exact* if the kernel of the homomorphism on each group is identical with the image of the preceding group.

If $0 \to A \to B$ occurs as part of an exact sequence then the map $A \to B$ is an isomorphism onto a subgroup of B. If $A \to B \to 0$ occurs, then the image of A covers B. Thus if $0 \to A \to B \to 0$ occurs, the homomorphism of A into B is an isomorphism.

Homomorphism sequences whose groups consist of a single element outside a finite interval or on a half infinite interval are sometimes of special interest. If a homomorphism sequence is defined with a finite or half infinite range for its index and is termed exact it will be understood to mean that the groups consist of a single element for all other values of the index. This implies, in particular, that if there is a first map, it is an isomorphism onto a subgroup and if there is a last map, it is a homomorphism onto.

The fundamental construction of this paper is embodied in the exact sequence theorem for which we shall now prepare. Let M denote a Mayer chain complex and N a subcomplex. There are three basic homomorphisms defined and denoted as follows:

	$\alpha: \text{ for } A \in H_r(M),$	$\alpha A = A + C_r(N) \epsilon H_r(M \mod M)$	d N)
(3.2)	β : for $A \in H_r(M \mod N)$,	$\beta A \qquad \epsilon H_{r-1}(N)$	
	γ : for $A \in H_{r-1}(N)$,	$\gamma A = A + \beta C_r(M) \epsilon H_{r-1}(M)$	•

Then the theorem is as follows.

THEOREM 3.3. If M is a Mayer chain complex and N a subcomplex then the homomorphism sequence

$$\cdots \to H_r(M) \xrightarrow{\sim} H_r(M/N) \xrightarrow{\sim} H_{r-1}(N) \xrightarrow{\sim} H_{r-1}(M) \to \cdots$$

is exact. The homomorphisms α and γ are induced by the identity map and the homomorphisms β by the boundary map.

The proof consists of verifying that the respective kernel-images are

$$K [\alpha, H_r(M)] = \gamma H_r(N) = [Z_r(N) + B_r(M)]/B_r(M) K [\beta, H_r(M/N)] = \alpha H_r(M) = [Z_r(M) + C_r(N)]/B_r(M \mod N) K [\gamma, H_{r-1}(N)] = \beta H_r(M/N) = Z_{r-1}(N) \cap B_{r-1}(M)/B_{r-1}(N).$$

Advantage has been taken of the identification of $H_r(M/N)$ and $H_r(M \mod N)$. A consequence of Theorem 3.3 is the analogue for cohomology.

THEOREM 3.4. If M is a Mayer cochain complex and N a subcomplex then the homomorphism sequence

$$\cdots \to H^{r}(M) \xrightarrow[\alpha]{} H^{r}(M/N) \xrightarrow[\beta]{} H^{r+1}(N) \xrightarrow[\gamma]{} H^{r+1}(M) \to \cdots$$

is exact. The homomorphisms α' and γ' are induced by the identity and the homomorphism β' is induced by the coboundary homomorphism.

Examining the case of Theorem 3.3 where M is a cyclic in two successive dimensions, that is, where $H_r(M) = H_{r-1}(M) = 0$, we can state the following theorem.

THEOREM 3.5. If M is a Mayer chain complex with $H_r(M) = H_{r-1}(M) = 0$ then for any subcomplex N, $H_r(M/N)$ and $H_{r-1}(N)$ are isomorphic under the map induced by the boundary homomorphism β .

Theorem 3.3 has a corollary about the ranks of groups. Writing $\rho[G]$ for the rank of the group G and abbreviating $\rho[H_r(M)]$ to $\rho_r(M)$ we state the corollary.

COROLLARY 3.6. If the ranks on the right hand side in one of the inequalities

$$\rho_r(N) \leq \rho_{r+1}(M/N) + \rho_r(M)$$

$$\rho_r(M) \leq \rho_r(N) + \rho_r(M/N)$$

$$\rho_r(M/N) \leq \rho_r(M) + \rho_{r-1}(N)$$

are finite then so is the rank on the left, and the inequality holds. If all ranks used in the following inequality are finite and $\rho_{-1}(N) = 0$ then

$$\sum_{r=0}^{q} (-1)^{q+r} \left[\rho_r(N) - \rho_r(M) + \rho_r(M/N) \right] \ge 0.$$

If in addition $\rho_{q+1}(M/N) = 0$ the inequality is an equality. If the homology groups of M and N have finite rank and vanish for |r| large and χ denotes Euler characteristic then

 $\chi(M) = \chi(N) + \chi(M/N).$

The corollary follows from the relation

$$\rho[G] = \rho[K[f, G]] + \rho[fG]$$

which applies to any map f on a group G when two of the ranks are finite. Of course the finiteness of rank of a group in an exact sequence may be inferred from that of its two immediate neighbors.

We close this section with the remark that an acyclic Mayer complex is an exact sequence and vice versa. Thus Theorem 3.5 can be used to examine a subsequence of an exact sequence. If $\{A_r, \varphi\}$ is a homomorphism sequence, $\{B_r, \psi\}$ is a subsequence if $B_r \subset A_r$ and $\psi = \varphi | B_r$. The quotient sequence is $\{C_r, \theta\}$ where $C_r = A_r/B_r$ and θ is induced by φ . Theorem 3.5 then has the following corollary.

COROLLARY 3.7. A subsequence of an exact sequence is exact if and only if the quotient sequence is exact.

688

HOMOLOGY THEORY

4. Direct systems of homomorphism sequences

It is necessary to set up a limiting process for homomorphism sequences. This is an extension of the usual process for the construction of limit groups. In formulating the ideas of a directed system of homomorphism sequences we shall presume familiarity with the notions of directed set, product, weak product, inverse limit and direct limit as given in [L]. In this section we shall formulate the idea of direct systems of exact sequences while the idea of inverse system is postponed to section 7.

Let Λ denote a directed set. Suppose that for each $\lambda \in \Lambda$ there is a homomorphism sequence $\psi_{\lambda}: A_{\lambda}^{r} \to A_{\lambda}^{r+1}$ of discrete groups. Suppose that there are homomophisms π_{μ}^{λ} , to be termed *projections*, of $\{A_{\lambda}^{r}, \psi_{\lambda}\}$ into $\{A_{\mu}^{r}, \psi_{\mu}\}$ whenever $\mu > \lambda$ such that

$$\pi^{\mu}_{\nu} \pi^{\lambda}_{\mu} = \pi^{\lambda}_{\nu}$$

whenever $\nu > \mu > \lambda$. Then the system consisting of the directed set, the sequences and the projections is called a *direct system of homomorphism sequences*. (Note that the projections are homomorphisms of a sequence into a sequence; i.e. commutativity relations are assumed.)

The weak products $\mathbf{P}^{w}A_{\lambda}^{r}$ form a homomorphism sequence under homomorphisms

$$\Psi: \mathbf{P}^{\boldsymbol{w}} A_{\boldsymbol{\lambda}}^{\boldsymbol{r}} \to \mathbf{P}^{\boldsymbol{w}} A_{\boldsymbol{\lambda}}^{\boldsymbol{r+1}}$$

defined coordinate-wise; if $a \in \mathbf{P}^{\nu} A_{\lambda}^{\tau}$ then $(\Psi a)_{\lambda} = \psi_{\lambda} a_{\lambda}$ where the subscript λ is adjoined to the symbol for an element of a product to denote the coordinate of the element in the group with index λ .

The direct limits Dir Lim A_{λ}^{r} , which are factor groups of the groups $\mathbf{P}^{\boldsymbol{\psi}}A_{\lambda}^{r}$, form a homomorphism sequence under the homomorphism $\boldsymbol{\psi}$ induced by the homomorphisms Ψ .

We have now the following theorem.

THEOREM 4.1. Let $\{A_{\lambda}^{r}, \psi_{\lambda}\}$ be a direct system of exact homophism sequences. Let Ψ be defined coordinate-wise on $\mathbf{P}^{v}A_{\lambda}^{r}$ and let ψ be the induced map on Dir Lim A_{λ}^{r} . Then $\{\mathbf{P}^{v}A_{\lambda}^{r}, \Psi\}$ and $\{\text{Dir Lim } A_{\lambda}^{r}, \psi\}$ are exact sequences.

We shall prove only the exactness of the direct limit sequence. Any element $a \in \text{Dir Lim } A'_{\lambda}$ is carried into 0 by application of two consecutive homomorphisms ψ . On the other hand suppose $\psi a = 0$. The coset a contains an element a' with only one non-zero coordinate, say a'_{β} . Then for some α , $\pi^{\beta}_{\alpha}\psi_{\beta}a'_{\beta} = 0$. Let $b_{\alpha} = \pi^{\beta}_{\alpha}a'_{\beta}$ and let b be the element of $\mathbf{P}^{w}A'_{\lambda}$ which has coordinates zero save in A'_{α} and there has coordinate b_{α} . Then $b \in a$ also. Since $\{A'_{\alpha}, \psi_{\alpha}\}$ is exact, there is an element $c_{\alpha} \in A^{r-1}_{\alpha}$, with $\psi_{\alpha}c_{\alpha} = b_{\alpha}$. Let c have coordinates zero save in A^{r-1}_{α} , and have coordinate c_{α} there. Then the element of Dir Lim A^{r-1}_{α} containing c maps onto a.

It would be convenient if the inverse limit of a system of exact homomorphism sequences were exact. This, as has been noticed by Eilenberg and Steenrod, is in general not true. It is however true in case the groups are compact and can then be deduced from Theorem 4.1. Accordingly we shall state the necessary facts from the theory of character groups and return to inverse limits in section 7.

5. Character groups and dual homomorphisms

For reference we shall collect facts about character groups and dual homomorphisms, most of which are familiar. All groups considered here will be either compact (bicompact) or discrete and subgroup will mean closed subgroup.

A homomorphism of a group A into the group of real numbers mod 1 is a *character* of A. The group of such homomorphisms is the character group of A and is denoted by A^* . The character group is given in the familiar compactopen topology; i.e. all characters carrying a fixed compact set of A into a fixed open set of the reals mod 1 form an open set in A^* , and all open sets in A^* are finite intersections or arbitrary unions of such sets. If A is discrete (compact) then A^* is compact (discrete). The notation $ba, b \in A^*, a \in A$, will be functional notation for such a homomorphism consistent with our general functional notation. For a subgroup $B \subset A$ the set of characters of A which carry B into zero form a subgroup of A^* which is called the *annihilator of B in A**. Explicitly, we define

Annih
$$B = \{a \mid a \in A^* \text{ and if } b \in B \text{ then } ab = 0\}.$$

Similarly, if B is a subgroup of A^* , Annih B is the subgroup of A consisting of all elements carried into zero by every element of B. The results needed in the rest of this paper can be summarized briefly. The reader is referred to [L, pp. 59–72].

THEOREM 5.1. a. If B is a subgroup of A and $a \in A$, $a \in B$, there exists $c \in Annih B$ in A^* with $ca \neq 0$.

b. If B is a subgroup of A^* , and $c \in A^*$, $c \in B$, there exists a ϵ Annih B in A with $ca \neq 0$.

c. If B is a subgroup of A or of A^* , Annih B = B.

d. For any group A, $A^{**} \approx A$, the element $a \in A$ corresponding to the function whose value at b, $b \in A^*$, is ba.

e. If A and B are subgroups of C and $A \subset B$ then $(B/A)^* \approx \text{Annih } A/\text{Annih}$ B where the annihilators are subgroups of C^* .

Actually parts c, d, e of this theorem can be derived easily from parts a and b. It will be convenient for future use to display explicitly the isomorphism whose existence is asserted in part e.

LEMMA 5.2. Let C be a group and A and B be subgroups such that $A \subset B \subset C$. Then for $D \in B/A$ and $E \in Annih A/Annih B$, the union of all images of D under elements of E consists of a single element of the group of real numbers mod 1. Each $E \in Annih A/Annih B$ corresponds in this fashion to a unique element of $(B/A)^*$. This correspondence is an isomorphism.

The isomorphism whose existence is affirmed in Theorem 5.1e will be used to establish equivalences in the sense defined in Section 1. In these cases, for example, Lemmas 5.5 and 5.6, and Theorem 6.5, the isomorphism of the equivalence will not be specifically named.

We will assume implicitly that each group A is identified with A^{**} under the isomorphism of Theorem 6.1d, so that each element of A is considered as a character of A^* .

If f is a homomorphism of the group A into the group B, a homomorphism f^* on B^* into A^* is define as follows. Corresponding to each element $a \in A$ and $b \in B^*$, bfa is an element of the group of reals mod 1. Thus corresponding to an element $b \in B^*$ an element $f^*b = bf$ in A^* has been designated. The homomorphism f^* on B^* to A^* is termed the homomorphism dual to f. It is readily established that $f^{**} = f$ from the identity $f^*ba = bfa$ in elements $a \in A$ and $b \in B^*$.

The dual of the operational product of functions is identified in the following lemma.

LEMMA 5.3. If $f: A \to B$ and $g: B \to C$ then the dual of $gf: A \to C$ is $(gf)^* = f^*g^*$.

The lemma follows from repeated use of the definition of dual homomorphism. Supposing $f:A \to B$ it will be useful to compute K[f, A] in terms of the dual map.

$$K[f, A] = \{a \mid fa = 0\} = \{a \mid af^* = 0\}$$
$$= \{a \mid af^*B^* = \{0\}\} = \text{Annih } f^*B^* \text{ in } A.$$

Now Theorem 5.1 on character groups leads to the following result, due to Alexandroff; see [A].

ALEXANDROFF LEMMA 5.4. Let $f: A \to B$ and $f^*: B^* \to A^*$ be dual maps. Then the following statements are true.

a.
$$K[f, A] = \operatorname{Annih} f^*B^*$$
 and $K[f^*, B^*] = \operatorname{Annih} fA$
b. $K[f, A]^* \approx A^*/f^*B^*$ and $K[f^*, B^*]^* \approx B/fA$

c. $(fA)^* \approx f^*B^*$.

For convenience we identify the dual map in three special cases in the lemmas which follow.

LEMMA 5.5. If B, C, D, E are subgroups of a group A with $B \supset C \supset E$ and $B \supset D \supset E$ and f is the homomorphism induced by the identity on B carrying C/E into B/D, then the dual homomorphism f^* carrying $(B/D)^*$ into $(C/E)^*$ is equivalent to the homomorphism induced by the identity on Annih C carrying Annih D/Annih B into Annih E/Annih C.

A lemma implying Lemma 5.5 is the following.

LEMMA 5.6. If groups are related so that $A \supset B \supset C, A' \supset B' \supset C', g: A \to A',$ $gB \subset B', gC \subset C'$, then the dual f^* to the induced homomorphism f on B/C to B'/C' is equivalent to the homomorphism on Annih C'/Annih B' to Annih C/ Annih B induced by $g^*: A'^* \to A^*$.

LEMMA 5.7. If A is a subgroup of B and i is the identity map on B to B, the dual to the map i | A on A to B is equivalent to the map of B^* into $B^*/Annih A$. If $f: B \to C$ then

Annih
$$K[f, A] = f^*C^* + \text{Annih A}$$

Annih $fA = f^{*-1}$ Annih A.

These lemmas are consequences of Theorem 5.1 and Lemmas 5.2 and 5.4 and the details of proof will be omitted.

If $\{A_r, \varphi\}$ is a homomorphism sequence then the character groups under the dual homomorphisms form a homomorphism sequence $\{A_r^*, \varphi^*\}$ termed the dual homomorphism sequence.

The following theorem permits application of character theory to exact sequences.

THEOREM 5.8. The homomorphism sequence dual to an exact sequence is exact.

Let $\{A_r, \varphi\}$ denote an arbitrary exact homomorphism sequence and consider the homomorphism sequence $\{A_r^*, \varphi^*\}$. One has in succession

 $K[\varphi^*, A_r^*] = \text{Annih } \varphi A_{r+1} = \text{Annih } K[\varphi, A_r] = \varphi^* A_{r-1}.$

These equalities follow respectively from Lemma 5.4a, the exactness of $\{A_r, \varphi\}$ and Lemma 5.4a again. This establishes the theorem.

6. Algebraic duality in Mayer complexes

Suppose M is a Mayer chain complex $\{C_r, \beta\}$. We assume all C_r are either compact or discrete. Then the character groups $C^r = C_r^*$ under the dual homomorphism form a Mayer cochain complex. We shall call it the *cochain complex* of M and use the notation $H^r(M)$ for its cohomology groups. This corresponds to the fact that in applications the homology and cohomology theories for a geometric complex arise together.

If N is a subcomplex of the chain complex M then the groups $C^{r}(M - N) =$ Annih $C_{r}(N)$ under the coboundary homomorphism β^{*} , suitably restricted in domain, form a subcomplex of the cochain complex of M. The truth of this statement rests on the fact that $\beta^{*}C^{r}(M - N) \subset C^{r+1}(M - N)$. This can be established from the relation $\beta C_{r+1}(N) \subset C_{r}(N)$ by taking annihilators of both sides, using Lemma 5.7 and applying β^{*} .

If M is the system of chains over a geometric complex and N the chains over a subcomplex then $C^{r}(M - N)$ is the group of *r*-cochains which vanish identically on the subcomplex or of *r*-cochains on the complement.

We recall that the groups $H_r(M)$, $H_r(N)$, $H_r(M/N) \approx H_r(M \mod N)$, $H^r(M)$, $H^r(M - N)$, $H^r(M/N - N) \approx H_r(M \mod M - N)$ are all well defined.

We prepare for the principal theorem of this section.

Since $\beta: C_{r+1}(M) \to C_r(M)$ and $\beta^*: C^r(M) \to C^{r+1}(M)$ are dual mappings Lemma 5.4a implies that

(6.1)
Annih
$$Z_r(M) = B^r(M)$$

Annih $B_r(M) = Z^r(M)$.

By use of Lemma 5.7 the annihilators in $C_r(M)$ of $Z^r(M - N)$ and $B^r(M - N)$ are shown to be

(6.2)
Annih
$$Z^{r}(M - N) = B_{r}(M) + C_{r}(N) = B_{r}(M \mod N)$$

Annih $B^{r}(M - N) = \beta^{-1}C_{r-1}(N) = Z_{r}(M \mod N).$

692

It is the equality of the annihilators of the groups of (6.2) which will be used. By use of Lemma 5.7 again the annihilators in $C^{r}(M)$ of $Z_{r}(N)$ and $B_{r}(N)$ are shown to be

(6.3)
Annih
$$Z_r(N) = B^r(M) + C^r(M-N) = B^r(M \mod M - N)$$

Annih $B_r(N) = \beta^{*-1}C^{r+1}(M-N) = Z^r(M \mod M - N).$

We are now prepared to compute character groups of $H_r(M)$, $H_r(M/N)$ and $H_r(N)$. Before doing so we note a fact we shall not use in the next theorem, namely that

(6.4)
$$H'(N) \approx H'(M \mod M - N).$$

For from (6.1) with M replaced by N and (6.3) each side is seen to be isomorphic with the dual to $H_r(N)$. The isomorphism can be identified more precisely but we shall not do so since the viewpoint "cohomology mod an open set" will be the one desired in further developments.

Our principle theorem of this section follows.

DUALITY THEOREM 6.5. The dual of the exact sequence

 $\cdots \to H_r(M) \xrightarrow{\sim} H_r(M \mod N) \xrightarrow{\sim} H_{r-1}(N) \xrightarrow{\sim} H_{r-1}(M) \to \cdots$

is equivalent to the exact sequence

$$\cdots \leftarrow H^{r}(M) \xleftarrow[]{}{}_{a^{\bullet}} H^{r}(M-N) \xleftarrow[]{}_{\beta^{\bullet}} H^{r-1}(M \mod M-N) \xleftarrow[]{}_{\gamma^{\bullet}} H^{r-1}(M) \leftarrow \cdots$$

The homomorphisms α^* and γ^* are induced by the identity and β^* is induced by the dual to the boundary homomorphism.

We have used α^* , heretofore the notation for the dual to α , for the correspondent to the dual in the equivalence inasmuch as we shall in practice identify the dual sequence and its equivalent.

The groups isomorphic to the dual groups are identified by use of (6.1), (6.2), (6.3) and Theorem 5.1e. The exactness of the dual sequence is stated in Theorem 5.8. The homomorphisms α^* and γ^* are identified in Lemma 5.7 while β^* is identified in Lemma 5.6. This completes the proof.

Statement c of the Alexandroff Lemma 5.4 shows that under the isomorphic equivalence the duals to the image-kernel groups in the given sequence are the image-kernel groups in the dual sequence. These groups have topological significance and it seems proper to display them in descriptive form. This is done in the following theorem.

DUALITY THEOREM 6.6. Let M be a Mayer complex and N a subcomplex. Then the following groups are dual pairs.

a.	The subgroup of $H_r(M)$ of ele- ments containing elements of $H_r(N)$	and	The subgroup of $H^{\tau}(M \mod M - N)$ of elements containing cocycles of M
b.	The subgroup of $H_r(M \mod N)$ of elements containing cycles of M	and	The subgroup of $H'(M)$ of ele- ments containing elements of $H'(M-N)$

The subgroup of $H_r(N)$ of elec. ments containing cycles bounding in M and the subgroup of $H^{r+1}(M-N)$ of elements containing cocycles cobounding in M.

There are, in view of the kernel-image identity, alternate descriptions of each of the above groups.

7. Inverse systems of exact sequences

As we remarked in Section 5, we shall not develop inverse systems independently but shall use character theory and the development of direct systems already carried out. We recall again that the ideas of products and limits used here are explained in [L].

Suppose Λ is a directed set and $\varphi_{\lambda}: A_{r\lambda} \to A_{r-1,\lambda}$, $\lambda \in \Lambda$, a set of homomorphism sequences of compact groups admitting homomorphisms, to be termed *projections*,

$$\pi^{\mu}_{\lambda}: \{A_{r\mu}, \varphi_{\mu}\} \to \{A_{r\lambda}, \varphi_{\lambda}\}$$

whenever $\mu > \lambda$ which have the property

$$\pi^{\mu}_{\lambda} \pi^{\nu}_{\mu} = \pi^{\nu}_{\lambda}$$

whenever $\nu > \mu > \lambda$. Then the directed set, the homomorphism sequences and the projections form an *inverse system of homomorphism sequences*.

The dual homomorphism sequences $\{A_{r\lambda}^*, \varphi_{\lambda}^*\}$ under the dual projections $\pi_{\lambda}^{\mu*}$ form a direct system which dualizes in turn to the given inverse system. Either system completely determines the other.

The products $\mathbf{P}A_{r\lambda}$ and $\mathbf{P}^{\boldsymbol{v}}A_{r\lambda}^{*}$ admit homomorphisms

$$\Phi: \mathbf{P}A_{r+1,\lambda} \to \mathbf{P}A_{r\lambda}$$
$$\Phi^{\times}: \mathbf{P}^{w}A_{r\lambda}^{*} \to \mathbf{P}^{w}A_{r+1,\lambda}^{*}$$

defined coordinate-wise; if $a \in \mathbf{P}A_{r+1,\lambda}$ then $(\Phi a)_{\lambda} = \varphi_{\lambda}a_{\lambda}$ and if $b \in \mathbf{P}^{\nu}A_{r+1,\lambda}^{*}$ then $(\Phi^{\times}a)_{\lambda} = \varphi_{\lambda}^{*}b_{\lambda}$, where a symbol x_{λ} denotes the coordinate of the element x in the group with index λ .

Special homomorphisms

(7.1)
$$f_r: \mathbf{P}^{\boldsymbol{w}} A_{r\boldsymbol{\lambda}}^* \to (\mathbf{P} A_{r\boldsymbol{\lambda}})^*$$

are defined as follows. If $a \in \mathbf{P}A_{r\lambda}$ and $b \in \mathbf{P}^{v}A_{r\lambda}^{*}$ then the value of $f_{r}b$ at a is

(7.2)
$$(f_r b)a = \sum_{\lambda} a_{\lambda} b_{\lambda} .$$

Its usefulness is in the following lemma.

LEMMA 7.3. The special homomorphisms f_r are isomorphisms under which the sequence $\{\mathbf{P}^{\boldsymbol{v}}A_{r\lambda}^*, \boldsymbol{\Phi}^*\}$ is equivalent to the sequence $\{\mathbf{P}A_{r\lambda}\}^*, \boldsymbol{\Phi}^*\}$.

The fact that f_r is an isomorphism is established in [L]. To prove the identity $f_{r+1}\Phi^{\times} = \Phi^* f_r$, let a denote an element of $\mathbf{P}A_{r+1,\lambda}$ and b an element of $\mathbf{P}^{\nu}A_{r\lambda}^*$. Then either member at b is a homomorphism on $\mathbf{P}A_{r+1,\lambda}$ to the reals mod 1. The left member is the homomorphism $f_{r+1}(\Phi^{\times}b)$ whose value at a is $\sum_{\lambda} a_{\lambda}(\varphi_{\lambda}^{\times}b_{\lambda}) =$ $\sum_{\lambda} (\varphi_{\lambda} a_{\lambda}) b_{\lambda}.$ The right member is the homomorphism $\Phi^{*}(f_{r}b)$ whose value at a is $[\Phi^{*}(f_{r}b)] a = (f_{r}b) (\Phi a) = \sum_{\lambda} (\varphi_{\lambda} a_{\lambda}) b_{\lambda}.$

REMARK 7.4. At this point we identify $\{\mathbf{P}^{\boldsymbol{\nu}}A_{r\lambda}^*, \Phi^{\times}\}$ with $\{\mathbf{P}A_{r\lambda}^*, \Phi^{\times}\}$ using the mixed notation $\{\mathbf{P}^{\boldsymbol{\nu}}A_{r\lambda}^*, \Phi^{*}\}$.

We shall let A_{r0} denote the subgroup of $\mathbf{P}^{\boldsymbol{v}}A_{r\lambda}^{\star}$ consisting of elements b such that $\sum_{\lambda} \pi_{\alpha}^{\lambda} b_{\lambda} = 0$ for some α greater than the indices of all non-null coordinates of b. Then Dir Lim $A_{r\lambda}^{\star} = \mathbf{P}^{\boldsymbol{v}}A_{r\lambda}^{\star}/A_{r0}$. The following lemma is proved in [L]. LEMMA 7.5. The annihilator of A_{r0} is Inv Lim $A_{r\lambda}$.

Let φ denote the homomorphism on Inv Lim $A_{r\lambda}$ obtained by restricting Φ and let φ^* denote the homomorphism on Dir Lim $A_{r\lambda}^*$ induced by Φ^* . We can state the following theorem.

THEOREM 7.6. Each of the homomorphism sequences {Inv Lim $A_{r\lambda}$, φ } and {Dir Lim $A_{r\lambda}^*$, φ^* } is equivalent to the dual of the other.

The isomorphism of the groups is the one established in Theorem 5.1e and Lemma 5.2 and under this isomorphism the equivalence is established in Lemma 5.6.

REMARK 7.7. We shall refer to the sequences of Theorem 7.6 as dual sequences, making the identification corresponding to the equivalence.

Theorem 7.6 with Theorem 5.8 provides an indirect proof of the following theorem.

THEOREM 7.8. Let $\{A_{r\lambda}, \varphi_{\lambda}\}$ be an inverse system of exact sequences of compact groups. Let Φ be defined coordinate-wise on $\mathbf{P}A_{r\lambda}$ and let φ be defined on Inv Lim $A_{r\lambda}$ by restricting the domain and range of Φ . Then $\{\mathbf{P}A_{r\lambda}, \Phi\}$ and $\{\text{Inv Lim} A_{r\lambda}, \varphi\}$ are exact sequences.

8. Čech and singular homology sequences

As has been indicated, the inverse limit of exact sequences is not always exact. It is therefore not to be expected that for a space and a closed subspace the corresponding Čech homology group sequence should in general be exact. We shall show that it is exact if the coefficient group is compact. This is proved by means of character theory, and our primary concern is actually with cohomology over a discrete coefficient group. We give a formal presentation since some rather delicate considerations arise. All Čech cohomology groups are computed with a fixed discrete coefficient group J, and all homology groups with J^* .

Let X be a topological space and Y a closed subspace. Let T(X) be the set of all open sets of X. Then T(Y) is obtained as the intersections of Y with elements of T(X).

A covering of X is a map of the positive integers $\mu: I \to T(X)$ such that

a. For all but a finite number of integers μi is the null set.

b. $\bigcup \mu i = X$.

The set $\Lambda(X)$ of all coverings of X is made to be a directed set by the convention that $\mu > \lambda$ if for each j there is an i_j such that $\mu j \subset \lambda i_j$. We also say that μ is a refinement of λ .

For each λ we construct the *nerve* of λ , N_{λ} , which is a finite simplicial complex.

A simplex consists of a finite set of integers together with the map λ , written $(i_0, \dots, i_q; \lambda)$, and the requirements

c. $\bigcap \lambda i_j \neq 0$, and

d. The finite set of indices is non-vacuous.

If condition d is not imposed the nerve has a (-1)-dimensional simplex, and is *augmented* in the sense of [L], p. 130. We note that N_{λ} and N_{μ} are disjoint if $\lambda \neq \mu$.

Whenever $\mu > \lambda$ there exist projections carrying N_{μ} into N_{λ} corresponding to maps carrying each vertex $(i; \mu)$ into a vertex $(j; \lambda)$ with $\mu i \subset \lambda j$. It is known that any two such maps induce the same homomorphisms

 $\pi^{\lambda}_{\mu}: H^{r}(N_{\lambda}) \to H^{r}(N_{\mu}) \quad \text{and} \quad \pi^{\mu}_{\lambda}: H_{r}(N_{\mu}) \to H_{r}(N_{\lambda})$

and that if $\nu > \mu > \lambda$

$$\pi^{\mu}_{\nu}\pi^{\lambda}_{\mu}=\pi^{\lambda}_{\nu}$$
 and $\pi^{\mu}_{\lambda}\pi^{\nu}_{\mu}=\pi^{\nu}_{\lambda}$

The Čech homology group $H_r(X)$ is defined to be Inv Lim $H_r(N_{\lambda})$ under the projections π_{λ}^{μ} and the cohomology group to be Dir Lim $H^r(N_{\lambda})$.

For each complex N_{λ} there is a subcomplex N'_{λ} consisting of all simplices $(i_0, \dots, i_q; \lambda)$ for which Y intersects $\bigcap \lambda i_j$. The projections, for $\mu > \lambda$, always carry N'_{μ} into N'_{λ} . Thus unique projections are defined for $H_r(N_{\lambda} \mod N'_{\lambda})$ and $H'(N_{\lambda} - N'_{\lambda})$. The inverse and direct limits of these groups are defined to be the groups $H_r(X \mod Y)$ and H'(X - Y) respectively. Further, $H'(X \mod X - Y)$ is defined to be the direct limit of $H'(N_{\lambda} \mod N_{\lambda} - N'_{\lambda})$.

Theorem 4.1 then implies the following statement.

LEMMA 8.1. The sequence

$$\cdots \to H^{r}(X) \to H^{r}(X \mod X - Y) \to H^{r+1}(X - Y) \to H^{r+1}(X) \to \cdots$$

is exact.

Further, Theorem 7.6 serves to locate the dual sequence.

LEMMA 8.2. The sequence

$$\cdots \leftarrow H_r(X) \leftarrow \text{Inv Lim } H_r(N'_{\lambda}) \leftarrow H_{r+1}(X \mod Y) \leftarrow H_{r+1}(X) \leftarrow \cdots$$

is equivalent to the dual of the sequence of Lemma 8.1.

It is now necessary to identify Inv Lim $H_r(N'_\lambda)$, where $\lambda \in \Lambda(X)$, with $H_r(Y) =$ Inv Lim $H_r(N_\lambda)$, where $\lambda \in \Lambda(Y)$. To that end we note the existence of a map $\theta: \Lambda(X) \to \Lambda(Y)$ defined by the relation $\theta \lambda i = Y \cap \lambda i$, $\lambda \in \Lambda(X)$, $i \in I$. Since Y is closed, θ maps $\Lambda(X)$ onto $\Lambda(Y)$. The map θ preserves order relations in the sense that if $\mu > \lambda$ then $\theta \mu > \theta \lambda$.

There is an isomorphic correspondence between the subcomplex N'_{λ} and the complex $N_{\theta\lambda}$. For each $\lambda \in \Lambda(X)$ there is an isomorphic map of $H_r(N_{\theta\lambda})$ onto $H_r(N'_{\lambda})$, which we shall denote by φ_{λ} . It has the property that for $\mu > \lambda$ the projections π^{μ}_{λ} and $\pi^{\theta\mu}_{\theta\lambda}$ satisfy the relation $\pi^{\mu}_{\lambda}\varphi_{\mu} = \varphi_{\lambda}\pi^{\theta\mu}_{\theta\lambda}$ as an identity on elements $H_r(N_{\theta\mu})$.

The maps φ_{λ} are used to define a map $\varphi: \mathbf{P}H_r(N_{\gamma}) \to \mathbf{P}H_r(N_{\lambda}'), \lambda \in \Lambda(X),$

696

 $\gamma \in \Lambda(Y)$, the image φa being defined to have coordinates $(\varphi a)_{\lambda} = \varphi_{\lambda} a_{\theta \lambda}$. It will be shown, in three steps, that the homomorphism φ , suitably restricted in domain, establishes the required isomorphism between $H_r(Y) \subset \mathbf{P}H_r(N_{\gamma})$ and Inv Lim $H_r(N'_{\lambda}) \subset \mathbf{P}H_r(N'_{\lambda})$.

First, φ is an isomorphism into $\mathbf{P}H_r(N'_{\lambda})$. For if $a \neq 0$, $a \in \mathbf{P}H_r(N_{\gamma})$, then some coordinate $a_{\gamma} \neq 0$; $\gamma = \theta \lambda$ for some λ since θ is a map onto; and finally $\varphi_{\lambda}a_{\gamma} \neq 0$ since φ_{λ} is an isomorphism. Thus no non-null element is carried into the null element.

Second, if $a \in \text{Inv Lim } H_r(N_{\gamma}), \gamma \in \Lambda(Y)$, then $\varphi a \in \text{Inv Lim } H_r(N'_{\lambda}), \lambda \in \Lambda(X)$. For if $\mu > \lambda$ then $\pi^{\mu}_{\lambda}(\varphi a)_{\mu} = \pi^{\mu}_{\lambda}\varphi_{\mu}a_{\theta\mu} = \varphi_{\lambda}\pi^{\theta\mu}_{\theta\lambda}a_{\theta\mu} = \varphi_{\lambda}a_{\theta\lambda} = (\varphi a)_{\lambda}$.

Third, the image under φ of Inv Lim $H_r(N_\gamma)$, $\gamma \in \Lambda(Y)$, covers Inv Lim $H_r(N'_{\lambda})$, $\lambda \in \Lambda(X)$. To see this, consider a fixed element $b \in \text{Inv}$ Lim $H_r(N'_{\lambda})$. For each $\lambda \in \Lambda(X)$, let $a(\lambda)$ be that element of $H_r(N_{\theta\lambda})$ which is carried by φ_{λ} onto b_{λ} . We must now show that if $\theta \lambda = \theta \mu$ then $a(\lambda) = a(\mu)$, so that to each element of Inv Lim $H_r(N'_{\lambda})$ there corresponds an element of $PH_r(N_{\gamma})$, $\gamma \in \Lambda(Y)$. Then we must show that this element belongs to Inv Lim $H'(N_{\lambda})$. This demonstration is based on the fact that if $\alpha > \lambda$, α , $\lambda \in \Lambda(X)$, then $b_{\lambda} = \pi^{\alpha}_{\lambda} \varphi_{\alpha} a(\alpha) = \varphi_{\lambda} \pi^{\theta \alpha}_{\theta \lambda} a(\alpha)$, so that $a(\lambda) = \pi^{\theta \alpha}_{\theta \lambda} a(\alpha)$. Thus, if $\theta \lambda = \theta \mu$, then $a(\lambda) = a(\mu)$, since both are equal to $\pi^{\theta \alpha}_{\theta \lambda} a(\alpha)$ for any $\alpha > \lambda$, μ . We may therefore replace $a(\lambda)$ by $a_{\theta \lambda}$. The element a with coordinates $a_{\theta \lambda}$ surely maps onto b under φ , and it remains to show that $a \in \text{Inv}$ Lim $H_r(N_{\lambda})$. If $\theta \mu > \theta \lambda$, choose α so that $\alpha > \mu$ and $\alpha > \lambda$. Then $a_{\theta \mu} = \pi^{\theta \mu}_{\theta \mu} a_{\theta \alpha}$, and $a_{\theta \lambda} = \pi^{\theta \mu}_{\theta \lambda} a_{\theta \alpha} = \pi^{\theta \mu}_{\theta \lambda} a_{\theta \alpha$

In view of Lemmas 8.1 and 8.2 we have established the following theorem, the first part of which is due to Hurewicz [WH].

THEOREM 8.3. If Y is a closed subspace of X, then the following sequence on Čech homology groups, computed over a compact coefficient group J^* , is exact.

$$\cdots \to H_r(X) \to H_r(X \mod Y) \to H_{r-1}(Y) \to H_{r-1}(X) \to \cdots$$

The following sequence of \check{C} ech cohomology groups, computed with coefficient group J, is equivalent to the dual sequence

$$\cdots \leftarrow H^{r}(X) \leftarrow H^{r}(X - Y) \leftarrow H^{r-1}(X \mod X - Y) \leftarrow H^{r-1}(X) \leftarrow \cdots$$

This form implies that $H^{r-1}(X \mod X - Y)$ is a function only of Y, being isomorphic to $H^{r-1}(Y)$. The notation $H^r(X - Y)$ is misleading, in that this group is not in general a function of X - Y. However, it can be shown that $H_r(X \mod Y)$ is the Čech homology group of the space X_r obtained from X by shrinking Y to a point. (More precisely, the points of X_r are the set Y and the points of the complement of Y, X - Y. The open sets in X_r are the open sets in X - Y and the open sets in X which contain Y.) We shall not give the proof of this fact, which is straightforward but tedious. In case X is compact, the space X_r is determined by X - Y, since it is the compactification of X - Y by a single point. See for example [L], p. 23. We have then as a corollary to Theorem 8.3 the following.

J. L. KELLEY AND EVERETT PITCHER

ALEXANDER TYPE DUALITY THEOREM 8.4. Let X be a compact space whose rand (r-1)-dimensional Čech homology groups vanish. Then the (r-1)-dimensional Čech homology group of any closed subset Y of X is isomorphic with the r-dimensional Čech homology group of the compactification of X - Y by the addition of a single point. In particular, the (r-1)-dimensional homology group of Y is a function of the space X - Y.

It will be seen in Section 9 that if X is a manifold there is a much simpler way of computing this function of X - Y. In particular the Alexander-Pontrjagin theorem will be obtained. However it seems to us that Theorem 8.4 is the fundamental functional relation of Alexander duality. Compare with [B], Theorem 6.2.

REMARK. Theorem 8.7 as stated gives no duality for the dimensions 1, 0 since the 0-dimensional homology group of X cannot vanish. This may be remedied by using augmented complexes ([L], p. 130) in the definition of the Čech homology group, in which case the 0-dimensional homology group of a connected space vanishes. The same theorem is valid for these modified Čech groups.

With respect to singular homology groups, the situation is quite simple. The singular homology groups of a space X are the homology groups of a closure finite complex (see for example [E]). If Y is a subspace of X, the singular complex of Y is a subcomplex of that of X and the singular homology groups $\mathcal{K}_r(X)$, $\mathcal{K}_r(Y)$, $\mathcal{K}_r(X \mod Y)$ of finite chains with discrete coefficient group are well defined. On the other hand, the infinite cochains of the singular complex of X with arbitrary coefficient group define the cohomology group $\mathcal{K}^r(X)$. We also consider the cochains on those singular cells of X which are not cells of Y and so define $\mathcal{K}^r(X - Y)$, and $\mathcal{K}^r(X \mod X - Y)$. The group $\mathcal{K}^r(X - Y)$ is not a function of the space X - Y, but depends on both X and Y. It is easy to see the following isomorphism.

(8.5)
$$\mathcal{H}^{r}(X \mod X - Y) \approx \mathcal{H}^{r}(Y).$$

We do not use this fact in stating the following theorem.

THEOREM 8.6. If Y is an arbitrary subspace of a topological space X the following sequences are exact.

$$\cdots \to \mathcal{H}_{r}(X) \xrightarrow{\sim} \mathcal{H}_{r}(X \mod Y) \xrightarrow{\rightarrow} \mathcal{H}_{r-1}(Y) \xrightarrow{\sim} \mathcal{H}_{r-1}(X) \to \cdots$$
$$\cdots \leftarrow \mathcal{H}^{r}(X) \xleftarrow{\gamma} \mathcal{H}^{r}(X - Y) \xleftarrow{\beta} \mathcal{H}^{r-1}(X \mod X - Y) \xleftarrow{\alpha} \mathcal{H}^{r-1}(X) \leftarrow \cdots.$$

These sequences are on singular homology and cohomology groups respectively, the homology groups being computed with a discrete coefficient group and the cohomology groups with an arbitrary topological group.

There appears to be no general analogue of Theorem 8.4 for singular homology. The group $\mathcal{H}_r(X \mod Y)$ seems, even in rather special situations, to depend on both X and Y. Even on a manifold (see Section 9) $\mathcal{H}_r(X \mod Y)$, Y open, depends on X - Y and on the dimension of X.

9. The exact sequence for closed subsets of a manifold

In this section Poincaré duality will be utilized to obtain a generalization of the classical Alexander-Pontrjagin theorem. The basic task is to identify the exact Čech cohomology sequence for a manifold and an open set with a sequence on the singular homology groups of complementary dimension.

Let M be an absolute orientable *n*-dimensional manifold and let N be a closed subset. Unless otherwise stated the coefficient group J will always be a discrete group. Let M_i , $i = 1, 2, \cdots$, be successive barycentric subdivisions of M, and let N_i be the smallest closed subcomplex containing N. Then Theorem 3.3 asserts the following.

LEMMA 9.1. For each i, the sequence

$$\cdots \to H^{r}(M_{i}) \to H^{r}(M_{i} \mod M_{i} - N_{i}) \to H^{r+1}(M_{i} - N_{i}) \to H^{r+1}(M_{i}) \to \cdots$$

is exact.

For each *i* choose a simplicial map π of M_{i+1} into M_i which carries each vertex $v \in M_{i+1}$ into a vertex of the lowest dimensional simplex of M_i which contains it. The simplicial map generates a map ω on cochains of M_i to cochains on M_{i+1} which induces a homomorphism θ of the sequence of Lemma 9.1 on M_i , N_i into the sequence on M_{i+1} , N_{i+1} . We then have a directed system, the integers, with projections θ , such that the direct limits of the terms of the sequence of Lemma 9.1 are defined and, by virtue of Theorem 4.1, form an exact sequence.

LEMMA 9.2. The sequence

$$\cdots \rightarrow \text{Dir Lim } H'(M_i) \rightarrow \text{Dir Lim } H'(M_i \mod M_i - N_i).$$

$$\rightarrow$$
 Dir Lim $H^{r+1}(M_i - N_i) \rightarrow$ Dir Lim $H^{r+1}(M_i) \rightarrow \cdots$

is exact, this limit being taken under the map θ generated by the simplicial map π .

We now identify these limit groups. For a vertex $v \in M_{i+1}$ let St v be the union of v and the interiors of all simplices of M_{i+1} of which v is a vertex. The sets St v are open and $v_0 \cdots v_q$ is a simplex of M_{i+1} if and only if $\bigcap \operatorname{St} v_j$, j = 0, \cdots , q, is nonvacuous. Since N_{i+1} is the smallest complex containing N, $v_0 \cdots v_q$ is a simplex of N_{i+1} if and only if $N \cap (\bigcap \operatorname{St} v_j)$ is nonvacuous. That is, M_{i+1} and N_{i+1} are geometric realizations of the nerve of the covering of M and N by the sets St v. Furthermore, since π carries $v \in M_{i+1}$ into a vertex of the smallest dimensional simplex of M_i containing v, St $v \subset \operatorname{St} \pi v$. Thus π is a projection of the nerve of the covering by $\{\operatorname{St} v | v \in M_i\}$, carrying each set into a set containing it. We have thus shown the direct limit groups of Lemma 9.2 to be isomorphic to the Čech cohomology groups; they are formally distinct only in that they are derived from a cofinal family of coverings instead of from the family of all coverings. This is stated precisely in the following lemma.

LEMMA 9.3. The exact sequence of Lemma 9.2 is equivalent to the folloingw sequence of Čech cohomology groups.

 $\cdots \to H^{r}(M) \to H^{r}(M \mod M - N) \to H^{r+1}(M - N) \to H^{r+1}(M) \to \cdots$

It is now necessary to use Poincaré duality to modify the form of this sequence. We require the following results from manifold geometry.

LEMMA 9.4. Dual to the oriented complex M_i there is a subdivision M'_i of M into an oriented cellular complex with the following algebraic and geometric properties.

a. There is a 1 - 1 correspondence associating with each q-dimensional simplex σ_q of M_i an (n - q)-dimensional cell σ'_q of M'_i .

b. This correspondence carries cocycles into cycles and coboundaries into boundaries isomorphically.

c. M_{i+1} is a common subdivision of M_i and M'_i .

d. If $\sigma_q = v_0 \cdots v_q \in M_i$ then the closure of the cell σ'_q is the intersection of the closures of the stars in M_{i+1} of the v_j .

e. If L is a closed subcomplex of M_i the cells of L' form an open set containing L.

Making use of the ' isomorphism we shall replace the r-dimensional cohomology groups of Lemma 9.2 by (n - r)-dimensional homology groups on the dual subdivision. We first note that the map θ of 9.2, carrying the cohomology sequence of M_i , $M_i - N_i$ into that of M_{i+1} , $M_{i+1} - N_{i+1}$, can be written $\theta_2 \theta_1$, where θ_1 is an isomorphism of the cohomology sequence on M_i , $M_i - N_i$ to that on M_{i+1} , $M_{i+1} - \text{Sd } N_i$, with Sd = subdivision, and θ_2 is induced by the identity map of the pair M_{i+1} , $M_{i+1} - N_{i+1}$ into the pair M_{i+1} , $M_{i+1} - \text{Sd } N_i$. Using the ' isomorphism we may then state the following lemma.

LEMMA 9.5. The exact sequence of Lemma 9.3 is equivalent to the sequence

$$\cdots \leftarrow \text{Dir Lim } H_{n-r}(M'_{i+1}) \leftarrow \text{Dir Lim } H_{n-r}(M'_{i+1} - N_{i+1})$$

 $\leftarrow \text{ Dir Lim } H_{n-r+1}(M'_{i+1} \mod M'_{i+1} - N'_{i+1}) \leftarrow \text{ Dir Lim } H_{n-r+1}(M'_{i+1}) \leftarrow \cdots$

the direct limit being taken under $\theta'_2\theta'_1$. Here θ'_1 is an isomorphism of the sequence on M'_i , $M'_i - N'_i$ to that on M'_{i+1} , $M'_{i+1} - (\operatorname{Sd} N_i)'$ and θ'_2 is induced by the identity map of M'_{i+1} , $M'_{i+1} - N'_{i+1}$ to M'_{i+1} , $M'_{i+1} - (\operatorname{Sd} N_i)'$.

We now identify these groups. The sets $L_i = M'_i - N'_i$ are closed polyhedra in the complement of N and form a monotone increasing set of sets whose union is M - N. The map θ' is the map induced by the identity. An argument essentially the same as one which is given to show the singular homology groups of a polyhedron are identical with those obtained by subdivision then establishes the following lemma.

LEMMA 9.6. The exact sequence of Lemma 9.3 is equivalent to the sequence

$$\cdots \leftarrow \mathcal{H}_{n-r}(M) \leftarrow \mathcal{H}_{n-r}(M-N) \leftarrow \mathcal{H}_{n-r+1}(M \bmod M-N) \leftarrow \mathcal{H}_{n-r+1}(M) \leftarrow \cdots$$

where K, denotes the singular homology groups.

Collecting results, we have the following theorem.

THEOREM 9.7. Let M be an n-dimensional manifold and N a closed subset. Let H^r denote the Čech cohomology groups and \mathcal{K} , the singular homology groups, both being based on a discrete coefficient group J. Then the following sequences are equivalent.

700

$$\cdots \to H^{r}(M) \to H^{r}(M \mod M - N) \to H^{r+1}(M - N) \to H^{r+1}(M) \to \cdots$$
$$\cdots \to \mathcal{H}_{n-r}(M) \to \mathcal{H}_{n-r}(M \mod M - N) \to \mathcal{H}_{n-r-1}(M - N) \to \mathcal{H}_{n-r-1}(M) \to \cdots$$

This result, by means of Theorem 8.3, can be stated entirely in terms of homology. This leads to the following theorem.

DUALITY THEOREM FOR MANIFOLDS 9.8. Let M be an n-dimensional manifold and N a closed subset. Let H_r denote the Čech homology group, \mathcal{K}_r the singular and J an arbitrary discrete group. When one permits identification of equivalent sequences the following two exact sequences are dual to each other.

$$\cdots \to H_r(M, J^*) \to H_r(M \mod N, J^*) \to H_{r-1}(N, J^*) \to H_{r-1}(M, J^*) \to \cdots$$
$$\cdots \leftarrow \mathcal{H}_{n-r}(M, J) \leftarrow \mathcal{H}_{n-r}(M - N, J)$$
$$\leftarrow \mathcal{H}_{n-r+1}(M \mod M - N, J) \leftarrow \mathcal{H}_{n-r+1}(M, J) \leftarrow \cdots .$$

The Alexandroff lemma asserts that for any homomorphism $f:A \to B$, $(fA)^* \approx f^*B^*$. Thus in Theorem 9.8 the character group of the image-kernel in each group of the second sequence can be identified with an image-kernel from the first sequence. In the following theorem we have stated these dualities, using in each case the description of the groups which appears intuitively simplest.

THEOREM 9.9. Each group in one of the following pairs is isomorphic to the character group of the other. All mappings mentioned are induced by the identity.

a. The image in $H_r(M \mod N, J^*)$ and The image in $\mathcal{K}_{n-r}(M, J)$ of $H_r(M, J^*)$ and The image in $\mathcal{K}_{n-r}(M, J)$ of $\mathcal{K}_{n-r}(M - N, J)$ b. The kernel of the map $H_{r-1}(N, J^*) \to H_{r-1}(M, J^*)$ and The kernel of the map $\mathcal{K}_{n-r}(M - N, J) \to \mathcal{K}_{n-r}(M, J)$ c. The image in $H_r(M, J^*)$ and The image in $\mathcal{K}_{n-r}(M \mod M - N, J)$ of $\mathcal{K}_{n-r}(M, J)$.

An immediate corollary of Theorem 9.8 is the following.

ALEXANDER-PONTRJAGIN DUALITY THEOREM 9.10. If M has vanishing r- and (r-1)-dimensional homology groups, then

$$H_{r-1}(N, J^*) \approx [\mathcal{H}_{n-r}(M - N, J)]^*.$$

10. Chain Mappings

The principal theorem of this section is a more elegant and inclusive formulation of a theorem announced several years ago by one of the authors [EP 2]. Let M and N denote Mayer complexes and $f: M \to N$ a chain mapping of Minto N, characterized by the fact that it commutes with the boundary operator in the sense that $f\beta = \beta f$ is an identity on chains of M. We distinguish a subcomplex of M, namely

$$M_0 = \{K[f, C_r(M)], \beta\}$$

whose groups are the kernels of the map of r-chains of M into r-chains of N, and a subcomplex of N, namely

$$fM = \{fC_r(M), \beta\}$$

whose groups are the set of images. The relations $\beta C_{r+1}(M_0) \subset C_r(M_0)$ and $\beta C_{r+1}(fM) \subset C_r(fM)$ follow from the commutativity. By applying Theorem 3.4 to M and its subcomplex and to N and its subcomplex we arrive at the following theorem.

THEOREM 10.1. A chain mapping f of the chains of one Mayer complex M into another Mayer complex N gives rise to the following pair of exact homomorphism sequences.

$$\cdots \to H_r(M) \xrightarrow{f} H_r(fM) \xrightarrow{gf^{-1}} H_{r-1}(M_0) \xrightarrow{i} H_{i-1}(M) \to \cdots$$
$$\cdots \to H_r(N) \xrightarrow{a} H_r(N \mod fM) \xrightarrow{g} H_{r-1}(fM) \xrightarrow{\gamma} H_{r-1}(N) \to \cdots$$

The homomorphism marked f is induced by the chain mapping, the one marked βf^{-1} by the inverse of the isomorphism on $H_r(M \mod M_0)$ to $H_r(fM)$ induced by f followed by the boundary homomorphism and the one marked i by the identity. The homomorphisms marked α and γ are induced by the identity and β by the boundary homomorphism.

In applying Theorem 3.4 we have used the fact that $H_r(M \mod M) \approx H_r(fM)$ under the isomorphism induced by f.

Our principal application of this theorem occurs in the following section. However, some applications will be cited in the following examples. It will be convenient to use the vocabulary of simplicial complexes, the application in the present formulation being to the Mayer complexes of the groups of chains.

EXAMPLE 10.2. Let N be a simplicial complex covered by two closed subcomplexes N_1 and N_2 . Let M be a complex consisting of two disjoint subcomplexes, M_1 and M_2 , copies respectively of N_1 and N_2 . Let $f: M \to N$ be the map identifying M_1 and M_2 with their copies. Then

$$\begin{aligned} H_r(M) &= H_r(M_1) + H_r(M_2) \approx H_r(N_1) \times H_r(N_2) \\ H_r(M_0) \approx H_r(N_1 \cap N_2) \\ H_r(fM) &= H_r(N). \end{aligned}$$

The kernel-image in $H_r(N)$ consists of homology classes containing a cycle which can be written as the sum of a cycle in N_1 and a cycle in N_2 . The kernel-image in $H_r(M_0)$ corresponds to homology classes of cycles of the intersection which bound on N_1 and N_2 . The first exact sequence of Theorem 10.1 is readily identified with the formulation of the Mayer-Vietoris formulas in [A-H, pp. 297-299] while the second is trivial.

EXAMPLE 10.3. Let M be a simplicial complex containing two disjoint subcomplexes which are copies of a complex which we denote by M_0 , since this will

702

HOMOLOGY THEORY

presently agree with previous notation. We identify the two complexes to obtain a complex N. (One must require that no simplex have a vertex in each subcomplex or achieve this by subdivision in order that N be a simplicial complex.) The first exact sequence of Theorem 6.1 provides a formulation of the extension of the Mayer-Vietoris formulas obtained by one of the authors [EP 1].

EXAMPLE 10.4. Let X and Y be topological spaces and f a continuous function on X to Y. Then a map also denoted by f is defined on the complex M of singular chains of X to the complex N of singular chains of Y. Then the groups $H_r(M_0)$, $H_r(fM)$, $H_r(N \mod fM)$ are homology invariants of the function f on X to Y. Conditions for the isomorphism of $\mathcal{K}_r(M)$ and $\mathcal{K}_r(N)$ can be stated in terms of these groups.

11. Coverings of Complexes

Let M be an oriented cellular complex [L, p. 89] and L_i , $i = 1, \dots, n$, a collection of subcomplexes whose union is M. It is natural, as an extension of the Mayer-Vietoris relations, to investigate the relations among the homology groups of M, L_i and the intersections of the L_i . Helly's theorem [A-H, p. 295] is an acyclic case of such a relation. This problem could be investigated under various sets of assumptions of which the following is an adequate set although by no means the only reasonable one. We shall assume that M and L_i are augmented in the sense of [L, p. 130] and is closure finite [L, p. 91]. We shall use finite chains with coefficients from a discrete coefficient group J.

We remark explicitly that the sets L_i need not be distinct. We construct the nerve N of the covering of M by the sets L_i . A q-dimensional simplex $\sigma_q \in N$ is an ordered subset L_{i_0} , \cdots , L_{i_q} of the elements of the covering whose intersection contains cells of non-negative dimension. We impose an orientation by requiring that $i_0 < i_1 < \cdots < i_q$. In particular the subset consisting of no element of the covering is the (-1)-dimensional cimplex $\sigma_{-1} \in N$. The incidence number $[\sigma_q:\sigma_{q+1}]$ is then defined for any two simplices of neighboring dimension. Notably $[\sigma_{-1}:\sigma_0] = +1$ for all $\sigma_0 \in N$. The complex N is augmented in the sense of [L, p. 130].

Let $C_r(M)$ be the r-dimensional chain group of M with coefficients in J. For any subcomplex L of M we denote by $C_r(L)$, $r = -1, 0, \cdots$, the subgroup of $C_r(M)$ consisting of chains which vanish outside L. For

$$\sigma_q = L_{i_0} L_{i_1} \cdots L_{i_n} \epsilon N \qquad (q = -1, 0, \cdots)$$

 $(j=0,1,\cdots,q).$

we define

(11.1)

$$C_r(\sigma_{-1}) = C_r(M).$$

 $C_r(\sigma_q) = C_r(\bigcap L_{i_i})$

We now define a Mayer complex M_q by defining $C_r(M_q)$ to be the direct product $\mathbf{P}C_r(M_q)$ for $\sigma_q \in N$. The complex M_q has a realization as disjoint copies of all the intersections of q + 1 of the L_i . In particular the following lemma holds.

LEMMA 11.2. $H_r(M_q)$ is the direct product of the r-dimensional homology groups of intersections of q + 1 sets of the covering.

We now construct a map D of M_q into M_{q-1} , which can be interpreted as an analogue of the boundary in N and also as a step in a process of constructing M from the intersections of the L_i . Since $C_r(M_q) = \mathbf{P}C_r(\sigma_q)$, $a \in C_r(M_q)$ has the form $a = (a_1, \dots, a_s)$, where $a_i \in C_r(\sigma_q^i)$ and σ_q^i are q-simplices of N, s in number. The chain Da is to belong to $C_r(M_{q-1})$. We define the j^{th} coordinate of Da to be

$$(Da)_{j} = \sum_{i} [\sigma_{q-1}^{j} : \sigma_{q}^{i}]a_{i}.$$

The principal property of this map, and in fact the reason for its construction, is given in the following theorem.

THEOREM 11.3. For $r = 0, 1, \cdots$, the sequence

(11.4)
$$\cdots \to C_r(M_q) \xrightarrow{D} C_r(M_{q-1}) \xrightarrow{D} C_r(M_{q-2}) \to \cdots$$

is exact. Further, there is an isomorphism between $C_{-1}(M_q)$ and $C_q(N)$ such that the sequence (11.4) with r = -1 is equivalent to the sequence

 $\cdots \rightarrow C_q(N) \rightarrow C_{q-1}(N) \rightarrow C_{q-2}(N) \rightarrow \cdots$

in which the homomorphism is the boundary homomorphism.

We shall prove the second of these statements first. The group $C_{-1}(M_q)$ is the direct product of groups isomorphic with the coefficient group J, one factor for each $\sigma_q \in N$. The isomorphism whose existence is affirmed is clear. A simple inspection of the definition of D shows the equivalence of the two sequences.

To prove the first statement, let Δ be a fixed r-dimensional cell of M. For each $q \, \text{let } C_r(M_q)_{\Delta}$ be the set of all $a \in C_r(M_q)$, such that for each $\sigma_q \in N$ the corresponding coordinate of a vanishes at cells of M other than Δ . Clearly $C_r(M_q)$ is isomorphic to the weak product of the groups $C_r(M_q)_{\Delta}$ for all r-dimensional cells Δ . Further, D carries $C_r(M_q)_{\Delta}$ into $C_r(M_{q-1})_{\Delta}$ so that it is only necessary to show D exact on the homomorphism sequence $\cdots \rightarrow C_r(M_q)_{\Delta} \rightarrow C_r(M_{q-1})_{\Delta} \rightarrow$ $C_r(M_{q-2})_{\Delta} \rightarrow \cdots$ generated by restricting the domain of the homomorphism D. Let $L_{i_0}, \cdots, L_{i_m}, i_0 < \cdots < i_m$, be the set of all L_i containing Δ and let these form the simplex σ_m . If $a \in C_r(M_q)_{\Delta}$ is of the form (a_1, \cdots, a_i) where $a_i \in C_r(\sigma_q^i)$, σ_q^i being the q-simplices of N, s in number, then a_i is zero unless σ_q^i is a face of σ_m . Thus $C_r(M_q)_{\Delta}$ is isomorphic with the r-dimensional chain group on σ_m . Further, under this isomorphism D on $C_r(M_q)$ is the boundary operator on σ_m .

It is clear if the definition is written out explicitly that D commutes with the boundary homomorphism β on the complexes M_q so that D is a chain mapping on M_q to M_{q-1} . Let K_q be the Mayer complex of the kernels of the homomorphisms $D:C_r(M_q) \rightarrow C_r(M_{q-1})$. Except for r = -1, $C_r(M_q/K_q) = C_r(M_q)/C_r(K_q) \approx C_r(K_{q-1})$, the isomorphism being that of the Noether theorem. From Theorem 3.3 the truth of the following lemma is seen.

LEMMA 11.5. The sequence

 $\cdots \to H_r(K_q) \to H_r(M_q) \to H_r(M_q/K_q) \to$ $\cdots \to H_0(K_q) \to H_0(M_q) \to H_0(M_q/K_q)$ $\to H_{-1}(K_q) \to H_{-1}(M_q) \to H_{-1}(M_q/K_q)$

is exact. Homomorphisms which preserve dimension are induced by the identity and those which reduce dimension by the boundary homomorphism.

It is clear from the construction of $C_{-1}(M_q)$ that $H_{-1}(M_q)$ vanishes, so that the last non-trivial term in the sequence is generally $H_{-1}(K_q)$. Computing, we find that $Z_{-1}(K_q) = C_{-1}(K_q)$ and $B_{-1}(K_q) = \beta C_0(K_q) = \beta D C_0(M_{q+1}) =$ $D\beta C_0(M_{q+1}) = D C_{-1}(M_{q+1})$. In view of the isomorphism affirmed in Theorem 11.3, one sees the truth of the following lemma.

LEMMA 11.6. The groups $H_{-1}(K_q)$ and $H_q(N)$ are isomorphic.

It is not hard to see, in view of Theorem 11.3, that for $r \ge 0$, $H_r(M_q/K_q) \approx H_r(K_{q-1})$. Then Lemmas 11.5 and 11.6 imply the following theorem.

THEOREM 11.7. For each integer q the following sequence is exact.

 $\cdots \to H_r(K_q) \to H_r(M_q) \to H_r(K_{q-1}) \to$

$$\to H_0(K_q) \to H_0(M_q) \to H_0(K_{q-1}) \to H_q(N).$$

If the complexes L_i and all their intersections have vanishing homology groups the theorem above shows that $H_r(K_{q-1}) \approx H_{r-1}(K_q)$ for $r \ge 0$. In particular, $H_r(M) = H_r(K_{-1}) \approx H_{-1}(K_r) \approx H_r(N)$. The following corollaries are then clear.

COROLLARY 11.8. If the complex M is covered by complexes L_i such that the homology groups of all intersections of the L_i vanish, then the homology groups of M are isomorphic with the homology groups of the nerve of the covering.

In this corollary, homology groups are homology groups of the augmented complexes. The covering of a geometric complex by its cells meets the conditions of the corollary trivially. Thus, in particular, it follows that the homology groups of finite geometric cell complex are isomorphic with those of the simplicial complex obtained by subdividing its cells. Helly's theorem [A-H, p. 295] is a special case of Corollary 11.8.

REMARK. A somewhat stronger theorem has actually been established. A sufficient condition that $H_r(M) \approx H_r(N)$ is that $H_{r-q}(M_q)$ and $H_{r-q-1}(M_q)$ vanish for each q.

COROLLARY 11.9. If $f: M \to M'$ is a simplicial map of M onto M' in which the inverse of every simplex is acyclic, then f maps the homology groups of M isomorphically onto those of M'.

A situation which is in a sense the reverse of that of Corollary 11.8 is the following. Suppose that all intersections of two or more sets L_i are acyclic and

that the nerve is acyclic. Then $H_r(M_q) = 0$ and $H_r(K_{q-1}) \approx H_{r-1}(K_q)$ for q > 0. On the other hand, the vanishing of $H_0(M_q)$ and $H_q(N)$ implies that $H_0(K_{q-1}) = 0$ for q > 0. Thus follows this corollary.

COROLLARY 11.10. If M is covered by complexes L_i in such manner that the homology groups of the intersections of two or more of the sets L_i and the homology groups of the nerve vanish, then $H_r(M)$ is isomorphic to the direct product of the groups $H_r(L_i)$.

REMARK. As in the case of Corollary 11.8 a slightly stronger theorem has been proved. A sufficient condition that $PH_r(L_i) \approx H_r(M)$ under the natural mapping induced by the identity is that $H_r(K_0)$ and $H_{r-1}(K_0)$ vanish.

The general case in which the nerve of the covering is one dimensional, that is, the case of coverings in which no three sets intersect is of interest. It is handled in the following corollary.

COROLLARY 11.11. If the nerve of the covering of M by L_i is one dimensional, then there is a group G such that the following sequences are exact.

a.
$$\dots \rightarrow \mathbf{P}H_r(L_i \cap L_j) \rightarrow \mathbf{P}H_r(L_i) \rightarrow H_r(M) \rightarrow$$

 $\dots \rightarrow \mathbf{P}H_1(L_i \cap L_j) \rightarrow \mathbf{P}H_1(L_i) \rightarrow H_1(M)$
 $\rightarrow \mathbf{G} \rightarrow \mathbf{P}H_0(L_i) \rightarrow H_0(M) \rightarrow H_0(N).$
b. $0 \rightarrow \mathbf{P}H_0(L_i) \rightarrow \mathbf{G} \rightarrow H_1(N) \rightarrow 0.$

Actually, $G = H_0(K_0)$. We phrase the corollary in this fashion since each group except G permits geometric interpretation if the Mayer complex is a geometric complex. Of course, if $H_1(N) = 0$, $G \approx \mathbf{P}H_0(L_i)$. If the nerve is acyclic, this reduces to the form of the Mayer-Vietoris formulas.

REMARK: Let X be a topological space, covered by the open sets U_1, \dots, U_n , and let S(X) and its subcomplexes $S(U_i)$ be the singular complexes of the spaces X and U_i . It can be shown that S(X) has the same homology groups as the union of the $S(U_i)$, using essentially the methods of Chap. IV of [E]. The results of this section can then be applied to $L_i = S(U_i)$ and $M = \bigcup S(U_i)$ to yield relations between the singular homology groups of X and those of the sets U_i .

12. Remarks on the theory of critical levels

A part of the theory of critical levels can be deduced easily from the exact sequence construction, and it appears to us that this construction furnishes a combinatorial basis for a substantial part of the theory. See [MM 1, 2, 3]. We discuss critical level theory briefly, deriving inequalities analogous to those of Morse for a simple case. A somewhat similar method has been used earlier by Mayer [WM 3] for the same purpose.

The plan of statement and proof are valid for any homology theory in the sense of [E-S], i.e., any homology theory for which an exact sequence theorem

HOMOLOGY THEORY

can be proved. In particular, singular homology theory with discrete coefficients or Čech or Vietoris homology theory with a coefficient group which is compact or a field could be used. Variations would occur from case to case, notably in that the singular theory can be developed on an arbitrary topological space with no restriction resembling semi-continuity on the function, the restriction being in the critical levels only. For this reason the development will be sketched here for the case of singular homology.

Suppose X is a topological space and $\mathcal{K}_r(A)$ denotes the *r*-dimensional singular homology group of a subset A of X. The two basic lemmas are the statement of the exact sequence construction and the description of another exact sequence.

LEMMA 12.1. If $B \subset A \subset X$ then the homomorphism sequence

 $\cdots \to \mathcal{K}_{r}(A) \xrightarrow{\bullet} \mathcal{K}_{r}(A \mod B) \xrightarrow{\bullet} \mathcal{K}_{r-1}(B) \xrightarrow{\bullet} \mathcal{K}_{r-1}(A) \to \cdots$

is exact.

LEMMA 12.2. If $C \subset B \subset A \subset X$ then the homomorphism sequence $\cdots \to \mathcal{H}_r(A \mod C) \xrightarrow{\bullet} \mathcal{H}_r(A \mod B) \xrightarrow{\bullet} \mathcal{H}_{r-1}(B \mod C)$

 $\xrightarrow{\mathbf{r}} \mathcal{H}_{r-1}(A \mod C) \to \cdots$

is exact. The maps α' and γ' are induced by the identity while β' is induced by the boundary homomorphism.

It has been shown by Eilenberg and Steenrod [E-S] that Lemma 12.2 is an algebraic consequence of Lemma 12.1. For our purposes we remark that if L is a Mayer complex, M a subcomplex of L and N a subcomplex of M, then M/N is a subcomplex of L/N and the quotient complex is equivalent to M/L. This establishes Lemma 12.2.

Now let f be a real valued function defined over X. For convenience we assume 0 < f < B but do not initially restrict f further. We define the set $f_t = \{a \mid a \in X \text{ and } f \mid a \leq t\}$ and consider the homology groups $\mathcal{H}_r(f_t)$ and $\mathcal{H}_r(f_t \mod f_s), s < t$, which we abbreviate to $\mathcal{H}_r(t)$ and $\mathcal{H}_r(t, s)$. We shall compare $\mathcal{H}_r(t, s)$ with $\mathcal{H}_r(t + \epsilon, s)$ for $\epsilon > 0$ and with $\mathcal{H}_r(t, s + \epsilon)$ for $t - s > \epsilon > 0$. Lemma 12.2 implies the following lemma.

LEMMA 12.3. a. $\mathcal{K}_r(t, s) \approx \mathcal{K}_r(t + \epsilon, s), \epsilon > 0$, for all r under the map induced by the identity if and only if $\mathcal{K}_r(t + \epsilon, t)$ vanishes for all r.

b.
$$\mathcal{H}_r(t,s) \approx \mathcal{H}_r(t,s+\epsilon), t-s > \epsilon > 0$$
, for all r if and only if $\mathcal{H}_r(s+\epsilon,s)$

vanishes for all r.

We shall say that a level t is ordinary if $\mathcal{K}_r(t + \epsilon_1, t - \epsilon_2)$ vanishes for all r and for all positive numbers ϵ_1 and ϵ_2 sufficiently small. Levels of t not ordinary are termed critical. We assume for the moment that the levels of t which are critical are isolated, which implies that they are finite in number. Lemma 12.3 then enables one to say that the relative homology groups $\mathcal{K}_r(t + \epsilon_1, t - \epsilon_2)$ are independent of ϵ_1 , ϵ_2 when these are positive and sufficiently small and accordingly we shall use the notation $\mathcal{K}_r(t^+, t^-)$. Likewise $\mathcal{K}_r(t + \epsilon)$ is independent of ϵ for ϵ positive and sufficiently small and the notation is replaced by $\mathcal{K}_r(t^+)$. Under sufficiently restrictive conditions, which include assumptions permitting one to replace $\mathcal{K}_r(t^+, t^-)$ with $\mathcal{K}_r(t, t - \epsilon), \epsilon > 0$, the ranks of these groups are the type numbers assigned to the level in the theory of Morse [MM 1, 2, 3]. However, we shall not make additional restrictions.

The intimate connection between the relative homology groups at critical levels and the homology groups of the sets f_t is seen in observing that the vanishing of $\mathcal{H}_r(t,s)$ and $\mathcal{H}_{r-1}(t,s)$ implies that $\mathcal{H}_{r-1}(t) \approx \mathcal{H}_{r-1}(s)$.

We now derive a general set of inequalities. To that end let t_1, t_2, \dots, t_n be any increasing sequence of values of t. Let $C_r = \mathbf{P}\mathcal{H}_r(t_i, t_{i-1}), i = 2, \dots, n$, and let $G_r = \mathbf{P}\mathcal{H}_r(t_i), i = 2, \dots, n-1$. Since the sequences

$$\cdots \to \mathcal{H}_r(t_i) \to \mathcal{H}_r(t_i, t_{i-1}) \to \mathcal{H}_{r-1}(t_{i-1}) \to \mathcal{H}_{r-1}(t_i) \to \cdots \qquad (i = 2, \cdots, n)$$

are exact, so is the sequence of products. This is stated in the following theorem together with the consequence on ranks which follows from Corollary 3.6.

THEOREM 12.4. If f is a bounded function on a topological space and t_1, t_2, \dots, t_n is an arbitrary finite increasing sequence of real numbers then the homomorphism sequence.

$$\cdots \to G_r \times \mathcal{H}_r(t_n) \to C_r \to \mathcal{H}_{r-1}(t_1) \times G_{r-1} \to G_{r-1} \times \mathcal{H}_{r-1}(t_n) \to \cdots$$

is exact, where

$$C_r = \mathbf{P}\mathcal{H}_r(t_i, t_{i-1}) \qquad (i = 2, \cdots, n)$$

$$G_r = \mathbf{P}\mathcal{H}_r(t_i) \qquad (i = 2, \cdots, n-1).$$

If the ranks of the groups C_r are finite then so are those of the rest of the groups and $(-1)^q \left\{ \sum_{r=0}^q (-1)^r \rho[C_r] - \sum_{r=0}^q (-1)^r \rho[\mathcal{K}_{r-1}(t_1)] + \sum_{r=0}^q (-1)^r \rho[\mathcal{K}_{r-1}(t_n)] \right\} \ge 0.$ The inequality is an equality for q = m if $C_r = 0$ for r > m.

This theorem is a theorem about a finite number of nested but otherwise arbitrary subsets of a topological space X, it being possible to construct a function which generates the sets in the prescribed manner.

From this follows a theorem about critical levels. Suppose the critical levels are isolated and denoted by t_1, \dots, t_n in increasing order. Suppose $t'_1 < t_1 < t'_2 < t_2 < \dots < t'_n < t_n < t'_{n+1}$. Then $\mathcal{H}_r(t'_i, t'_{i-1}) \approx \mathcal{H}_r(t^+_{i-1}, t^-_{i-1})$, $\mathcal{H}_r(t'_i) \approx \mathcal{H}_r(t^+_{i-1}), \mathcal{H}_r(t'_{n+1}) \approx \mathcal{H}_r(X)$ and $\mathcal{H}_r(t'_1) = 0$. Then Theorem 12.4 applied to the values t'_i yields the following result.

THEOREM 12.5. If f is a bounded function on a topological space X whose critical levels in increasing order are t_1, t_2, \dots, t_n then the sequence

$$\cdots \to G_r \times \mathcal{H}_r(X) \to C_r \to G_{r-1} \to G_{r-1} \times \mathcal{H}_{r-1}(X) \to \cdots$$

is exact, where

$$C_r = \mathbf{P}\mathcal{H}_r(t_i^+, t_i^-) \qquad (i = 1, \dots, n)$$

$$G_r = \mathbf{P}\mathcal{H}_r(t_i^+) \qquad (i = 1, \dots, n-1).$$

If the groups C_r are of finite rank then so are all the groups in the sequence. Setting $M_r = \rho[C_r]$ and $R_r = \rho[\mathcal{K}_r(X)]$, then

$$M_0 \geq R_0$$

$$M_1 - M_0 \ge R_1 - R_0$$

 $M_r - M_{r-1} + \cdots + (-1)^r M_0 \ge R_r - R_{r-1} + \cdots + (-1)^r R_0$ with equality. for r = m if $M_r = 0$ for r > m.

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