Classifying triples of Lagrangians in a Hermitian vector space

Elisha Falbel, Jean-Pierre Marco, Florent Schaffhauser

Abstract

The purpose of this paper is to study the diagonal action of the unitary group $U(n)$ on triples of Lagrangian subspaces of $\mathbb{C}^n$. The notion of angle of Lagrangian subspaces is presented here, and we show how pairs (L_1, L_2) of Lagrangian subspaces are classified by the eigenvalues of the unitary map $\sigma_{L_2} \circ \sigma_{L_1}$ obtained by composing certain anti-symplectic involutions relative to L_1 and L_2. We then study the case of triples of Lagrangian subspaces of $\mathbb{C}^2$ and give a complete description of the orbit space. As an application of the methods presented here, we give a way of computing the inertia index of a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^n$ from the measures of the angles (L_j, L_k), and relate the classification of Lagrangian triples of $\mathbb{C}^2$ to the classification of two-dimensional unitary representations of the fundamental group $\pi_1(S^2\backslash\{s_1, s_2, s_3\})$.

1 Introduction

Let (V, ω) be a $2n$-dimensional real symplectic vector space. A *complex structure* on V is an automorphism J of V such that $J^2 = -Id$. The complex structure J is said to be *adapted to* ω if the bilinear form $g(u, v) := \omega(u, Jv)$ is a Euclidean scalar product on V. Given a complex structure J, the vector space V can be endowed with a complex vector space structure by setting $i.v = Jv$ for all $v \in V$. If J is adapted to ω, the form $h = g - i\omega$ then is a Hermitian scalar product on V endowed with its complex structure and the map J is both orthogonal and symplectic. Let $U(V)$ be the unitary group of V relative to h, $O(V)$ the orthogonal group of V relative to g and $Sp(V)$ the symplectic group of V relative to ω. Then by definition of h, we have $U(V) = O(V) \cap Sp(V)$.

A real subspace L of V is said to be *Lagrangian* if its orthogonal $L^{\perp_\omega}$ with respect to ω is L itself. Equivalently, L is Lagrangian if and only if its orthogonal with respect to g is $L^{\perp_g} = JL$. In particular, given a Lagrangian subspace L of V, we have the g-orthogonal decomposition $V = L \oplus JL$. Denoting by $\mathcal{L}(V)$ the set of all Lagrangian subspaces of V (the *Lagrangian Grassmannian* of V), the above decomposition enables us to associate to every Lagrangian L an $\mathbb{R}$-linear map

$$\begin{array}{cccc} \sigma_L : & V & \longrightarrow & V \\ & x + Jy & \longmapsto & x - Jy \end{array}$$

called the *Lagrangian involution associated to* L (see also [5],[3]). Observe that σ_L is *anti-holomorphic*: $\sigma_L \circ J = -J \circ \sigma_L$ and that the map

$$\begin{array}{ccc} \mathcal{L}(V) & \longrightarrow & GL(V) \\ L & \longmapsto & \sigma_L \end{array}$$

is continuous.

The purpose of this paper is to use these Lagrangian involutions, some properties of which are stated in section 2.4, to study the diagonal action of the unitary group $U(V)$ on $\mathcal{L}(V) \times \mathcal{L}(V) \times \mathcal{L}(V)$. We shall first recall the unitary classification result for pairs of Lagrangian subspaces of V, then describe the diagonal action of $U(V)$ on $\mathcal{L}(V) \times \mathcal{L}(V) \times \mathcal{L}(V)$, and give a complete classification and description of the orbit space in the case where V is of complex dimension 2 (theorems 3.1, 3.5 and 3.6). We then deduce a way to compute the inertia index of a triple of Lagrangian subspaces of V (theorem 4.4). Finally, we relate these results to the classification of two-dimensional unitary representations of the fundamental group $\pi_1(S^2\backslash\{s_1, s_2, s_3\})$ (corollary 4.10).

By choosing a unitary basis of V, we may identify V with the Hermitian vector space $(\mathbb{C}^n, h(z, z') = \sum_{k=1}^n z_k \overline{z'_k})$ and denote $\mathcal{L}(V)$ by $\mathcal{L}(n)$, $U(V)$ by $U(n)$, $O(V)$ by $O(2n)$ and $Sp(V)$ by $Sp(n)$. We set $g = \mathrm{Re}\ h$ and $\omega = -\mathrm{Im}\ h$ and denote by J the $\mathbb{R}$-endomorphism of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ corresponding to multiplication by $i \in \mathbb{C}$ in $\mathbb{C}^n$, and we have indeed $g = \omega(., J.)$.

2 Pairs of Lagrangian subspaces

The unitary group $U(n)$ acts smoothly and transitively on the Lagrangian Grassmannian $\mathcal{L}(n)$. Fixing a Lagrangian L in $\mathcal{L}(n)$, its stabilizer can be identified to $O(n)$, and $\mathcal{L}(n)$ therefore is a compact homogenous space diffeomorphic to $U(n)/O(n)$. We shall here be concerned with the diagonal action of $U(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n)$. Observe that requiring $\psi(L)$ to be Lagrangian when L is Lagrangian and $\psi \in O(2n)$ is equivalent to requiring that ψ be unitary (since $L \oplus JL = \mathbb{C}^n$, a g-orthogonal basis B for L over $\mathbb{R}$ is a unitary basis for $\mathbb{C}^n$ over $\mathbb{C}$, and if L is Lagrangian and ψ orthogonal, $\psi(B)$ then also is a unitary basis, so that ψ is a unitary map). Equivalently, the orbit of a pair (L_1, L_2) of Lagrangian subspaces under the diagonal action of $U(n)$ is the intersection with $\mathcal{L}(n) \times \mathcal{L}(n)$ of the orbit of (L_1, L_2) under the diagonal action of $O(2n)$. The orbit $[L_1, L_2]$ of the pair (L_1, L_2) under the diagonal action of $U(n)$ may therefore be called the *Lagrangian angle* formed by L_1 and L_2. In the following, we shall simply speak of the angle (L_1, L_2) to designate the orbit $[L_1, L_2]$. We now wish to find complete numerical invariants for this action: to each angle (L_1, L_2) we shall associate a *measure*, denoted by $\mathrm{meas}(L_1, L_2)$, in a way that two pairs (L_1, L_2) and (L_1', L_2') lie in the same orbit of the action of $U(n)$ if and only if $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L_1', L_2')$. This can be done in three equivalent ways, which we shall describe and compare.

2.1 Projective properties of Lagrangian subspaces of $\mathbb{C}^n$

A real subspace W of $\mathbb{C}^n$ is said to be *totally real* if $h(u, v) \in \mathbb{R}$ for all $u, v \in W$. A real subspace L of V therefore is Lagrangian if and only if it is totally real and of maximal dimension with respect to this property. Let p be the projection $p : \mathbb{C}^n \backslash \{0\} \to \mathbb{C}\mathrm{P}^{n-1}$ on the $(n-1)$-dimensional complex projective space, and for any real subspace W of $\mathbb{C}^n$, let $p(W)$ be the image of $W \backslash \{0\}$.

When L is a Lagrangian subspace of $\mathbb{C}^n$, recall that we denote by σ_L the only anti-holomorphic involution of $\mathbb{C}^n$ leaving L pointwise fixed (called the Lagrangian involution associated to L). The map σ_L being anti-holomorphic, it induces a map

$$\begin{array}{rccc} \widehat{\sigma_L}: & \mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}^{n-1} \\ & [z] & \longmapsto & [\sigma_L(z)] \end{array}$$

If we endow $\mathbb{C}\mathrm{P}^{n-1}$ with the Fubiny-Study metric, $\widehat{\sigma_L}$ becomes an isometry, and $p(L)$ is the fixed point set of that isometry. Therefore, for any Lagrangian L of $\mathbb{C}^n$, the subspace $l = p(L)$ of $\mathbb{C}\mathrm{P}^{n-1}$, called a *projective Lagrangian*, is a totally geodesic embedded submanifold of $\mathbb{C}\mathrm{P}^{n-1}$. More generally, every totally real subspace W of $\mathbb{C}^n$ is sent by p to a closed embedded submanifold of $\mathbb{C}\mathrm{P}^{n-1}$ which is diffeomorphic to $\mathbb{R}\mathrm{P}(W)$ (see [8], p. 73). These projective properties can be used to prove the first diagonalization lemma (theorem 2.1), as shown in [8]. They will also be important to us in the study of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$.

2.2 First diagonalization lemma and unitary classification of Lagrangian pairs

We state here the results obtained by Nicas in [8]. Let (L_1, L_2) be a pair of Lagrangian subspaces of $\mathbb{C}^n$ and let $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ be orthonormal bases for L_1 and L_2 respectively. Let A be the $n \times n$ complex matrix with coefficients $A_{ij} = h(v_j, u_i)$.

Definition 1 (Souriau matrix, [8]). The matrix AA^t, where A^t is the transpose of A, is called the *Souriau matrix* of the pair (L_1, L_2) with respect to the bases B_1 and B_2.

The matrix AA^t is both unitary and symmetric. As shown in [8], p. 73, two Souriau matrices of the pair (L_1, L_2) are conjugate. We can therefore make the following definition.

Definition 2 ([8]). The *characteristic polynomial* of the pair (L_1, L_2), denoted by $P(L_1, L_2)$, is the characteristic polynomial of any Souriau matrix of the pair (L_1, L_2).

Theorem 2.1 (First diagonalization lemma,[8]). *Let (L_1, L_2) be a pair of Lagrangian subspaces of $\mathbb{C}^n$. Then there exists an orthonormal basis $(u_1, \ldots, u_n)$ for L_1 and unit complex numbers $e^{i\lambda_1}, \ldots, e^{i\lambda_n}$ such that $(e^{i\lambda_1}u_1, \ldots, e^{i\lambda_n}u_n)$ is an orthonormal basis for L_2. Furthermore, the squares $e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}$*

of these numbers are the roots of the characteristic polynomial of the pair (L_1, L_2), counted with their multiplicities.

Theorem 2.2 (Unitary classification of Lagrangian pairs of $\mathbb{C}^n$, [8]). *Let (L_1, L_2) and (L'_1, L'_2) be two pairs of Lagrangian subspaces of $\mathbb{C}^n$. A necessary and sufficient condition for the existence of a unitary map $\psi \in U(n)$ such that $\psi(L_1) = L'_1$ and $\psi(L_2) = L'_2$ is that the characteristic polynomials $P(L_1, L_2)$ and $P(L'_1, L'_2)$ are equal.*

2.3 Second diagonalization lemma

It is possible to express the result of the first diagonalization lemma in terms of unitary maps sending L_1 to L_2, in a way that generalizes the situation of real lines in $\mathbb{C}$.

Proposition 2.3 (Second diagonalization lemma). *Given two Lagrangian subspaces of L_1 and L_2 of $\mathbb{C}^n$, then there exists a unique unitary map $\varphi_{12} \in U(n)$ sending L_1 to L_2 and verifying the following diagonalization conditions:*

- (i) *the eigenvalues of φ_{12} are unit complex numbers $e^{i\lambda_1}, \ldots, e^{i\lambda_n}$ verifying $\pi > \lambda_1 \geq \ldots \geq \lambda_n \geq 0$;*
- (ii) *there exists an orthonormal basis $(u_1, \ldots, u_n)$ for L_1 such that u_k is an eigenvector for φ_{12} (with eigenvalue $e^{i\lambda_k}$) .*

Proof. The existence is a consequence of the first diagonalization lemma. As for unicity, observe that two such unitary maps have the same eigenspaces and the same corresponding eigenvalues, and therefore are equal. □

It is also possible to give a direct proof of this result, which then proves the first diagonalization lemma without making use of projective geometry (see [7] or [1]). Observe that condition (i) is essential for the unicity part: for any Lagrangian L, the two maps J and $-J$ are both unitary, they both send L to $JL = (-J)L$ and verify condition (ii) for any orthonormal basis of L, but J is the only one of these two maps whose eigenvalues are located in the upper half of the unit circle of $\mathbb{C}$.

Observe that the Souriau matrix of the pair (L_1, L_2) with respect to the bases $(u_1, \ldots, u_n)$ and $(e^{i\lambda_1}u_1, \ldots, e^{i\lambda_n}u_n)$ is the diagonal matrix $\mathrm{diag}(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n})$. Therefore, the roots of the characteristic polynomial $P(L_1, L_2)$ are the squares of the eigenvalues of φ_{12}.

At last, observe that if (L_1, L_2) and (L'_1, L'_2) are located in the same orbit of the diagonal action of $U(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n)$, then the two associated unitary maps φ_{12} and φ'_{12} are conjugated in $U(n)$. Indeed, if $\psi(L_1) = L'_1$ and $\psi(L_2) = L'_2$ with $\psi \in U(n)$, then $\psi \circ \varphi_{12} \circ \psi^{-1}$ sends L'_1 to L'_2 and verifies the conditions of the second diagonalization lemma, hence by unicity $\psi \circ \varphi_{12} \circ \psi^{-1} = \varphi'_{12}$. The unitary maps φ_{12} will be very useful in the study of the diagonal action of $U(2)$ on triples of Lagrangian subspaces of $\mathbb{C}^2$ (see section 3).

2.4 Lagrangian involutions

We give here the properties of Lagrangian involutions that we shall need in the following. The finite groups generated by such involutions are studied in [4].

Proposition 2.4. *Let $L \in \mathcal{L}(n)$ be a Lagrangian subspace of $\mathbb{C}^n$. Then:*

- (i) *There exists a unique anti-holomorphic map σ_L whose fixed point set is exactly L.*
- (ii) *If L' is a Lagrangian subspace such that $\sigma_L = \sigma_{L'}$, then $L = L'$: there is a one-to-one correspondence between Lagrangian subspaces and Lagrangian involutions.*
- (iii) *σ_L is* anti-unitary*: for all $z, z' \in \mathbb{C}^n$, $h(\sigma_L(z), \sigma_L(z')) = \overline{h(z, z')}$.*

Observe then that the composite map $\sigma_{L_2} \circ \sigma_{L_1}$ of two Lagrangian involutions is unitary. As a direct consequence of the definition of a Lagrangian involution, we have, for any Lagrangian L and any unitary map ψ, $\sigma_{\psi(L)} = \psi \circ \sigma_L \circ \psi^{-1}$ (since $\psi \circ \sigma_L \circ \psi^{-1}$ is anti-holomorphic and leaves $\psi(L)$ pointwise fixed). The following result establishes the relation between Lagrangian involutions and angles of Lagrangian subspaces.

Proposition 2.5. *Let L_1 and L_2 be two Lagrangian subspaces of $\mathbb{C}^n$. The eigenvalues of $\sigma_{L_2} \circ \sigma_{L_1}$ are the roots of the characteristic polynomial $P(L_1, L_2)$ of the pair (L_1, L_2), with the same multiplicity. Equivalently, since $P(L_1, L_2)$ is monic, it is the characteristic polynomial of the holomorphic map $\sigma_{L_2} \circ \sigma_{L_1}$.*

Proof. By the first diagonalization lemma, there exists an orthonormal basis $(u_1, \ldots, u_n)$ for L_1 and unit complex numbers $\alpha_1, \ldots, \alpha_n$ such that $(\alpha_1 u_1, \ldots, \alpha_n u_n)$ is an orthonormal basis for L_2 and $\alpha_1^2, \ldots, \alpha_n^2$ are the roots of $P(L_1, L_2)$, counted with their multiplicities. Let ψ be the unitary morphism sending u_k to $\alpha_k u_k$ for $k = 1, \ldots, n$. Then $\alpha_1^2, \ldots, \alpha_n^2$ are the eigenvalues of ψ^2, counted with their multiplicities, and it is therefore sufficient to prove that $\sigma_{L_2} \circ \sigma_{L_1} = \psi^2$.
The map $\psi \circ \sigma_{L_1} \circ \psi^{-1}$ is anti-holomorphic and leaves L_2 pointwise fixed, hence $\sigma_{L_2} = \psi \circ \sigma_{L_1} \circ \psi^{-1}$. Furthermore, for all $j = 1, \ldots, n$, we have $\sigma_{L_1} \circ \psi^{-1}(u_j) = \sigma_{L_1}(\frac{1}{\alpha_j} u_j) = \alpha_j \sigma_{L_1}(u_j) = \psi(u_j) = \psi \circ \sigma_{L_1}(u_j)$ □

In particular, setting $\psi = \varphi_{12}$ in the above proof, we obtain the following corollary.

Corollary 2.6. *Let φ_{12} be the only unitary map sending L_1 to L_2 and verifying the conditions of proposition 2.3. Then $\varphi_{12}^2 = \sigma_{L_2} \circ \sigma_{L_1}$.*

2.5 Measure of a Lagrangian angle

Keeping the classification theorem in mind, we are then led to the following definition.

Definition 3 (Measure of a Lagrangian angle). Let L_1 and L_2 be two Lagrangians of $\mathbb{C}^n$ and let $e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}$ be the eigenvalues of $\sigma_{L_2} \circ \sigma_{L_1}$, counted with their multiplicities. The symmetric group $\mathfrak{S}_n$ acts on $S^1 \times \cdots \times S^1$ by permuting the elements of the n-tuples of unit complex numbers, and we denote by $[e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}]$ the equivalence class of $(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}) \in S^1 \times \cdots \times S^1$, and call it the *measure* of the angle formed by L_1 and L_2: $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}] \in (S^1 \times \cdots \times S^1)/\mathfrak{S}_n$.

As $\sigma_{\psi(L)} = \psi \circ \sigma_L \circ \psi^{-1}$ for any unitary map $\psi \in U(n)$, we have $\mathrm{meas}(\psi(L_1), \psi(L_2)) = \mathrm{meas}(L_1, L_2)$, so this notion is well-defined. This definition of a measure does not extend the usual one (in the case $n = 1$, we obtain $e^{i2\lambda}$, where $\lambda \in [0, \pi[$ is the usual measure). It will nonetheless prove to be relevant.

Observe that, since $\sigma_{L_1} \circ \sigma_{L_2} = (\sigma_{L_2} \circ \sigma_{L_1})^{-1}$, if $e^{i2\lambda}$ is an eigenvalue of $\sigma_{L_2} \circ \sigma_{L_1}$ then $e^{-i2\lambda}$ is an eigenvalue of $\sigma_{L_1} \circ \sigma_{L_2}$. As a consequence, if $(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n})$ is a representative of $\mathrm{meas}(L_1, L_2)$ then $(e^{i2\xi_1}, \ldots, e^{i2\xi_n})$, where $\xi_k = \pi - \lambda_k \mod \pi$, is a representative of $\mathrm{meas}(L_2, L_1)$. In particular, $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$ if and only if $\mathrm{meas}(L_2, L_1) = \mathrm{meas}(L'_2, L'_1)$.

In the following, we shall identify $S^1 \times \cdots \times S^1$ with the n-torus $\mathbb{T}^n = \mathbb{R}^n/\pi\mathbb{Z}^n$, to which it is homeomorphic. The measure of the angle (L_1, L_2) will be denoted by $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}] \in \mathbb{T}^n/\mathfrak{S}_n$, where $(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n})$ is a representative of $\mathrm{meas}(L_1, L_2)$ verifying $\pi > \lambda_1 \geq \ldots \geq \lambda_n \geq 0$. In view of proposition 2.5 above, we may now reformulate the classification theorem for Lagrangian pairs in the following way.

Proposition 2.7. *Given two pairs of Lagrangian subpaces (L_1, L_2) and (L'_1, L'_2) of Lagrangian subspaces of $\mathbb{C}^n$, a necessary and sufficient condition for the existence of a unitary map $\psi \in U(n)$ such that $\psi(L_1) = L'_1$ and $\psi(L_2) = L'_2$ is that* $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$. *Equivalently, the map*

$$\begin{array}{cccc} \chi : & (\mathcal{L}(n) \times \mathcal{L}(n))/U(n) & \longrightarrow & \mathbb{T}^n/\mathfrak{S}_n \\ & [L_1, L_2] & \longmapsto & \mathrm{meas}(L_1, L_2) \end{array}$$

is one-to-one.

The map χ is in fact a bijection : given $[e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}] \in \mathbb{T}^n/\mathfrak{S}_n$, consider any Lagrangian $L_1 \in \mathcal{L}(n)$, $(u_1, \ldots, u_n)$ an orthonormal basis for L_1 and let L_2 be the real subspace of $\mathbb{C}^n$ generated by $(e^{i\lambda_1} u_1, \ldots, e^{i\lambda_n} u_n)$. Since $(e^{i\lambda_1} u_1, \ldots, e^{i\lambda_n} u_n)$ is a unitary basis of $\mathbb{C}^n$ over $\mathbb{C}$, L_2 is Lagrangian and $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}]$.

Corollary 2.8. *The angle space $(\mathcal{L}(n) \times \mathcal{L}(n))/U(n)$, endowed with the quotient topology, is homeomorphic to the quotient space $\mathbb{T}^n/\mathfrak{S}_n$, both being Hausdorff and compact.*

As a final remark, observe that the corresponding symplectic problem admits a simple answer: a necessary and sufficient condition for the existence of a symplectic map $\psi \in Sp(n)$ such that $\psi(L_1) = L'_1$ and $\psi(L_2) = L'_2$ is that $\dim(L_1 \cap L_2) = \dim(L'_1 \cap L'_2)$; the measure of the symplectic angle formed by two Lagrangian subspaces of $\mathbb{C}^n$ simply is the dimension of their intersection.

2.5.1 Orthogonal decomposition of L_1 associated to meas(L_1, L_2)

The presentation given here follows that of [8]. This notion will enable us to classify triples of Lagrangian subspaces of $\mathbb{C}^2$ (theorem 3.1).

Let (L_1, L_2) be a pair of Lagrangian subspaces of $\mathbb{C}^n$, and let $(\alpha_1^2, \ldots, \alpha_n^2)$ be a representative of $\mathrm{meas}(L_1, L_2) \in \mathbb{T}^n/\mathfrak{S}_n$. The unit complex numbers $\alpha_1^2, \ldots, \alpha_n^2$ then are the roots of the characteristic polynomial $P(L_1, L_2)$ of the pair (L_1, L_2) (proposition 2.5). Let $\alpha_{j_1}^2, \ldots, \alpha_{j_m}^2$ be the distinct roots of $P(L_1, L_2)$. For $k = 1, \ldots, m$, define the real subspace W_k of L_1 by $W_k = \{u \in L_1/\ \alpha_{j_k} u \in L_2\}$ Observe that W_k is independent of the choice of the square root of $\alpha_{j_k}^2$, and that $W_1 \oplus \cdots \oplus W_m$ is independent, up to permutation of the subspaces, of the choice of the representative $(\alpha_1^2, \ldots, \alpha_n^2)$ of $\mathrm{meas}(L_1, L_2) \in \mathbb{T}^n/\mathfrak{S}_n$.

Proposition 2.9 ([8]). *L_1 decomposes as an orthogonal direct sum: $L_1 = W_1 \oplus \cdots \oplus W_m$, the dimension of W_k being the multiplicity of the root $\alpha_{j_k}^2$ of $P(L_1, L_2)$.*

Observe that L_2 then also decomposes as an orthogonal direct sum: $L_2 = \alpha_{j_1} W_1 \oplus \cdots \oplus \alpha_{j_m} W_m$. Furthermore, by considering the representative $(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n})$ of $\mathrm{meas}(L_1, L_2)$, where $e^{i\lambda_1}, \ldots, e^{i\lambda_n}$ are the eigenvalues of the unitary map φ_{12}, we see that the subspace W_k of L_1 is the intersection with L_1 of the eigenspace of φ_{12} with respect to the eigenvalue $e^{i\lambda_{j_k}}$. Given a Lagrangian triple (L_1, L_2, L_3), the unitary maps φ_{12} and φ_{13} therefore have the same eigenspaces if and only if the orthogonal decompositions of L_1 associated to $\mathrm{meas}(L_1, L_2)$ and $\mathrm{meas}(L_1, L_3)$ are the same (see definition 5).

3 Triples of Lagrangian subspaces

3.1 First classification result for triples of Lagrangian subspaces of $\mathbb{C}^2$

The following remark is valid for any n. If (L_1, L_2, L_3) and (L'_1, L'_2, L'_3) are two triples of Lagrangian subspaces of $\mathbb{C}^n$ that lie in the same orbit of the diagonal action of $U(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n) \times \mathcal{L}(n)$, it follows from section 2 that we have in particular $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$ and $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L'_1, L'_3)$. Let $L_1 = W_1 \oplus \cdots \oplus W_m$ be the orthogonal decomposition of L_1 associated to $\mathrm{meas}(L_1, L_2)$ and let $L_1 = Z_1 \oplus \cdots \oplus Z_p$ be the orthogonal decomposition of L_1 associated to $\mathrm{meas}(L_1, L_3)$. Define $L'_1 = W'_1 \oplus \cdots \oplus W'_m$ and $L'_1 = Z'_1 \oplus \cdots \oplus Z'_p$ similarly. Since $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$ and $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L'_1, L'_3)$, the respective numbers of factors m and p in the above decompositions are indeed pairwise the same, and furthermore $\dim W_k = \dim W'_k$ for $k = 1, \ldots, m$ and $\dim Z_l = \dim Z'_l$ for $l = 1, \ldots, p$. More specifically, if the unitary map $\psi \in U(n)$ sends L_j to L'_j for $j = 1, 2, 3$, then $\psi(W_k) = W'_k$ for $k = 1, \ldots, m$ and $\psi(Z_l) = Z'_l$ for $l = 1, \ldots, p$, as follows from the definition of W_k and Z_l. Since ψ is unitary, we even have $\psi(W_k \oplus JW_k) = W'_k \oplus JW'_k$ for all k and $\psi(Z_l \oplus JZ_l) = Z'_l \oplus JZ'_l$ for all l.

When $n = 2$, the above remark admits an easy converse, which gives a first classification result for triples of Lagrangians of $\mathbb{C}^2$. We shall use the following notations: given two triples (L_1, L_2, L_3) and (L'_1, L'_2, L'_3) of Lagrangian subspaces of $\mathbb{C}^2$, let φ_{12} be the only unitary map sending L_1 to L_2 and verifying the conditions of the second diagonalization lemma (theorem 2.3), and let $(e^{i\lambda_{12}}, e^{i\mu_{12}})$ be its eigenvalues, where $\pi > \lambda_{12} \geq \mu_{12} \geq 0$, and define $\varphi_{13}, \varphi'_{12}, \varphi'_{13}$ and $(e^{i\lambda_{13}}, e^{i\mu_{13}}), (e^{i\lambda'_{12}}, e^{i\mu'_{12}}), (e^{i\lambda'_{13}}, e^{i\mu'_{13}})$ similarly. As a preliminary remark to the statement of the classification result, observe that when both φ_{12} and φ_{13} have two distinct eigenvalues, respectively denoted by $(e^{i\lambda_{12}}, e^{i\mu_{12}})$ and by $(e^{i\lambda_{13}}, e^{i\mu_{13}})$, where $\pi > \lambda_{12} > \mu_{12} \geq 0$ and $\pi > \lambda_{13} > \mu_{13} \geq 0$, then $W_1 = \{u \in L_1/\ e^{i\lambda_{12}} u \in L_2\}$ and $Z_1 = \{u \in L_1/\ e^{i\lambda_{13}} u \in L_3\}$ are one-dimensional real subspaces of the Euclidean space L_1, and therefore form a (non-oriented) angle measured by a real number $\theta \in [0, \frac{\pi}{2}]$, that will be denoted by $\mathrm{meas}(W_1, Z_1)$. A real number θ' may be defined similarly in L'_1, since W'_1 are Z'_1 are also one-dimensional.

Theorem 3.1 (Unitary classification of Lagrangian triples of $\mathbb{C}^2$). *Given two triples (L_1, L_2, L_3) and (L'_1, L'_2, L'_3) of Lagrangian subspaces of $\mathbb{C}^2$, a necessary and sufficient condition for the existence of* *a unitary map $\psi \in U(n)$ such that $\psi(L_1) = L_1', \psi(L_2) = L_2'$ and $\psi(L_3) = L_3'$ is that either*

$$\text{(A)} \qquad \lambda_{12} \neq \mu_{12}, \lambda_{13} \neq \mu_{13} \text{ and } \begin{cases} (\lambda_{12}, \mu_{12}) &= (\lambda_{12}', \mu_{12}') \\ (\lambda_{13}, \mu_{13}) &= (\lambda_{13}', \mu_{13}') \\ \theta &= \theta' \end{cases}$$

where $\theta = \mathrm{meas}(W_1, Z_1) \in [0, \frac{\pi}{2}]$ and $\theta' = \mathrm{meas}(W_1', Z_1')$ are defined as above, or

$$\text{(B) } \lambda_{12} = \mu_{12} \text{ or } \lambda_{13} = \mu_{13} \text{ and } \begin{cases} (\lambda_{12}, \mu_{12}) &= (\lambda_{12}', \mu_{12}') \\ (\lambda_{13}, \mu_{13}) &= (\lambda_{13}', \mu_{13}') \end{cases}$$

Observe that, in each case, the condition $(\lambda_{jk}, \mu_{jk}) = (\lambda_{jk}', \mu_{jk}')$ is equivalent to the condition $\mathrm{meas}(L_j, L_k) = \mathrm{meas}(L_j', L_k')$.

Proof. Suppose that such a $\psi \in U(2)$ exists. Then, as we have seen earlier, $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L_1', L_2')$ and $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L_1', L_3')$. Furthermore, $\psi(W_1) = W_1'$ and $\psi(Z_1) = Z_1'$, so that if φ_{12} and φ_{13} both have distinct eigenvalues (that is, we are in the situation of (A) above), we have $\theta = \theta'$ since $\psi|_{L_1} : L_1 \to L_1'$ is an orthogonal map.
Conversely, suppose first that conditions (A) are fulfilled. Let $w_1 \in L_1$ be a generator of W_1 and let $w_1' \in L_1'$ be a generator of W_1'. By choosing w_2 in L_1 orthogonal to w_1 and w_2' in L_1' orthogonal to w_1', we may define an orthogonal map $\nu : L_1 \to L_1'$ sending W_1 to W_1' (and therefore $W_2 = W_1^\perp$ to $W_2' = (W_1')^\perp$). Then the measure of the angle $(W_1', \nu(Z_1)) = (\nu(W_1), \nu(Z_1))$ is $\theta = \theta'$, so there exists an orthogonal map $\xi \in O(L_1')$ such that $\xi \circ \nu(W_1) = W_1'$ and $\xi \circ \nu(Z_1) = Z_1'$. The subspace L_1 being Lagrangian, the orthogonal map $\xi \circ \nu$ can be extended $\mathbb{C}$-linearly to a unitary transformation $\psi \in U(2)$ of $\mathbb{C}^2 = L_1 \oplus JL_1$ sending L_1 to L_1' by construction. But $L_2 = e^{i\lambda_{12}} W_1 \oplus e^{i\mu 12} W_2$ and $L_3 = e^{i\lambda_{13}} Z_1 \oplus e^{i\mu_{13}} Z_2$ (corollary of proposition 2.9, hence $\psi(L_2) = e^{i\lambda_{12}} W_1' \oplus e^{i\mu 12} W_2' = L_2'$ and $\psi(L_3) = e^{i\lambda_{13}} Z_1' \oplus e^{i\mu_{13}} Z_2' = L_3'$.
If now the conditions (B) are fulfilled, then for instance $L_2 = e^{i\lambda} L_1$ and the result is a consequence of the classification of pairs. $\square$

Observe that, given real numbers $(\lambda_{12}, \mu_{12}, \lambda_{13}, \mu_{13}, \theta)$ as in (A), it is always possible to find a triple (L_1, L_2, L_3) such that $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_{12}}, e^{i2\mu_{12}}]$, $\mathrm{meas}(L_1, L_3) = [e^{i2\lambda_{13}}, e^{i2\mu_{13}}]$ and $\mathrm{meas}(W_1, Z_1) = \theta$. Indeed, let L_1 be any lagrangian of $\mathbb{C}^2$ and let (u_1, u_2) be an orthonormal basis for L_1, let $d_1 = \mathbb{R}u_1$, $d_2 = \mathbb{R}u_2$, and let d be the image of d_1 by the rotation of the euclidean space L_1 with matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in the basis (u_1, u_2), and set $L_2 = e^{i\lambda_{12}} d_1 \oplus e^{i\mu_{12}} d_2$ and $L_3 = e^{i\lambda_{13}} d \oplus e^{i\mu_{13}} d^\perp$. Given numbers $(\lambda_{12} = \mu_{12} = \lambda, \lambda_{13}, \mu_{13})$ as in (B), we only need to set $L_2 = e^{i\lambda} L_1$ and $L_3 = e^{i\lambda_{13}} d_1 \oplus e^{i\mu_{13}} d_2$.

Thus, the orbits of the diagonal action of $U(2)$ on $\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2)$ are generically characterized by the five invariants $\lambda_{12}, \mu_{12}, \lambda_{13}, \mu_{13}$ and θ.

3.2 Geometric study of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

The aim of this section is to study the space $(\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2))/U(2)$ of the orbits of the diagonal action of $U(2)$ on triples of Lagrangians subspaces of $\mathbb{C}^2$, and more specifically to describe it in terms of the map

$$\begin{array}{rccc} \rho : & (\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2))/U(2) & \longrightarrow & \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \\ & [L_1, L_2, L_3] & \longmapsto & (\mathrm{meas}(L_1, L_2), \mathrm{meas}(L_2, L_3), \mathrm{meas}(L_3, L_1)) \end{array}$$

which will enable us to state the classification result for Lagrangian triples in a way (theorem 3.6) that is similar to the corresponding result for Lagrangian pairs (theorem 2.7). We shall see in paragraph 3.3 that this way of doing things is equivalent to considering orthogonal decompositions of one of the three subspaces. We are first going to describe the image of the map ρ and then prove that it is one-to-one. This will also give a topological description of the orbit space. Our main tool to characterize the image of ρ will be the study of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$.

3.2.1 Configurations of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

In the following, we shall constantly identify the complex projective line $\mathbb{C}\mathrm{P}^1$, endowed with the Fubini-Study metric, with the Euclidean sphere $S^2 \subset \mathbb{R}^3$ endowed with its usual structure of oriented Riemannian manifold. We will denote by p the projection

$$\begin{array}{rccc} p: & \mathbb{C}^2\backslash\{0\} & \longrightarrow & \mathbb{C}\mathrm{P}^1 \\ & z=(z_1,z_2) & \longmapsto & p(z)=[z]=[z_1,z_2] \end{array}$$

As seen in 2.1, the image of a Lagrangian subspace of $\mathbb{C}^2$ is a totally geodesic submanifold of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ that is diffeomorphic to $\mathbb{R}\mathrm{P}^1 \simeq S^1$. Therefore $l = p(L)$ is a great circle of S^2, and the isometry $\widehat{\sigma_L}$ of $\mathbb{C}\mathrm{P}^1$, induced by the Lagrangian involution σ_L, acts on S^2 as the reflexion with respect to the plane of $\mathbb{R}^3$ containing the great circle $l = p(L)$. Recall that the unitary group $U(2)$ acts transitively on the Lagrangian Grassmannian $\mathcal{L}(2)$. The action of $U(2)$ on $\mathbb{C}\mathrm{P}^1$ is the same as the action of the special unitary group $SU(2)$, which acts on S^2 by the 2-folded universal covering map $h : SU(2) \to SO(3)$. The map $L \in \mathcal{L}(2) \mapsto l = p(L) \subset \mathbb{C}\mathrm{P}^1 \simeq S^2$ is equivariant for these actions. For any $\varphi \in GL(2,\mathbb{C})$, we shall denote by $\widehat{\varphi}$ the induced map of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ into itself: $\widehat{\varphi}.[z] = [\varphi(z)]$. If $\varphi \in U(2)$ then $\widehat{\varphi}$ acts on S^2 as an element of $SO(3)$: indeed $\varphi = e^{i\frac{\delta}{2}}\psi$, where $e^{i\delta} = \det\varphi$ and $\psi \in SU(2)$, and then $\widehat{\varphi} = \widehat{\psi}$ in $\mathrm{Aut}(CP^1)$, the action on S^2 being obtained by considering $h(\psi)$, which we shall from now on simply denote by $\widehat{\psi}$.

In the following, let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$ and let (l_1, l_2, l_3) be the triple of corresponding great circles of S^2: $l_j = p(L_j)$ for $j = 1, 2, 3$. As above, we denote by φ_{jk} the only unitary map sending L_j to L_k and verifying the conditions of the second diagonalization lemma. Let $(e^{i\lambda_{jk}}, e^{i\mu_{jk}})$ be its eigenvalues, where $\pi > \lambda_{jk} \geq \mu_{jk} \geq 0$, and let (u_{jk}, v_{jk}) be an orthonormal basis for L_j formed by eigenvectors of φ_{jk}: $\varphi_{jk}(u_{jk}) = e^{i\lambda_{jk}}u_{jk}$ and $\varphi(v_{jk}) = e^{i\mu_{jk}}v_{jk}$. Recall that $(e^{i2\lambda_{jk}}, e^{i2\mu_{jk}})$ then is a representative of $\mathrm{meas}(L_j, L_k) \in \mathbb{T}^2/\mathfrak{S}_2$. We denote by L_0 the Lagrangian subspace $L_0 = \{(x, y) \in \mathbb{C}^2 \mid x, y \in \mathbb{R}\}$ of $\mathbb{C}^2$. We denote its projection on $\mathbb{C}\mathrm{P}^1$ by $l_0 = p(L_0)$.

We are now going to relate the angles of projective Lagrangians of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ with the Lagrangian angles defined in section 2. Furthermore, in order to study configurations of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$, we are going to define a notion of sign of a projective Lagrangian triple. To do so, we shall first define such a notion in a generic case and then extend it to the remaining cases. At last, we shall see that there is also a notion of sign for Lagrangian triples of $\mathbb{C}^n$ and that in the case $n = 2$, the triples (L_1, L_2, L_3) and (l_1, l_2, l_3) have same sign.

Proposition 3.2 (Projection of a Lagrangian pair). *Let (L_1, L_2) be a pair of Lagrangian subspaces of $\mathbb{C}^2$ and let $(e^{i\lambda_{12}}, e^{i\mu_{12}})$ be the eigenvalues of φ_{12}. Then $l_1 = l_2$ if and only if $\lambda_{12} = \mu_{12}$. Furthermore, if $\lambda_{12} \neq \mu_{12}$, then l_2 is the image of l_1 by the (direct) rotation of angle $\alpha_{12} = \lambda_{12} - \mu_{12} \in]0, \pi[$ around the point $[v_{12}] \in \mathbb{C}\mathrm{P}^1 \simeq S^2 \subset \mathbb{R}^3$ ($[v_{12}] = \mathbb{C}v_{12}$ being the complex eigenline of φ_{12} associated to the eigenvalue $e^{i\mu_{12}}$ of lowest argument).*

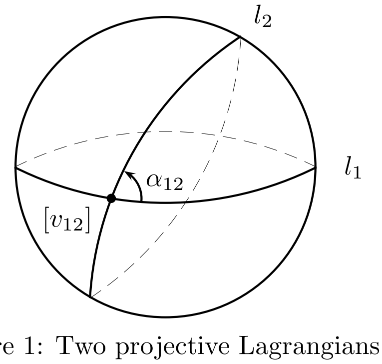


Figure 1: Two projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

Proof. If $\lambda_{12} = \mu_{12} = \lambda$ then $L_2 = e^{i\lambda}L_1$ and therefore $l_2 = l_1$ in $\mathbb{C}\mathrm{P}^1$.
If now $\lambda_{12} \neq \mu_{12}$, suppose first that $L_1 = L_0$ and that (u_{12}, v_{12}) is the standard basis of $\mathbb{C}^2$. Then L_2 is the image of L_1 by the unitary map whose matrix in the standard basis of $\mathbb{C}^2$ is $\begin{pmatrix} e^{i\lambda_{12}} & 0 \\ 0 & e^{i\mu_{12}} \end{pmatrix}$

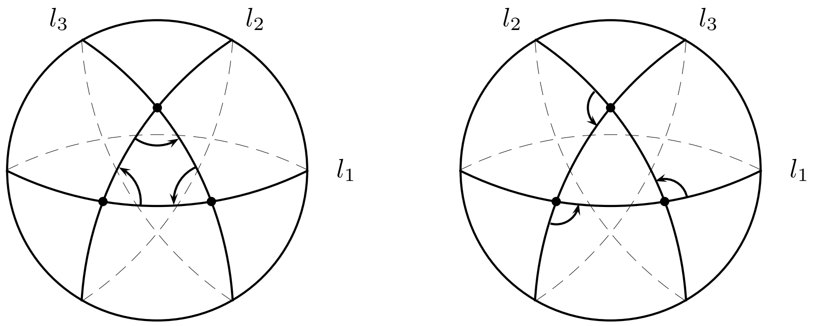
so that $L_2 = \{(e^{i\lambda_{12}}x, e^{i\mu_{12}}y) \mid x, y \in \mathbb{R}\}$ and $l_2 = p(L_2) = \{[e^{i\lambda_{12}}x, e^{i\mu_{12}}y] \mid x, y \in \mathbb{R}\}$. Therefore, in the chart $[z_1, z_2] \mapsto \frac{z_1}{z_2}$ of $\mathbb{C}\mathrm{P}^1$ containing $[v_{12}] = [0, 1]$, l_2 is sent diffeomorphically onto the real line $\{e^{i(\lambda_{12}-\mu_{12})}\frac{x}{y} \mid x, y \in \mathbb{R},\ y \neq 0\} = e^{i(\lambda_{12}-\mu_{12})}d_0$ of the plane $\mathbb{C} \simeq \mathbb{R}^2$, where d_0 is the image of $l_0 = l_1$ in this same chart. Thus, l_2 and l_1 intersect at $a_{12} = [u_{12}]$ and $b_{12} = [v_{12}]$, and l_2 is the image of l_1 by the rotation of angle $\alpha_{12} = \lambda_{12} - \mu_{12} \in]0, \pi[$ around the point $b_{12} = v_{12}$, which means that the oriented angle formed by l_1 and l_2 at b_{12} is of measure $\alpha_{12} = \lambda_{12} - \mu_{12}$. Note that the oriented angle at a_{12} is of measure $\pi - \alpha_{12} \in]0, \pi[$, since in the chart $[z_1, z_2] \mapsto \frac{z_2}{z_1}$, l_2 is diffeomorphic to the real line $e^{i(\mu_{12}-\lambda_{12})}d_0 = e^{i(\pi-(\lambda_{12}-\mu_{12}))}d_0$.
If now (u_{12}, v_{12}) is not the standard basis of $\mathbb{C}^2$, consider the unitary map $\psi \in U(2)$ sending the standard basis (e, f) of $\mathbb{C}^2$ to (u_{12}, v_{12}). Then $L_0 = \psi^{-1}(L_1)$, and let $L = \psi^{-1}(L_2)$. Then $[v_{12}] = \widehat{\psi}.[f]$, $l_2 = \widehat{\psi}(l)$ and $l_1 = \widehat{\psi}(l_0)$. Since then $\mathrm{meas}(L_0, L) = \mathrm{meas}(L_1, L_2)$, we deduce from the above paragraph that $l = p(L)$ is the image of l_0 by the rotation of angle the α_{12} around the point $[f]$. Since $\widehat{\psi} \in SO(3)$, the oriented angle between l_1 and l_2 at $b_{12} = [v_{12}] \in l_1 \cap l_2$ therefore also has measure α_{12}. □

Observe that this proof also provides an elementary way to see why L_0, and therefore every Lagrangian subspace of $\mathbb{C}^2$, projects to a great circle of $S^2 \simeq \mathbb{C}\mathrm{P}^1$. We shall state a converse to the above result later (see proposition 3.4).

Note that the preceeding result gives a complete description of the relative position of the projective Lagrangians l_1 and l_2 only by means of the unitary map φ_{12}. In particular, the rotation described above is no other that the map $\widehat{\varphi_{12}}$ of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ into itself: $l_2 = \widehat{\varphi_{12}}(l_1)$. The axis of that rotation is the real line of $\mathbb{R}^3$ generated by any of the antipodal points $a_{12} = [u_{12}]$ or $b_{12} = [v_{12}]$ of S^2, (u_{12}, v_{12}) being a unitary basis of $\mathbb{C}^2$ into which the matrix of φ_{12} is diagonal.

We are now going to describe all the possible configurations of the projective Lagrangians l_1, l_2 and l_3 of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ satisfying the following condition for $(j, k) = (1, 2), (2, 3), (3, 1)$: if $l_j \neq l_k$ then l_k is the image of l_j by the direct rotation φ_{jk} of $S^2 \subset \mathbb{R}^3$ of angle $\alpha_{jk} \in]0, \pi[$ around a specified point $b_{jk} \in l_j \cap l_k$.

First case: l_1, l_2 and l_3 are pairwise distinct.
(a) Suppose first that the three points b_{12}, b_{23}, b_{31} are pairwise distinct (that is, l_1, l_2, l_3 do not have a common diameter). We may then consider the spherical triangle (b_{12}, b_{23}, b_{31}), whose sides $[b_{12}, b_{23}]$, $[b_{23}, b_{31}]$, $[b_{31}, b_{12}]$ are respectively contained in the geodesics l_2, l_3, l_1. Since l_k is the image of l_j by a direct rotation around b_{jk}, the only possible configurations are the following ones:



A negative triple

A positive triple

Figure 2: Triples of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$ in general position

On each sphere, we represent the angles α_{jk} around the point b_{jk} and we shall continue to do so in the following. We call the first triangle *negative* and the second triangle *positive*. Let us explain this terminology and prove that these cases are indeed the only possible ones when the b_{jk} are pairwise distinct.
Let $\varphi = \varphi_{31} \circ \varphi_{23} \circ \varphi_{12} \in U(2)$. Then $\varphi(L_1) = L_1$ and therefore $\widehat{\varphi}(l_1) = l_1$. There are only two possible cases: either $\widehat{\varphi}$ preserves the orientation induced on l_1 by the orientation of S^2, or it reverses that orientation. But $\widehat{\varphi} = \widehat{\varphi_{31}} \circ \widehat{\varphi_{23}} \circ \widehat{\varphi_{12}}$ is the map obtained by composing the three rotations φ_{jk} around the b_{jk}'s. When $\widehat{\varphi}$ reverses the orientation of l_1, which we will call the *negative case*, then $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ are the angles of the spherical triangle (b_{12}, b_{23}, b_{31}). When $\widehat{\varphi}$ preserves the orientation

of l_1, which we will call the *positive case*, then the angles of the triangle (b_{12}, b_{23}, b_{31}) are β_{jk}, where $\beta_{jk} = \pi - \alpha_{jk} \in]0, \pi[$. Observe that this gives a series of necessary conditions for the existence of a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ projecting onto a triple (l_1, l_2, l_3) of great circles of S^2 that do not have a common diameter. Namely, for instance if the triangle (b_{12}, b_{23}, b_{31}) has angles α_{jk} (negative case), we necessarily have

$$(\Delta) \quad \left\{ \begin{array}{ccc} \alpha_{12}, \alpha_{23}, \alpha_{31} & \in &]0, \pi[\\ \alpha_{12} + \alpha_{23} + \alpha_{31} & > & \pi \\ \alpha_{12} + \pi & > & \alpha_{23} + \alpha_{31} \\ \alpha_{23} + \pi & > & \alpha_{31} + \alpha_{12} \\ \alpha_{31} + \pi & > & \alpha_{12} + \alpha_{23} \end{array} \right.$$

since $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ are the angles of a spherical triangle. In the positive case, the same conditions apply to $(\beta_{12}, \beta_{23}, \beta_{31})$. In the following we shall write $(\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \Delta$ to say that $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ verify this set of conditions. Δ is an open subset of $\mathbb{R}^3$ and its closure $\overline{\Delta}$ in $\mathbb{R}^3$ is a tetrahedron, and is therefore endowed with a 3-dimensional cell complex structure (see figure 3).

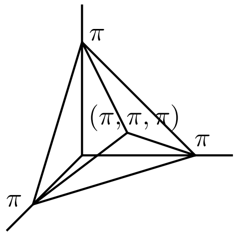


Figure 3: The tetrahedron (Δ)

(b) Suppose now that b_{12}, b_{23} and b_{31} are not pairwise distinct: either $b_{12} = b_{23} = b_{31}$ or, for instance, $b_{12} = b_{23}$ and $b_{31} \neq b_{12}$. Since $l_1, l_2 . l_3$ are still supposed to be pairwise distinct and satisfying $l_k = \widehat{\varphi_{jk}}(l_j)$, the only possible configurations are the ones shown in figure 4.
These 4 cases correspond to degenerate spherical triangles, so that we have the following respective necessary conditions:

$$\begin{array}{c|c} \begin{array}{ccc} \alpha_{12}, \alpha_{23}, \alpha_{31} & \in &]0, \pi[\\ \alpha_{12} + \alpha_{23} + \alpha_{31} & = & \pi \\ \alpha_{12} + \pi & > & \alpha_{23} + \alpha_{31} \\ \alpha_{23} + \pi & > & \alpha_{31} + \alpha_{12} \\ \alpha_{31} + \pi & > & \alpha_{12} + \alpha_{23} \end{array} & \begin{array}{ccc} \beta_{12}, \beta_{23}, \beta_{31} & \in &]0, \pi[\\ \beta_{12} + \beta_{23} + \beta_{31} & = & \pi \\ \beta_{12} + \pi & > & \beta_{23} + \beta_{31} \\ \beta_{23} + \pi & > & \beta_{31} + \beta_{12} \\ \beta_{31} + \pi & > & \beta_{12} + \beta_{23} \end{array} \\ \hline \begin{array}{ccc} \beta_{12}, \beta_{23}, \beta_{31} & \in &]0, \pi[\\ \beta_{12} + \beta_{23} + \beta_{31} & > & \pi \\ \beta_{12} + \pi & > & \beta_{23} + \beta_{31} \\ \beta_{23} + \pi & > & \beta_{31} + \beta_{12} \\ \beta_{31} + \pi & = & \beta_{12} + \beta_{23} \end{array} & \begin{array}{ccc} \alpha_{12}, \alpha_{23}, \alpha_{31} & \in &]0, \pi[\\ \alpha_{12} + \alpha_{23} + \alpha_{31} & > & \pi \\ \alpha_{12} + \pi & > & \alpha_{23} + \alpha_{31} \\ \alpha_{23} + \pi & > & \alpha_{31} + \alpha_{12} \\ \alpha_{31} + \pi & = & \alpha_{12} + \alpha_{23} \end{array} \end{array}$$

This means that either the α_{jk} or the β_{jk}, depending on the negativity or positivity of the triple (l_1, l_2, l_3), are located in an open 2-cell of the 3-dimensional complex $\overline{\Delta}$ (see figure 3). The remaining 2-cells are obtained when $b_{23} = b_{31}$ and $b_{12} \neq b_{23}$, and when $b_{31} = b_{12}$ and $b_{23} \neq b_{31}$.

Second case: l_1, l_2 and l_3 are not pairwise distinct.
(a) Suppose first, for instance, that $l_1 = l_2$ and $l_3 \neq l_1$. Since $l_1 = l_2$, we may consider either that $\alpha_{12} = 0$ or that $\alpha_{12} = \pi$ and that it is the angle of a direct rotation around $b_{23} \in l_2 \cap l_3 = l_1 \cap l_3$, *so that the notion of negative and positive triples is still valid.* Then the only possible configurations of l_1, l_2, l_3 are the ones shown in figure 5.

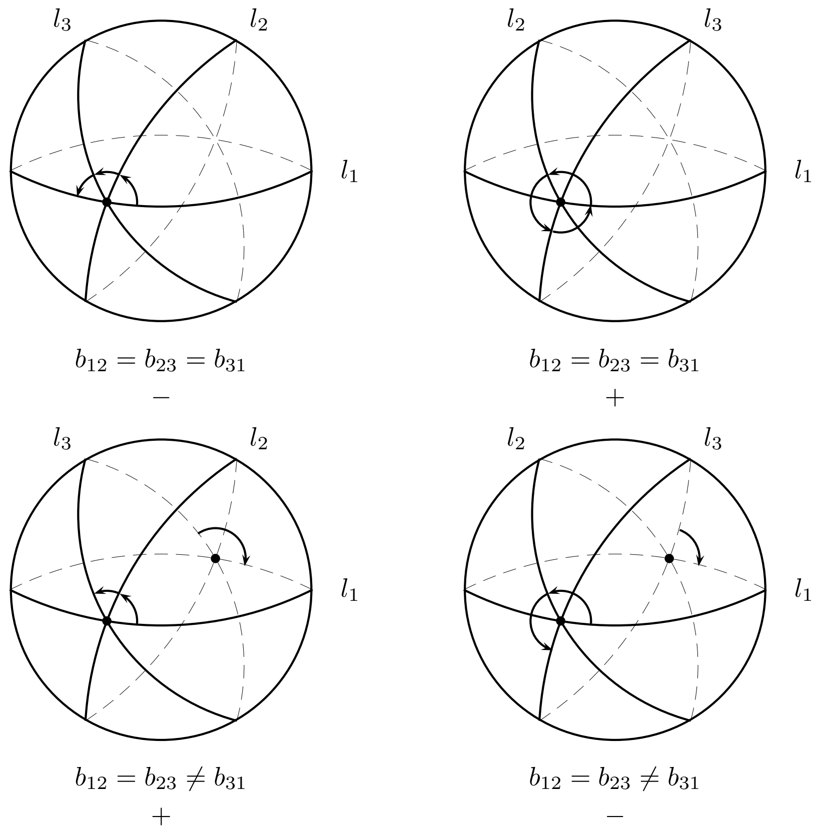


Figure 4: Exceptional triples of pairwise distinct projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

Those configurations correspond to open 1-cells of $\overline{\Delta}$ (see figure 3):

$\alpha_{12} = 0$		$\alpha_{23}, \alpha_{31} \in]0, \pi[$	$\beta_{12} = \pi$		$\beta_{23}, \beta_{31} \in]0, \pi[$
$\alpha_{12} + \alpha_{23} + \alpha_{31}$	$=$	π	$\beta_{12} + \beta_{23} + \beta_{31}$	$>$	π
$\alpha_{12} + \pi$	$=$	$\alpha_{23} + \alpha_{31}$	$\beta_{12} + \pi$	$>$	$\beta_{23} + \beta_{31}$
$\alpha_{23} + \pi$	$>$	$\alpha_{31} + \alpha_{12}$	$\beta_{23} + \pi$	$=$	$\beta_{31} + \beta_{12}$
$\alpha_{31} + \pi$	$>$	$\alpha_{12} + \alpha_{23}$	$\beta_{31} + \pi$	$=$	$\beta_{12} + \beta_{23}$
$\beta_{12} = 0$		$\beta_{23}, \beta_{31} \in]0, \pi[$	$\alpha_{12} = \pi$		$\alpha_{23}, \alpha_{31} \in]0, \pi[$
$\beta_{12} + \beta_{23} + \beta_{31}$	$=$	π	$\alpha_{12} + \alpha_{23} + \alpha_{31}$	$>$	π
$\beta_{12} + \pi$	$=$	$\beta_{23} + \beta_{31}$	$\alpha_{12} + \pi$	$>$	$\alpha_{23} + \alpha_{31}$
$\beta_{23} + \pi$	$>$	$\beta_{31} + \beta_{12}$	$\alpha_{23} + \pi$	$=$	$\alpha_{31} + \alpha_{12}$
$\beta_{31} + \pi$	$>$	$\beta_{12} + \beta_{23}$	$\alpha_{31} + \pi$	$=$	$\alpha_{12} + \alpha_{23}$

The remaining 1-cells are obtained when $l_2 = l_3$ and $l_1 \neq l_2$ and when $l_3 = l_1$ and $l_2 \neq l_3$.
(b) Suppose at last that $l_1 = l_2 = l_3$. The notion of negative and positive triples remains valid by considering either that $\alpha_{jk} = 0$ or that $\alpha_{jk} = \pi$, and that the b_{jk}'s all are a same b chosen arbitrarily in $l_1 = l_2 = l_3$. Then the possible configurations on S^2 correspond to the 0-cells of $\overline{\Delta}$; that is, in the negative case, $(\alpha_{12}, \alpha_{23}, \alpha_{31}) = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi)$ or (π, π, π), and in the positive case: $(\beta_{12}, \beta_{23}, \beta_{31}) = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi)$ or (π, π, π). Observe that in the cases where the three rotations $\widehat{\varphi_{jk}}$ occur around a same point b_{jk} or around two diametrically opposed points, then the negative case corresponds to $(\alpha_{12} + \alpha_{23} + \alpha_{31}) \equiv \pi \pmod{2\pi}$, and the positive case corresponds to $(\beta_{12} + \beta_{23} + \beta_{31}) \equiv \pi \pmod{2\pi}$, that is to $(\alpha_{12} + \alpha_{23} + \alpha_{31}) \equiv 0 \pmod{2\pi}$. Also note that if l_1, l_2, l_3 are pairwise distinct great circles of S^2 that do not have a common diameter, the mutual intersections $l_j \cap l_k$ determine 6 points on S^2,

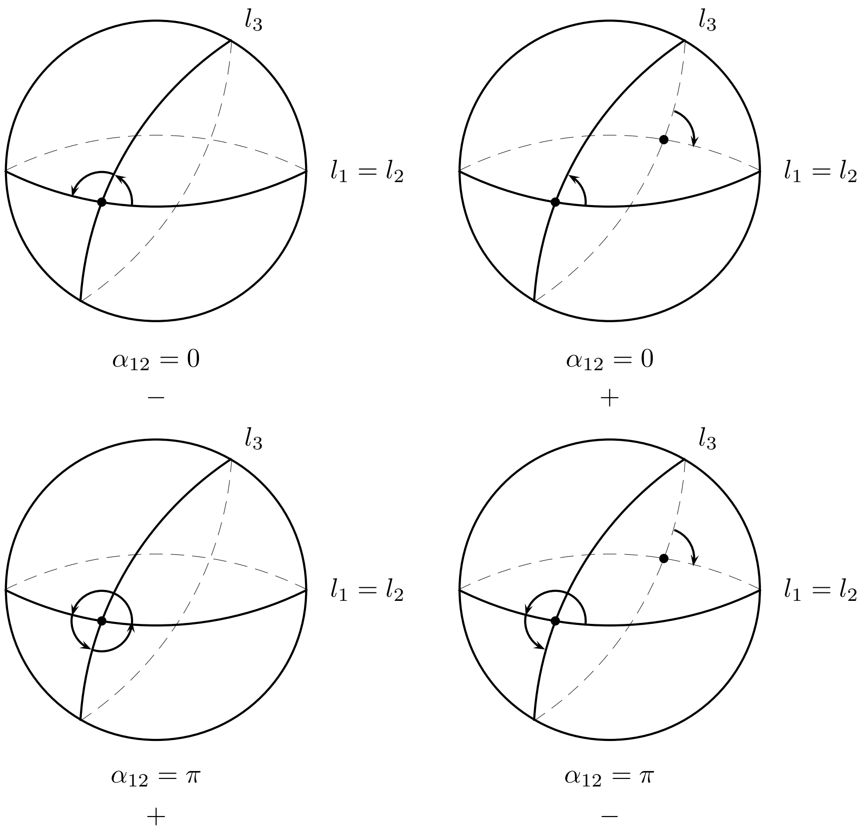


Figure 5: Triples of non pairwise distinct projective Lagrangians of $\mathbb{CP}^1$

which in turn give rise to 8 spherical triangles, four of which are negative, the four others being positive. Two triangles with a common edge have opposite sign, whereas two triangles with only a common vertex have same sign.

From the study above, we deduce that a Lagrangian triple (L_1, L_2, L_3) projects on a triple (l_1, l_2, l_3) of great circles of S^2, that is either positive with $(\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \overline{\Delta}$ or negative with $(\beta_{12}, \beta_{23}, \beta_{31}) \in \overline{\Delta}$. In particular, these conditions are necessary conditions for $([e^{i2\lambda_{12}}, e^{i2\mu_{12}}], [e^{i2\lambda_{23}}, e^{i2\mu_{23}}], [e^{i2\lambda_{31}}, e^{i2\mu_{31}}])$ to be the triple of measures of a Lagrangian triple. Before showing that these conditions are sufficient, we shall give another way of determining if a triple (l_1, l_2, l_3) is negative or positive.

Proposition 3.3. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$, and set $\varphi = \varphi_{31} \circ \varphi_{23} \circ \varphi_{12}$. Write $\det \varphi = e^{i\delta}$, where $\delta = (\lambda_{12} + \mu_{12}) + (\lambda_{23} + \mu_{23}) + (\lambda_{31} + \mu_{31})$. Then $\delta \equiv 0 \pmod{\pi}$, and (l_1, l_2, l_3) is negative if $\delta \equiv \pi \pmod{2\pi}$ and positive if $\delta \equiv 0 \pmod{2\pi}$.*

Observe that when $\delta \equiv 0 \pmod{2\pi}$ we have $\det \varphi = 1$, so that we might also say that the triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ is positive. Similarly, when $\delta \equiv \pi \pmod{2\pi}$, $\det \varphi = -1$ and (L_1, L_2, L_3) will then be said to be negative. The above proposition then says that the triples (L_1, L_2, L_3) and (l_1, l_2, l_3) have same sign. Note that the notion of sign of a Lagrangian triple (L_1, L_2, L_3) is also valid for Lagrangian subspaces of $\mathbb{C}^n$.

Proof. Suppose first that $L_1 = L_0$ and that (u_{12}, v_{12}) is the standard basis of $\mathbb{C}^2$. Write $\varphi_{jk} = e^{i\frac{\lambda_{jk}+\mu_{jk}}{2}} \psi_{jk}$ where $\psi_{jk} \in SU(2)$ and $e^{i(\lambda_{jk}+\mu_{jk})} = \det \varphi_{jk}$. Set $\psi = \psi_{31} \circ \psi_{23} \circ \psi_{12}$, so that $\varphi = e^{i\frac{\delta}{2}} \psi$, where $\delta = \sum_{j,k} (\lambda_{jk} + \mu_{jk})$. Note that $\widehat{\varphi_{jk}} = \widehat{\psi_{jk}}$ and $\widehat{\varphi} = \widehat{\psi}$. In particular, $\widehat{\psi}(l_0) = l_0$. But the matrix of ψ in the standard basis of $\mathbb{C}^2$ is of the form $A = \begin{pmatrix} s & -\overline{t} \\ t & \overline{s} \end{pmatrix}$ where $s, t \in \mathbb{C}$ and verify $|s|^2 + |t|^2 = 1$. Since $l_0 = \{[x, y] \in \mathbb{CP}^1, x, y \in \mathbb{R}\}$, $\widehat{\psi}(l_0) = l_0$ if and only if $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ or $A = \begin{pmatrix} ia & ib \\ ib & -ia \end{pmatrix}$ where $a, b \in \mathbb{R}$

and verify $a^2 + b^2 = 1$.
In the first case $\psi(L_0) = L_0$, so that $L_0 = \varphi(L_0) = e^{i\frac{\delta}{2}}$, and since L_0 is totally real we have $\frac{\delta}{2} \equiv 0 \pmod{\pi}$, that is $\delta \equiv 0 \pmod{2\pi}$. In the second case $\psi(L_0) = i.L_0$, so that $L_0 = \varphi(L_0) = e^{i\frac{\delta}{2}} i.L_0$ and therefore $\frac{\delta}{2} \equiv \frac{\pi}{2} \pmod{\pi}$, that is $\delta \equiv \pi \pmod{2\pi}$. Now recall that $\widehat{\varphi}(l_0) = (l_0)$. When $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, the orientation on l_0 is preserved by $\widehat{\psi}$ (since, in the chart $[z_1, z_2] \mapsto \frac{z_1}{z_2}$, the map $x \in \mathbb{R} \mapsto \frac{ax-b}{bx+a}$ is increasing), so that the triple (l_0, l_2, l_3) is positive. When $A = \begin{pmatrix} ia & ib \\ ib & -ia \end{pmatrix}$, the orientation on l_0 is reversed by $\widehat{\psi}$ (since, in the chart $[z_1, z_2] \mapsto \frac{z_1}{z_2}$, the map $x \in \mathbb{R} \mapsto \frac{ax+b}{bx-a}$ is decreasing), so that the triple (l_0, l_2, l_3) is negative.
Suppose now that (u_{12}, v_{12}) is not the standard basis of $\mathbb{C}^2$, and define the unitary map $\nu \in U(2)$ sending the standard basis to (u_{12}, v_{12}). Let $L'_2 = \nu^{-1}(L_2)$, $L'_3 = \nu^{-1}(L_3)$, $l'_2 = p(L'_2)$ and $l'_3 = p(L_3)$. Then $\nu^{-1} \circ \varphi \circ \nu$ sends L_0 to L_0 and $\det(\nu^{-1} \circ \varphi \circ \nu) = \det \varphi = e^{i\delta}$. From the study above, the triple (l_0, l'_2, l'_3) is positive if and only if $\delta \equiv 0 \pmod{2\pi}$, and negative if and only if $\delta \equiv \pi \pmod{2\pi}$. But since $l_1 = \widehat{\nu}(l_0)$, $l_2 = \widehat{\nu}(l'_2)$ and $l_3 = \widehat{\nu}(l'_3)$ with $\widehat{\nu} \in SO(3)$, the triples (l_1, l_2, l_3) and (l'_0, l'_2, l'_3) have same sign. □

3.2.2 Second classification result for triples of Lagrangian subspaces of $\mathbb{C}^2$

As a converse to proposition 3.2, it is possible, given two distinct great circles $l_1 \neq l_2$ of $S^2 \simeq \mathbb{CP}^1$, to describe the measure of the angle (L_1, L_2) between two Lagrangians of $\mathbb{C}^2$ that project respectively to l_1 and l_2. Recall that two distinct great cirles $l_1 \neq l_2$ intersect along two antipodal points a, b, and that $\alpha \in]0, \pi[$ is said to be the measure of the oriented angle between l_1 and l_2 at $b \in l_1 \cap l_2$ if l_2 is the image of l_1 by the (direct) rotation of angle α around b.

Proposition 3.4 (Lifting lemma). *Let $l_1 \neq l_2$ be two distinct projective Lagrangians of $\mathbb{CP}^1 \simeq S^2$, let $b \in l_1 \cap l_2$ and let $\alpha \in]0, \pi[$ be the measure of the oriented angle (l_1, l_2) at b. Then, given λ and μ such that $\pi > \lambda > \mu \geq 0$, and given a Lagrangian subspace $L_1 \in p^{-1}(l_1)$, there exists a unique Lagrangian subspace $L_2 \in p^{-1}(l_2)$ such that* $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda}, e^{i2\mu}]$.

Proof. Let $v \in L_1$ such that $p(v) = b$. We may choose v such that $\|v\| = h(v, v) = 1$. Let then $u \in L_1$ such that (u, v) is an orthonormal basis for L_1. Since L_1 is Lagrangian, (u, v) is a unitary basis for $\mathbb{C}^2$. Let ψ be the unitary transformation of $\mathbb{C}^2$ whose matrix in the basis (u, v) is $\begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{i\mu} \end{pmatrix}$, and let $L = \psi(L_1)$. Then L is Lagrangian and $\mathrm{meas}(L_1, L) = [e^{i2\lambda}, e^{i2\mu}]$. Therefore, by proposition 3.2, $l = p(L)$ is a great circle of S^2, distinct of l_1 since $\lambda \neq \mu$, that intersects l_1 at $p(v) = b$ and the measure of the oriented angle between l_1 and l at b is $\lambda - \mu = \alpha$, so that $l = l_2$.
As for unicity, if $L' \in p^{-1}(l_2)$, then, again by proposition 3.2, we know that $L' = e^{i\theta}.L$, where $\theta \in]0, \pi[$. The unitary map $e^{i\theta}.\psi$ then sends L_1 to L', and its matrix in the unitary basis (u, v), which is an orthonormal basis for L_1 is $\begin{pmatrix} e^{i(\theta+\lambda)} & 0 \\ 0 & e^{i(\theta+\mu)} \end{pmatrix}$ so that $\mathrm{meas}(L_1, L') = [e^{i2((\theta+\lambda) \mod \pi)}, e^{i((\theta+\mu) \mod \pi)}]$, with $\pi > (\theta + \lambda) \mod \pi > (\theta + \mu) \mod \pi \geq 0$. Since $\mathrm{meas}(L_1, L) = \mathrm{meas}(L_1, L')$, we have in particular $(\theta + \lambda) \mod \pi = \lambda$, hence $\theta \mod \pi = 0$ (and so $\theta = 0$) and $L' = e^{i\theta}.L = L$. □

The next theorem is our main result: it completely describes the image of the map ρ and lays the ground for the second classification theorem for triples of Lagrangian subspaces of $\mathbb{C}^2$.

Theorem 3.5. *Given a triple of measures* $([e^{i2\lambda_{12}}, e^{i2\mu_{12}}], [e^{i2\lambda_{23}}, e^{i2\mu_{23}}], [e^{i2\lambda_{31}}, e^{i2\mu_{31}}])$ *satisfying the conditions $\pi \geq \lambda_{jk} \geq \mu_{jk} \geq 0$, set $\alpha_{jk} = \lambda_{jk} - \mu_{jk} \in [0, \pi]$, $\beta_{jk} = \pi - \alpha_{jk} \in [0, \pi]$ and $\delta = (\lambda_{12} + \mu_{12}) + (\lambda_{23} + \mu_{23}) + (\lambda_{31} + \mu_{31})$. Then, a necessary and sufficient condition for the existence of a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ such that* $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_{12}}, e^{i2\mu_{12}}]$, $\mathrm{meas}(L_2, L_3) = [e^{i2\lambda_{23}}, e^{i2\mu_{23}}]$, *and* $\mathrm{meas}(L_3, L_1) = [e^{i2\lambda_{31}}, e^{i2\mu_{31}}]$ *is that*

$$\delta \equiv \pi \pmod{2\pi} \quad \text{and} \quad (\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \overline{\Delta} \text{ (negative case)}$$

or

$$\delta \equiv 0 \pmod{2\pi} \quad \text{and} \quad (\beta_{12}, \beta_{23}, \beta_{31}) \in \overline{\Delta} \text{ (positive case)}$$

(Here we allow $\lambda_{jk} = \pi$ so that we may have $\alpha_{jk} = \pi$ and $\beta_{jk} = 0$).

Proof. The study made in paragraph 3.2.1 shows that these conditions are necessary.
Conversely, suppose first that $\delta \equiv \pi \pmod{2\pi}$ and that $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ lie in the open set Δ. Then there exists a negative triple $(l_1, l_2 l_{,3})$ of pairwise distinct great circles of S^2 such that l_k is the image of l_j by the direct rotation of angle α_{jk} around a certain point $b_{jk} \in l_j \cap l_k$ for $(j,k) = (1,2), (2,3), (3,1)$, and we may suppose that $l_1 = l_0$. Let $L_1 = L_0$. Then, by proposition 3.4, there exists a unique Lagrangian $L_2 \in p^{-1}(l_2)$ such that $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_{12}}, e^{i2\mu_{12}}]$. Again by proposition 3.4, there exists a unique Lagrangian $L_3 \in p^{-1}(l_3)$ such that $\mathrm{meas}(L_2, L_3) = [e^{i2\lambda_{23}}, e^{i2\mu_{23}}]$, and a unique Lagrangian $L_4 \in p^{-1}(l_1)$ such that $\mathrm{meas}(L_3, L_4) = [e^{i2\lambda_{31}}, e^{i2\mu_{31}}]$. Let φ_{34} be the unique unitary map sending L_3 to L_4 and satisfying the conditions of the second diagonalization lemma 2.3, and let $\varphi = \varphi_{34} \circ \varphi_{23} \circ \varphi_{12}$. Then $\varphi(L_1) = L_4$ and $\det \varphi = e^{i\delta}$. Write $\varphi = e^{i\frac{\delta}{2}}\psi$, where $\psi \in SU(2)$. Then $\widehat{\psi}(l_1) = l_1$, and since (l_1, l_2, l_3) is negative, we have, from the study made in paragraph 3.2.1, that $\psi(L_1) = i.L_1$, hence, as $\delta \equiv \pi \pmod{2\pi}$, we have $L_4 = \varphi(L_1) = e^{i\frac{\delta}{2}} i.L_1 = L_1$.
Suppose now that $(\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \partial\Delta$. If $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ lay in an open 2-cell of $\overline{\Delta}$, there exists a negative triple (l_1, l_2, l_3) of pairwise distinct great circles of S^2 such that l_k is the image of l_j by the direct rotation of angle α_{jk} around a certain point $b_{jk} \in l_j \cap l_k$ for $(j,k) = (1,2), (2,3), (3,1)$, and we can therefore conclude as earlier. If now $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ lay in an open 1-cell of $\overline{\Delta}$, there exists a negative triple, for instance of the form $(l_1, l_2 = l_1, l_3 \neq l_2)$, such that l_3 is the image of l_2 by the rotation of angle α_{23} around $b_{23} \in l_2 \cap l_3$ and such that l_1 is the image of l_3 by the rotation of angle α_{31} around $b_{31} \in l_3 \cap l_1$. Since $l_2 = l_1$, α_{12} is either 0 or π, and by setting $b_{12} = b_{23}$ (or $b_{12} = b_{31}$), we have that l_2 is the image of l_1 by the rotation of angle α_{12} around $b_{12} \in l_1 \cap l_2$ (see figure 5). Let $L_1 = L_0$. If $\alpha_{12} = 0$, then $\lambda_{12} = \mu_{12}$ and we set $L_2 = e^{i\lambda_{12}}.L_1$. If $\alpha_{12} = \pi$, then $\lambda_{12} = \pi$ and $\mu_{12} = 0$, and we set $L_2 = L_1$. In both cases $L_2 \in p^{-1}(l_2) = p^{-1}(l_1)$ and $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_{12}}, e^{i2\mu_{12}}]$. Since $l_1 = l_2 \neq l_3$, there exists, by proposition 3.4, a unique Lagrangian $L_3 \in p^{-1}(l_3)$ such that $\mathrm{meas}(L_2, L_3) = [e^{i2\lambda_{23}}, e^{i2\mu_{23}}]$, and a unique Lagrangian $L_4 \in p^{-1}(l_1)$ such that $\mathrm{meas}(L_3, L_4) = [e^{i2\lambda_{31}}, e^{i2\mu_{31}}]$. As earlier, since the triple (l_1, l_2, l_3) is negative, we have $L_4 = e^{i\frac{\delta}{2}} i.L_1 = L_1$.
At last, if $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ is a 0-cell of $\overline{\Delta}$, that is, if $(\alpha_{12}, \alpha_{23}, \alpha_{31}) = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi)$ or (π, π, π), then $L_1 = L_2 = L_3 = L_0$ meet the required conditions.
If now, $\delta \equiv 0 \pmod{2\pi}$, the condition $(\beta_{12}, \beta_{23}, \beta_{31}) \in \overline{\Delta}$ implies the existence of a positive triple (l_1, l_2, l_3) of pairwise distinct great circles of S^2, with angles α_{jk} as required. Reasoning the same way, we find 4 Lagrangians L_1, L_2, L_3 and L_4 with prescribed angles $[e^{i2\lambda_{jk}}, e^{i2\mu_{jk}}]$, and since (l_1, l_2, l_3) is positive we have: $L_4 = \varphi(L_1) = e^{i\frac{\delta}{2}}.L_1$, and therefore, as $\delta \equiv 0 \pmod{2\pi}$, $L_4 = L_1$.
The other cases are treated identically. $\square$

We now obtain the following classification theorem for triples of Lagrangian subspaces of $\mathbb{C}^2$.

Theorem 3.6 (Unitary classification of Lagrangian triples of $\mathbb{C}^2$, second version). *Given two triples (L_1, L_2, L_3) and (L_1', L_2', L_3') of Lagrangian subspaces of $\mathbb{C}^2$, a necessary and sufficient condition for the existence of a unitary map $\varphi \in U(2)$ such that $\varphi(L_1) = L_1'$, $\varphi(L_2) = L_2'$ and $\varphi(L_3) = L_3'$ is that* $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L_1', L_2')$, $\mathrm{meas}(L_2, L_3) = \mathrm{meas}(L_2', L_3')$, *and* $\mathrm{meas}(L_3, L_1) = \mathrm{meas}(L_3', L_1')$.
Equivalently, the map $\rho : (\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2))/U(2) \longrightarrow \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2$ is one-to-one and is therefore a homeomorphism from the orbit space $(\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2))/U(2)$ onto a closed subset of the measure space $\mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2$.

Proof. It only remains to prove that the above conditions are sufficient. Let (L_1, L_2, L_3) and (L_1', L_2', L_3') be two Lagrangian triples such that $\mathrm{meas}(L_j, L_k) = \mathrm{meas}(L_j', L_k')$ for all j, k. Then, the (generalized) triangles (b_{12}, b_{23}, b_{31}) and $(b_{12}', b_{23}', b_{31}')$ have the same angles, so there exists a map $\psi \in SU(2)$ such that $\widehat{\psi}(b_{jk}) = b_{jk}'$ for all j, k. Since moreover $\delta = \delta'$, the triples (l_1, l_2, l_3) and (l_1', l_2', l_3') have same sign and we therefore even have $\widehat{\psi}(l_j) = l_j'$ for all j. Equivalently, $p(\psi(L_j)) = \widehat{\psi}(p(L_j)) = \widehat{\psi}(l_j) = l_j' = p(L_j')$. In particular, by proposition 3.2, we have $L_1' = e^{i\theta}.\psi(L_1)$ for some $\theta \in [0, \pi[$. Set $\varphi = e^{i\theta}.\psi \in U(2)$. Then $\varphi(L_1) = L_1'$ and $p(\varphi(L_2)) = \widehat{\varphi}(p(L_2)) = \widehat{\psi}(l_2) = l_2'$ and $\mathrm{meas}(L_1', \varphi(L_2)) = \mathrm{meas}(\varphi(L_1), \varphi(L_2)) = \mathrm{meas}(L_1, L_2) = \mathrm{meas}(L_1', L_2')$, hence, by unicity in proposition 3.4, $\varphi(L_2) = L_2'$. Likewise, $p(\varphi(L_3)) = l_3'$ and $\mathrm{meas}(L_2', \varphi(L_3)) = \mathrm{meas}(L_2', L_3')$, therefore $\varphi(L_3) = L_3'$. $\square$

The above study suggests using trigonometry in $\mathbb{C}\mathrm{P}^{n-1}$ to classify triples of Lagrangian subspaces of $\mathbb{C}^n$.

3.3 Equivalence of the two classification results

We now wish to explain why the two classification results that we have obtained (theorems 3.1 and 3.6) are indeed equivalent.

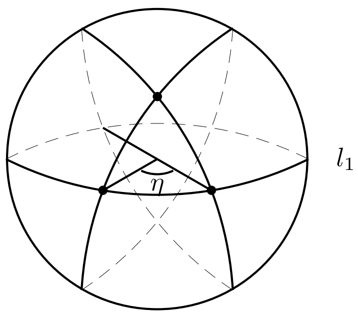
Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$. If one of the unitary maps φ_{jk} is of the form $e^{i\lambda}Id$ (for instance if $L_2 = e^{i\lambda}.L_1$), and if (L'_1, L'_2, L'_3) is a triple of Lagrangian subspaces such that $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$ and $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L'_1, L'_3)$ (or equivalently $\mathrm{meas}(L_3, L_1) = \mathrm{meas}(L'_3, L'_1)$), we necessarily have $\mathrm{meas}(L_2, L_3) = \mathrm{meas}(L'_2, L'_3)$, which proves that in this case the two classification results are indeed the same.

If now each unitary map φ_{jk} has two distinct eigenvalues $e^{i\lambda_{jk}}$ and $e^{i\mu_{jk}}$, where $\pi > \lambda_{jk} > \mu_{jk} \geq 0$, set $d_{12} = \mathbb{R}u_{12} \subset L_1$ and $d_{13} = \mathbb{R}u_{13} \subset L_1$ (where u_{12} and u_{13} are defined as earlier by means of φ_{12} and φ_{13}), and let $\theta = \mathrm{meas}(d_{12}, d_{13}) \in [0, \frac{\pi}{2}]$ be the measure of the non-oriented angle formed by the real lines d_{12} and d_{13} in the Euclidean space L_1. Recall that $L_1 = d_{12} \oplus d_{12}^{\perp} = d_{13} \oplus d_{13}^{\perp}$, where $d_{12}^{\perp} = \mathbb{R}v_{12}$ and $d_{13}^{\perp} = \mathbb{R}v_{13}$, and observe that θ is also the measure of the angle $(d_{12}^{\perp}, d_{13}^{\perp})$. As earlier, define $b_{jk} = [v_{jk}] \in l_j \cap l_k \subset \mathbb{C}\mathrm{P}^1 \simeq S^2$.

Observe now that $b_{31} \in l_1 \cap l_3$ is one of the two antipodal points a_{13} or $b_{13} \in l_1 \cap l_3$. One can then check the following remarks:

- The measure of the non-oriented angle formed by the two vectors b_{12} and b_{13} of $S^2 \subset \mathbb{R}^3$ is $2\theta \in [0, \pi]$ (in particular, two orthogonal vectors of L_1 project onto antipodal points of S^2).
- If $\mu_{13} = 0$ then $b_{31} = b_{13}$ and therefore $\mathrm{meas}(b_{12}, b_{31}) = 2\theta$. If $\mu_{13} \neq 0$ then $b_{31} = a_{13}$ and therefore $\mathrm{meas}(b_{12}, b_{31}) = \pi - 2\theta$.

Let now $(\gamma_{12}, \gamma_{23}, \gamma_{31})$ be the measures of the angles of the spherical triangle (b_{12}, b_{23}, b_{31}) (from the study of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$, we know that either $(\gamma_{12}, \gamma_{23}, \gamma_{31}) = (\alpha_{12}, \alpha_{23}, \alpha_{31})$ or $(\gamma_{12}, \gamma_{23}, \gamma_{31}) = (\beta_{12}, \beta_{23}, \beta_{31})$, where $\alpha_{jk} = \lambda_{jk} - \mu_{jk}$ and $\beta_{jk} = \pi - \alpha_{jk}$). Let $\eta \in [0, \pi]$ be the measure of the non-oriented angle (b_{12}, b_{31}) (from the study above, we know that either $\eta = 2\theta$ or $\eta = \pi - 2\theta$). Then we know from spherical trigonometry that $\cos\gamma_{23} = \sin\gamma_{12}\sin\gamma_{31}\cos\eta - \cos\gamma_{12}\cos\gamma_{31}$.



$\eta = 2\theta$ or $\pi - 2\theta$

Figure 6: Relation between the two classication results

The next proposition completes the explanation why our two classification results are indeed equivalent.

Proposition 3.7. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$ such that $\varphi_{12}, \varphi_{23}$ and φ_{31} have distinct eigenvalues. Let (L'_1, L'_2, L'_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$ such that* $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$ *and* $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L'_1, L'_3)$ *(this last condition being equivalent to* $\mathrm{meas}(L_3, L_1) = \mathrm{meas}(L'_3, L'_1)$*). Let* $\theta = \mathrm{meas}(d_{12}, d_{13}) \in [0, \frac{\pi}{2}]$ *be the measure of the non-oriented angle (d_{12}, d_{13}) in L_1 and define* $\theta' = \mathrm{meas}(d'_{12}, d'_{13}) \in [0, \frac{\pi}{2}]$ *in L'_1 similarly. Then* $\mathrm{meas}(L_2, L_3) = \mathrm{meas}(L'_2, L'_3)$ *if and only if $\theta = \theta'$.*

Proof. Assume first that $\mathrm{meas}(L_2, L_3) = \mathrm{meas}(L'_2, L'_3)$. Since we also have $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$ and $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L'_1, L'_3)$, we get $\delta = \delta'$: the triples (l_1, l_2, l_3) and (l'_1, l'_2, l'_3) have same sign. As a consequencce, the spherical triangles (b_{12}, b_{23}, b_{31}) and $(b'_{12}, b'_{23}, b'_{31})$ have the same angles: $\gamma_{jk} = \gamma'_{jk} \in]0, \pi[$ for all j, k. Since $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L'_1, L'_3)$ we have $\mu_{13} = \mu'_{13}$, therefore, by the above

remarks, either $b_{31} = b_{13}$ and $b'_{31} = b'_{13}$ (when μ_{13} and μ'_{13} equal zero) or $b_{31} = a_{13}$ and $b'_{31} = a'_{13}$ (when $\mu_{13} = \mu'_{13} \neq 0$), so either $\eta = 2\theta$ and $\eta' = 2\theta'$ or $\eta = \pi - 2\theta$ and $\eta' = \pi - 2\theta'$. But then from the relation from spherical trigonometry recalled above, since $\sin \gamma_{jk} \neq 0$ for all j, k, we have $\cos \eta = \cos \eta'$, and since $\eta, \eta' \in [0, \pi]$ we get $\eta = \eta'$, therefore $\theta = \theta'$.
Assume now that $\theta = \theta'$. Then, as in proposition 3.1, there exists a unitary $\psi \in U(2)$ such that $\psi(L_j) = L'_j$ for $j = 1, 2, 3$, so that $\text{meas}(L'_2, L'_3) = \text{meas}(L_2, L_3)$. □

4 Applications

4.1 Computation of the inertia index of a Lagrangian triple

4.1.1 Basic properties of the inertia index

In contrast with the corresponding situation for pairs of Lagrangian subspaces, the orbit of a triple (L_1, L_2, L_3) of Lagrangian subspaces of a $2n$-dimensional symplectic vector space (V, ω) under the diagonal action of the symplectic group $Sp(V)$ is not characterized by the integers $n_{12} = \dim(L_1 \cap L_2)$, $n_{23} = \dim(L_2 \cap L_3)$, $n_{31} = \dim(L_3 \cap L_1)$ and $n_0 = \dim(L_1 \cap L_2 \cap L_3)$, which are invariants of this action. To classify the orbits, one introduces the notion of *inertia index* (sometimes called *Maslov index*, or simply index, or signature) of a Lagrangian triple (L_1, L_2, L_3). For the following definition and properties of the inertia index, we refer to Kashiwara ([6], p.486 sqq).

Definition 4 (Inertia index). The *inertia index* of the Lagrangian triple (L_1, L_2, L_3), denoted by $\tau(L_1, L_2, L_3)$, is the signature of the quadratic form q defined on the $3n$-dimensional vector space $L_1 \oplus L_2 \oplus L_3$ by: $q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$.

In a suitable basis of $L_1 \oplus L_2 \oplus L_3$, one can represent q by a diagonal matrix whose entries consist of r terms $+1$, s terms -1 and $3n - r - s$ terms 0, the integers r and s being independent from the choice of the basis. What is called *signature* of q here, and denoted $\text{sgn}(q)$ is the integer $sgn(q) = r - s$. From the definition, we see that for any symplectic map $\psi \in Sp(n)$, we have $\tau(\psi(L_1), \psi(L_2), \psi(L_3)) = \tau(L_1, L_2, L_3)$. We summarize here some of the properties of the inertia index that we will need in the following.

Proposition 4.1. *The inertia index has the following properties:*

(i) $\tau(L_1, L_2, L_3) \equiv n - (n_{12} + n_{23} + n_{31}) \mod 2\mathbb{Z}$

(ii) $|\tau(L_1, L_2, L_3)| \leq n + 2n_0 - (n_{12} + n_{23} + n_{31})$

We may now state the theorem of symplectic classification of triples of Lagrangian subspaces of (V, ω), which is due to Kashiwara. For $d = (n_0, n_{12}, n_{23}, n_{31}, \tau) \in \mathbb{N}^4 \times \mathbb{Z}$, we set:

$$O_d = \left\{ (L_1, L_2, L_3) \in \mathcal{L}(V) \times \mathcal{L}(V) \times \mathcal{L}(V) \;\middle|\; \begin{array}{l} \dim(L_1 \cap L_2 \cap L_3) = n_0, \\ \dim(L_1 \cap L_2) = n_{12}, \\ \dim(L_2 \cap L_3) = n_{23}, \\ \dim(L_3 \cap L_1) = n_{31}, \\ \tau(L_1, L_2, L_3) = \tau \end{array} \right\}$$

Theorem 4.2 (Symplectic classification of Lagrangian triples,[6], p.493)). *O_d is non-empty if and only if $d = (n_0, n_{12}, n_{23}, n_{31}, \tau)$ satisfies the conditions:*

$$\begin{array}{ll} \text{(i)} & 0 \leq n_0 \leq n_1, n_2, n_3 \leq n \\ \text{(ii)} & n_{12} + n_{23} + n_{31} \leq n + 2n_0 \\ \text{(iii)} & |\tau| \leq n + 2n_0 - (n_{12} + n_{23} + n_{31}) \\ \text{(iv)} & \tau \equiv n - (n_{12} + n_{23} + n_{31}) \mod 2\mathbb{Z} \end{array}$$

If (L_1, L_2, L_3) and (L'_1, L'_2, L'_3) are two triples of Lagrangian subspaces of V, there exists a symplectic map $\psi \in Sp(V)$ such that $\psi(L_1) = L'_1$, $\psi(L_2) = L'_2$ and $\psi(L_3) = L'_3$ if and only if $n_0 = n'_0$, $n_{12} = n'_{12}$, $n_{23} = n'_{23}$, $n_{31} = n'_{31}$ and $\tau = \tau'$.

Thus, the diagonal action of $Sp(V)$ on $\mathcal{L}(V) \times \mathcal{L}(V) \times \mathcal{L}(V)$ has only finitely many orbits and these orbits are the O_d's, where d satisfies conditions (i) to (iv) above.

We now specialize to the case where $V = \mathbb{C}^n$, and make the following definition.

Definition 5. A triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^n$ is said to be an *exceptional triple* if the unitary maps φ_{12} and φ_{13} have the same eigenspaces.

As can be seen from the case $n = 2$, a triple (L_1, L_2, L_3) is generically not exceptional, which justifies the terminology.

We now extract from theorem 4.2 the following proposition, which will be very important to us. It shows the interest of the notion of exceptional Lagrangian triple: every Lagrangian triple is symplectically equivalent to an exceptional triple.

Proposition 4.3. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^n$. Then there exists an exceptional triple (L'_1, L'_2, L'_3) and a symplectic map $\psi \in Sp(n)$ such that $L'_j = \psi(L_j)$ for $j = 1, 2, 3$; that is, each orbit O_d of the diagonal action of the symplectic group $Sp(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n) \times \mathcal{L}(n)$ contains at least one exceptional triple.*

4.1.2 From angles to inertia index

We saw earlier (proposition 3.3) that the quantity $\delta = (\lambda_{12} + \mu_{12}) + (\lambda_{23} + \mu_{23}) + (\lambda_{31} + \mu_{31})$ defined for Lagrangian subspaces of $\mathbb{C}^2$, verifies $\delta \equiv 0 \pmod{\pi}$ and contains information about the triple (L_1, L_2, L_3). Namely, if $\delta \equiv 0 \pmod{2\pi}$ the triple (L_1, L_2, L_3) is positive (that is, setting $\varphi = \varphi_{31} \circ \varphi_{23} \circ \varphi_{12}$, we have $\det \varphi = e^{i\delta} = 1 > 0$), and if $\delta \equiv \pi \pmod{2\pi}$ the triple is negative (that is $\det \varphi = e^{i\delta} = -1 < 0$). The interest of that notion was that the triple (l_1, l_2, l_3) of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$ had same sign as (L_1, L_2, L_3): if $\delta \equiv 0 \pmod{2\pi}$ the transformation $\widehat{\varphi} = \widehat{\varphi_{31}} \circ \widehat{\varphi_{23}} \circ \widehat{\varphi_{12}}$ of $\mathbb{C}\mathrm{P}^1$ preserves the orientation on l_1 (the triple (l_1, l_2, l_3) is then said to be positive), and if $\delta \equiv \pi \pmod{2\pi}$ then $\widehat{\varphi}$ reverses that orientation (the triple (l_1, l_2, l_3) is said to be negative); and this enabled us to distinguish between positive and negative spherical triangles, which was essential in order to determine the image of the map $\rho : (\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2))/U(2) \to \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2$. But δ can actually be defined for a triple of Lagrangian subspaces of $\mathbb{C}^n$ for any integer n. For such a triple (L_1, L_2, L_3), since $\sigma_{L_j}^2 = Id$, we have the following relation: $(\sigma_{L_1} \circ \sigma_{L_3}) \circ (\sigma_{L_3} \circ \sigma_{L_2}) \circ (\sigma_{L_2} \circ \sigma_{L_1}) = Id$, and the determinant of this unitary map therefore is of the form $e^{i2\delta}$ with $\delta \equiv 0 \pmod{\pi}$. When $n = 2$, the eigenvalues of the unitary map $\sigma_{L_k} \circ \sigma_{L_j}$ are $e^{i2\lambda_{jk}}$ and $e^{i2\mu_{jk}}$, so that we have indeed $\delta = (\lambda_{12} + \mu_{12}) + (\lambda_{23} + \mu_{23}) + (\lambda_{31} + \mu_{31})$.

In the following, we shall consider a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^n$, for arbitrary n. We shall denote the measures of the angles (L_1, L_2), (L_2, L_3) and (L_3, L_1) by $\mathrm{meas}(L_1, L_2) = [e^{i2\alpha_1}, \ldots, e^{i2\alpha_n}]$, $\mathrm{meas}(L_2, L_3) = [e^{i2\beta_1}, \ldots, e^{i2\beta_n}]$ and $\mathrm{meas}(L_3, L_1) = [e^{i2\gamma_1}, \ldots, e^{i2\gamma_n}]$, where $\pi > \alpha_1 \geq \ldots \geq \alpha_n \geq 0$, $\pi > \beta_1 \geq \ldots \geq \beta_n \geq 0$ and $\pi > \gamma_1 \geq \ldots \geq \gamma_n \geq 0$. We then have $\delta = \sum_{j=1}^n (\alpha_j + \beta_j + \gamma_j)$, where $e^{i2\delta} = 1$ is the determinant of the unitary map $(\sigma_{L_1} \circ \sigma_{L_3}) \circ (\sigma_{L_3} \circ \sigma_{L_2}) \circ (\sigma_{L_2} \circ \sigma_{L_1}) = Id$, so that $\delta \equiv 0 \pmod{\pi}$. Since δ, which we shall also denote $\delta(L_1, L_2, L_3)$ to avoid confusion, is defined by means of the measures of the angles (L_j, L_k) (that is, up to permutation, the eigenvalues of the unitary maps $\sigma_{L_k} \circ \sigma_{L_j}$), δ is invariant under the diagonal action of the unitary group $U(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n) \times \mathcal{L}(n)$: if $\varphi \in U(n)$, then $\delta(\varphi(L_1, L_2, L_3)) = \delta(L_1, L_2, L_3)$.

The next theorem is the main result of this paragraph.

Theorem 4.4. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^n$, and set $n_{jk} = \dim(L_j \cap L_k)$, $\tau = \tau(L_1, L_2, L_3)$ and $\delta = \delta(L_1, L_2, L_3)$. Then $\tau = 3n - \frac{2\delta}{\pi} - (n_{12} + n_{23} + n_{31})$.*

Lemma 4.5. *If $c : t \in [0, 1] \mapsto (L_1(t), L_2(t), L_3(t)) \in \mathcal{L}(n) \times \mathcal{L}(n) \times \mathcal{L}(n)$ is a continuous map such that the dimensions $n_{jk}(t) = \dim(L_j(t) \cap L_k(t))$ of the pairwise intersections are constant functions of t, then the map $\delta : t \longmapsto \delta(L_1(t), L_2(t), L_3(t))$ is a constant map.*

Observe that this result is also true for the inertia index (see [6], pp. 487-488).

Proof of lemma 4.5. Since the n_{jk}'s remain constant along the deformation, the non-zero $\alpha_j(t), \beta_j(t)$ and $\gamma_j(t)$ vary continuously. Therefore, $\delta(L_1, L_2, L_3)$ varies continuously. As $\delta(t) \equiv 0 \pmod{\pi}$, δ is a constant map. □

Lemma 4.6. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^n$ and let $\psi \in Sp(n)$ be a symplectic map. Then $\delta(\psi(L_1), \psi(L_2), \psi(L_3)) = \delta(L_1, L_2, L_3)$, that is: δ is a symplectic invariant.*

Proof of lemma 4.6. Since the symplectic group is connected, there exists a continuous path $t \in [0,1] \mapsto \psi_t \in Sp(n)$ such that $\psi_0 = Id$ and $\psi_1 = \psi$. For all $t \in [0,1]$, set $L_j(t) = \psi_t(L_j)$ for $j = 1,2,3$. As ψ_t is invertible, $n_{12}(t), n_{23}(t)$ and $n_{31}(t)$ are constant, and by the above lemma so is $\delta(t)$, so that $\delta(\psi(L_1), \psi(L_2), \psi(L_3)) = \delta(1) = \delta(0) = \delta(L_1, L_2, L_3)$. □

Lemma 4.7. *Let (L_1, L_2, L_3) be an exceptional triple of Lagrangian subspaces of $\mathbb{C}^n$ and let $(u_1, \ldots, u_n)$ be an orthonormal basis for L_1 formed of eigenvectors of φ_{12}: $\varphi_{12}(u_k) = e^{i\alpha_k} u_k$, where $[e^{i2\alpha_1}, \ldots, e^{i2\alpha_n}] =$ $\mathrm{meas}(L_1, L_2)$. For all k, set $d_1^k = L_1 \cap \mathbb{C}^{(k)}$, $d_2^k = L_2 \cap \mathbb{C}^{(k)}$ and $d_3^k = L_3 \cap \mathbb{C}^{(k)}$. Then d_1^k, d_2^k and d_3^k are real lines of $\mathbb{C}^{(k)}$ and, if we denote by $\mathrm{meas}(d_1^k, d_2^k), \mathrm{meas}(d_k^2, d_k^3), \mathrm{meas}(d_3^k, d_1^k) \in [0, \pi[$ the measures of the oriented angles $(d_1^k, d_2^k), (d_2^k, d_3^k), (d_3^k, d_1^k)$ in $\mathbb{C}^{(k)}$, then*

$$\delta(L_1, L_2, L_3) = \sum_{k=1}^{n} (\mathrm{meas}(d_1^k, d_2^k) + \mathrm{meas}(d_k^2, d_k^3) + \mathrm{meas}(d_3^k, d_1^k))$$

Proof of lemma 4.7. Set $\mathrm{meas}(L_1, L_3) = [e^{i2\varepsilon_1}, \ldots, e^{i2\varepsilon_n}]$. Observe first that L_1 intersects the complex line $\mathbb{C}^{(k)} = \mathbb{C}u_k$ because $u_k \in L_1$. Since $(u_1, \ldots, u_n)$ is a basis of L_1 formed of eigenvectors of φ_{12}, and since φ_{12} and φ_{13} have the same eigenspaces, there exists a permutation $g \in \mathfrak{S}_n$ such that, for all $k \in \{1, \ldots, n\}$, $\varphi_{13}(u_k) = e^{i\varepsilon_{g(k)}} u_k \in L_3$. Therefore, we have $e^{i\alpha_k} u_k \in L_2$ and $e^{i\varepsilon_k} u_k \in L_3$, so that $\mathbb{C}^{(k)}$ also intersects both L_2 and L_3. But if $u \in \mathbb{C}^n \backslash \{0\}$ is contained in a Lagrangian subspace L of $\mathbb{C}^n$ then $L \cap \mathbb{C}u = \mathbb{R}u$. Indeed, if $v \in L \cap \mathbb{C}u$ then $v = \lambda u + \mu J u$ with $\lambda, \mu \in \mathbb{R}$, and since L is Lagrangian $\omega(u,v) = 0$. But $\omega(u,v) = \lambda\omega(u,u) + \mu\omega(u, Ju) = \mu g(u,u)$ with $g(u,u) \neq 0$, therefore $v = \lambda u \in \mathbb{R}u$. Therefore, since $\varphi_{12}(u_k) = e^{i\alpha_k} u_k \in L_2$, we have $d_1^k = L_1 \cap \mathbb{C}u_k = \mathbb{R}u_k$ and $d_k^2 = L_2 \cap \mathbb{C}u_k = \mathbb{R}(e^{i\alpha_k} u_k) = e^{i\alpha_k} d_1^k$, hence $\mathrm{meas}(d_1^k, d_2^k) = \alpha_k \in [0, \pi[$. Likewise, since $e^{i\varepsilon_{g(k)}} u_k \in L_3$, we have $d_3^k = e^{i\varepsilon_{g(k)}} d_1^k$, so that $\mathrm{meas}(d_1^k, d_3^k) = \varepsilon_{g(k)}$, hence, setting $\xi_k = \pi - \varepsilon_{g(k)} \mod \pi$, $\mathrm{meas}(d_3^k, d_1^k) = \xi_k \in [0, \pi[$. Setting $w_k = e^{i\varepsilon_{g(k)}} u_k \in L_3$, we have $e^{i\xi_k} w_k = \pm e^{i(\pi - \varepsilon_{g(k)})} w_k = \pm u_k \in L_1$. The $(e^{i\xi_k})$ therefore are the roots of the characteristic polynomial $P(L_3, L_1)$ of the pair (L_3, L_1), hence $[e^{i2xi_1}, \ldots, e^{i2\xi_n}] = \mathrm{meas}(L_3, L_1) = [e^{i2\gamma_1}, \ldots, e^{i2\gamma_n}]$, and since $\xi_k, \gamma_k \in [0, \pi[$, there exists a permutation $g_3 \in \mathfrak{S}_n$ such that, for all k, $\xi_k = \gamma_{g_3(k)}$. Similarly, setting $v_k = e^{i\alpha_k} u_k \in L_2$ and $\zeta_k = (\varepsilon_{g(k)} - \alpha_k) \mod \pi$, we have $e^{i\zeta_k} v_k = \pm e^{i\varepsilon_{g(k)}} u_k \in L_1$, hence $[e^{i2\zeta_1}, \ldots, e^{i2\zeta_n}] = \mathrm{meas}(L_2, L_3) = [e^{i2\beta_1}, \ldots, e^{i2\beta_n}]$, and since $\zeta_k, \beta_k \in [0, \pi[$, there exists $g_2 \in \mathfrak{S}_n$ such that, for all k, $\zeta_k = \beta_{g_2(k)}$. Furthermore, since $d_2^k = \mathbb{R}v_k$ and $d_3^k = \mathbb{R}e^{i\varepsilon_{g(k)}} u_k = \mathbb{R}e^{i\zeta_k} v_k$, we have $\mathrm{meas}(d_k^2, d_3^k) = \zeta_k$. Hence

$$\begin{array}{rcl}
\sum_{k=1}^{n} (\mathrm{meas}(d_1^k, d_2^k) + \mathrm{meas}(d_2^k, d_3^k) + \mathrm{meas}(d_3^k, d_1^k)) & = & \sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \zeta_k + \sum_{k=1}^{n} \xi_k \\
& = & \sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \beta_{g_2(k)} + \sum_{k=1}^{n} \gamma_{g_3(k)} \\
& = & \sum_{k=1}^{n} (\alpha_k + \beta_k + \gamma_k) \\
& = & \delta(L_1, L_2, L_3)
\end{array}$$

□

We now have all the material we need to relate δ to τ and show that the inertia index can be computed from the measures of the Lagrangian angles (L_1, L_2), (L_2, L_3) and (L_3, L_1); that is, from the eigenvalues of the unitary maps $\sigma_{L_k} \circ \sigma_{L_j}$, where σ_{L_j} is the Lagrangian involution associated to L_j.

Proof of theorem 4.4. By proposition 4.3, there exists a symplectic map $\psi \in Sp(n)$ such that $(\psi(L_1),$ $\psi(L_2),$ $\psi(L_3))$ is an exceptional triple. Since such a transformation leaves τ, δ and the n_{jk}'s invariant, we may assume that (L_1, L_2, L_3) is itself exceptional. Let us recall the notations $\mathrm{meas}(L_1, L_2) =$ $[e^{i2\alpha_1}, \ldots, e^{i2\alpha_n}]$ $\mathrm{meas}(L_1, L_3) = [e^{i2\varepsilon_1}, \ldots, e^{i2\varepsilon_n}]$ where $\pi > \alpha_1 \geq \ldots \geq \alpha_n \geq 0$ and $\pi > \varepsilon_1 \geq \ldots \geq \varepsilon_n \geq 0$. Then, since (L_1, L_2, L_3) is exceptional, there exists an orthonormal basis $(u_1, \ldots, u_n)$ for L_1 and a permutation $g \in \mathfrak{S}_n$ such that $(e^{i\alpha_1} u_1, \ldots, e^{i\alpha_n} u_n)$ is an orthonormal basis for L_2 and $(e^{i\varepsilon_1} u_1, \ldots, e^{i\varepsilon_n} u_n)$ is an orthonormal basis for L_3. By abandoning the condition $\pi > \varepsilon_1 \geq \ldots \geq \varepsilon_n \geq 0$, we may suppose that $g = Id$. Set $d_1^k = \mathbb{R}u_k$, $d_2^k = e^{i\alpha_k} d_1^k$, $d_3^k = e^{i\varepsilon_k} d_1^k$, and $\tau_k = \tau(d_1^k, d_2^k, d_3^k)$ in the symplectic space $\mathbb{C}u_k$. Set $\delta_k = \text{meas}(d_1^k, d_2^k) + \text{meas}(d_2^k, d_3^k) + \text{meas}(d_3^k, d_1^k)$ and set, as in lemma 4.7, $\zeta_k = (\varepsilon_k - \alpha_k) \mod \pi$ and $\xi_k = (\pi - \varepsilon_k) \mod \pi$, so that $\delta_k = \alpha_k + \zeta_k + \xi_k$. Observe that $\delta_k = \delta(d_1^k, d_2^k, d_3^k)$ in the symplectic space $\mathbb{C}u_k$. In particular, this implies that $\delta_k \equiv 0 \mod \pi$. If $d_1^k = d_2^k = d_3^k$, which happens n_0 times, then $\tau_k = 0$ and $\delta_k = 0$. If either $d_1^k = d_2^k \neq d_3^k$ or $d_2^k = d_3^k \neq d_1^k$ or $d_3^k = d_1^k \neq d_2^k$, which happens $(n_{12} - n_0) + (n_{23} - n_0) + (n_{31} - n_0)$ times, then $\tau_k = 0$ and $0 < \delta_k = \alpha_k + \zeta_k + \xi_k < 2\pi$ (since one of these numbers is 0 and since all of them are $< \pi$ and two of them are non-zero), but $\delta_k \equiv 0 \mod \pi$ so $\delta_k = \pi$. If $d_1^k \neq d_2^k \neq d_3^k \neq d_1^k$, which happens $n + 2n_0 - (n_{12} + n_{23} + n_{31})$ times, then either $\tau_k = 1$ and $\delta_k = \pi$ or $\tau_k = -1$ and $\delta_k = 2\pi$, so that $\tau_k = 3 - \frac{2\delta_k}{\pi}$ (see figure 7).

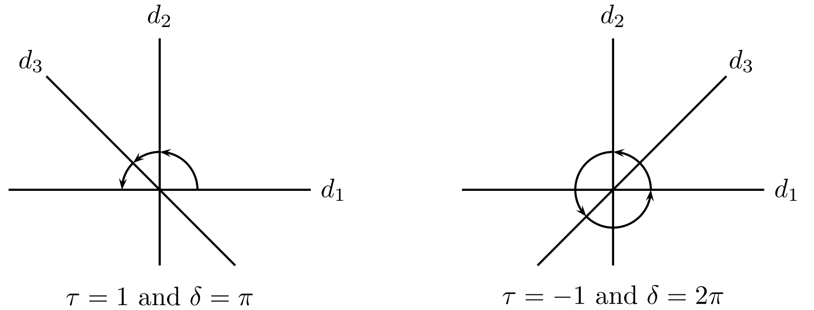


Figure 7: Relation between δ and τ for exceptional triples of Lagrangians

Since (L_1, L_2, L_3) is an exceptional triple, we have, by proposition 4.7, $\delta = \sum_{k=1}^n \delta_k$. Likewise, $\tau = \sum_{k=1}^n \tau_k$, so that we have:

$$\begin{aligned}
\tau &= \textstyle\sum_{k=1}^{n+2n_0-(n_{12}+n_{23}+n_{31})} (3 - \frac{2\delta_k}{\pi}) \\
&= 3(n + 2n_0 - (n_{12} + n_{23} + n_{31})) - \tfrac{2}{\pi}(\delta - \pi((n_{12} - n_0) + (n_{23} - n_0) + (n_{31} - n_0))) \\
&= 3n - \tfrac{2\delta}{\pi} - (n_{12} + n_{23} + n_{31})
\end{aligned}$$

□

4.2 Two-dimensional unitary representations of $\pi_1(S^2 \backslash \{s_1, s_2, s_3\})$

Let $s_1, \ldots, s_n$ be n distinct points of the Euclidean sphere S^2. Then, for any $z \in S^2 \backslash \{s_1, \ldots, s_n\}$, the fundamental group of the sphere minus these n points has finite presentation

$$\pi_1(S^2 \backslash \{s_1, \ldots, s_n\}, z) = < g_1, \ldots, g_n \mid g_n \ldots g_1 = 1 >$$

where g_k is the homotopy class of a loop around s_k for $k = 1, \ldots, n$. Giving a two-dimensional unitary representation $\rho : \pi_1(S^2 \backslash \{s_1, \ldots, s_n\}) \to U(2)$ of this group is therefore equivalent to giving n unitary matrices $U_1, \ldots, U_n \in U(2)$ satisfying $U_n \ldots U_1 = Id$ and setting $\rho(g_k) = U_k$ for all k. Two such representations ρ and ρ' are equivalent if there exists a unitary transformation $\varphi \in U(2)$ of $\mathbb{C}^2$ such that $U'_k \circ \varphi = \varphi \circ U_k$, or equivalently $U'_k = \varphi \circ U_k \circ \varphi^{-1}$ for $k = 1, \ldots, n$. Determining and classifying two-dimensional unitary representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_n)\}$ up to equivalence therefore is equivalent to finding, given conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_n$ in the unitary group $U(2)$, necessary and sufficient conditions for the existence of unitary matrices $U_k \in \mathcal{C}_k$ verifying $U_n \ldots U_1 = Id$.

These conditions have been determined by Biswas in [2] after reformulating the question in terms of parabolic vector bundles of rank two over the projective line $\mathbb{C}\mathrm{P}^1$. The methods presented in section 3 above provide an elementary proof of the same result in the case $n = 3$. The relation between the result of existence obtained by Biswas ([2], p.524) and the result of existence of theorem 3.5 above is a consequence of the following proposition.

Proposition 4.8. *Given three unitary matrices $U_{12}, U_{23}, U_{31} \in U(2)$ verifying $U_{31}U_{23}U_{12} = Id$, there exists a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ such that $\sigma_{L_k} \circ \sigma_{L_j} = U_{jk}$ for all j, k. If (L_1, L_2, L_3) and (L'_1, L'_2, L'_3) are two such triples, then there exists a unitary map $\varphi \in U(2)$ such that $\varphi(L_j) = L'_j$ for $j = 1, 2, 3$.*

Lemma 4.9. *Let $A \in U(2)$ be a unitary matrix and let $(e^{i2\lambda}, e^{i2\mu})$ be the eigenvalues of A, where $\pi > \lambda \geq \mu \geq 0$. Let $v \in \mathbb{C}^2$ be an eigenvector of A with respect to the eigenvalue $e^{i2\mu}$ and such that $h(v,v) = 1$. Let l_1 be a projective Lagrangian of $\mathbb{C}\mathrm{P}^1$ containing $[v] \in \mathbb{C}\mathrm{P}^1 \simeq S^2$, and let l_2 be the projective Lagrangian image of l_1 by the rotation of angle $(\lambda - \mu) \in [0, \pi[$ around $[v]$. Then, given a Lagrangian $L_1 \in p^{-1}(l_1)$, there exists a Lagrangian $L_2 \in p^{-1}(l_2)$ such that $\sigma_{L_2} \circ \sigma_{L_1} = A$.*

Proof of lemma 4.9. Let u be an eigenvector of A with respect to the eigenvalue $e^{i2\lambda}$ and such that $h(u,u) = 1$. Set $L_0 = \mathbb{R}u \oplus \mathbb{R}v$. Then (u, v) is a unitary basis of $\mathbb{C}^2$ and therefore L_0 is Lagrangian. Furthermore, $[v] \in l_0 \cap l_1$, where $l_0 = p(L_0)$. Therefore, l_1 is the image of l_0 by a rotation $\widehat{\psi} \in SO(3)$ around $[v] \in S^2$, where $\psi \in SU(2)$ is a special unitary map having u and v as eigenvectors (since $\widehat{\psi}$ is a rotation around $[v] = -[u] \in \mathbb{R}^3$): $\psi(u) = \alpha u$ and $\psi(v) = \beta v$, where $\alpha, \beta \in \mathbb{C}$. Set $L = \psi(L_0)$. Then L is a Lagrangian subspace of $\mathbb{C}^2$ and $p(L) = \widehat{\psi}(l_0) = l_1$. Then, by proposition 3.2, $L_1 = e^{i\theta} L$ for some $\theta \in [0, \pi[$. Set $u_{12} = e^{i\theta}\psi(u)$ and $v_{12} = e^{i\theta}\psi(v)$. Then (u_{12}, v_{12}) is an orthonormal basis for L_1. Set $L_2 = \mathbb{R}e^{i\lambda}u_{12} \oplus \mathbb{R}e^{i\mu}v_{12}$. Then L_2 is Lagrangian and, by proposition 3.2, $p(L_2)$ is the image of $p(L_1) = l_1$ by the rotation of angle $(\lambda - \mu)$ arouns $[v_{12}] = [v]$, so $p(L_2) = l_2$. At last

$$\sigma_{L_2} \circ \sigma_{L_1}(u_{12}) = \sigma_{L_2}(u_{12}) = \sigma_{L_2}(e^{-i\lambda}e^{i\lambda}u_{12}) = e^{i\lambda}\sigma_{L_2}(e^{i\lambda}u_{12}) = e^{i\lambda}e^{i\lambda}u_{12} = e^{i2\lambda}u_{12}$$

so that $\sigma_{L_2} \circ \sigma_{L_1}(e^{i\theta}\alpha u) = e^{i2\lambda}e^{i\theta}\alpha u$ hence, since $\sigma_{L_2} \circ \sigma_{L_1}$ is holomorphic, $\sigma_{L_2} \circ \sigma_{L_1}(u) = Au$. For the same reasons, $\sigma_{L_2} \circ \sigma_{L_1}(v) = Av$, and therefore $\sigma_{L_2} \circ \sigma_{L_1} = A$. □

Proof of proposition 4.8. For fixed j, k, let (u_{jk}, v_{jk}) be a unitary basis of $\mathbb{C}^2$ formed of eigenvectors of U_{jk}:$U_{jk}u_{jk} = e^{i2\lambda_{jk}}u_{jk}$ and $U_{jk}v_{jk} = e^{i2\mu_{jk}}v_{jk}$, where $\pi > \lambda_{jk} \geq \mu_{jk} \geq 0$.There exists a great circle l_2 of S^2 containing both $[v_{23}] \in \mathbb{C}\mathrm{P}^1 \simeq S^2$ and $[v_{12}] \in \mathbb{C}\mathrm{P}^1 \simeq S^2$. Let l_1 be the great circle of S^2 containing $[v_{12}]$ and such that l_2 be the image of l_1 by the rotation of angle $(\lambda_{12} - \mu_{12})$ around v_{12}. Fix a Lagrangian $L_1 \in p^{-1}(l_1)$ arbitrarily. By the lemma above, there exists a Lagrangian $L_2 \in p^{-1}(l_2)$ such that $\sigma_{L_2} \circ \sigma_{L_1} = U_{12}$. Now let l_3 be the image of l_2 by the rotation of angle $(\lambda_{23} - \mu_{23})$ around $[v_{23}]$. By the lemma above, there exists a Lagrangian $L_3 \in p^{-1}(l_3)$ such that $\sigma_{L_3} \circ \sigma_{L_2} = U_{23}$. Then $\sigma_{L_1} \circ \sigma_{L_3} = (\sigma_{L_3} \circ \sigma_{L_1})^{-1} = (\sigma_{L_3} \circ \sigma_{L_2} \circ \sigma_{L_2} \circ \sigma_{L_1})^{-1} = (U_{23} \circ U_{12})^{-1} = U_{31}$. At last, if (L_1, L_2, L_3) and (L_1', L_2', L_3') are two Lagrangian triples verifying $\sigma_{L_k} \circ \sigma_{L_j} = U_{jk} = \sigma_{L_k'} \circ \sigma_{L_j'}$ for all j, k, then $\mathrm{meas}(L_j, L_k) = \mathrm{meas}(L_j', L_k')$ for all j, k, hence, by theorem 3.6, there exists a unitary map $\varphi \in U(2)$ such that $\varphi(L_j) = L_j'$ for $j = 1, 2, 3$. □

Corollary 4.10. *Given three conjugacy clases $\mathcal{C}_{12}, \mathcal{C}_{23}, \mathcal{C}_{31}$ in the unitary group $U(2)$ (that is, given eigenvalues $[e^{i2\lambda_{12}}, e^{i2\mu_{12}}], [e^{i2\lambda_{23}}, e^{i2\mu_{23}}]$ and $[e^{i2\lambda_{31}}, e^{i2\mu_{31}}]$, where $\pi > \lambda_{jk} \geq \mu_{jk} \geq 0$), there exists unitary matrices $U_{jk} \in \mathcal{C}_{jk}$ verifying $U_{31}U_{23}U_{12} = Id$ if and only if there exists a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ such that* $\mathrm{meas}(L_j, L_k) = [e^{i2\lambda_{jk}}, e^{i2\mu_{jk}}]$ *for all j, k, that is, setting $\delta = \sum_{j,k}(\lambda_{jk} + \mu_{jk})$, $\alpha_{jk} = \lambda_{jk} - \mu_{jk}$ and $\beta_{jk} = \pi - \alpha_{jk}$, if and only if*

$$\begin{array}{cl} & \delta \equiv \pi \pmod{2\pi} \text{ and } (\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \overline{\Delta} \\ \text{or} & \\ & \delta \equiv 0 \pmod{2\pi} \text{ and } (\beta_{12}, \beta_{23}, \beta_{31}) \in \overline{\Delta} \end{array}$$

Proof. Given such a triple (L_1, L_2, L_3), the maps $\sigma_{L_2} \circ \sigma_{L_1}$, $\sigma_{L_3} \circ \sigma_{L_2}$ and $\sigma_{L_1} \circ \sigma_{L_3}$ are unitary, have the prescribed eigenvalues, and verify $(\sigma_{L_1} \circ \sigma_{L_3}) \circ (\sigma_{L_3} \circ \sigma_{L_2}) \circ (\sigma_{L_2} \circ \sigma_{L_1}) = Id$.
Conversely, given three unitary matrices U_{12}, U_{23}, U_{31} verifying $U_{31}U_{23}U_{12} = Id$, there exists , by the above proposition, a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ such that $\sigma_{L_k} \circ \sigma_{L_j} = U_{jk}$ for all j, k, so that $\mathrm{meas}(L_j, L_k) = [e^{i2\lambda_{jk}}, e^{i2\mu_{jk}}]$.
The conditions on δ, the α_{jk}'s and the β_{jk}'s then follow from theorem 3.5. □

References

[1] Vladimir Igorevich Arnold. On a characteristic class entering into conditions of quantization. *Functional Analysis and its applications*, 1:1–13, 1967.

[2] Indranil Biswas. A criterion for the existence of a parabolic stable bundle of rank two over the projective line. *International Journal of Mathematics*, 9:523–533, 1998.

[3] E. Falbel and V. Zocca. A poincaré polyhedron theorem for complex hyperbolic geometry. *J. Reine Ang. Math.*, 516:133–158, 2000.

[4] Elisha Falbel. Finite groups generated by involutions on lagrangian planes in $\mathbb{C}^2$. *Canadian Mathematical Bulletin*, 44:408–419, 2001.

[5] W. Goldman. *Complex Hyperbolic Geometry.* Oxford Mathematical Monographs. Oxford Science Publications, 1999.

[6] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds.* Grundlehren der mathematischen Wissenschaften 292. Springer-Verlag, 1990.

[7] Pierre Lochak, Jean-Pierre Marco, and David Sauzin. *On the splitting of invariant manifolds in multi-dimensional Hamiltonian systems.* AMS Notices, 2003. (To appear).

[8] Andrew J. Nicas. Classifying pairs of lagrangians in a hermitian vector space. *Topology and its applications*, 42:71–81, 1991.