C. Kearton

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The classification of simple knots

M.Sh. Farber

CONTENTS

Introduction

0.1. As is well known, classical knot theory studied the methods of embedding a circle in three-dimensional Euclidean space [4]. Higher-dimensional knots, that is, embeddings in the sphere S^{n+q} of manifolds homeomorphic to S^n, were first intensively investigated in the early 60's, when modern methods of analysis of manifolds were created. By the mid-60's knots of codimension $q > 2$ had been completely classified: Zeeman [56] and Stallings [49] had proved that there are no non-trivial topological or piecewise linear knots, and the set of smooth knots had been computed by Haefliger [24] and Levine [36]. From these computations it followed that there are not too many smooth knots either. For example, it was shown that for fixed n and $q > 2$ the smooth knots of S^n in S^{n+q} form a finitely generated Abelian group.

The phenomenon of knottedness is most clearly exhibited in codimension 2. The isotopy types of knots of codimension 2 form an infinitely generated Abelian semigroup, and an idea of its size can be gained from the realization theorems of Kervaire [33] and Levine [38], and also from the classification Theorems 3.1, 8.13, 9.3, and 10.13 of the present paper. This infinite variety of types is clearly the reason why the study of knots of codimension 2 turns out to be so difficult.

Interest in knots of codimension 2 arose not only because they include the classical knots; it was also stimulated by connections discovered by Brieskorn with the study of singularities of complex hypersurfaces [5]. Another reason for the preoccupation with such knots was the fact that, as it turned out, every homotopy sphere bounding a parallelizable manifold can be realized as a smooth knot in codimension 2. Moreover, it turned out that it is quite natural to construct and to study smooth structures on spheres in the framework of knot theory [38].

The study of simple knots was initiated by Kervaire and Levine: they occupy a special place in the theory of knots of codimension 2. There are several reasons for this. In the first place, the class of simple knots includes all classical knots in S^3 and all two-dimensional knots in S^4. Furthermore, the applications of knot theory of which we spoke above, in algebraic geometry and the theory of exotic spheres, reduce precisely to the simple knots. Finally, these knots are the simplest from the point of view of existing methods of investigation (hence their name).

0.2. In the present paper simple knots of all dimensions except 1, 2, 3, 4, and 6 are classified in terms of their algebraic invariants.

The most explicit results in the paper are about odd-dimensional knots, the theory of which is not overburdened with technical details and seems to be more lucid. This part of the paper is written in the style of a text-book: references to the literature are kept to a minimum and are replaced by proofs, but references to original papers, comments, alternative approaches, etc. are collected in "Notes" at the end of each section. Many of the results about odd-dimensional knots are presented as a unified and simplified account of a number of classical theorems of Kervaire, Levine, Trotter, Kearton, and others. Of the new results we must single out first of all the introduction of the notion of R-equivalence and the clarification of its role (cf. Theorems 1.5, 3.1, and 5.1). Thanks to this the structure of the theory is completely changed (Seifert matrices and S-equivalence are omitted from the discussion), and many proofs have been simplified.

Another significant advantage of this new scheme of reasoning consists in its universality. Using the same circle of ideas, we give an algebraic classification also of even-dimensional simple knots. This result is new; in previous papers only particular classes of such knots have been described.

0.3. The terminology of the paper is that of differential-topology. For example, the term "submanifold" means smooth submanifold etc.

Let us define more precisely the fundamental concept of the paper—that of a knot. Since we study here only knots of codimension 2 and knots of other codimensions do not appear at all, we move away from the general definition of knots given above and adopt the following: an *n-dimensional knot* is a pair (S^{n+2}, k^n) consisting of the sphere S^{n+2} and an n-dimensional, closed, oriented submanifold k of it that is homeomorphic (but not necessarily diffeomorphic) to the n-dimensional sphere S^n. Two knots (S^{n+2}, k_1) and (S^{n+2}, k_2) are said to be *equivalent* (or of the same *isotopy type*) if there exists an orientation-preserving diffeomorphism from S^{n+2} onto itself that takes k_1 to k_2 with the orientations preserved. A *trivial knot* is a knot equivalent to (S^{n+2}, S^n) where $S^n \subset S^{n+2}$ is the standard embedding.

To motivate the definition of a simple knot (see 0.7) we begin by quoting some fundamental facts about the existence of Seifert manifolds. We recall that a *Seifert manifold* of a knot (S^{n+2}, k) is any compact connected orientable $(n+1)$-dimensional submanifold $V \subset S^{n+2}$ with $\partial V = k$.

0.4. *Theorem.* *Every knot has a Seifert manifold.*

For classical knots this was discovered by Frankl and Pontryagin [22]. Seifert [48] reproved this theorem and constructed a whole theory in which a surface spanned on the knot is used to compute knot invariants. The general higher-dimensional case was proved independently by Kervaire [33], Zeeman [57], and Levine [37].

The following question now arises: under what conditions does a knot bound an r-connected Seifert manifold? Interest in this question is explained by the fact that an answer to it must give a criterion for being unknotted. For if the knot bounds a contractible manifold, then this manifold is a disc, hence, the knot is trivial.

0.5. *Theorem.* *If a knot (S^{n+2}, k) has an r-connected Seifert manifold, then $\pi_i(S^{n+2} - k) = \pi_i(S^1)$ for all $i \leqslant r$. Conversely, if $n \neq 2$ and $\pi_i(S^{n+2} - k) = \pi_i(S^1)$ for all $i \leqslant r$, then the knot (S^{n+2}, k) bounds an r-connected Seifert manifold.*

The first assertion of this theorem, and also the second for $n \geqslant 4$, was proved by Levine [37]. The case $n = 1$ follows from Dehn's lemma, and the case $n = 3$ from work of Levine [39] and Trotter [53].

A consequence of Theorem 0.5 and of Smale's theorem [9] on the characterization of the disc is the following criterion of Levine [37] for being unknotted.

0.6. *Theorem.* *For $n \neq 2$, 4 a knot (S^{n+2}, k) is trivial provided that $\pi_i(S^{n+2} - k) = \pi_i(S^1)$ for all $i \leqslant (n+1)/2$. This also holds for $n = 4$ under the additional condition that k is diffeomorphic to the standard sphere.*

0.7. After these theorems the definition of a simple knot will be understandable. A knot (S^{n+2}, k) is called *simple* if $\pi_i(S^{n+2} - k) = \pi_i(S^1)$ for all $i \leqslant (n-1)/2$. As follows from Theorem 0.5, an n-dimensional knot is simple if and only if it bounds an $[(n-1)/2]$-connected Seifert manifold.

Such manifolds admit rather simple decompositions into handles. For example, a Seifert manifold for a simple $(2q-1)$-dimensional knot can be obtained from the disc by attaching a certain number of handles of index q. The arrangement of these handles in the ambient sphere is completely determined by their mutual linkage coefficients. This gives a method of an algebraic classification of the isotopy type of the embedding of the Seifert manifold, hence, also of the knot bounding it; however, all this fits into the framework of the present paper and will be discussed in detail later.

As Novikov [7] has shown, every locally flat topological knot (S^{n+2}, k) with $n \geqslant 5$ is carried by a homeomorphism into some smooth knot. Thus, the principal results of this paper are also true for topological locally flat knots.

I wish to thank A.V. Chernavskii and O.Ya. Viro for reading the paper in manuscript form and for making a number of valuable comments.

CHAPTER I

HOMOLOGICAL INVARIANTS OF KNOTS

§1. The homology of Seifert manifolds

In what sense is the homology of a Seifert manifold an invariant of the knot bounding it? The point is that a Seifert manifold of a knot is far from being unique: isotopies and surgeries along embedded handles, whose effect is illustrated in Fig. 1, can modify the Seifert manifold so as to change its homotopy type and its homology, but not its boundary.

In this section the homology groups of a Seifert manifold are equipped with the structure of modules over the ring P of integral polynomials. It turns out that if the homology is interpreted as a P-module, then it can be shown that in some sense it is not changed at all by surgeries on the Seifert manifold, and consequently is an invariant of the knot itself. Formally, what happens is this: a certain equivalence relation is introduced on the set of P-modules and it is shown that the equivalence class of the homology module of a Seifert manifold is an invariant of the knot.

If the Seifert manifold is even-dimensional, then its homology module of middle-dimension, together with the intersection index form, is an object first studied by Kervaire [34] under the name of an isometric structure.

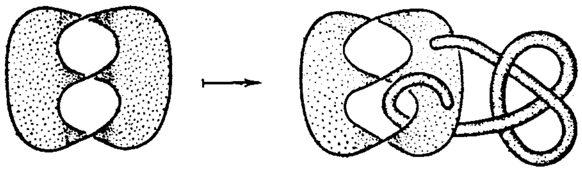


Fig. 1

We also introduce an equivalence relation on isometric structures and prove an analogous theorem for them.

1.1. Let $K = (S^{n+2}, k^n)$ be some knot and $V^{n+1} \subset S^{n+2}$ some Seifert manifold of it. Since the orientation of the knot $k = \partial V$ is fixed, V has a canonical orientation.

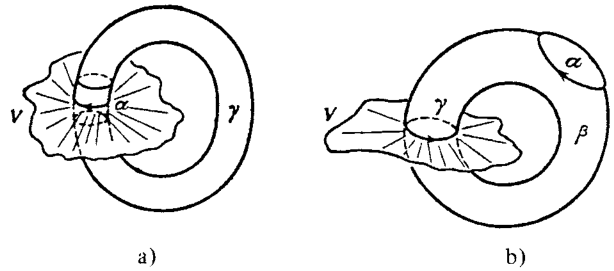


Fig. 2

We denote by $i_+, i_-: V \to S^{n+2} - V$ maps given by small displacements along the positive and negative normal directions to V, respectively. It is not hard to check that *the homomorphism* $H_rV \to H_r(S^{n+2} - V)$ *that takes a class* $a \in H_rV$ *to* $i_{+*}(a) - i_{-*}(a)$ *is an isomorphism for all* r. (Throughout this paper the symbol H_r denotes the reduced integral homology group.) To prove this we have to consider the corresponding Mayer-Vietoris sequence, but we can also use the following intuitive argument. If a class $a \in H_rV$, realized by a cycle α lying in int V, has the property that $i_{+*}(a) - i_{-*}(a) = 0$, then there exists a chain β in $S^{n+2} - V$ with $\partial\beta = i_{+*}(\alpha) - i_{-*}(\alpha)$. Adjoining to β the cylindrical chain over α we obtain an $(r+1)$-dimensional cycle γ in $S^{n+2} - k$ whose intersection with V gives α. By the Alexander duality theorem $H_{r+1}(S^{n+1} - k) = 0$ for $r > 0$. Consequently, γ bounds a chain δ lying in $S^{n+2} - k$, and the intersection of δ with V gives a chain spanning α in int V. Thus, $a = 0$. See Fig. 2a).

To prove surjectivity we assume that we are given a class $\{\alpha\} \in H_r(S^{n+2} - V)$. The cycle α bounds some chain β in $S^{n+2} - k$ (for $r > 1$ this follows from the fact that $H_r(S^{n+2} - k) = 0$, and for $r = 1$ from the fact that $H_1(S^{n+2} - k)$ is isomorphic to $\mathbf{Z}$ and is generated by any cycle whose linking number with k is 1: the cycle α does not intersect V, hence, its linking number with the knot is 0. If γ is the intersection of β with int V, then clearly, $i_{+*}(\gamma) - i_{-*}(\gamma)$ is homologous to α in $S^{n+2} - V$. See Fig. 2b).

Let $P = \mathbf{Z}[z]$ be the ring of integral polynomials. To define a P-module structure on H_rV it is sufficient to specify a class $za \in H_rV$ for each class $a \in H_rV$. We do this in such a way that

$$(i_{+*} - i_{-*})(za) = i_{+*}(a). \tag{1}$$

Since $i_{+*} - i_{-*}$ is an isomorphism, this formula gives a well-defined action of $z \in P$ on $H_r V$. Geometrically, za is the homology class of the cycle obtained as follows: we choose a cycle representing a and lift it above V; the resulting cycle bounds in $S^{n+2} - k$ a film whose intersection with V also represents za. Let us now explain the connection between the module structure on $H_r V$ and the intersection index of cycles.

1.2. ***Proposition.*** *If $a \in H_r V$, $b \in H_s V$, $r+s = n+1$, then*

$$(2) \qquad \langle za, b \rangle = \langle a, \bar{z}b \rangle,$$

where $\bar{z}$ denotes $1 - z \in P$ and the brackets $\langle , \rangle$ denote the intersection index on V.

■ To prove this, we use the formula

$$(3) \qquad (i_{-*} - i_{+*})(\bar{z}a) = i_{-*}(a),$$

which is derived from (1), and also the following properties of the linking numbers $L\colon H_r V \otimes H_s(S^{n+2} - V) \to \mathbf{Z}$ (see [3], [38]):

$$(4) \qquad L(a \otimes i_{+*}(b)) = (-1)^{rs+1} L(b \otimes i_{-*}(a)),$$

$$(5) \qquad L(a \otimes (i_{+*} - i_{-*})(b)) = \langle a, b \rangle.$$

We have

$$\langle za, b \rangle = (-1)^{rs} \langle b, za \rangle = (-1)^{rs} L(b \otimes (i_{+*} - i_{-*})(za)) =$$
$$= (-1)^{rs} L(b \otimes i_{+*}(a)) = -L(a \otimes i_{-*}(b)) =$$
$$= L(a \otimes (i_{+*} - i_{-*})(\bar{z}b)) = \langle a, \bar{z}b \rangle,$$

which proves (2). ■

1.3. An *isometric structure* is a P-module A equipped with a $\mathbf{Z}$-bilinear form $\langle , \rangle : A \times A \to \mathbf{Z}$ such that

(a) the module A is finitely generated as an Abelian group;

(b) the form $\langle , \rangle$ is ε-symmetric, where $\varepsilon = \pm 1$;

(c) the two homomorphisms $A \to \mathrm{Hom}_{\mathbf{Z}}(A; \mathbf{Z})$ associated with $\langle , \rangle$ are epimorphisms and their kernels are $T(A) = \mathrm{Tors}_{\mathbf{Z}} A$;

(d) $\langle za, b \rangle = \langle a, \bar{z}b \rangle$, where $a, b \in A$.

The number ε in (b) is called the *parity* of the isometric structure A. Two isometric structures A and B are *isomorphic* if there exists a P-isomorphism $A \to B$ preserving the scalar product.

If V is a Seifert manifold of a knot (S^{n+2}, k) and $n = 2q - 1$ is odd, then there is a scalar product on the P-module $H_q V$ given by the intersection index, and *$H_q V$ is an isometric structure of parity* $(-1)^q$. For (a) and (b) are clear, (c) follows from Poincaré's duality theorem, and (d) was proved in §1.2. $H_q V$ is called *the isometric structure of the manifold V.*

1.4. Let m be a non-negative integer. Two P-modules A and B are called *m-adjoining* if there exist P-homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that each of the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ coincides with multiplication by $(\bar{z}z)^m \in P$. Similarly, two isometric structures A and B will be called *m-adjoining* if there exist P-homomorphisms $\varphi\colon A \to B$, $\psi\colon B \to A$ such that each of the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ coincides with multiplication by $(\bar{z}z)^m \in P$ and for any $a \in A$, $b \in B$

$$\langle a, \psi(b) \rangle = \langle \varphi(a), b \rangle.$$

It is clear that P-modules and isometric structures are 0-adjoining if and only if they are isomorphic. If A m-adjoins B and B l-adjoins C, then A $(m+l)$-adjoins C.

The equivalence relation generated by 1-adjoining is called *R-equivalence*. In more detail, two P-modules A and B are said to be *R-equivalent* if there exists a finite sequence of P-modules $C_1, \ldots, C_m$, such that $A = C_1$, $B = C_m$ and C_i 1-adjoins C_{i+1} for all $i = 1, \ldots, m-1$. Similarly, two isometric structures of the same parity are said to be *R-equivalent* if they can be joined by a finite chain of isometric structures of the same parity in which consecutive structures are 1-adjoining. The principal result of this section consists in the following assertion.

1.5. ***Theorem.*** *Let $V_\nu \subset S^{n+2}$ be Seifert manifolds of knots $K_\nu = (S^{n+2}, k_\nu)$ ($\nu = 1$ or 2). If K_1 and K_2 are equivalent, then for any i the P-modules $H_i V_1$ and $H_i V_2$ are R-equivalent. If, in addition, $n = 2q - 1$ is odd, then the isometric structures $H_q V_1$ and $H_q V_2$ are R-equivalent.*

Theorem 1.5 is a trivial consequence of Theorem 1.7 and Proposition 1.8 below.

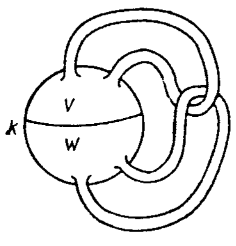


Fig. 3

1.6. Let (S^{n+2}, k^n) be a knot and $V, W \subset S^{n+2}$ two Seifert manifolds of it. We call them *adjoining* if $\mathrm{int}\, V \cap \mathrm{int}\, W = \varnothing$ (see Fig. 3). To explain the significance of this concept we note that $V \cup W$ is a closed manifold of dimension $n+1$, bounding some $(n+2)$-dimensional body in S^{n+2}. This body may be regarded as a cobordism from V to W. Decomposing it into handles, we obtain a sequence of embedded handles such that surgeries along them transform V into W.

The property of adjoining is symmetric, but not-transitive and non-reflexive. We call the equivalence relation on the set of Seifert manifolds of a given knot generated by adjoining *R-equivalence*. In other words, two Seifert manifolds V and W are said to be R-equivalent if there exists a finite sequence $U_0, U_1, \ldots, U_N$ of Seifert manifolds of the knot in question such that $U_0 = V$, $U_N = W$ and U_i adjoins U_{i+1} for all $i = 0, 1, \ldots, N-1$.

1.7. *Theorem.* *Any two Seifert manifolds of a knot are R-equivalent.*

Now, to deduce Theorem 1.5 it is sufficient to observe the following.

1.8. *Proposition.* *If $V, W \subset S^{n+2}$ are adjoining Seifert manifolds of a knot (S^{n+2}, k), then for all i the modules H_iV and H_iW are 1-adjoining. If, moreover, $n = 2q-1$ is odd, then the isometric structures H_qV and H_qW are also 1-adjoining.*

In the remaining part of this section we prove Theorem 1.7 (§§1.9–1.14) and Proposition 1.8 (§§1.15–1.17).

The proof of Theorem 1.7 consists in successively removing parts of the intersection $V \cap W$ by replacing V and W by R-equivalent Seifert manifolds. In the first step we reduce the general case to the situation when there is no intersection in a neighbourhood of the boundary, that is, when the set $\operatorname{int} V \cap \operatorname{int} W$ is compact. After that we remove all remaining intersections.

We assume that a Riemannian metric is fixed on the sphere S^{n+2}. Every Seifert manifold V of a knot (S^{n+2}, k) determines a normal vector field v on k, which is formed of unit vectors tangent to V and pointing into V.

1.9. *Lemma.* *The normal vector fields on k determined by any pair of Seifert manifolds are homotopic (in the class of unit normal vector fields).*

■ For $n \neq 1$ all unit normal vector fields are homotopic (this follows from obstruction theory). For $n = 1$ the obstruction to homotopy between two such fields is the difference in their winding numbers [8] (the winding number of a normal field is defined as the linking number of the knot with a small displacement of it along the field). It is clear that the winding number of the normal field defined by any Seifert manifold is 0. ■

1.10. *Lemma.* *Let X be a compact connected topological space and $f, g: X \to S^1$ two homotopic continuous maps. Then there exists a finite sequence of continuous maps $h_i: X \to S^1$ $(i = 0, 1, \ldots, N)$ such that $h_0 = f$, $h_N = g$, and $h_i(x) \neq h_{i+1}(x)$ for each $i = 0, 1, \ldots, N-1$ and each point $x \in X$.*

■ We identify S^1 with the unit circle in the complex plane. We begin by proving the lemma in the case when $g(x) = 1$ for all $x \in X$. Let $p: \mathbf{R} \to S^1$, $p(t) = \exp(2\pi i t)$, be the universal covering. Since f is homotopic to a constant map, it has a lifting $\tilde{f}: X \to \mathbf{R}$. The space X is compact and connected, therefore, $\tilde{f}(X) = [a, b]$. We may assume that $0 < a \leqslant 1$ (if this condition is not satisfied, we can consider another lifting $x \mapsto \tilde{f}(x) + k$ for a suitable integer k). Let N be an arbitrary integer greater than b. We define $h_i: X \to S^1$ by the formula

$$h_i(x) = p((1 - i/N)\tilde{f}(x)), \quad x \in X, \quad i = 0, 1, \ldots, N.$$

It is clear that $h_0(x) = f(x)$ and $h_N(x) = 1 = g(x)$. Moreover, if $h_i(x) = h_{i+1}(x)$ for some $x \in X$, then

$$(1 - i/N)\tilde{f}(x) - (1 - (i+1)/N)\tilde{f}(x) = l$$

is an integer. But then $l = \tilde{f}(x)/N$ and $0 < a/N \leqslant l \leqslant b/N < 1$, which is a contradiction.

In the general case, if $g \neq 1$, then we can apply the above argument to $f(x)g(x)^{-1}$ and 1 and multiply the resulting system of functions by $g(x)$. ■

1.11. *Lemma.* *Suppose that $V \subset S^{n+2}$ is a Seifert manifold of a knot (S^{n+2}, k), that v is the normal field on k determined by it, and w another smooth unit normal vector field on k such that $v(x) \neq w(x)$ for all $x \in k$. Then in any neighbourhood of V there exists a Seifert manifold W of (S^{n+2}, k) that adjoins V and has the normal field w.*

■ The vector field w on k can be extended to a unit field $\overline{w}$ transverse to V and defined on some neighbourhood of V. Let g^t be the local one-parameter group of diffeomorphisms generated by $\overline{w}$. Taking a sufficiently small $\varepsilon > 0$, we can form

$$g^\varepsilon(V) \cup \Big(\bigcup_{t \in [0, \varepsilon]} g^t(k)\Big)$$

and by smoothing corners along $g^\varepsilon(k)$ we obtain the required manifold W. ■

1.12. *Corollary.* *If V and W are Seifert manifolds of a knot (S^{n+2}, k), then there exists a Seifert manifold U of this knot that is R-equivalent to V and such that $u(x) \neq w(x)$ for all $x \in k$, where u and w are the normal fields on k determined by U and W, respectively. In particular, the set $\operatorname{int} U \cap \operatorname{int} W$ is compact.*

■ Let v be the normal field on k determined by V. By 1.9 and 1.10 there is a finite sequence $u_0, u_1, \ldots, u_N$ of unit normal vector fields on k with $u_0 = v$, $u_N = w$ and $u_i(x) \neq u_{i+1}(x)$ for all $x \in k$, $i = 0, 1, \ldots, N-1$. By Lemma 1.11 we may assume that the u_i are the normal fields of some Seifert manifolds U_i, where $U_0 = V$ and U_i adjoins U_{i+1} for all $i = 0, 1, \ldots, N-1$. The manifold U_{N-1} clearly satisfies the required conditions. ■

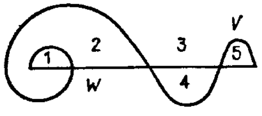


Fig. 4

The following drawing shows that closures of components of $S^{n+2} - (V \cup W)$ are not always manifolds. Figure 4 shows a zero-dimensional knot in the plane and two of its Seifert manifolds, the components 2 and 3 are "bad", while the closures of the other components are manifolds, although with corners.

1.13. ***Lemma.*** *Suppose that V and W are Seifert manifolds of a knot (S^{n+2}, k) in general position such that* $\operatorname{int} V \cap \operatorname{int} W$ *is compact. Then the closure of at least one component of $S^{n+2} - (V \cup W)$ is a manifold (with corners).*

■ Let K and L be two distinct components of $\operatorname{int} V \cap \operatorname{int} W$. Let ω be a path in V connecting some point of K to some point of L. Let η be a path in W with $\eta(0) = \omega(1)$ and $\eta(1) = \omega(0)$. Then $\omega\eta$ is a loop disjoint from the knot. We denote its linkage coefficient with K by $d(K, L)$. This number does not depend on the choices made, but only on the components K and L.

Let m be an integer so large that $|d(K, L)| + 1 < m$ for any components K and L of the intersection $\operatorname{int} V \cap \operatorname{int} W$. We consider the m-sheeted cyclic cover $p: \tilde{X} \to S^{n+2}$ branched over k (definitions and constructions of a branched cyclic covering are given, for example, in [2]). Let $t: \tilde{X} \to \tilde{X}$ be a generator of the group of covering transformations. We denote lifts of V and W to $\tilde{X}$ by $\tilde{V}$ and $\tilde{W}$, respectively. If the intersection $t^i\tilde{V} \cap \tilde{W}$ were non-empty for all $i = 0, 1, \ldots, m-1$, then there would be components K and L of the intersection $\operatorname{int} V \cap \operatorname{int} W$ with $d(K, L) = m - 1$, which is impossible. Consequently, without loss of generality we may assume that $\operatorname{int} \tilde{V} \cap \operatorname{int} \tilde{W} = \varnothing$.

We cut $\tilde{X}$ along $\operatorname{int} \tilde{V}$. As a result we obtain a compact manifold Y (with corners) and a map $\pi: Y \to \tilde{X}$ that maps $\operatorname{int} Y$ homeomorphically onto $\tilde{X} - \tilde{V}$. The boundary ∂Y contains k, and $\partial Y - k$ consists of two components, each mapped by π homeomorphically onto $\operatorname{int} \tilde{V}$. We denote the closures of these components by $\partial_0 Y$ and $\partial_1 Y$.

The manifold $\pi^{-1}(t\tilde{V})$ divides Y into two components. One of them contains all the sets $\pi^{-1}(t^i \operatorname{int} \tilde{V})$ for $i = 2, 3, \ldots, m-1$. We denote the closure of the other component by Z. (For $m = 2$ this condition does not determine Z uniquely, and in this case we take Z to denote either component.) The manifold Z contains precisely one of the sets $\partial_0 Y$ or $\partial_1 Y$. Suppose to be definite that $\partial_0 Y \subset Z$.

The manifold $\pi^{-1}(\tilde{W})$ also divides Y into two components. We denote by N the closure of the component containing $\operatorname{int} \partial_0 Y$. Let j be the least natural number for which $T = \operatorname{int}(N \cap \pi^{-1}(t^j \pi Z))$ is non-empty. Then the set $M = p(\pi(T)) \subset S^{n+2}$ is open, and its closure is $p(\pi(N \cap \pi^{-1}(t^j \pi Z)))$, which is homeomorphic to $N \cap \pi^{-1}(t^j \pi Z)$, that is, it is a manifold. On the other hand, M is clearly the union of some components of $S^{n+2} - (V \cup W)$.■

1.14. *Proof of Theorem* 1.7. Let V and W be two Seifert manifolds of a knot (S^{n+2}, k). We may assume that the normal vector fields on k determined by V and W do not coincide at any point

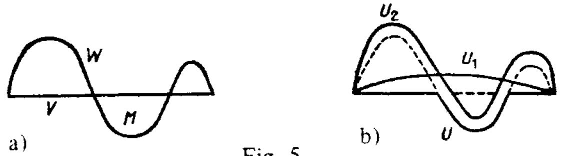


Fig. 5

(see Corollary 1.12) and that the interiors of V and W intersect transversely. By Lemma 1.13 there is a component of $S^{n+2} - (V \cup W)$ whose closure M is a manifold. This M adjoins V only on one side. Taking the corresponding vector field on V which points outward from M on $V \cap M$, we can construct a sufficiently small displacement of V as a result of which we obtain a Seifert manifold $U_1 \subset S^{n+2}$ with the following properties: (a) U_1 adjoins V; (b) U_1 does not intersect M. Similarly, by taking the normal field on $W \cap M$ pointing into M and by extending it accordingly to the whole of W, and then to some neighbourhood of W, we obtain a local one-parameter group of diffeomorphisms that displaces W to a new Seifert manifold U_2 with the following properties: (a) U_2 adjoins W; (b) U_2 and W define the same normal field on k; (c) $V \cap W$ and $V \cap U_2$ have the same number of components; (d) $U_2 \cap V \cap M \subset \operatorname{int}(\partial M \cap V)$.

Finally, let U be obtained from $(V \cup (\partial M \cap W)) - \operatorname{int}(\partial M \cap V)$ by smoothing corners along $\partial M \cap V \cap W$. Then U_1 adjoins U, hence, U is R-equivalent to V, while the intersection $\operatorname{int} U \cap \operatorname{int} U_2$ has fewer components than $\operatorname{int} V \cap \operatorname{int} W$. The proof is now completed by induction. ■

1.15. In the proof of Proposition 1.8 we need the following two remarks. The module structure of a Seifert manifold depends on the orientation of the knot. If the orientation of the knot is switched, then so is the orientation of the Seifert manifold, hence, the maps i_+ and i_- change places. Comparing (1) and (3) (see §1.1) we see that this leads to changing multiplication by z to multiplication by $\bar{z} = 1 - z \in P$. Moreover, the intersection index form changes sign.

The other remark is that a P-module structure can also be introduced on the homology of closed submanifolds of the sphere of codimension 1. For let $U^{n+1} \subset S^{n+2}$ be a closed connected oriented $(n+1)$-dimensional submanifold. Taking an arbitrary disc $D^{n+1} \subset U$, we can form $U_1 = U - \operatorname{int} D$, where ∂U_1 is a sphere. Hence, as was shown in §1.1, the groups $H_i U_1$ are defined as P-modules. On the other hand, the embedding $U_1 \to U$ induces an isomorphism of all homology groups of dimension $\leqslant n$. Consequently, the groups $H_i U$ are defined as P-modules for $i \leqslant n$. Similarly, if $n = 2q - 1$ is odd, then $H_q U$ is an isometric structure. Clearly, a different choice of the disc D yields isomorphic objects.

The homology modules of closed manifolds have the following special property.

1.16. ***Lemma.*** *If $U^{n+1} \subset S^{n+2}$ is a closed connected oriented submanifold, then for all $a \in H_iU$, where $i \leqslant n$, the product $(\bar{z}z)a$ vanishes. (We recall that $\bar{z}$ denotes $1-z \in P$.)*

■ The submanifold U divides S^{n+2} into two components. Let M and N be their closures and $r: U \to M$ and $s: U \to N$ the embeddings. Then $N \cap M = U$, $N \cup M = S^{n+2}$, and it follows from the Mayer-Vietoris sequence that the map

$$H_iU \xrightarrow{r_*\oplus s_*} H_iM \oplus H_iN$$

is an isomorphism for $i \leqslant n$. Thus, each element $a \in H_iU$ can be written uniquely in the form $a = a_1 + a_2$, where $a_1, a_2 \in H_iU$, $r_*a_1 = 0$, and $s_*a_2 = 0$. Suppose that U is oriented so that its positive normal points into M. Then $r_*a_1 = 0$ implies that $za_1 = 0$ (see (1)), and $s_*a_2 = 0$ implies $\bar{z}a_2 = 0$ (see (3)). If U has the opposite orientation, then $\bar{z}a_1 = 0$, $za_2 = 0$. In either case, $z\bar{z}a = z\bar{z}a_1 + z\bar{z}a_2 = 0$. ■

1.17. *Proof of Proposition* 1.8. Let $V, W \subset S^{n+2}$ be adjoining Seifert manifolds, $i_+, i_-: V \to S^{n+2} - V$ the maps of §1.1, and j: $\mathrm{int}\, W \to S^{n+2} - V$ the embedding. We define ψ: $H_iW \to H_iV$ by requiring that the composite map

$$H_i(\mathrm{int}\, W) \underset{\approx}{\to} H_iW \xrightarrow{\psi} H_iV \underset{\approx}{\xrightarrow{i_{+*}-i_{-*}}} H_i(S^{n+2}-V)$$

coincides with j_*. Since the end-maps are isomorphisms, ψ is a well-defined **Z**-homomorphism. Geometrically, the action of ψ can be described as follows. Let α be a cycle in W. By moving it slightly we may assume that α lies in int W. This cycle bounds some chain in S^{n+2} whose intersection with V represents the class $\psi(\{\alpha\})$.

Similarly we define a homomorphism φ: $H_iV \to H_iW$.

To prove that φ and ψ satisfy the condition defining 1-adjoining in §1.4, we consider the closed manifold $V \cup W = U$. In general, U is not smooth because of the corner along $\partial V = \partial W$. But clearly we can construct an isotopy h_t of S^{n+2} that is constant on k and outside some neighbourhood of k and such that $h_1(V) \cup W$ is a smooth manifold. Since the modules and isometric structures of V and $h_1(V)$ are isomorphic, we may assume from the outset that $U = V \cup W$ is smooth.

The embeddings $V \to U$ and $W \to U$ induce a group isomorphism $H_iV \oplus H_iW \to H_iU$, hence, each $x \in H_iU$ uniquely determines a pair (a, b), with $a \in H_iV$ and $b \in H_iW$. To simplify the notation we write $x = (a, b)$. We orientate U consistently with V. Then it is clear from geometric arguments that

$$z(a, 0) = (za, \varphi(a)), \quad a \in H_iV,$$
$$z(0, b) = (\psi(b), \bar{z}b), \quad b \in H_iW.$$

The second formula is quite analogous to the first: the difference is that the orientation of W is opposite to that of V (see the first remark in §1.15). We have

$$z\bar{z}(a, 0) = z(\bar{z}a, -\varphi(a)) = z(\bar{z}a, 0) - z(0, \varphi(a)) =$$
$$= (z\bar{z}a, \varphi(\bar{z}a)) - (\psi\varphi(a), \bar{z}\varphi(a)) = (z\bar{z}a - \psi\varphi(a), \varphi(\bar{z}a) - \bar{z}\varphi(a)).$$

But by the previous lemma, $z\bar{z}x = 0$ for any $x \in H_iU$. Consequently, for any $a \in H_iV$

$$\psi\varphi(a) = z\bar{z}a, \quad \varphi(\bar{z}a) = \bar{z}\varphi(a).$$

The second equation means that φ is a P-homomorphism. Similarly, by putting $z\bar{z}(0, b) = 0$ we obtain

$$\varphi\psi(b) = z\bar{z}b, \quad \psi(zb) = z\psi(b),$$

for any $b \in H_iW$. This means that the modules H_iV and H_iW are 1-adjoining for $i \leqslant n$. But for $i > n$ they vanish.

Let $n = 2q-1$ be odd. Then clearly, the intersection index form on U acts as follows:

$$\langle(a_1, b_1), (a_2, b_2)\rangle_U = \langle a_1, a_2\rangle_V - \langle b_1, b_2\rangle_W.$$

Using Proposition 1.2 we obtain for $a \in H_qV$ and $b \in H_qW$

$$\langle\varphi(a), b\rangle_W = -\langle z(a, 0), (0, b)\rangle_U = -\langle(a, 0), \bar{z}(0, b)\rangle_U = \langle a, \psi(b)\rangle_V.$$

Hence the isometric structures H_qV and H_qW are 1-adjoining. ■

Notes to §1. The map of a homology group of a Seifert manifold into itself acting (in the notation of §1.1) by the formula $a \to za$, $a \in H_iV$, is not new. The matrix describing this map already occurs in the classical paper of Seifert [48], where it is denoted by Γ. (Incidently, in [48] this matrix is used far more than the matrix that is now known as the Seifert matrix.) However, the homology of Seifert manifolds as P-modules is considered for the first time in the present paper. This seemingly small change of the point of view turns out to be very useful: not only does it allow us to describe surgeries of Seifert manifolds in the language of 1-adjoining (see 1.4 and Theorem 1.5), but it also makes it possible to compute Alexander modules in explicit transparent form (see §2) and to find a verifiable criterion for R-equivalence (see Theorem 5.1).

The notion of an isometric structure was introduced by Kervaire [34]. The only difference is that here we do not require that $T(A) = 0$. The term "isometric structure" is explained by the condition (d) in 1.3. When multiplication by $z \in P$ is an isomorphism $A \to A$, we can define a homomorphism $t: A \to A$ by putting $ta = a - z^{-1}a$, and (d) means that t is an isometry: $\langle ta, tb\rangle = \langle a, b\rangle$ for all $a, b \in A$. Conversely, given an automorphism t of an Abelian group A that preserves some scalar product $\langle , \rangle: A \times A \to \mathbf{Z}$ and such that $1-t: A \to A$ is an automorphism, A can be endowed with a P-module structure by putting $za = (1-t)^{-1}a$, and then (d) holds.

The relation of adjoining of modules and isometric structures is an algebraic analogue of the more general homotopy-theoretical concepts, presented in §8. Theorem 1.5 is generalized by Corollary 8.10 (see below).

Theorem 1.7 can hardly be regarded as new, although it is nowhere stated in this form. It is well known that all modifications of Seifert manifolds can be obtained by isotopies and surgery along embedded handles. This fact follows from arguments of Levine [39], 186–187, and a theorem of Kearton and Lickorish [29] on embeddings with critical levels. For classical knots a theorem of this kind was proved by Rice [46], Theorem 2.2, but as is clear from his proof, he mistakenly believed that all the components of the complement $S^{n+2} - (V \cup W)$ satisfy the conclusion of our Lemma 1.13.

A stronger assertion than Theorem 1.7 was proved (but not stated explicitly) in the present author's paper [21], 201–207. There it is assumed that the initial Seifert manifolds V and W are r-connected, and it is proved that they can be connected by a chain of adjoining r-connected Seifert manifolds. However, the arguments of [21] are valid only for $r \geqslant 2$, $n \geqslant 4$.

The problem of constructing knot invariants which, like the invariants of this section, can be calculated from any Seifert manifold, is classical. In knot theory there are many other invariants of a similar kind. This area was opened up by Seifert, paper [48], where it is shown how homology and linkage coefficients in finite-sheeted cyclic branched covers can be computed using a Seifert manifold. The next important step was taken in 1970 by Levine [39], [40], where he showed that the S-equivalence class of the Seifert matrix is a complete invariant of an odd-dimensional simple knot. The relation of S-equivalence had been considered by Trotter [52] and Murasugi [45] even before these papers of Levine; it describes the changes in a Seifert matric resulting from surgery. A somewhat different approach to the construction of matrix invariants of knots was proposed by Rice [46]: he also investigated the changes resulting from attaching an individual handle.

§2. Alexander modules

In this section we establish a connection between the knot invariants constructed in §1 and Alexander modules, which have long been one of the tools of knot theory. Namely, we derive a simple formula that expresses an Alexander module in terms of a homology P-module of an arbitrary Seifert manifold (see Theorem 2.6). There is also a reverse relationship: the Alexander module determines the R-equivalence class of the P-module of the Seifert manifold (see Theorem 2.7).

2.1. Let (S^{n+2}, k) be an n-dimensional knot and $X = S^{n+2} - k$ its complement. The *universal Abelian cover* $p: \widetilde{X} \to X$ is the cover defined by the derived subgroup of $\pi = \pi_1(X)$. According to the theory of covering spaces, the group of covering transformations of this cover is the Abelianized group $\pi/[\pi, \pi] = H_1X$. By Alexander's duality theorem, $H_1X = \mathbf{Z}$, hence, the cover $p: \widetilde{X} \to X$ has an infinite cyclic group of covering transformations. The orientations of S^{n+2} and k determine one of the two generators of this group corresponding to the class in H_1X that has the linking number $+1$ with k. Let $t: \widetilde{X} \to \widetilde{X}$ be this generator. The homeomorphism t acts on the homology $H_*\widetilde{X}$, making it into modules over the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$. The Λ-modules $H_i\widetilde{X}$ are called the *Alexander modules* of the knot.

Let $V^{n+1} \subset S^{n+2}$ be a Seifert manifold for (S^{n+2}, k). Since every cycle in int V has zero linkage coefficient with k, the embedding int $V \to X$ can be lifted to $\widetilde{X}$. Fixing some lifting f: int $V \to \widetilde{X}$ and identifying H_i(int V) with $H_i(V)$, we obtain an induced homomorphism $f_*: H_iV \to H_i\widetilde{X}$.

2.2. *Proposition.* (1) *The formula* $(1 - t)f_*(zv) = f_*(v)$ *holds for any* $v \in H_iV$; (2) *when* $v \in H_iV$, *then* $f_*(v) = 0$ *if and only if* $(z\bar{z})^m v = 0$ *for some* $m \geqslant 0$; (3) *for each class* $x \in H_i\widetilde{X}$ *there are integers* $l \geqslant 0$ *and* $m \geqslant 0$ *such that* $x = (1 - t)^l t^{-m} f_*(v)$ *for some* $v \in H_iV$.

We postpone the proof (see 2.8) and first derive some consequences. First we obtain the following well known fact [33]:

2.3. *Corollary.* *The Alexander modules* $H_i\widetilde{X}$ *of every knot have the property that multiplication by* $(1 - t) \in \Lambda$ *is an automorphism* $H_i\widetilde{X} \to H_i\widetilde{X}$.

■ It follows from Proposition 2.2, (1) and (3), that multiplication by $1 - t$ is an epimorphism from $H_i\widetilde{X}$ onto itself. On the other hand $H_i\widetilde{X}$ is finitely generated over Λ (since X retracts onto a finite complex and Λ is Noetherian), and every surjective endomorphisms of a finitely generated module over a Noetherian ring is known to be an automorphism. ■

Owing to Corollary 2.3 $H_i\widetilde{X}$ can be regarded as a module over the ring $L = \mathbf{Z}[t, t^{-1}, (1 - t)^{-1}]$. Let z denote $(1 - t)^{-1} \in L$. The subring of L generated by z is isomorphic to $P = \mathbf{Z}[z]$.

2.4. *Corollary.* *The map* $f_*: H_iV \to H_i\widetilde{X}$ *is a P-homomorphism.*

This follows from Proposition 2.2 (1). ■

2.5. *Lemma.* *Let A be a P-module. Then the kernel of the homomorphism* $A \to L \otimes_P A$ *taking* $a \in A$ *to* $1 \otimes a$ *is the set of* $a \in A$ *such that* $(z\bar{z})^m a = 0$ *for some integer* $m \geqslant 0$.

■ If $(z\bar{z})^m a = 0$, then $1 \otimes a = (z\bar{z})^{-m} \otimes (z\bar{z})^m a = 0$. The reverse assertion follows from the fact that L, regarded as a P-module, is isomorphic to the direct limit of $P \to P \to P \to \ldots$, where all the homomorphisms are multiplication by $z\bar{z}$, hence, $L \otimes_P A$ is isomorphic (over P) to the direct limit $A \to A \to \ldots$ of multiplications by $z\bar{z}$. ■

The following fact is a result of this lemma and Proposition 2.2.

2.6. *Theorem.* *Let* $\widetilde{X}$ *be the infinite cyclic cover of the complement of a knot,* V *some Seifert manifold of it, and* $f_*: H_iV \to H_i\widetilde{X}$ *the homomorphism induced by some lifting* int $V \to \widetilde{X}$. *Then the formula* $F(q \otimes v) = qf_*(v)$, $q \in L$, $v \in H_iV$, *gives a well-defined L-homomorphism*

$$F: L \otimes_P H_iV \to H_i\widetilde{X},$$

and this is an isomorphism.

■ It is easy to see that for any finite set $q_1, \ldots, q_r \in L$ there is an integer $m \geqslant 0$ such that $q_j = (z\bar{z})^{-m} p_j$ for some polynomials $p_j \in P$ $(j = 1, 2, \ldots, r)$. If $v_j \in H_iV$, then

$$\sum_{j=1}^{r} q_j \otimes v_j = (z\bar{z})^{-m} \otimes \left(\sum_{j=1}^{r} p_j v_j\right)$$

hence, every element of $L \otimes_P H_iV$ can be written in the form $(z\bar{z})^{-m} \otimes v$ for some $m \geqslant 0$ and $v \in H_iV$. As follows from Lemma 2.5,

$$(z\bar{z})^{-m} \otimes v = (z\bar{z})^{-l} \otimes w$$

if and only if

$$(z\bar{z})^{l+k} v = (z\bar{z})^{m+k} w$$

for some $k > 0$. Thus

$$F((z\bar{z})^{-m} \otimes v) = (z\bar{z})^{-m} f_*(v) = (z\bar{z})^{-m-l-k} f_*((z\bar{z})^{l+k} v) =$$

$$= (z\bar{z})^{-m-l-k} f_*((z\bar{z})^{m+k} w) = F((z\bar{z})^{-l} \otimes w),$$

so that F is well-defined.

If $F((z\bar{z})^{-m} \otimes v) = 0$, then $(z\bar{z})^{-m} f_*(v) = 0$ and $f_*(v) = 0$. But by Proposition 2.2 (2) it follows that $(z\bar{z})^k v = 0$ for some $k \geqslant 0$. Hence, $(z\bar{z})^{-m} \otimes v = (z\bar{z})^{-m-k} \otimes (z\bar{z})^k v = 0$ and the fact that F is monomorphic is proved.

That F is epimorphic is easy to deduce from Proposition 2.2 (3). ■

As a supplement to Theorem 2.6 we state the following algebraic result.

2.7. Theorem. (1) *Every finitely generated Λ-module $\tilde{A}$ for which multiplication by $1-t$ is an isomorphism $\tilde{A} \to \tilde{A}$ is isomorphic to $L \otimes_P A$ for some P-module A that is finitely generated over* $\mathbf{Z}$; (2) *for any pair A and B of P-modules that are finitely generated over $\mathbf{Z}$ the following are equivalent*:

(a) *A and B are R-equivalent*;

(b) *A and B are m-adjoining for some* $m \geqslant 0$;

(c) *the L-modules $L \otimes_P A$ and $L \otimes_P B$ are isomorphic.* ■

In the definition of R-equivalence there occurs a certain chain which makes it not effectively verifiable. By way of contrast, (b) and (c) have the great advantage of being verifiable (at least to the extent that an isomorphism is). This is the significance of the second statement of Theorem 2.7. We do not give a proof here, since later we prove analogous statements for the more complicated case of isometric structures.

2.8. *Proof of Proposition* 2.2. Let $V^{n+1} \subset S^{n+2}$ be a Seifert manifold of the knot $k = \partial V \subset S^{n+2}$, $p: \tilde{X} \to X$ an infinite cyclic cover of the complement $X = S^{n+2} - k$, $t: \tilde{X} \to \tilde{X}$ the generator of the translation group constructed in 2.1, and j: int $V \to \tilde{X}$ a lifting of the embedding int $V \subset X$. We denote the image of j by W. The complement $\tilde{X} - (W \cup tW)$ consists of three components, one of which is mapped homeomorphically by p onto $S^{n+2} - V$. We denote this component by Y, and let $\mathbf{v}: Y \to \tilde{X}$ be the embedding. The maps i_+, i_-: $V \to S^{n+2} - V$ (see 1.1), composed with the homeomorphism $p^{-1}: S^{n+2} - V \to Y$, give maps $j_+, j_-: V \to Y$. From geometric arguments it is clear that $t \circ \mathbf{v} \circ j_+$ and $\mathbf{v} \circ j_-$ are homotopic maps $V \to \tilde{X}$. If $v \in H_i V$, then $\mathbf{v}_* j_{+*}(v) = f_*(v)$, and it follows from what we have just said that $\mathbf{v}_* j_{-*}(v) = tf_*(v)$. In accordance with the definition of the P-module structure of $H_i V$ we have $(i_{+*} - i_{-*})(zv) = i_{+*}(v)$. Consequently,

$$(1) \qquad (j_{+*} - j_{-*})(zv) = j_{+*}(v).$$

Applying $\mathbf{v}_*$ to each side of this equality we obtain

$$f_*(zv) - tf_*(zv) = f_*(v),$$

which proves (1).

Now (1) clearly implies the analogous formula

$$(2) \qquad (1 - t^{-1}) f_*(\bar{z}v) = f_*(v),$$

where $v \in H_i V$. Therefore, if $(z\bar{z})^m v = 0$, then

$$f_*(v) = (1-t)^m (1-t^{-1})^m f_*((z\bar{z})^m v) = 0,$$

which proves half of (2).

We consider the exact homology sequence

$$\ldots \to H_{i+1}(\tilde{X}, \tilde{Y}) \to H_i \tilde{Y} \to H_i \tilde{X} \to H_i(\tilde{X}, \tilde{Y}) \to \ldots,$$

where $\tilde{Y}$ is the union of the sets $t^k Y$, $k \in \mathbf{Z}$. Since $t^k Y \cap t^j Y = \varnothing$ for $k \neq j$, the space $\tilde{Y}$ is the disjoint union of the sets $t^k Y$, and $H_i \tilde{Y}$ can be regarded as $\Lambda \otimes_{\mathbf{Z}} H_i Y$, identifying $H_i(t^k Y)$ with $t^k \otimes H_i Y$. If N is a small regular neighbourhood of W in $\tilde{X}$ of the form $W \times [-1, 1]$ and $\tilde{N} = \bigcup t^k N$, $\tilde{W} = \bigcup t^k W$, then $\tilde{Y} = \tilde{X} - \tilde{W}$ and $H_{i+1}(\tilde{X}, \tilde{Y}) \approx H_{i+1}(\tilde{N}, \tilde{N} - \tilde{W})$ by the excision axiom. As above, the latter module can be identified with $\Lambda \otimes_{\mathbf{Z}} H_{i+1}(N, N - W)$, and

$$H_{i+1}(N, N - W) \approx H_{i+1}(W \times ([-1, +1], [-1, 1] - \{0\})) \approx H_i W = H_i V.$$

Consequently, we obtain a Λ-isomorphism $H_{i+1}(\tilde{X}, \tilde{Y}) \approx \Lambda \otimes_{\mathbf{Z}} H_i V$, and the exact sequence above gives

$$(3) \qquad \ldots \to \Lambda \otimes_{\mathbf{Z}} H_i V \xrightarrow{d} \Lambda \otimes_{\mathbf{Z}} H_i Y \xrightarrow{e} H_i \tilde{X} \to \Lambda \otimes_{\mathbf{Z}} H_{i-1} V \to \ldots$$

It is not hard to see that d and e act according to the formulae

$$d(t^k \otimes v) = t^k \otimes j_{+*}(v) - t^{k-1} \otimes j_{-*}(v), \quad e(t^k \otimes y) = t^k \mathbf{v}_*(y),$$

where $v \in H_i V$, $y \in H_i Y$.

Suppose now that $v \in H_i V$ is such that $f_*(v) = 0$. Then the element $1 \otimes j_{+*}(v)$ lies in the kernel of e, hence there is an element

$$w = \sum_k t^k \otimes v_k \in \Lambda \otimes_{\mathbf{Z}} H_i V$$

with $d(w) = 1 \otimes j_{+*}(v)$. Here k ranges over the integers, and all but finitely many of the v_k are zero. For the classes v_k we obtain the following system of equations:

$$j_{+*}(v_k) - j_{-*}(v_{k+1}) = 0, \quad k \in \mathbf{Z}, \quad k \neq 0,$$
$$j_{+*}(v_0) - j_{-*}(v_1) = j_{+*}(v).$$

Applying (1) and the analogous formula $(j_{+*} - j_{-*})(\bar{z}v) = -j_{-*}(v)$ (see (3) in §1) and also the fact that $j_{+*} - j_{-*}$ is an isomorphism, we obtain the equivalent system of equations:

$$zv_k + \bar{z}v_{k+1} = 0, \quad k \in \mathbf{Z}, \quad k \neq 0, \qquad zv_0 + \bar{z}v_1 = zv.$$

Let m be a positive number so large that $v_k = 0$ for all $|k| \geqslant m$. Then

$$z^{m-1}v_1 = -\bar{z}z^{m-2}v_2 = \ldots = (-1)^{m-1}\bar{z}^{m-1}v_m = 0,$$
$$\bar{z}^{m}v_0 = -z\bar{z}^{m-1}v_{-1} = \ldots = (-1)^m z^m v_{-m} = 0.$$

Consequently,

$$z^m\bar{z}^{m}v = z^m\bar{z}^{m}v_0 + z^{m-1}\bar{z}^{m+1}v_1 = 0.$$

This completes the proof of Proposition 2.2 (2).

To prove (3) we note that the homomorphism d in the exact sequence (3) is a monomorphism. For if $\sum_k t^k \otimes v_k \in \Lambda \otimes_{\mathbf{Z}} H_iV$ belongs to ker(d), then by arguments analogous to the above we see that $zv_k + \bar{z}v_{k+1} = 0$ for all $k \in \mathbf{Z}$. Hence, as above, it follows that $z^m v_k = 0$ and $\bar{z}^m v_k = 0$ for a sufficiently large m. However, there exist integer polynomials $p(z)$ and $q(z)$ such that $z^m p(z) + \bar{z}^m q(z) = 1$, hence,

$$v_k = z^m p(z)v_k + \bar{z}^m q(z)v_k = 0.$$

That e is an epimorphism follows from the fact that d is a monomorphism, hence, every class $x \in H_i\tilde{X}$ can be represented in the form

$$x = \sum_k t^k \nu_*(y_k),$$

where $y_k \in H_iY$ and $y_k = 0$ whenever $|k|$ is greater than some m. Since $j_{+*} - j_{-*}\colon H_iV \to H_iY$ is an isomorphism, there exist classes $v_k \in H_iV$ such that

$$y_k = (j_{+*} - j_{-*})(v_k).$$

Then $\nu_* y_k = (1-t)f_*(v_k)$ and for $|k| \leqslant m$, using Proposition 2.2 (1) we obtain

$$f_*(v_k) = (1-t)^{m-k}(1-t^{-1})^{m+k} f_*(z^{m-k}\bar{z}^{m+k}v_k),$$
$$t^k f_*(v_k) = t^{-m}(1-t)^{2m} f_*((-1)^{m+k}\bar{z}^{m+k}z^{m-k}v_k),$$
$$x = \sum_k t^k \nu_*(y_k) = (1-t)\sum_k t^k f_*(v_k) = t^{-m}(1-t)^{2m+1} f_*(v),$$

where $v = \sum_k (-1)^{m+k} z^{m-k}\bar{z}^{m+k} v_k$. This completes the proof of Proposition 2.2. ■

Notes to §2. Proposition 2.2 is a technical result we will need and is easy to derive from known properties of Alexander modules. The exact sequence (3) in 2.8 is taken from Levine [38]. This is the classical method of recovering the Alexander module from information about the Seifert manifold. By comparison, the formula $H_i\tilde{X} \approx L \otimes pH_iV$ of Theorem 2.6 is much more powerful. With its help one can study a large class of functors of knot modules (see [16], [17]), which leads to new constructions of forms on Alexander modules. Among the forms constructed by the methods of [16] and [17] are all known forms, and also some new ones. Below in §§6-7 the Milnor form and the Blanchfield form are constructed by similar methods.

A proof of Theorem 2.7 is in [16], [17].

CHAPTER II

THE CLASSIFICATION OF ODD-DIMENSIONAL SIMPLE KNOTS

§3. Reduction to algebra

We call an isometric structure A *special* if $T(A) = \mathrm{Tors}_{\mathbf{Z}}A = 0$. We call a Seifert manifold *special* if it is $(n+1)$-dimensional and $[(n-1)/2]$-connected. Simple knots and only they admit special Seifert manifolds (this follows from Theorem 0.5, or for $n = 2$ from Theorem 0.4). It is clear that the isometric structure of every even-dimensional special Seifert manifold is special.

The main result of this section is the following assertion, which supplements Theorem 1.5.

3.1. ***Theorem.*** *Let $n \geqslant 5$ be an odd number. The map that associates with a knot the R-equivalence class of the isometric structure of some special Seifert manifold is a bijection from the set of types of n-dimensional simple knots to the set of R-equivalence classes of special isometric structures of parity* $(-1)^{(n+1)/2}$.

The proof (see 3.5) uses Lemmas 3.2–3.4 below.

3.2. ***Lemma.*** *Let $n \geqslant 5$ be odd and $V, W \subset S^{n+2}$ be two special Seifert manifolds whose isometric structures are 1-adjoining. Then there is an isotopy of the sphere S^{n+2} that takes V to an oriented submanifold V_1 such that* $\mathrm{int}\, V_1 \cap \mathrm{int}\, W = \varnothing$, $\partial V = \partial W$ *and V_1 and W induce the same orientation on* $\partial V_1 = \partial W$.

3.3. ***Lemma.*** *Let $n \geqslant 5$ be odd. Every special isometric structure of parity* $(-1)^{(n+1)/2}$ *is the isometric structure of some special Seifert manifold* $V^{n+1} \subset S^{n+2}$.

3.4. ***Lemma.*** *If two special isometric structures of the same parity are R-equivalent, then they can be connected by a finite chain of special isometric structures of the same parity in which all consecutive structures are 1-adjoining.*

3.5. The proof of Theorem 3.1 follows from Lemmas 3.2–3.4 in the obvious way. For if two simple n-dimensional knots have Seifert manifolds with R-equivalent isometric structures, then (by Theorem 1.5) the isometric structures of any Seifert manifolds of these knots are R-equivalent. Let V be a special Seifert manifold of one of the knots and W of the other. By Lemma 3.4, the isometric structures of V and W can be connected by a chain of special isometric structures in which all consecutive structures are 1-adjoining. By Lemma 3.3, each isometric structure in this chain can be realized as that of a special Seifert manifold $V_i \subset S^{n+2}$. By Lemma 3.2, the knot $(S^{n+2}, \partial V_i)$ is ambient-isotopic to $(S^{n+2}, \partial V_{i+1})$ for all i, and this proves that the original knots are equivalent. ■

3.6. The purely algebraic Lemma 3.4 is proved in §5 (see 5.7). The remainder of this section is devoted to proofs of Lemmas 3.2 and 3.3.

We make use of Wall's theory of thickenings [55]. We recall that an *m-dimensional thickening* of a k-dimensional cell complex K (it is assumed that $m \geqslant k+3$) is the name for a simple homotopy equivalence $\varphi: (K, *) \to (V, *)$, where V is a compact m-dimensional manifold equipped with a base point and an orientation of the tangent space at the base point and such that the embedding $\partial V \to V$ induces an isomorphism of the fundamental groups. Two m-dimensional thickenings $\varphi_1: (K, *) \to (V_1, *)$ and $\varphi_2: (K, *) \to (V_2, *)$ are called *equivalent* if there exists a diffeomorphism $h: V_1 \to V_2$ preserving the base points and the orientations on them and such that

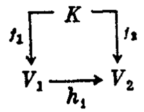
$$h \circ \varphi_1 \sim \varphi_2: (K, *) \to (V_2, *).$$

3.7. ***Theorem.*** *Let U^m be a manifold, K^k a cell complex with $m \geqslant k+3$ and $f: K \to U$ a $(2k-m+1)$-connected map. Then there exists a compact submanifold $V^m \subset U^m$ with $\pi_1(\partial V) = \pi_1(V)$ and a simple homotopy equivalence $g: K \to V$ that is homotopic* **rel** $*$ *to f in U.* ■

3.8. ***Theorem.*** *Suppose that two compact m-dimensional manifolds V_1 and V_2 lie in some manifold U^m and that $\pi_1(\partial V_\nu) = \pi_1(V_\nu)$ ($\nu = 1$ or 2). Let K be a finite k-dimensional cell complex and $f_\nu: K \to V_\nu$ two homotopy equivalences such that the compositions*

$$K \xrightarrow{f_\nu} V_\nu \xrightarrow{\subset} U$$

are homotopic and $(2k-m+2)$-connected ($\nu = 1$ or 2). If $m \geqslant 6$ and $k \leqslant m-3$, then there exists a smooth isotopy $h_t: U \to U$ such that $h_0 =$ id, $h_t(*) = *$, $h_1(V_1) = V_2$, and the diagram



commutes up to homotopy. ■

Theorem 3.7 is the first part of Wall's embedding theorem ([55], 76). Theorem 3.8 is a refinement of the second part of this theorem of Wall, taking account of results of Hudson [25] and Rourke [47] that concordance implies isotopy.

The simple homotopy equivalence $g: K \to V$ whose existence is asserted by Theorem 3.7 is called *the thickening induced by the map $f: K \to U$.* Theorem 3.8 essentially asserts the uniqueness of an induced thickening.

3.9. To prove Lemmas 3.2 and 3.3 we apply Theorems 3.7 and 3.8 in the following situation. Let U^{2q} be a closed $(q-1)$-connected manifold, where $q \geqslant 3$, and $V^{2q} \subset U$ an almost closed submanifold (that is, one bounded by a homotopy sphere). Then V has the homotopy type of a bouquet of q-spheres, and the embedding $i: V \to U$ induces a monomorphism $H_qV \to H_qU$. Clearly, i_* preserves intersection indices. Consequently, the restriction to $i_*(H_qV) \subset H_qU$ of the intersection index form $\langle , \rangle_U$ is unimodular.

Conversely, suppose that $A \subset H_qU$ is a subgroup such that the restriction of the form $\langle , \rangle_U$ to A is unimodular. We take a bouquet of r q-spheres K, where r is the rank of A. We can construct a continuous map $f: K \to U$ that induces a monomorphism on the q-dimensional homology with image A. By Theorem 3.7, f defines a thickening $V \subset U$. Obviously, V is almost closed and the image of $H_qV \to H_qU$ is A.

If V^{2q}, $W^{2q} \subset U^{2q}$ are two almost closed submanifolds with $i_*(H_qV) = j_*(H_qW)$, where i and j are the embeddings in U, then V and W are ambient-isotopic in U. For let K be a bouquet of q-spheres and $f: K \to V$ an arbitrary homotopy equivalence. Since $i_*(H_qV) = j_*(H_qW)$ we can construct a homotopy equivalence $g: K \to W$ with $i \circ f \sim g$, and our assertion follows from Theorem 3.8.

Hence *there is a bijection between the subgroups $A \subset H_qU$ for which $\langle , \rangle|_A$ is unimodular and the ambient-isotopy classes of almost closed submanifolds $V^{2q} \subset U$.*

3.10. ***Lemma.*** *Let $n = 2q-1$, $q \geqslant 3$, and let $V, W \subset S^{2q+1}$ be special Seifert manifolds whose isometric structures are 0-adjoining. Then there exists an isotopy of S^{2q+1} taking V to W with orientations preserved.*

■ Let $\varphi: H_qV \to H_qW$ be a P-isomorphism preserving the intersection index forms. Let $f: V \to W$ be a map inducing φ. We consider an embedding $V \times [0, 1] \to S^{2q+1}$ that extends the embedding of $V \times 0 = V$ and has the property that the curve $t \mapsto (v, t)$, where $v \in V$ and $t \in [0, 1]$, leaves $v = (v, 0)$ in the direction of the negative normal to V. We denote the image of $V \times [0, 1]$ by $N(V)$ and define $N(W)$ similarly. Let $\alpha: V \to N(V)$ and $\beta: W \to N(W)$ be the canonical embeddings. The maps α and $\beta \circ f$ are thickenings of the complex V. By Theorem 3.8, there exists a homotopy h_t of S^{2q+1} taking $N(V)$ to $N(W)$ and such that $h_1 \circ \alpha: V \to N(W)$ is homotopic to $\beta \circ f$.

Hence, we may assume from the very beginning that $N(V) = N(W) = N$. We write $M = S^{2q+1} - \text{int}\, N$, $U = \partial N$, and $i: V \to U$, $j: W \to U$, $r: U \to M$, $s: U \to N$ for the embeddings. By construction, $s_* i_* v = s_* j_* f_* v$ for all $v \in H_qV$.

The homomorphism $i_*: H_qV \to H_qU$ is not, in general, a P-homomorphism. But it is easy to see that for any $v \in H_qV$ the difference $i_*(zv) - zi_*(v)$ belongs to the annihilator of $i_*(H_qV)$ in H_qU. In other words,

$$\langle i_*(zv), i_*v_1\rangle_U = \langle zi_*(v), i_*v_1\rangle_U$$

for any $v, v_1 \in H_qV$. The homomorphism j_* has the analogous property.

We claim that $r_* i_* v = r_* j_* f_* v$ for all $v \in H_q V$. If $v \in H_q V$, then

$$L(s_* i_* v_1, r_*(i_* v - j_* f_* v)) = L(s_* i_* v_1, r_* i_* v) - L(s_* j_* f_* v_1, r_* j_* f_* v) =$$
$$= \langle i_* v_1, z i_* v\rangle_U - \langle j_* f_* v_1, z j_* f_* v\rangle_U =$$
$$= \langle v_1, zv\rangle_V - \langle f_* v_1, z f_* v\rangle_W = 0,$$

where L denotes the linking number. Since each element of $H_q N$ can be expressed in the form $s_* i_* v_1$, it follows that $r_* a = 0$, where $a = i_* v - j_* f_* v$. We have already pointed out above that $s_* a = 0$. Consequently, $a = 0$ (see proof of Lemma 1.16), that is, we have proved that $i_* v = j_* f_* v$ for all $v \in H_q V$. But then it follows that $i_*(H_q V) = j_*(H_q W)$, and so V and W are ambient-isotopic in U (see 3.9). By means of a tubular neighbourhood of U in S^{n+2}, this isotopy can be extended to an isotopy of the whole sphere S^{n+2}. ■

3.11. ***Lemma.*** *Let A be a special isometric structure of parity $(-1)^q$ with the property that $z\bar{z}a = 0$ for all $a \in A$. Then there exists a closed oriented $2q$-dimensional $(q-1)$-connected submanifold $U \subset S^{2q+1}$ whose isometric structure is isomorphic to A.*

■ We write $A_+ = \{a \in A;\ za = 0\}$ and $A_- = \{a \in A;\ \bar{z}a = 0\}$. It is easy to see that the module A is isomorphic to the direct sum $A_+ \oplus A_-$ and that the restrictions of the scalar product to A_+ and to A_- vanish. It then follows that the isomorphism class of the isometric structure A is completely determined by the parity $(-1)^q$ and the rank r of the free Abelian group A_+.

We consider an arbitrary embedding of the disjoint union of r copies of the sphere S^q in S^{2q+1}. Let U be the connected sum of the boundaries of their tubular neighbourhoods. We orientate U so that the positive normal is directed away from the tubular neighbourhoods. Then, as was established in the proof of Lemma 1.16, there is a group isomorphism $(H_q U)_+ \approx A_+$, and the lemma follows from what was said above and from Lemma 1.16. ■

3.12. *Proof of Lemma* 3.2. Let $\varphi: H_q V \to H_q W$ and $\psi: H_q W \to H_q V$ be P-homomorphisms satisfying the definition of 1-adjoining (see 1.4). We define a new isometric structure on the group $A = H_q V \oplus H_q W$ by setting

$$z(a, b) = (za + \psi(b), \bar{z}b + \varphi(a)),$$
$$\langle (a_1, b_1), (a, b)\rangle = \langle a_1, a\rangle_V - \langle b_1, b\rangle_W$$

for $(a, b), (a_1, b_1) \in A$. It is easy to check that $z\bar{z}A = 0$, therefore, by Lemma 3.11, A can be realized by a $(q-1)$-connected oriented submanifold $U^{2q} \subset S^{2q+1}$. Let $A_1 \subset A$ be the subgroup of elements of the form $(a, 0)$, $a \in H_q V$, and $A_2 \subset A$ the subgroup of elements of the form $(0, b)$, $b \in H_q W$. By what was said in 3.9, there exists an almost closed submanifold $V_1^{2q} \subset U$ corresponding to A_1. Let $W_1 = U - \operatorname{int} V_1$. Then W_1 corresponds to the subgroup A_2. The isometric structure of V_1, oriented compatibly with U, is isomorphic to the isometric structure of V, and by Lemma 3.10.

V_1 is ambient-isotopic to V with orientations preserved. Similarly W_1, oriented compatibly with U, is ambient-isotopic to W with orientations preserved. ■

3.13. *Proof of Lemma* 3.3. Let A be a special isometric structure. Let $\varphi: A \to A$ be multiplication by z and $\psi: A \to A$ multiplication by $\bar{z}$. As in 3.12 we define a new isometric structure on the group $B = A \oplus A$. Then $z\bar{z}B = 0$ and by Lemma 3.11, B can be realized by a $(q-1)$-connected closed oriented submanifold $U^{2q} \subset S^{2q+1}$. Let $A_1 \subset B$ be the subgroup of elements of the form $(a, 0)$. By what was said in 3.9, there exists an almost closed submanifold $V \subset U$ corresponding to A_1. It is clear that the isometric structure of V is isomorphic to A. ■

Notes to §3. This section is an alternative account of the results of Levine [39]. If we bear in mind that the Seifert pairing and the isometric structure of a special Seifert manifold determine each other uniquely, Lemma 3.10 can be identified with Lemma 3 of Levine's paper [39], and Lemma 3.2 with an assertion proved by Kervaire [33] in the course of the proof of Theorem II.3. The proofs presented here are based on the theory of thickenings; they are modelled on arguments in [21], where a much more general situation is studied.

As follows from Theorem 3.1, for simple knots (S^{2q+1}, k^{2q-1}) the isometric structure A determines the embedding type completely, hence also the differentiable structure on k. Since a homotopy sphere embedded in a sphere with codimension 2 bounds a parallelizable manifold, the type of an exotic sphere k is determined for even q by the signature of the form $\langle\,,\rangle: A \times A \to \mathbf{Z}$ (see [32]), and for odd q by the Arf-invariant of the quadratic function $a \mapsto \langle a, za\rangle \pmod 2$, $a \in A$ (see [38], Proposition 3.3). Levine [38] also showed that this Arf-invariant c is given by the formulae

$$c = \begin{cases} 0 & \text{if } \Delta(-1) \equiv \pm 1 \pmod 8, \\ 1 & \text{if } \Delta(-1) \equiv \pm 3 \pmod 8), \end{cases}$$

where $\Delta(t)$ is the Alexander polynomial in q-dimensional homology.

§4. Minimal isometric structures

We say that a P-module A is *minimal* if multiplication by $z\bar{z} \in P$ is a monomorphism. We recall that $\bar{z}$ denotes $1 - z \in P$.) An isometric structure A is said to be *minimal* if the module A is minimal. The significance of minimal modules is explained by Lemma 2.5.

4.1. ***Theorem.*** *Every isometric structure is R-equivalent to a minimal one.*

This theorem is the main result of the section. To prove it we introduce the notion of a discriminant structure; this allows us to survey all the isometric structures adjoining a given one. The technique of discriminant structures is used essentially in the next section.

4.2. ***Lemma.*** *Every isometric structure A is R-equivalent to an isometric structure B with minimal module $T(B) = \mathrm{Tors}_{\mathbf{Z}} B$.*

■ Let $K = \{a \in T(A);\ za = 0\}$ and $A_1 = A/K$ with the natural P-module structure. We define a scalar product on A_1 by the formula $\langle \pi a, \pi b\rangle = \langle a, b\rangle$, where $a, b \in A$ and $\pi: A \to A_1$ is the projection. Then A_1 is an isometric structure of the same parity as A. We define homomorphisms $\varphi: A \to A_1$

and $\psi: A_1 \to A$ by the formulae

$$\varphi(a) = \pi(\bar{z}a), \quad \psi(\pi a) = za, \quad a \in A.$$

These homomorphisms satisfy all the conditions in the definition of 1-adjoining (see 1.4), hence, A_1 1-adjoins A. Applying the construction just described to A_1 we obtain an isometric structure A_2. Similarly we obtain A_3 from A_2 and so on. It is clear that for sufficiently large N the isometric structure A_N has the property that multiplication by z determines a monomorphism $T(A_N) \to T(A_N)$. Moreover, A_N is R-equivalent to A.

Now we change the modification process somewhat. Let $L_N = \{a \in T(A_N);\ \bar{z}a = 0\}$ and $A_{N+1} = A_N/L_N$ with the natural P-module structure and the scalar product defined as above. Then A_{N+1} is an isometric structure and 1-adjoins A_N. Similarly we obtain an isometric structure A_{N+2} 1-adjoining A_{N+1}, and then A_{N+3} and so on. It is clear that for sufficiently large M the P-module $T(A_{N+M})$ is minimal, and we may put $B = A_{N+M}$. ∎

4.3. A *periodic isometric structure* [50] is a P-module A equipped with a $\mathbf{Z}$-bilinear form $\{,\}$: $A \times A \to \mathbf{Q}/\mathbf{Z}$, where it is assumed that (a) the module A is finite; (b) the form $\{,\}$ is ε-symmetric, where $\varepsilon = \pm 1$; (c) the form $\{,\}$ is non-degenerate, that is, both the associated homomorphisms $A \to \mathrm{Hom}(A;\ \mathbf{Q}/\mathbf{Z})$ are isomorphisms; (d) $\{za, b\} = \{a, \bar{z}b\}$ for all $a, b \in A$. The number ε is called the parity of A.

4.4. We now show that every isometric structure A with minimal module $T(A)$ determines a series of periodic isometric structures.

Let $m > 0$ be an integer and let $[A]_m$ denote the set of all elements $a \in A$ for which there is an integer $N \neq 0$ such that $Na \in (z\bar{z})^m A$. We denote by A_m the group $[A]_m/(z\bar{z})^m A$ with the natural P-module structure and the form $\{,\}$: $A_m \times A_m \to \mathbf{Q}/\mathbf{Z}$ acting as follows. If $\alpha, \beta \in A_m$ and $a, b \in [A]_m$ are respective representatives with $Na = (z\bar{z})^m a_1$ for some $N \in \mathbf{Z}$, $N \neq 0$, $a_1 \in A$, then we set by definition

$$\{\alpha, \beta\} = \langle a_1, b\rangle/N \pmod{\mathbf{Z}}.$$

Let us prove that this is well-defined. If b' is another representative of the class β, then $b - b' = (z\bar{z})^m x$ for some $x \in A$ and

$$\langle a_1, b\rangle/N - \langle a_1, b'\rangle/N = \langle a_1, (z\bar{z})^m x\rangle/N = \langle a, x\rangle = 0 \pmod{\mathbf{Z}}.$$

If $Mb = (z\bar{z})^m b_1$ for $b_1 \in A$, $M \in \mathbf{Z}$, $M \neq 0$, then

$$\langle a_1, b\rangle/N = \langle a_1, Mb\rangle/NM = \langle a_1, (z\bar{z})^m b_1\rangle/NM = \\ = \langle (z\bar{z})^m a_1, b_1\rangle/NM = \langle a, b_1\rangle/M,$$

which shows that the definition of $\{\alpha, \beta\}$ is independent of the choice of N and a_1, and that it is independent of the choice of a in the class α follows from arguments analogous to the above.

4.5. *Proposition.* *If A is an isometric structure of parity ε with a minimal module $T(A)$, then the module A_m with the form $\{,\}$ is a periodic isometric structure of parity ε.*

∎ It is trivial to verify the conditions of Definition 4.3. For example, if $\{\alpha, \beta\} = 0$ for all $\beta \in A_m$, then $\langle a_1, b\rangle$ is divisible by N for all $b \in [A]_m$ (where b is a representative of β, a is a representative of α, and $Na = (z\bar{z})^m a_1$). The homomorphism $b \mapsto \langle a_1, b\rangle/N \in \mathbf{Z}$ given on $[A]_m$ can be extended to A, hence, there is an element $a_2 \in A$ such that $\langle a_2, b\rangle = \langle a_1, b\rangle/N$ for all $b \in [A]_m$. It now follows that for any $b \in A$

$$\langle (z\bar{z})^m a_2, b\rangle = \langle a_2, (z\bar{z})^m b\rangle = \langle a_1, (z\bar{z})^m b\rangle/N = \langle a, b\rangle,$$

Consequently, the element $a - (z\bar{z})^m a_2 = a_3$ belongs to $T(A)$. Since $T(A)$ is minimal, there is an $a_4 \in T(A)$ with $(z\bar{z})^m a_4 = a_3$, hence, $a = (z\bar{z})^m (a_2 + a_4) \in \in (z\bar{z})^m A$ and $\alpha = 0$. ∎

The periodic isometric structure A_m is called the *m-th discriminant structure* of A.

4.6. Let A be an isometric structure with a minimal module $T(A)$, A_m its m-th discriminant structure, and $L \subset A_m$ a *metabolic submodule* [50], that is, a submodule that is its own annihilator $L^\perp$ relative to the form $\{,\}$. We now show that *L determines a new isometric structure A_L that m-adjoins A.*

Let π: $[A]_m \to A_m$ be the natural projection and $A_L = \pi^{-1}(L)$ with the induced P-module structure. We specify a scalar product $\langle,\rangle_1$ on A_L as follows: if $a, b \in A_L$ and $Na = (z\bar{z})^m a_1$, where $a_1 \in A$, $N \in \mathbf{Z}$, $N \neq 0$, then

$$\langle a, b\rangle_1 = \langle a_1, b\rangle/N.$$

If $Mb = (z\bar{z})^m b_1$, then

$$\langle a_1, b\rangle/N = \langle a_1, Mb\rangle/NM = \langle (z\bar{z})^m a_1, b_1\rangle/NM = \langle a, b_1\rangle/M,$$

which shows that $\langle a, b\rangle_1$ is independent of the choice of N and a_1. To check that $\langle a, b\rangle_1$ is an integer we note that $\langle a, b\rangle_1 \equiv \{\pi(a), \pi(b)\} \equiv 0 \pmod{\mathbf{Z}}$, since $\pi(a), \pi(b) \in L$ and $L \subset L^\perp$.

The module A_L, with the scalar product $\langle,\rangle_1$, obviously satisfies the conditions (a), (b), and (d) in the definition of an isometric structure in 1.3. To check condition (c), suppose that $\langle a, b\rangle_1 = 0$ for all $a \in A_L$. But then $0 = \langle a, b\rangle = \langle (z\bar{z})^m a, b\rangle_1$ for all $a \in A$, hence, $b \in T(A) = T(A_L)$. Given a $\mathbf{Z}$-homomorphism $f: A_L \to \mathbf{Z}$, we consider $g: A \to \mathbf{Z}$, where $g(a) = f((z\bar{z})^m a)$, $a \in A$. There is an element $b \in A$ such that $g(a) = \langle a, b\rangle$ for all $a \in A$. Since g vanishes on $K_m = \{a \in A;\ (z\bar{z})^m a \in T(A)\}$, we see that b belongs to the annihilator $K_m^\perp$ with respect to $\langle,\rangle$. On the other hand, $K_m = [A]_m^\perp$, consequently, $K_m^\perp = [A]_m^{\perp\perp}$. But $[A]_m$ is a pure subgroup hence, $[A]_m^{\perp\perp} = [A]_m$, and $b \in K_m^\perp = [A]_m$. If $a \in A_L$ and $Na = (z\bar{z})^m a_1$, then

$$f(a) = f((z\bar{z})^m a_1)/N = g(a_1)/N = \langle a_1, b\rangle/N \equiv \{\pi(a), \pi(b)\} \pmod{\mathbf{Z}}.$$

Since $f(a) \in \mathbf{Z}$ for $a \in A_L$, it follows that $\pi(b) \in L^\perp = L$, hence, $b \in A_L$. Moreover, $\langle a, b\rangle_1 = f(a)$ for all $a \in A_L$. This proves (d), hence, $(A_L, \langle,\rangle_1)$ is an isometric structure of the same parity as A.

Suppose that $\varphi: A_L \to A$ is the embedding, and that $\psi: A \to A_L$ acts as follows: $\psi(a) = (z\bar{z})^m a$, $a \in A$. Then all the conditions of the definition of m-adjoining are satisfied (see 1.4), hence A and A_L are m-adjoining.

4.7. *Proof of Theorem* 4.1. Let A be an isometric structure with minimal module $T(A)$ (see Lemma 4.2). Then the first discriminant structure A_1 (see 4.4) is defined. We choose some metabolic submodule $L \subset A_1$, for example, $L = \{a \in A_1;\ za = 0\}$. Then the isometric structure A_L is also defined (see 4.6) and 1-adjoins A. It is clear from the construction of 4.6 that if the module A is not minimal, then the rank of A_L is less than that of A, while $T(A_L)$ is still minimal. If A_L is also not minimal, then similarly we can construct an isometric structure of smaller rank that 1-adjoins A_L. Clearly, by iterating this process we obtain after finitely many steps a minimal isometric structure that is R-equivalent to the original one. ■

Notes to §4. As already remarked, an isometric structure determines a Seifert form. If the isometric structure is minimal, then the corresponding Seifert matrix is non-degenerate. Therefore, Theorem 4.1 has the same significance as Trotter's theorem [52] that every Seifert matrix is S-equivalent to a non-degenerate one.

The notion of periodic isometric structures was introduced by Stoltzfus [50]. He used this concept to describe the obstruction to a rational isometric structure being cobordant to an integral one.

The construction of the discriminant structure is obtained by applying the standard construction of a discriminant bilinear form (see, for example, [6]) to the form $(a, b) \mapsto \langle z^m a, z^m b\rangle$ defined by the isometric structure. The quadratic discriminant form (see [6]) vanishes in our case, owing to the fact that $\langle a, a\rangle = (1+\varepsilon)\langle a, za\rangle \equiv 0 \pmod 2$ for all $a \in A$, where ϵ is the parity of the isometric structure.

§5. A criterion for R-equivalence of isometric structures

In this section the study of the technique of discriminant structures is continued, and it is later applied in the proof of the following criterion for R-equivalence.

5.1. ***Theorem.*** *Two isometric structures are R-equivalent if and only if they are m-adjoining for some m.*

The significance of this theorem is that it allows us to replace the relation of R-equivalence, which is difficult to verify, by the relation of m-adjoining, which is expressed in terms of homomorphisms between the original isometric structures.

Lemma 3.4 is also proved in the course of this section.

We begin with the following assertion.

5.2. ***Proposition.*** *Let A and B be minimal m-adjoining isometric structures. Then B is isomorphic to A_L for some metabolic submodule $L \subset A_m$.*

■ Let $\varphi: A \to B$ and $\psi: B \to A$ be two P-homomorphisms such that $\varphi\circ\psi$ and $\psi\circ\varphi$ are multiplication by $(z\bar{z})^m$ and $\langle a, \psi(b)\rangle_A = \langle\varphi(a), b\rangle_B$ for any $a \in A$, $b \in B$. It follows from these conditions and the minimality of A and B that φ and ψ are monomorphisms. Moreover, $[A]_m = A$, also by minimality. Let $\pi: A \to A_m$ be the natural projection with kernel $(z\bar{z})^m A$ and let $L = \pi(\psi(B)) \subset A_m$. We claim that L is a metabolic submodule. If $x, y \in B$ and $N\psi(x) = (z\bar{z})^m a_1$ for some $a_1 \in A$ and $N \in \mathbf{Z}$, $N \neq 0$, then $Nx = \varphi(a_1)$ and

$$\{\pi\psi(x), \pi\psi(y)\} = \langle a_1, \psi(y)\rangle/N = \langle\varphi(a_1), y\rangle/N = \langle x, y\rangle \equiv 0 \pmod{\mathbf{Z}}.$$

This proves that $L \subset L^\perp$. Suppose now that $a \in A$ is such that $\{\pi a, \pi\psi b\} = 0$ for all $b \in B$. If $Na = (z\bar{z})^m a_1$, where $a_1 \in A$, $N \neq 0$, then

$$\langle a_1, \psi(b)\rangle/N = \langle\varphi(a_1), b\rangle/N$$

is an integer for all $b \in B$, hence, there is a $y \in B$ such that $\langle y, b\rangle = \langle\varphi(a_1), b\rangle/N$ for all $b \in B$. But then $t = \varphi(a_1) - Ny$ belongs to $T(B)$, so that $Na = (z\bar{z})^m a_1 = \psi\varphi(a_1) = N\psi(y) + \psi(t)$, and consequently, $a - \psi(y) \in T(A)$. But $T(A) \subset \operatorname{im}\psi$ by the minimality of A and B, hence $a \in \operatorname{im}\psi$. Thus, $\pi(a) \in L$ and $L^\perp \subset L$.

Since $\operatorname{im}\psi \supset (z\bar{z})^m A$, we see that $A_L = \pi^{-1}(L)$ coincides with $\psi(B)$. If $x, y \in B$, then arguments similar to the above show that $\langle\psi(x), \psi(y)\rangle_1 = \langle x, y\rangle_B$, so that the map $\psi: B \to A_L$ is an isomorphism preserving the scalar product. ■

5.3. Here we examine the formation of discriminant structures in more detail.

Let A be a periodic isometric structure and $m > 0$ an integer such that $(z\bar{z})^m a = 0$ for any $a \in A$ (this holds if A is an m-th discriminant structure, see 4.4 and 4.5). We introduce the notation

$$A^+ = \{a \in A;\ z^m a = 0\},\quad A^- = \{a \in A;\ \bar{z}^m a = 0\}$$

and consider the homomorphisms $\tau_+, \tau_-: A \to A$, where

$$\tau_+(a) = (1 - z^m)^m a,\quad \tau_-(a) = (1 - \bar{z}^m)^m a,\quad a \in A.$$

It is easy to verify that $\operatorname{im}\tau_+ = A^+$, $\operatorname{im}\tau_- = A^-$ and that τ_+ and τ_- are mutually complementary projections: $\tau_+^2 = \tau_+$, $\tau_-^2 = \tau_-$, $\tau_+ + \tau_- = 1$. From this it follows that the map $a \mapsto (\tau_+(a), \tau_-(a))$ is a P-isomorphism $A \to A^+ \oplus A^-$.

If $a, b \in A$, then $\{\tau_+(a), b\} = \{a, \tau_-(b)\}$. Hence, $(A^+)^\perp = A^+$, $(A^-)^\perp = A^-$.

Suppose that $L \subset A$ is a submodule and $L^+ = L \cap A^+$, $L^- = L \cap A^-$. Then $L = L^+ \oplus L^-$ and $L^\perp = (L^+)^\perp \cap (L^-)^\perp$. Since $(L^+)^\perp \supset A^+$ and $(L^-)^\perp \supset A^-$, the submodule L is metabolic if and only if

$$L^- = (L^+)^\perp \cap A^-,\quad L^+ = (L^-)^\perp \cap A^+.$$

It follows from these equalities that a metabolic submodule $L \subset A$ is determined by either of its submodules $L^+ \subset A^+$ or $L^- \subset A^-$. Conversely, if $X \subset A^+$ is an arbitrary submodule, then the submodule

$$K = X + (X^\perp \cap A^-) \subset A$$

is metabolic and $K^+ = X$.

5.4. ***Proposition.*** *Let A be a minimal isometric structure and $L \subset A_m$ a metabolic submodule of the m-th discriminant structure. Let $L^+ = L \cap A_m^+$, $X = zL^+$, and $K = X + (X^\perp \cap A_m^-)$. Then $K \subset A_m$ is a metabolic submodule, and the isometric structures A_K and A_L corresponding to K and L, respectively, are 1-adjoining.*

■ We remark first that $K^+ = X$ and

$$K^- = X^\perp \cap A_m^- = \{a \in A_m^-;\ (X, a) = 0\} = \{a \in A_m^-;\ \{zL^+, a\} = 0\} =$$
$$= \{a \in A_m^-;\ \{L^+, \bar{z}a\} = 0\} = \{a \in A_m^-;\ \bar{z}a \in L^-\},$$

where $L^- = L \cap A_m^-$. Let $\pi\colon A \to A_m$ be the natural projection. Then according to Definition 4.6, $A_L = \pi^{-1}(L)$ and $A_K = \pi^{-1}(K)$. We define two homomorphisms $\varphi\colon A_L \to A_K$ and $\psi\colon A_K \to A_L$ by

$$\varphi(a) = za, \quad a \in A_L, \qquad \psi(b) = \bar{z}b, \quad b \in A_K.$$

If $a \in A_L$, then $\pi a = \alpha_1 + \alpha_2$, here $\alpha_1 \in L^+$ and $\alpha_2 \in L^-$. Then $\pi(za) = z\alpha_1 + z\alpha_2$, and $z\alpha_1 \in K^+ = zL^+$ and $z\alpha_2 \in L^- \subset K^-$ (see the remark at the beginning of the proof). Thus, $\varphi(a) \in A_K$ for $a \in A_L$. Similarly, if $b \in A_K$ and $\pi(b) = \beta_1 + \beta_2$, where $\beta_1 \in K^+$ and $\beta_2 \in K^-$, then $\pi(\bar{z}b) = \bar{z}\beta_1 + \bar{z}\beta_2$, where $\bar{z}\beta_1 \in K^+ \subset L^+$ and $\bar{z}\beta_2 \in L^-$ (by the same remark). Hence, $\psi(b) \in A_L$ for $b \in A_K$.

It is clear that $\varphi \circ \psi$ and $\psi \circ \varphi$ are multiplication by $z\bar{z}$. The relation $\langle a, \psi(b)\rangle_{A_L} = \langle \varphi(a), b\rangle_{A_K}$ for $a \in A_L$, $b \in A_k$, is trivial to verify. ■

5.5. ***Proposition.*** *Let A_m be the discriminant structure of a minimal isometric structure A and let $L = A_m^-$. Then the isometric structure A_L is isomorphic to A.*

■ By definition (see 4.6), A_L consists of those $a \in A$ for which $\bar{z}^m a = (z\bar{z})^m b$ for some $b \in A$. It then follows from the minimality that $a = z^m b$, and so $A_L = z^m A$. Let $\varphi\colon A \to A_L$ act by the formula $\varphi(b) = z^m b$, $b \in A$. Then φ is an isomorphism. If $b, c \in A$ and $Nb = \bar{z}^m b_1$ for some $b_1 \in A$, $N \in \mathbf{Z}$, $N \neq 0$, then $N\varphi(b) = (z\bar{z})^m b_1$ and

$$\langle \varphi(b), \varphi(c)\rangle_{A_L} = \langle b_1, z^m c\rangle / N = \langle \bar{z}^m b_1, c\rangle / N = \langle b, c\rangle.$$

This completes the proof of the proposition. ■

5.6. *Proof of Theorem* 5.1. If A, B, and C are isometric structures such that A m-adjoins B and B l-adjoins C, then A $(m+l)$-adjoins C. It follows that if there is a chain of $(m+1)$ isometric structures in which consecutive structures are 1-adjoining, then the end-structures are m-adjoining. Hence, R-equivalence implies m-adjoining for some $m \geqslant 0$.

Conversely, suppose that two isometric structures A and B are m-adjoining for some $m \geqslant 0$. By Theorem 4.1, we may assume without loss of generality that A and B are minimal. Then by Proposition 5.2, the isometric structure B is isomorphic to A_L for some metabolic submodule $L \subset A_m$. For $i = 0, 1, \ldots, m$ we define a metabolic submodule $L_i \subset A_m$ by the formula (see 5.3)

$$L_i = X_i + (X_i^\perp \cap A_m^-), \quad X_i = z^i L^+.$$

Then the isometric structure $A_{L_0} = A_L$ is isomorphic to B, also A_{L_m} to A (by Proposition 5.5), and for each $i = 0, 1, \ldots, m-1$ A_{L_i} and $A_{L_{i+1}}$ are 1-adjoining (by Proposition 5.4). ■

5.7. *Proof of Lemma* 3.5. To every special isometric structure we can attach by a finite chain of 1-adjoining special isometric structures one that is minimal (this is clear from the proof of Theorem 4.1, see 4.7). Therefore, to prove the lemma it is enough to show that any two special minimal m-adjoining isometric structures can be linked by a finite chain of 1-adjoining special minimal isometric structures. But this follows from the proof of Theorem 5.1 (see 5.6), since each A_{L_i} is minimal and special. ■

Notes to §5. Theorem 5.1 would follow immediately from a conjecture about R-equivalence (see [15]), but the latter has not yet been either proved or disproved.

The proof given here for Theorem 5.1 has a conceptual connection with the work of Trotter [53]; it arose from an attempt to simplify the proof of the main theorem of [53].

§6. The Milnor form

By Theorem 2.7, two P-modules A and B are R-equivalent if and only if the L-modules $\tilde{A} = L \otimes_P \tilde{A}$ and $\tilde{B} = L \otimes_P B$ are isomorphic. If A and B are isometric structures, then R-equivalence of A and B, of course, implies an isomorphism $\tilde{A} \approx \tilde{B}$, but the converse is false. In this section a form $\tilde{A} \times \tilde{A} \to \mathbf{Q}$ (the Milnor form) is constructed for every isometric structure A, and it is shown that isometric structures A and B are R-equivalent if and only if there is an L-isomorphism $\tilde{A} \to \tilde{B}$ preserving the Milnor form.

6.1. Let A be a P-module. We say that a submodule $B \subset A$ is *basic* if

(a) B contains $(z\bar{z})^k A$ for some $k \geqslant 0$;

(b) if $a \in A$ and $Na \in B$ for some $N \neq 0$, then $a \in B$;

(c) the kernel of the homomorphism $B \to B$ given by multiplication by $z\bar{z} \in P$ is contained in $T(B) = \mathrm{Tors}_{\mathbf{Z}} B$.

6.2. ***Lemma.*** *Every P-module that is finitely generated over* $\mathbf{Z}$ *has a unique basic submodule.*

■ If a P-module A is finitely generated over $\mathbf{Z}$, then in the descending series of subgroups

$$A \supset (z\bar{z})A \supset (z\bar{z})^2A \supset \dots$$

the ranks of the groups $(z\bar{z})^sA$ are independent of s for all s from some k onwards. Let B be the set of those $a \in A$ for which some integer multiple belongs to $(z\bar{z})^kA$. Then (a) and (b) are satisfied. If B contains an element x of infinite order with $z\bar{z}x = 0$, then $Nx \in (z\bar{z})^kA$ for some $N \neq 0$, now Nx has infinite order and $z\bar{z}(Nx) = 0$. Hence, the rank of the image of the homomorphism $(z\bar{z})^kA \to (z\bar{z})^kA$ given by multiplication by $z\bar{z} \in P$ is less than that of $(z\bar{z})^kA$. But this contradicts the fact that the rank of $(z\bar{z})^{k+1}A$ is equal to that of $(z\bar{z})^kA$.

If $B_1 \subset A$ is another basic submodule, then it follows from (a) and (b) that $B_1 \supset B$. On the other hand, as follows from (c), the rank of $(z\bar{z})^sB_1$ is equal to that of B_1 for all $s \geqslant 0$, so that

$$\operatorname{rank} B_1 = \operatorname{rank} (z\bar{z})^kB_1 \leqslant \operatorname{rank} (z\bar{z})^kA = \operatorname{rank} B.$$

Hence, applying (b) a second time, we obtain $B = B_1$. ■

Note that if A is minimal, then $B = A$.

6.3. *Lemma.* *Let A be a P-module that is finitely generated over $\mathbf{Z}$, $B \subset A$ its basic submodule, and $\nu: A \to \tilde{A}$ the map that takes $a \in A$ to $\nu(a) = 1 \otimes a$. Then (1) the kernel of $\nu|_B$ is contained in $T(B)$; (2) every $x \in \tilde{A}$ can be written in the form $x = (z\bar{z})^{-n}\nu(b)$ for some $b \in B$ and $n \geqslant 0$; (3) for each $x \in \tilde{A}$ there exists an $N \in \mathbf{Z}$, $N \neq 0$, such that $Nx \in \operatorname{im}(\nu|_B)$.*

■ (1) follows from Lemma 2.5 and Definition 6.1, (c). As was shown in the proof of Theorem 2.6, each $x \in \tilde{A}$ can be written in the form $x = (z\bar{z})^{-m} \otimes a$, where $a \in A$, $m \geqslant 0$. Then $x = (z\bar{z})^{-m-k} \otimes (z\bar{z})^ka$, and $(z\bar{z})^ka \in B$ for sufficiently large k, which proves (2). Now (3) follows from (2) and Definition 6.1, (c). ■

6.4. We can now construct the Milnor form. Let A be an isometric structure, $B \subset A$ its basic submodule, $\tilde{A} = L \otimes_P A$ and $\nu: A \to \tilde{A}$ the map of Lemma 6.3. We assume that $x, y \in \tilde{A}$. By Lemma 6.3 (3) there exist non-zero integers N and M and $a, b \in B$ with $Nx = \nu(a)$, $My = \nu(b)$. Then we put

$$[x, y] = \frac{1}{NM}\langle a, b\rangle \in \mathbf{Q}.$$

A trivial check shows that this gives a well-defined bilinear map $[\,,\,]: \tilde{A} \times \tilde{A} \to \mathbf{Q}$, which is called *the Milnor form of the isometric structure A.*

Of the properties of the Milnor form we need only the following.

6.5. *Lemma.* *If $x, y \in \tilde{A}$ and $f \in L = \mathbf{Z}[z, z^{-1}, \bar{z}^{-1}]$, then*

$$[f(z)x, y] = [x, f(\bar{z})y].$$

■ Let $Nx = \nu(a)$ and $My = \nu(b)$, where $a, b \in B$, $N, M \in \mathbf{Z}$, $N \neq 0 \neq M$, and $B \subset A$ is the basic submodule. We write f in the form $f = (z\bar{z})^{-n}p(z)$, where $p \in P$ is a polynomial and $n \geqslant 0$. Then $Ka = (z\bar{z})^na_1$ and $Lb = (z\bar{z})^nb_1$ for some $a_1, b_1 \in B$ and $K \neq 0 \neq L$. Consequently, $NK(f(z)x) = \nu(p(z)a_1)$, and by definition,

$$\begin{aligned}[f(z)x, y] &= \langle p(z)a_1, b\rangle/NKM = \langle p(z)a_1, (z\bar{z})^nb_1\rangle/NKML = \\ &= \langle (z\bar{z})^na_1, p(\bar{z})b_1\rangle/NKML = \langle a, p(\bar{z})b_1\rangle/NML = [x, f(\bar{z})y],\end{aligned}$$

since $ML(f(\bar{z})y) = \nu(p(\bar{z})b_1)$. ■

The principal result of this section is the following.

6.6. *Theorem.* *Two isometric structures A_1 and A_2 are R-equivalent if and only if there is an L-isomorphism $\tilde{A}_1 = L \otimes_P A_1 \to \tilde{A}_2 = L \otimes_P A_2$ preserving the Milnor form.*

■ We use Theorem 5.1. Suppose that A_1 and A_2 are m-adjoining and that $\varphi: A_1 \to A_2$ and $\psi: A_2 \to A_1$ are P-homomorphisms such that $\varphi\circ\psi$ and $\psi\circ\varphi$ are multiplication by $(z\bar{z})^m \in P$ and $\langle a, \psi(b)\rangle = \langle\varphi(a), b\rangle$ for $a \in A_1$, $b \in A_2$. Suppose that the L-homomorphism $\Phi: \tilde{A}_1 \to \tilde{A}_2$ is the tensor product of φ with the homomorphism $L \to L$ given by multiplication by z^{-m}. Similarly, let Ψ be $\bar{z}^{-m} \otimes \psi$. Then Φ and Ψ are mutually inverse isomorphisms. We claim that Φ preserves the Milnor form. Let $\nu_i: A_i \to \tilde{A}_i$ ($i = 1$ or 2) be the P-homomorphism that takes $a \in A_i$ to $1 \otimes a$ and let $B_i \subset A_i$ be the basic submodule. Let $x, y \in \tilde{A}_1$ and $Nx = \nu_1(a)$ and $My = \nu_1(b)$, where $a, b \in B_1$, $N, M \in \mathbf{Z}$, $N \neq 0 \neq M$. It follows from Definition 6.1 that the numbers N and M can be chosen so that, in addition, $a, b \in \psi(B_2)$. If $a = \psi(c)$, $b = \psi(d)$, then

$$N\Phi(x) = z^{-m}\nu_2(\varphi(a)) = z^{-m}\nu_2(\varphi\psi(c)) = \nu_2(\bar{z}^mc)$$

and similarly $M\Phi(y) = \nu_2(\bar{z}^md)$. Then

$$\begin{aligned}[\Phi(x), \Phi(y)] &= \langle\bar{z}^mc, \bar{z}^md\rangle_{A_2}/NM = \langle\varphi\psi(c), d\rangle_{A_2}/NM = \\ &= \langle\psi(c), \psi(d)\rangle_{A_1}/NM = \langle a, b\rangle_{A_1}/NM = [x, y].\end{aligned}$$

As follows from Theorem 4.1 and what has just been proved, to prove the converse assertion we may assume without loss of generality that A_1 and A_2 are minimal, so that ν_1 and ν_2 are injective (see Lemma 2.5). Let $\Phi: \tilde{A}_1 \to \tilde{A}_2$ be an L-isomorphism preserving the Milnor form. It follows from Lemma 6.3 (2) that

$$(z\bar{z})^s\,\Phi(\operatorname{im}\nu_1) \subset \operatorname{im}\nu_2 \quad\text{and}\quad (z\bar{z})^s\,\Phi^{-1}(\operatorname{im}\nu_2) \subset \operatorname{im}\nu_1$$

for large enough s. We can define P-homomorphisms $\varphi: A_1 \to A_2$ and $\psi: A_2 \to A_1$:

$$\varphi(a) = \nu_2^{-1}[(z\bar{z})^s\,\Phi\nu_1(a)], \quad a \in A_1,$$

$$\psi(b) = \nu_1^{-1}[(z\bar{z})^s\,\Phi^{-1}\nu_2(b)], \quad b \in A_2.$$

Then, clearly, $\varphi\circ\psi$ and $\psi\circ\varphi$ are multiplication by $(z\bar{z})^{2s} \in P$ and if $a \in A_1$, $b \in A_2$, then

$$\langle a, \psi(b)\rangle = [\nu_1(a), \nu_1\psi(b)] = [\Phi\nu_1(a), \Phi\nu_1\psi(b)] =$$
$$= [(z\bar{z})^{-s}\nu_2\varphi(a), (z\bar{z})^{-s}\nu_2\varphi\psi(b)] = [\nu_2\varphi(a), \nu_2(b)] = \langle\varphi(a), b\rangle.$$

Thus, A_1 and A_2 are $2s$-adjoining, and our assertion follows from Theorem 5.1. ∎

Notes to §6. We recall Milnor's original geometric construction [43]. Let X be the complement of an open tubular neighbourhood of a knot (S^{n+2}, k) and $p: \tilde{X} \to X$ the infinite cyclic covering. As Milnor has shown [43], the vector space $H^{n+1}(\tilde{X}, \partial\tilde{X};\ \mathbf{Q})$ is one-dimensional, and for any i and j with $i+j = n+1$ the cup-product defines a non-degenerate pairing

$$H^i(\tilde{X}, \partial\tilde{X}; \mathbf{Q}) \otimes H^j(\tilde{X}; \mathbf{Q}) \to H^{n+1}(\tilde{X}, \partial\tilde{X}; \mathbf{Q}).$$

This pairing allows us to identify $H^i(\tilde{X}, \partial\tilde{X};\ \mathbf{Q})$ with $H_j(\tilde{X};\ \mathbf{Q}) \approx \mathrm{Hom}_{\mathbf{Q}}(H^j(\tilde{X};\ \mathbf{Q});\ \mathbf{Q})$. Similarly, $H^j(\tilde{X};\ \mathbf{Q})$ can be identified with $H_i(\tilde{X};\ \mathbf{Q})$, and the original pairing takes the form

$$H_j(\tilde{X}; \mathbf{Q}) \otimes H_i(\tilde{X}; \mathbf{Q}) \to \mathbf{Q}.$$

Composing with the canonical embedding of the integral in the rational homology, we obtain for $i = j = q$, $n = 2q-1$, the form

$$H_q\tilde{X} \otimes H_q\tilde{X} \to \mathbf{Q}.$$

It can be shown that to within the isomorphism $H_q\tilde{X} \approx L \otimes_P A$ given by Theorem 2.6 the latter form coincides with the form $[,]: \tilde{A} \otimes \tilde{A} \to \mathbf{Q}$ constructed algebraically in the text.

In the case of fibred knots the Milnor form is particularly simple to construct: it is the same as the intersection index pairing on the fibre and takes values in $\mathbf{Z} \subset \mathbf{Q}$. In this form it is often used in papers on the theory of singularities (see, for example, [1], [5]).

Trotter [53] has given another algebraic construction of the Milnor form (which he called the rational scalar form). As definition he used the connection with the Blanchfield form, which is presented in §7 below.

§7. The Blanchfield form

In this section we give an algebraic construction of the Blanchfield form, using the isometric structure of a knot. The trace function of Trotter [53] establishes a connection between the Milnor form and the Blanchfield form, and it turns out that these forms mutually determine each other. Combined with the results of the preceding section, this gives the theorem of Trotter–Kearton [53], [26], [27] that simple knots are equivalent if and only if their Blanchfield forms are isometric. This theorem completes our study of odd-dimensional simple knots.

7.1. Let A be an isometric structure of parity ε and let $\tilde{A} = L \otimes_P A$. Let $Q(L)$ be the field of fractions of L. In L and $Q(L)$ there is an involution $f \mapsto \bar{f}$, where $\bar{f}(z) = f(\bar{z})$. The L-module $Q(L)/L$ inherits the anti-homomorphism $^-: Q(L)/L \to Q(L)/L$.

There is an exact sequence of L-modules

$$0 \to L \otimes_{\mathbf{Z}} A \xrightarrow{d} L \otimes_{\mathbf{Z}} A \xrightarrow{e} \tilde{A} \to 0,$$

where $d(q \otimes a) = q \otimes za - zq \otimes a$, $e(q \otimes a) = q \otimes a$ for $q \in L$, $a \in A$.

For example, to check that d is a monomorphism, we note that each $w \in L \otimes_{\mathbf{Z}} A$ can be uniquely expressed in the form

$$w = \sum z^i\bar{z}^j \otimes a_{ij},$$

where $a_{ij} \in A$, i ranges over $\mathbf{Z}$, j ranges over the negative integers, and $a_{ij} \neq 0$ for only finitely many pairs (i, j). If $d(w) = 0$, then $za_{ij} = a_{i+1,j}$, hence, $a_{ij} = z^k a_{i-k,j}$ for any k. But $a_{i-k,j} = 0$ for sufficiently large k, and so $a_{ij} = 0$ and $w = 0$.

We define a pairing $|\,,\,|: (L \otimes_{\mathbf{Z}} A) \times (L \otimes_{\mathbf{Z}} A) \to L$ by the formula

$$|(1 \otimes a), (1 \otimes b)| = \langle a, b\rangle, \quad a, b \in A,$$

and by requiring additivity in each variable separately, L-linearity in the first argument:

$$|qx, y| = q\,|x, y|$$

and L-skew-linearity in the second argument:

$$|x, qy| = \bar{q}\,|x, y|,$$

$x, y \in L \otimes_{\mathbf{Z}} A$, $q \in L$. It is easy to check that then

$$|x, y| = \varepsilon\,\overline{|y, x|}, \quad |x, d(y)| = -\,|d(x), y|.$$

We can now define *the Blanchfield form*

$$\cdot: \tilde{A} \times \tilde{A} \to Q(L)/L.$$

Let $\alpha, \beta \in \tilde{A}$ and let $\alpha = e(x)$ and $\beta = e(y)$, where $x, y \in L \otimes_{\mathbf{Z}} A$. Let $\mu(z)$ be a non-zero element of L such that $\mu(z)\alpha = 0$. (Such an element always exists.) Then the product $\mu(z)x$, evaluated in $L \otimes_{\mathbf{Z}} A$ using the L-module structure of the factor L, lies in the image of d, and we put

$$\alpha\cdot\beta = |d^{-1}(\mu(z)x), y|/\mu(z) \pmod{L}.$$

It is easy to see that this is well-defined.

The Blanchfield form obviously is:

L-linear: $(q\alpha)\cdot\beta = q(\alpha\cdot\beta)$, $q \in L$,

$(-\varepsilon)$-Hermitian: $\alpha\cdot\beta = -\,\varepsilon\overline{\beta\cdot\alpha}$, $\alpha, \beta \in \tilde{A}$.

7.2. *The Blanchfield form and the Milnor form determine each other.* To prove this we need the trace function

$$\chi: Q(L)/L \to \mathbf{Q},$$

which acts as follows. According to the theory of partial fractions, every rational function $f \in Q(L) = \mathbf{Q}(z)$ can be expressed uniquely in the form $f = f_1 + f_2$, where $f_1 \in \mathbf{Q}[z, z^{-1}, \bar{z}^{-1}]$, and f_2 is a proper fraction whose numerator and denominator are rational polynomials, and whose denominator is divisible neither by z nor by $\bar{z}$. By definition, $\chi(f)$ is the negative of the coefficient of z^{-1} in the Laurent series of f_2 at infinity. For example, if $f = (u_{n-1}z^{n-1} + \ldots + u_0)/(z^n + v_{n-1}z^{n-1} + \ldots + v_0)$ and the denominator is prime to z and to $\bar{z}$, then $\chi(f) = -\,u_{n-1}$.

Example. If $f = 1/(z-2)z$, then $\chi(f) = -1/2$.

7.3. *Proposition.* *Let A be an isometric structure and $\widetilde{A} = L \otimes_P A$. Then $[\alpha, \beta] = \chi(\alpha\cdot\beta)$ for any $\alpha, \beta \in \widetilde{A}$.*

■ By Lemma 6.3, it is sufficient to prove the formula for $\alpha, \beta \in \nu(B)$, where $\nu: A \to \widetilde{A}$ is the map $\nu(a) = 1 \otimes a$, $a \in A$, and $B \subset A$ is the basic submodule. Let $\alpha = \nu(a)$, $\beta = \nu(b)$, where $a, b \in B$. Then $\alpha = e(1 \otimes a)$ and $\beta = e(1 \otimes b)$ (here we use the notation of 7.1). Let $\Delta(z) \in P$ be an integer polynomial with leading coefficient 1 such that $\Delta(0) \neq 0 \neq \Delta(1)$ and there exist integers $k, l \geqslant 0$ with $\Delta(z)z^k\bar{z}^l a = 0$ for all $a \in A$. (For $\Delta(z)$ we can take the characteristic polynomial of the endomorphism $A \to A$, $a \mapsto za$, divided by $z^k\bar{z}^l$ for suitable k and l.) We define polynomials $p_0(z), p_1(z), \ldots, p_{n-1}(z)$ as follows:

$$p_0(z) = (\Delta(z) - \Delta(0))/z,$$
$$p_i(z) = (p_{i-1}(z) - p_{i-1}(0))/z \quad (i = 1, 2, \ldots, n-1),$$

where $n = \deg \Delta$. Then, as is easy to verify,

$$d\Big(\sum_{i=0}^{n-1} z^i \otimes p_i(z)a\Big) = 1 \otimes \Delta(z)a - \Delta(z) \otimes a$$

and $e(1 \otimes \Delta(z)a) = e(z^{-k}\bar{z}^{-l} \otimes \Delta(z)z^k\bar{z}^l a) = 0$. Consequently, by Definition 7.1,

$$\alpha\cdot\beta = -\Big|\sum_{i=0}^{n-1} z^i \otimes p_i(z)a,\ 1 \otimes b\Big|/\Delta(z) = -\Big(\sum_{i=0}^{n-1} \langle p_i(z)a,\ b\rangle z^i\Big)/\Delta(z) \pmod{L}.$$

Hence $\gamma(\alpha\cdot\beta) = \langle p_{n-1}(z)a, b\rangle = \langle a, b\rangle = [\alpha, \beta]$, since $p_{n-1}(z) \equiv 1$. ■

Let $\mathscr{K} \subset Q(L)$ be the set of those rational functions $f \in Q(L)$ that can be represented in the form $f = p/q$, where p and q are integer polynomials in z, and q has the leading coefficient 1. Clearly, $L \subset \mathscr{K}$. The proof of Proposition 7.3 has the following corollary.

7.4. *Corollary.* *The Blanchfield form of any isometric structure takes values in $\mathscr{K}/L \subset Q(L)/L$.*

7.5. *Lemma.* *An element $f \in \mathscr{K}$ belongs to L if and only if $\chi(\lambda f) = 0$ for all $\lambda \in L$.*

■ Suppose that $f \in \mathscr{K}$ and $f = p(z)/q(z)z^k\bar{z}^l$, where $q(z)$ is prime to z and to $\bar{z}$ and has the leading coefficient 1. Let $p(z) = a(z)q(z) + r(z)$, where $\deg r < \deg q$, and let $s = \deg q - \deg r - 1$. Then

$$z^{k+s}\bar{z}^l f = z^s a(z) + r(z)\, z^s/q(z)$$

and $\chi(z^{k+s}\bar{z}^l f)$ is the negative of the leading coefficient of $r(z)$. It follows that if $\chi(\lambda f) = 0$ for all $\lambda \in L$, then $r(z) = 0$ and $f = a(z)/z^k\bar{z}^l \in L$. The converse is obvious. ■

It follows from Corollary 7.4 and Lemma 7.5 that an L-isomorphism $\widetilde{A}_1 \to \widetilde{A}_2$ preserves the Milnor form if and only if it preserves the Blanchfield form. This fact together with Theorems 3.1, 5.1, and 6.6 gives the following result.

7.6. *Theorem.* *Let $K_\nu = (S^{2q+1}, k_\nu^{2q-1})$ be two simple knots, where $q \geqslant 3$, $\nu = 1$ or 2, and let $H_q\widetilde{X}_\nu$ be their respective Alexander modules (see 2.1). The following conditions are equivalent:*

(a) *the knots K_1 and K_2 are equivalent;*

(b) *the isometric structure of some Seifert manifold of K_1 m-adjoins that of some Seifert manifold of K_2 for some $m \geqslant 0$;*

(c) *there is an L-isomorphism $H_q\widetilde{X}_1 \to H_q\widetilde{X}_2$ preserving the Milnor form;*

(d) *there is an L-isomorphism $H_q\widetilde{X}_1 \to H_q\widetilde{X}_2$ preserving the Blanchfield form.* ■

Notes to §7. As already mentioned, the trace function was introduced by Trotter [53], [54], using an idea of Milnor [44].

The equivalence of Theorem 7.6 (a) and (c) is the content of the Trotter–Kearton theorem, which is rightly regarded as an apex of knot theory. The first proof of this theorem was published by Trotter [53]; it was purely algebraic and was based on topological results of Levine [39]. Kearton [27], see also [26], gave a direct geometric proof.

A geometric construction of the Blanchfield form is in [20], [27], [35], [23].

A realization theorem for the Blanchfield pairing was proved by Trotter [54] and Levine [42]. It is not hard to obtain a similar theorem for the Milnor pairing $[\,,]: \widetilde{A} \times \widetilde{A} \to \mathbf{Q}$. To do this we have to remark that this pairing satisfies the following non-degeneracy condition: there is a finitely generated subgroup $A \subset \widetilde{A}$ that is a P-submodule and generates $\widetilde{A}$ over L; it has the property that the form $[\,,]$ takes integer values and is unimodular on A.

The paper of Trotter [53] also investigates in detail the relationship between S-equivalence and congruence of Seifert matrices, which in the language of the present paper corresponds to R-equivalence and isomorphism of isometric structures. These matters are also studied in [40], [18].

CHAPTER III

THE CLASSIFICATION OF EVEN-DIMENSIONAL SIMPLE KNOTS

The algebraic classification of odd-dimensional simple knots presented in the preceding chapter is essentially achieved in two steps. The first step is a reduction to algebra (§3), and the second is the solution of the algebraic problem (that is, the classification of isometric structures relative to R-equivalence; three different solutions are given in §5, 6, and 7).

By comparison, the reduction to algebra of the classification problem for even-dimensional simple knots is not so elementary, and cannot be achieved at the first step. To find a complete system of algebraic invariants of a simple even-dimensional knot, we start out from a general reduction to stable homotopy of the knot classification problem, which was obtained by the present author in [12], [21], and restated in [15]. Then, using the computational apparatus of homotopy theory, we single out the necessary invariants. Thus, the investigation of even-dimensional simple knots proceeds in four steps: (1) reduction to stable homotopy; (2) homotopy-theoretical calculations; (3) reduction to algebra; (4) solution of the algebraic problem.

A complete exposition of all this material would at least double the length of the paper.(1) Therefore, we confine ourselves to a sketch of the reduction to stable homotopy, revealing the essence of the problem under study, and to an exposition of the algebraic technique (steps (3) and (4)), which has not been published before. Details of the homotopy-theoretic calculations and some of the proofs existing in the literature are not given here, but are replaced by references, mainly to the papers [13] and [21].

§8. Reduction to stable homotopy

8.1. Let (S^{n+2}, k^n) be a knot and $V^{n+1} \subset S^{n+2}$ some Seifert manifold of it. Let $i_+, i_-: V \to S^{n+2} - V$ be the maps considered in 1.1. We define a map $h: SV \to S(S^{n+2} - V)$ by

$$h[v, t] = \begin{cases} [i_+(v), 2t] & \text{if } 0 \leqslant t \leqslant 1/2, \\ [i_-(v), 2-2t] & \text{if } 1/2 < t \leqslant 1, \end{cases}$$

where $v \in V$, $t \in [0, 1]$, and S denotes unreduced suspension. By what was said in 1.1, the map h induces isomorphisms of all homology groups. Moreover, both spaces involved are simply-connected, so that h is a homotopy equivalence. Hence, there exists a map $z: SV \to SV$, unique up to homotopy, such that $h \circ z$ is homotopic to Si_+. It is called the *excision map*.

The excision map acts on the homology of V, and this action coincides with multiplication by the generator of the ring P, which was denoted in §1 by z. Using the same letter to denote the excision map and the generator $z \in P$ is convenient and does not cause any confusion.

8.2. Let Y be the complement of an open tubular neighbourhood of V in S^{n+2}. We fix base points in V and Y and consider the canonical Spanier-Whitehead duality $v: V \wedge Y \to S^{n+1}$. Regarding $i_+, i_-: V \to Y$ as S-maps, we can form an S-map $u: V \wedge V \to S^{n+1}$ by putting $u = v \circ (1 \wedge (i_+ - i_-))$. It is called the *intersection form* on V.

To state properties of the S-maps u and z we use the following concept. A *stable isometric structure* of dimension n is a triple (X, u, z) consisting of a finite cell-complex X with a distinguished point, and two S-maps $u: X \wedge X \to S^{n+1}$ and $z: X \to X$ such that (a) u is a quality map; (b) $u' = (-1)^{n+1} u$; (c) $u \circ (1 \wedge z) = u \circ (\bar{z} \wedge 1)$. In (b) u' denotes the composite with u of the map $X \wedge X \to X \wedge X$ interchanging the factors. In (c) $\bar{z}$ denotes the S-map $1 - z: X \to X$. This notation is used in what follows.

Two stable isometric structures (X_ν, u_ν, z_ν) ($\nu = 1$ or 2) of the same dimension are said to be *isomorphic* if there is an S-equivalence $f: X_1 \to X_2$ such that $f \circ z_1 = z_2 \circ f$ and $u_2 \circ (f \wedge f) = u_1$.

(1) For a full account of the topics of this chapter, see [62].

8.3. ***Theorem.*** (1) *If $V^{n+1} \subset S^{n+2}$ is a Seifert manifold of a knot (S^{n+2}, k^n), then the collection (V, u, z), where u is the intersection form on V and z is the stable class of the excision map, is a stable isometric structure.* (2) *For $n \geqslant 4$ every stable isometric structure (X, u, z) of dimension n with an $[(n+2)/3]$-connected space X is isomorphic to the stable isometric structure of some smooth compact simply-connected oriented submanifold $V^{n+1} \subset S^{n+2}$ whose boundary is a homotopy sphere.*

■ That the intersection form is a duality map follows immediately from its definition and the fact that $i_+ - i_-$ is an S-equivalence (see 1.1). This proves 8.2, (a). For the proof of (b) we use the following relationships (see [21], 188):

$$v \circ (1 \wedge i_+) = (-1)^n v' \circ (i_- \wedge 1), \quad v \circ (1 \wedge i_-) = (-1)^n v' \circ (i_+ \wedge 1),$$

where the sign = denotes equality of S-maps. We have

$$u' = v' \circ ((i_+ - i_-) \wedge 1) = (-1)^n v \circ (1 \wedge i_-) - (-1)^n v \circ (1 \wedge i_+) = (-1)^{n+1} u.$$

For the proof of (c) we note that, as follows from the definition of the excision map in 8.1, there is an equality between S-maps $(i_+ - i_-) \circ z = i_+$. Hence, $u \circ (1 \wedge z) = v \circ (1 \wedge (i_+ - i_-) \circ z) = v \circ (1 \wedge i_+)$. Similarly, $(i_+ - i_-) \circ \bar{z} = - i_-$, and $u \circ (\bar{z} \wedge 1) = (-1)^{n+1} u' \circ (\bar{z} \wedge 1) = (-1)^n v' \circ (i_- \wedge 1) = v \circ (1 \wedge i_+)$, which proves (c), and with it also (1).

Let (X, u, z) be a stable isometric structure of dimension $n \geqslant 4$ with an $[(n+2)/3]$-connected X. Since u is a duality map, $H^i(X) \approx H_{n+1-i}(X) = 0$ for $i > n - r$, where $r = [(n+2)/3]$, hence, we may assume that X is $(n-r)$-dimensional. We denote $u \circ (1 \wedge z)$ by $\xi: X \wedge X \to S^{n+1}$. By Theorem 1.3 of [21], there exists a compact oriented submanifold $V^{n+1} \subset S^{n+2}$ with simply-connected boundary ∂V and a homotopy equivalence $g: V \to X$ such that $\xi \circ (g \wedge g)$ is homotopic to the homotopy Seifert pairing (see [21]) of V. Since $[\xi + (-1)^{n+1} \xi'] \circ (g \wedge g) = u \circ (g \wedge g)$ is a duality map, ∂V is a homology sphere, by Theorem 1.4 of [21]. It remains to prove that g induces an isomorphism between the stable isometric structure of V and (X, u, z). But this follows from the fact that the homotopy Seifert pairing θ_V and the stable isometric structure (V, u_V, z_V) uniquely determine each other. For if θ_V is given, then $u_V = \theta_V + (-1)^{n+1} \theta_V'$ and the S-class $z_V: V \to V$ of the excision map is determined by the relation $u_V \circ (1 \wedge z_V) = \theta_V$. This also permits us to define θ_V in terms of u_V and z_V. ■

The arguments just given and Theorem 1.2 of [21] lead to the following result.

8.4. ***Theorem.*** *Let $n \geqslant 5$ and let V^{n+1} and $W^{n+1} \subset S^{n+2}$ be two smooth compact $[(n+3)/3]$-connected oriented submanifolds whose boundaries are homotopy spheres. If the stable isometric structures of these manifolds are isomorphic, then there exists an isotopy of the sphere S^{n+2} that takes V to W with orientations preserved.* ■

We need the following analogue of Theorem 1.7.

8.5. ***Theorem.*** *Let V^{n+1} and $W^{n+1} \subset S^{n+2}$ be two smooth compact r-connected oriented submanifolds whose boundaries are homotopy spheres. If the oriented knots $(S^{n+2}, \partial V)$ and $(S^{n+2}, \partial W)$ are equivalent and $r \geqslant 2$, $n \geqslant 4$, then there is a finite chain of smooth compact r-connected $(n+1)$-dimensional oriented submanifolds $U_0, U_1, \ldots, U_N$ such that (a) $U_0 = V$; (b) $\operatorname{int} U_i \cap \operatorname{int} U_{i+1} = \varnothing$ for $i = 0, 1, \ldots, N-1$, $\partial U_i = \partial U_{i+1}$, and the orientations of U_i and U_{i+1} agree on ∂U_i; (c) U_N is ambient-isotopic to W with orientations preserved.*

■ A proof is given in [21], 201–207, although this theorem is not stated there explicitly. ■

8.6. Let $m \geqslant 0$ be an integer and let (X_ν, u_ν, z_ν) ($\nu = 1$ or 2) be two stable isometric structures of dimension n. We say they are *m-adjoining* (compare with Definition 1.4) if there exist S-maps $\varphi: X_1 \to X_2$ and $\psi: X_2 \to X_1$ such that

$$\varphi \circ z_1 = z_2 \circ \varphi, \quad \psi \circ z_2 = z_1 \circ \psi, \quad u_1 \circ (1_{X_1} \wedge \psi) = u_2 \circ (\varphi \wedge 1_{X_2}),$$
$$\varphi \circ \psi = (z_2 \circ \bar{z}_2)^m, \quad \psi \circ \varphi = (z_1 \circ \bar{z}_1)^m.$$

The first two equalities mean that φ and ψ commute with the excision map, the third expresses the fact that φ and ψ adjoin, (in particular, φ determines ψ and vice versa). Clearly, 0-adjoining is equivalent to isomorphism. If $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are three stable isometric structures with $\mathscr{A}$ m-adjoining $\mathscr{B}$ and $\mathscr{B}$ l-adjoining $\mathscr{C}$, then $\mathscr{A}$ $(m+l)$-adjoins $\mathscr{C}$.

Two stable isometric structures $\mathscr{A}$ and $\mathscr{B}$ are said to be *R-equivalent* if there is a finite sequence of stable isometric structures $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_N$ such that $\mathscr{A} = \mathscr{C}_0$, $\mathscr{B} = \mathscr{C}_N$, and $\mathscr{C}_i$ 1-adjoint $\mathscr{C}_{i+1}$ for all $i = 0, 1, \ldots, N-1$.

8.7. ***Theorem.*** *Suppose that V^{n+1} and $W^{n+1} \subset S^{n+2}$ are two smooth compact oriented submanifolds whose boundaries are homotopy spheres.*
(1) If $\operatorname{int} V \cap \operatorname{int} W = \varnothing$, $\partial V = \partial W$, and the orientations of V and W agree on ∂V, then the stable isometric structures of V and W are 1-adjoining.
(2) Conversely, if the stable isometric structures of V and W are 1-adjoining, V and W are $[(n+3)/3]$-connected, and $n \geqslant 5$, then there exists an isotopy of S^{n+2} taking V onto a submanifold $U \subset S^{n+2}$ such that $\operatorname{int} U \cap \operatorname{int} W = \varnothing$, $\partial U = \partial W$, and the orientations of U and W agree on ∂W.

The first statement of this theorem is a generalization of Proposition 1.8 and the second of Lemma 3.2. Theorem 8.7 can be derived from Theorems 2.4 and 2.5 of [21] and the following Lemma.

8.8. ***Lemma.*** *Let (X, u, z) be a stable isometric structure of dimension n and let $\theta = u \circ (1 \wedge z): X \wedge X \to S^{n+1}$ be the corresponding homotopy Seifert pairing. The condition $z^2 = z$ is equivalent to the existence of an S-equivalence $f: K \vee L \to X$ and a duality map $v: L \wedge K \to S^{n+1}$ such that $\theta \circ (f \wedge f)$ is defined by the matrix*

$$\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}.$$

■ Let $z^2 = z$. By a theorem of Freyd [58], the idempotent z splits, that is, there exist a complex K and S-maps $i_1: K \to X$, $\pi_1: X \to K$ with $\pi_1 i_1 = 1_K$, $i_1 \pi_1 = z$. Similarly, $\bar{z}^2 = \bar{z}$, and there exists a complex L and S-maps $i_2: L \to X$, $\pi_2: X \to L$ with $\pi_2 i_2 = 1_L$, $i_2 \pi_2 = \bar{z}$. Let $f: K \vee L \to X$ and $g: X \to K \vee L$ be induced by i_1, i_2 and π_1, π_2, respectively. Then f and g are mutually inverse S-equivalences. The relations $zi_1 = i_1$, $zi_2 = 0$, $u \circ (i_1 \wedge i_1) = 0$ and $u \circ (i_2 \wedge i_2) = 0$ imply that $\theta \circ (f \wedge f)$ is given by the matrix $\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}$, where $v = u \circ (i_2 \wedge i_1)$, and it remains only to show that v is a duality map. But this follows from the fact that the pairing $u \circ (f \wedge f)$, which is obviously a duality map, is given by the matrix

$$\begin{bmatrix} 0 & (-1)^{n+1} v' \\ v & 0 \end{bmatrix}$$

The converse is obvious. ■

8.9. *Proof of Theorem* 8.7. Let (V, u_V, z_V) and (W, u_W, z_W) be the stable isometric structures, and $\theta_V = u_V \circ (1_V \wedge z_V)$, $\theta_W = u_W \circ (1_W \wedge z_W)$ the homotopy Seifert pairings, corresponding to V and W, respectively. By Theorem 2.4 of [21], in the situation of (1) there exists a pairing $\alpha: V \wedge W \to S^{n+1}$ such that the pairing $\xi: (V \vee W) \wedge (V \vee W) \to S^{n+1}$ given by the matrix

$$\begin{bmatrix} \theta_V & \alpha \\ (-1)^n \alpha' & (-1)^n \theta'_W \end{bmatrix},$$

is congruent to a pairing of the form considered in Lemma 8.8. Let $(V \vee W, U, Z)$ be the stable isometric structure corresponding to ξ. Then, clearly, U and Z are given by the matrices

$$\begin{bmatrix} u_V & 0 \\ 0 & -u_W \end{bmatrix} \text{ and } \begin{bmatrix} z_V & \varphi \\ \psi & \bar{z}_W \end{bmatrix}$$

respectively, where the S-maps φ and ψ are determined by the relations $\alpha = u_V \circ (1_V \wedge \psi) = u_W \circ (\varphi \wedge 1_W)$. By Lemma 8.8, $Z^2 = Z$, and when we equate the corresponding entries, we obtain the following four relations $\varphi \circ z_V = z_W \circ \varphi$, $\psi \circ z_W = z_V \circ \psi$, $\varphi \circ \psi = z_W \circ \bar{z}_W$, $\psi \circ \varphi = z_V \circ \bar{z}_V$, which mean that (V, u_V, z_V) and (W, u_W, z_W) are 1-adjoining.

To prove (2) we have to carry out the arguments in the opposite direction and apply Theorem 2.5 of [21]. If (V, u_V, z_V) and (W, u_W, z_W) are 1-adjoining and $\varphi: V \to W$, $\psi: W \to V$ are the corresponding S-maps, then we can construct a stable isometric structure $(V \vee W, U, Z)$, by defining U and Z by matrices as above. Then $Z^2 = Z$ and we can use Lemma 8.8; then Theorem 2.5 of [21] gives the required isotopy. ■

Theorems 1.7 and 8.7 (1) lead to the following corollary.

8.10. ***Corollary.*** *The R-equivalence class of the stable isometric structure of any Seifert manifold is a knot invariant.* ■

8.11. An n-dimensional knot (S^{n+2}, k^n) is said to be *r-simple*, where r is some integer, if $\pi_i(S^{n+2} - k) \approx \pi_i(S^1)$ for $i \leqslant r$. A stable isometric structure (X, u, z) is said to be *r-connected* if X is r-connected. To r-connected stable isometric structures there correspond r-simple knots, by Theorem 0.5. An n-dimensional knot is said to be *stable* if it is $[(n+3)/3]$-simple and $n \geqslant 5$.

8.12. ***Lemma.*** *Let $I^\nu = (X_\nu, u_\nu, z_\nu)$ ($\nu = 1$ or 2) be two R-equivalent r-connected stable isometric structures of dimension n. If $r \geqslant [(n+2)/3] > 1$, then they can be linked by a finite chain of 1-adjoining r-connected stable isometric structures of dimension n.*

■ Given an s-connected stable isometric structure $I = (X, u, z)$ of dimension n and an integer N, we can form a new stable isometric structure $\sigma^{2N}I = (S^{2N}X, v, S^{2N}z)$, where v is defined as the composite

$$S^{2N}X \wedge S^{2N}X \underset{\approx}{\rightarrow} S^{2N} \wedge S^{2N} \wedge (X \wedge X) \xrightarrow{1\wedge 1 \wedge u} S^{2N} \wedge S^{2N} \wedge S^{n+1} = S^{n+4N+1}.$$

Clearly, $\sigma^{2N}I$ is $(s+2N)$-connected and of dimension $n+4N$. It follows that for sufficiently large N the stable isometric structure $\sigma^{2N}I$ satisfies the conditions of Theorem 8.3 (2).

Since I^1 and I^2 are R-equivalent, we can find a finite sequence of stable isometric structures $I_1, \ldots, I_m$ such that $I_1 = I^1$, $I_m = I^2$, and I_j 1-adjoins I_{j+1} for all $j = 1, \ldots, m-1$. For any N the sequence $\sigma^{2N}I_1, \ldots, \sigma^{2N}I_m$ also has the latter property, and if N is large enough, then by Theorem 8.3 (2), $\sigma^{2N}I_j$ can be realized by a Seifert manifold $V_j \subset S^{n+4N+2}$. By Theorem 8.7 (2), the knots $(S^{n+4N+2}, \partial V_1)$ and $(S^{n+4N+2}, \partial V_m)$ are equivalent. Applying Theorem 8.5 we see that there exists a finite chain of $(r+2N)$-connected stable isometric structures $J_1, \ldots, J_k$ of dimension $n+4N$ such that $J_1 = \sigma^{2N}I^1$, $J_k = \sigma^{2N}I^2$, and J_i 1-adjoins J_{i+1} for all $i = 1, \ldots, k-1$. By the suspension theorem (see [59], 458, and also exercise D1 on 461; this is where the condition $r \geqslant [(n+2)/3] > 1$ is used), the isometric structure J_i is isomorphic to $\sigma^{2N}I'_i$ for some r-connected stable isometric structure I'_i of dimension n, and all consecutive structures in the chain $I^1, I'_2, \ldots, I'_{k-1}, I^2$ are 1-adjoining. ■

8.13. ***Theorem.*** *For $r \geqslant [(n+3)/3]$ and $n \geqslant 5$ the map that associates with a knot the R-equivalence class of some r-connected Seifert manifold is a bijection from the set of r-simple n-dimensional knot types onto the set of R-equivalence classes of r-connected stable isometric structures of dimension n.*

■ By Theorem 0.5 and Corollary 8.10, the map of Theorem 8.13 is well-defined. That it is surjective follows from Theorem 8.3, and that it is injective from Lemma 8.12 and Theorems 8.3 (2) and 8.7 (2). ■

Notes to §8. The results of this section were announced in the author's paper [15], where the reduction to stable homotopy of [21] is restated in a form more convenient for the later computations. It is clear that Theorems 1.5 and 3.1 can be deduced from Corollary 8.10 and Theorem 8.13 (Theorem 3.1 is proved in §3 by the methods of [21], but without using the results of [21]).

§9. Reduction to algebra

As will become clear later, the homological invariants constructed in §1 are not sufficient to obtain an algebraic classification of even-dimensional simple knots. For this purpose we use additional invariants, whose construction differs from that in §1 only in that in place of the ordinary homology one of the generalized homology theories is used: the theory of stable homotopy groups. As a result, we can associate with each even-dimensional simple knot an algebraic object called a P-quintet, which consists of two P-modules equipped with bilinear forms and a given homomorphism between them. A relation of R-equivalence is then introduced on the collection of P-quintets, which resembles the relation of the same name between isometric structures (see §1) and plays an analogous role.

9.1. We begin by describing the necessary algebraic concepts. A **Z**-*quintet of parity* ε $(=\pm 1)$ is a collection (A, B, α, l, ψ) consisting of a pair of Abelian groups A and B, a homomorphism $\alpha\colon A \otimes \mathbf{Z}_2 \to B$ and a pair of forms $l\colon T(A) \otimes T(A) \to \mathbf{Q}/\mathbf{Z}$ (where $T(A)$ denotes $\mathrm{Tors}_{\mathbf{Z}} A$) and $\psi\colon B \otimes B \to \mathbf{Z}_4$ satisfying the following conditions:

(a) l and ψ are non-degenerate and ε-symmetric;

(b) there is an exact sequence

$$0 \to A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B \xrightarrow{\beta} \mathrm{Hom}(A; \mathbf{Z}_2) \to 0,$$

where[(1)] $\beta(b)(a) = \psi(b \otimes \alpha(\pi(a)))$ for $b \in B$, $a \in A$, and $\pi\colon A \to A \otimes \mathbf{Z}_2$ is the projection;

(c) the composite $B \xrightarrow{\gamma} A \xrightarrow{\pi} A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B$ is multiplication by 2, where $\gamma(b)$ for $b \in B$ belongs to $T(A)$ and is defined so that $\psi(b \otimes \alpha(\pi(a))) = l(\gamma(b) \otimes a)$ for all $a \in T(A)$.

A **Z**-quintet with a finitely generated group A is called *finitely generated.*

Note that (c) completely describes the extension type in (b) in the finitely generated case. It means, in particular, that the group B must contain a summand $\mathbf{Z}_4$ for every summand $\mathbf{Z}_2$ of A, and a summand $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ for every summand $\mathbf{Z}$ or $\mathbf{Z}_{2^k}$ with $k > 1$ in A.

A *morphism* from a **Z**-quintet (A, B, α, l, ψ) to a **Z**-quintet $(A_1, B_1, \alpha_1, l_1, \psi_1)$ is a collection of homomorphisms $(\eta, \xi, \hat\eta, \hat\xi)$, where $\eta\colon A \to A_1$, $\xi\colon B \to B_1$, $\hat\eta\colon A_1 \to A$, and $\hat\xi\colon B_1 \to B$, such that

$$l(\hat\eta(a) \otimes b) = l_1(a \otimes \eta(b)) \qquad \text{for } a \in T(A_1),\ b \in T(A),$$
$$\psi(\hat\xi(a) \otimes b) = \psi_1(a \otimes \xi(b)) \qquad \text{for } a \in B_1,\ b \in B,$$

(1) We assume fixed embeddings $\mathbf{Z}_2 \subset \mathbf{Z}_4 \subset \mathbf{Q}/\mathbf{Z}$.

and that the diagrams

$$\begin{array}{ccc} A\otimes \mathbf{Z}_2 & \xrightarrow{\eta\otimes 1} & A_1\otimes \mathbf{Z}_2 \\ \alpha\downarrow & & \downarrow\alpha_1 \\ B & \xrightarrow[\xi]{} & B_1 \end{array}, \qquad \begin{array}{ccc} A\otimes \mathbf{Z}_2 & \xleftarrow{\hat\eta\otimes 1} & A_1\otimes \mathbf{Z}_2 \\ \alpha\downarrow & & \downarrow\alpha_1 \\ B & \xleftarrow[\hat\xi]{} & B_1 \end{array}$$

commute.

A *P-quintet* is a finitely generated $\mathbf{Z}$-qunitet (A, B, α, l, ψ), where A and B are endowed with the structure of left P-modules and it is assumed that α is a P-homomorphism and in addition,

(d) $\quad l(x \otimes zy) = l(\bar{z}x \otimes y) \quad$ for $\quad x, y \in T(A)$,

(e) $\quad \psi(x \otimes zy) = \psi(\bar{z}x \otimes y) \quad$ for $x, y \in B$.

Here $\bar{z}$ denotes $1 - z \in P$, as before. A *morphism of P-quintets* is a morphism of $\mathbf{Z}$-quintets consisting of P-homomorphisms. Two P-quintets of the same parity are said to be *m-adjoining* ($m \geqslant 0$) if there exists a P-morphism $(\eta, \xi, \hat\eta, \hat\xi)$ from one of the P-quintets to the other such that all four composites $\eta\circ\hat\eta$, $\hat\eta\circ\eta$, $\xi\circ\hat\xi$, and $\hat\xi\circ\xi$ are multiplication by $(z\bar{z})^m \in P$. Two P-quintets are said to be *R-equivalent* if they can be linked by a chain of P-quintets in which every two consecutive P-quintets are 1-adjoining. In other words, R-equivalence is the equivalence relation generated by 1-adjoining.

9.2. Let $\sigma_i(X)$ denote the stable homotopy group of the space X, that is, $\sigma_i(X) = \lim\limits_{N\to\infty} \pi_{i+N}(S^N X)$. Let (S^{2q+2}, k^{2q}) be a simple knot and $V \subset S^{2q+2}$ some special (that is, $(q-1)$-connected) Seifert manifold of it. We consider the intersection form $u\colon V \wedge V \to S^{2q+1}$ and the S-class $z: V \to V$ of the excision map (see 8.1 and 8.2). The S-map z induces endomorphisms of the groups $A = \sigma_q V = H_q V$ and $B = \sigma_{q+2}(V)$, making them into P-modules. We construct a collection (A, B, α, l, ψ), specifying α to be composition with a non-trivial element of $\sigma_{q+2}(S^q)$ and defining l and ψ by means of the S-map u, as indicated in [12], 109–110.

9.3. ***Theorem.*** (1) *The collection (A, B, α, l, ψ) thus constructed is a P-quintet of parity $(-1)^{q+1}$;* (2) *its R-equivalence class does not depend on the choice of the Seivert manifold V, but is determined by the type of the knot (S^{2q+2}, k^{2q});* (3) *the resulting map from the set of types of simple knot (S^{2q+2}, k^{2q}) to the set of R-equivalence classes of 2q-dimensional knots in S^{2q+2} to the set of R-equivalence classes of P-quintets of parity $(-1)^{q+1}$ is a bijection for $q > 3$.*

9.4. In the proof (see 9.6) we need the following auxiliary notion. Let $q > 0$ be a fixed number. A *dualized space* is a pair (X, u) consisting of a $(q-1)$-connected cell complex X with a distinguished base point and an S-duality $u\colon X \wedge X \to S^{2q+1}$ such that $u' = u\circ\gamma$ is stably homotopic to $-u$, where $\gamma\colon X\wedge X \to X \wedge X$ is the map that interchanges the factors.

With each dualized space we associate a collection (A, B, α, l, ψ) where $A = H_q X = \sigma_q X$ and $B = \sigma_{q+2}(X)$ are groups, and $\alpha\colon A\otimes\mathbf{Z}_2 \to B$, $l\colon T(A)\otimes T(A) \to \mathbf{Q}/\mathbf{Z}$, and $\psi\colon B\otimes B \to \mathbf{Z}_4$ are homomorphisms defined as in 9.2.

If (X, u) and (Y, v) are two dualized spaces, then every S-map $f: X \to Y$ determines a dual S-map $\hat f: Y \to X$ such that $u\circ(1_X \wedge \hat f) \sim v\circ(f \wedge 1_Y)$. We denote the induced homomorphisms by $H(f)\colon H_q X \to H_q Y$ and $\sigma(f)\colon \sigma_{q+2}(X) \to \sigma_{q+2}(Y)$. As a result we can associate with the S-map f the collection of four homomorphisms $(H(f), \sigma(f), H(\hat f), \sigma(\hat f))$.

9.5. ***Theorem.*** (1) *For every dualized space (X, u) the collection (A, B, α, l, ψ) constructed in 9.4 is a $\mathbf{Z}$-quintet of parity $(-1)^{q+1}$;* (2) *every $\mathbf{Z}$-quintet of parity $(-1)^{q+1}$ can be realized in this way;* (3) *for any two dualized spaces and any S-map $f: X \to Y$ the collection $(H(f), \sigma(f), H(\hat f), \sigma(\hat f))$ is a morphism between the corresponding $\mathbf{Z}$-quintets;* (4) *an S-map f is stably homotopic to zero if and only if it induces the zero morphism of $\mathbf{Z}$-quintets;* (5) *every morphism from a $\mathbf{Z}$-quintet of (X, u) to a $\mathbf{Z}$-quintet of (Y, v) can be realized by some S-map.*

■ For $q \geqslant 3$ the proof can be obtained from the results of [13], §§6, 7, by means of arguments analogous to those given in [13], 8.2–8.6. When $q = 1$ or 2 we must use the same arguments, after first suspending the space in question a suitable number of times. ■

9.6. The proof of Theorem 9.3 is easily obtained from the results of §8 and from Theorem 9.5. For if $V^{2q+1} \subset S^{2q+2}$ is a special Seifert manifold, and $u\colon V\wedge V \to S^{2q+1}$ and $z: V \to V$ are the intersection form and the S-class of the excision map (see 8.1 and 8.2), then the pair (V, u) is a dualized space and the S-map z defines an endomorphism of the $\mathbf{Z}$-quintet corresponding to it. Now 9.3 (1) follows. 9.3 (2) can be deduced from Corollary 8.10, Lemma 8.12, and Theorem 9.5.

Every P-quintet (A, B, α, l, ψ) can be regarded as a $\mathbf{Z}$-quintet equipped with a morphism $(\eta, \xi, \hat\eta, \hat\xi)$ into itself, where η and ξ are multiplication by $z \in P$ and $\hat\eta$ and $\hat\xi$ multiplication by $\bar z \in P$. By Theorem 9.5 (2) and (5) this $\mathbf{Z}$-quintet and its morphism can be realized by a dualized space (X, u) and an S-map $z: X \to X$, respectively, where it follows from 9.5 (4) that $\hat z = 1 - z$, hence, (X, u, z) is a $(q-1)$-connected stable isometric structure of dimension $2q$. Thus, to every P-quintet of parity $(-1)^{q+1}$ there corresponds some $(q-1)$-connected stable isometric structure of dimension $2q$, and vice versa. P-quintets are R-equivalent if and only if the corresponding stable isometric structures are R-equivalent. Now 9.3 (3) follows from Theorem 8.13 and Lemma 8.12. ■

Notes to §9. Kearton [28] proposed an algebraic classification of even-dimensional simple knots based on quite different ideas; he found a way of connecting this problem with an analogous problem for odd-dimensional knots, which had previously been solved by Levine [39]. However, in [28] a classification is obtained only for even-dimensional simple knots that admit Seifert manifolds without 2-torsion in homology. Moreover, our relation of R-equivalence is more algebraic than Kearton's

T-equivalence. Owing to this fact, our Theorem 9.3 allows us to obtain an answer in terms of invariants (see § 10), which is a completely new result.

Individual classes of fibred knots were classified by Kojima [30] and [31]. The general case was studied by the author [13] and [14].

§ 10. Criteria for R-equivalence of P-quintets

In this section we indicate two criteria for R-equivalence of P-quintets. The first is analogous to Theorem 5.1:

10.1. *Theorem.* *Two P-quintets are R-equivalent if and only if they are m-adjoining for some* $m \geqslant 0$.

The second criterion (see Theorem 10.11) plays a role analogous to the results of §6–7 on the Milnor and Blanchfield forms.

To prove Theorem 10.1 we begin by simplifying the P-quintets in question, replacing them by R-equivalent ones. The first step is minimization.

We will say that a P-quintet (A, B, α, l, ψ) is *minimal* if the P-module A is minimal (see §4).

10.2. *Proposition.* *Every P-quintet is R-equivalent to some minimal one.*

■ The proof is analogous to that of Theorem 4.1. But we can also proceed differently: using arguments of Levine [37], it is easy to show that every simple knot of dimension $n \geqslant 4$ has a special Seifert manifold V with the property that the maps i_+ and i_- both induce monomorphisms $H_qV \to H_q(S^{n+2} - V)$, where $q = [(n+1)/2]$: to this manifold there corresponds a minimal P-quintet, by Theorem 9.3. ■(1)

10.3. Let A be a finite P-module. For sufficiently large n the subgroup $(z\bar{z})^nA$, which we denote by $(A)_0$, does not depend on n. We consider the subgroups

$$(A)_+ = \{a \in A;\ \exists n \geqslant 0,\ z^n a = 0\},\quad (A)_- = \{a \in A;\ \exists n \geqslant 0,\ \bar{z}^n a = 0\}.$$

Clearly $(A)_0$, $(A)_+$, and $(A)_-$ are submodules, and every P-homomorphism $f: A \to B$ of finite P-modules maps $(A)_0$ into $(B)_0$, $(A)_+$ into $(B)_+$, and so on. We denote the corresponding restrictions by f_0, f_+, and f_-. Thus we obtain three functors on the category of finite P-modules. It is easy to see that *they are all exact.* (Note that the functor $(\ )_0$ is not defined on the category of all P-modules, whereas the functors $(\ )_+$ and $(\ )_-$ are defined, but are only left exact.)

The proofs of the following Lemmas are not complicated, and are, therefore, omitted.

10.4. *Lemma.* *Every finite P-module A is isomorphic to the direct sum* $(A)_0 \oplus (A)_+ \oplus (A)_-$, *where* $(A)_0$ *is isomorphic to* $L \otimes_P A$. ■

(1)Proposition 10.2 also follows from a general theorem in [63] on the existence of minimal Seifert manifolds.

10.5. *Lemma.* *Let A be a P-module that is finitely generated over* $\mathbf{Z}$, *and let* $\tilde{A} = L \otimes_P A$. *Let* $\nu: A \to \tilde{A}$ *be the homomorphism acting by* $\nu(a) = 1 \otimes a$, $a \in A$. *Then the kernel of the restriction of* ν *to* $T(A)$ *is the same as* $(T(A))_+ + (T(A))_-$, *and the restriction of* ν *to* $(T(A))_0$ *is an isomorphism* $(T(A))_0 \to T(\tilde{A})$. ■

10.6. We call a P-quintet (A, B, α, l, ψ) *splitting* if the exact sequence of P-modules(1) $0 \to (A \otimes \mathbf{Z}_2)_+ \xrightarrow{\alpha_+} (B)_+ \xrightarrow{\beta_+} (\mathrm{Hom}(A; \mathbf{Z}_2))_+ \to 0$ obtained by applying the functor $(\ )_+$ to the exact sequence of 9.1 (b) splits. Then the analogous "minus-sequence" also splits. As a supplement to Proposition 10.2 we establish the following result.

10.7. *Proposition.* *Every P-quintet is R-equivalent to some minimal splitting P-quintet.*

■ Let $\mathfrak{S} = (A, B, \alpha, l, \psi)$ be a P-quintet. By Proposition 10.2 we may assume that it is minimal. We construct a sequence of P-quintets $\mathfrak{S}_0 = \mathfrak{S}, \mathfrak{S}_1, \ldots$ in which all the $\mathfrak{S}_i$ are minimal and $\mathfrak{S}_i$ is splitting for large enough i.

We define the first quintet $\mathfrak{S}_1 = (A_1, B_1, \alpha_1, l_1, \psi_1)$ as follows. We put $A_1 = A$ and for $a, b \in T(A_1)$ we define $l_1(a \otimes b) = l((z\bar{z})^{-1}a \otimes b)$. Since the module A is minimal, so is $T(A_1)$, hence, l_1 is well-defined. We define the following P-modules:

$$B^1_+ = \{(b, f) \in (B)_+ \times (\mathrm{Hom}(A; \mathbf{Z}_2))_+;\ \beta(b) = z\bar{z}f\},$$

$$N_1 = \{(\alpha(a), -z\bar{z}a);\ a \in (A \otimes \mathbf{Z}_2)_-\} \subset (B)_- \oplus (A \otimes \mathbf{Z}_2)_-,$$

$$B^1_- = ((B)_- \oplus (A \otimes \mathbf{Z}_2)_-)/N_1,\quad B_1 = (B)_0 \oplus B^1_+ \oplus B^1_-.$$

We specify the homomorphism $\alpha_1: A_1 \otimes \mathbf{Z}_2 \to B_1$ as the direct sum of the following three homomorphisms

$$(\alpha_1)_0:\ (A_1 \otimes \mathbf{Z}_2)_0 \to (B)_0,\quad a \mapsto \alpha(a),$$

$$(\alpha_1)_+:\ (A_1 \otimes \mathbf{Z}_2)_+ \to (B_1)_+ = B^1_+,\quad a \mapsto (\alpha(a), 0),$$

$$(\alpha_1)_-:\ (A_1 \otimes \mathbf{Z}_2)_- \to (B_1)_- = B^1_-,\quad a \mapsto (0, a) + N_1.$$

We define the pairing $\psi_1: B_1 \otimes B_1 \to \mathbf{Z}_4$ by the matrix

$$\begin{bmatrix} \varkappa & 0 & 0 \\ 0 & 0 & \sigma \\ 0 & \varepsilon\sigma' & 0 \end{bmatrix},$$

where $\varkappa: (B)_0 \otimes (B)_0 \to \mathbf{Z}_4$ acts according to the formula $\varkappa(a \otimes b) = \psi((z\bar{z})^{-1}a \otimes b)$, $a, b \in (B)_0$, and $\sigma: B^1_+ \otimes B^1_- \to \mathbf{Z}_4$ according to the formula

$$\sigma((b, f) \otimes ((b', c) + N_1)) = \psi(b \otimes b') + f(c),$$

where $(b, f) \in B^1_+$, $((b', c) + N_1) \in B^1_-$, and ε is the parity of the original quintet.

(1)The P-module structure on $\mathrm{Hom}(A; \mathbf{Z}_2)$ is given by $(zf)(a) = f(\bar{z}a)$ for $f \in \mathrm{Hom}(A; \mathbf{Z}_2)$, $a \in A$; then β is a P-homomorphism.

We have obtained a collection $\mathfrak{G}_1 = (A_1, B_1, \alpha_1, l_1, \psi_1)$. and a simple verification shows that it is a P-quintet of parity ε. To show that it 1-adjoins $\mathfrak{G}$, we define the following homomorphisms:

$$\begin{array}{ll}
\eta\colon A \to A_1, & a \mapsto z\bar{z}a,\\
\hat{\eta}\colon A_1 \to A, & a \mapsto a,\\
\xi_0\colon (B)_0 \to (B_1)_0, & b \mapsto z\bar{z}b,\\
\xi_+\colon (B)_+ \to (B_1)_+, & b \mapsto (z\bar{z}b,\ \beta(b)),\\
\xi_-\colon (B)_- \to (B_1)_-, & b \mapsto (b,\ 0) + N_1,\\
\hat{\xi}_0\colon (B_1)_0 \to (B)_0, & b \mapsto b,\\
\hat{\xi}_+\colon (B_1)_+ \to (B)_+, & (b,\ f) \mapsto b,\\
\hat{\xi}_-\colon (B_1)_- \to (B)_-, & ((b,\ a) + N_1) \mapsto \alpha(a) + z\bar{z}b,
\end{array}$$

and take $\xi\colon B \to B_1$ to be the direct sum $\xi_0 \oplus \xi_+ \oplus \xi_-$, and $\hat{\xi}\colon B_1 \to B$ to be the direct sum $\hat{\xi}_0 \oplus \hat{\xi}_+ \oplus \hat{\xi}_-$. Then the collection $(\eta, \xi, \hat{\eta}, \hat{\xi})$ is easily seen to be a morphism from $\mathfrak{G}$ to $\mathfrak{G}_1$ and to satisfy the definition of 1-adjoining (see 9.1).

Let $\mathfrak{G}_2$ be the P-quintet obtained by applying to $\mathfrak{G}_1$ the same construction used to get $\mathfrak{G}_1$ from $\mathfrak{G}$. In general, let $\mathfrak{G}_i$ be obtained by applying this construction to $\mathfrak{G}_{i-1}$. Then the P-quintets $\mathfrak{G}_i$ are all minimal and R-equivalent to the original one. We claim that from some point onwards these P-quintets are all splitting. For, if $\mathfrak{G}_i = (A_i, B_i, \alpha_i, l_i, \psi_i)$, then B_i can be identified with the direct sum $(B)_0 \oplus B^i_+ \oplus B^i_-$, where

$$B^i_+ = \{(b,\ f) \in (B)_+ \times (\mathrm{Hom}(A;\ \mathbf{Z}_2))_+;\ \beta(b) = (z\bar{z})^i f\},$$

$$N_i = \{(\alpha(a),\ -(z\bar{z})^i a);\ a \in (A \otimes \mathbf{Z}_2)_-\},$$

$$B^i_- = ((B)_- \oplus (A \otimes \mathbf{Z}_2)_-)/N_i.$$

It follows that if i is large enough so that $(z\bar{z})^i\ (A \otimes \mathbf{Z}_2)_- = 0$, then $\mathfrak{G}_i$ is splitting. ■

10.8. *Proof of Theorem* 10.1. It is clear that R-equivalence implies m-adjoining, and we have to prove the converse. Let $\mathfrak{G}_i = (A_i, B_i, \alpha_i, l_i, \psi_i)$ $(i = 1$ or $2)$ be two P-quintets of parity ε and let $(\eta, \xi, \hat{\eta}, \hat{\xi})$ be a P-morphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$ such that the composites $\eta \circ \hat{\eta}$, $\hat{\eta} \circ \eta$, $\xi \circ \hat{\xi}$, $\hat{\xi} \circ \xi$ all are multiplication by $(z\bar{z})^m$. By Proposition 10.7 we may assume that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are minimal and splitting. The theorem will be proved if we can construct a minimal splitting P-quintet $\mathfrak{G}_3$ that 1-adjoins $\mathfrak{G}_1$ and $(m-1)$-adjoins $\mathfrak{G}_2$.

We begin the construction of the quintet $\mathfrak{G}_3 = (A_3, B_3, \alpha_3, l_3, \psi_3)$. Let A_3 be the set of those $a \in A_2$ for which $z\bar{z}a \in \operatorname{im} \eta$. It follows from the minimality of A_2 that $T(A_3) = T(A_2)$, consequently, we can define a form

$l_3\colon T(A_3) \otimes T(A_3) \to \mathbf{Q}/\mathbf{Z}$ by putting $l_3(x \otimes y) = l_2(x \otimes (z\bar{z})^{1-m}y)$ for $x, y \in T(A_3)$. We specify the module B_3 in the form of the direct sum:

$$B_3 = (B_2)_0 \oplus (A_3 \otimes \mathbf{Z}_2)_+ \oplus (A_3 \otimes \mathbf{Z}_2)_- \oplus (\mathrm{Hom}(A_3;\ \mathbf{Z}_2))_+ \oplus \oplus (\mathrm{Hom}(A_3;\ \mathbf{Z}_2))_-,$$

and the momomorphism $\alpha_3\colon A_3 \otimes \mathbf{Z}_2 \to B_3$ as the direct sum of the homomorphisms:

$$(\alpha_3)_0\colon\ (A_3 \otimes \mathbf{Z}_2)_0 \xrightarrow{(i\otimes 1)_0} (A_2 \otimes \mathbf{Z}_2)_0 \xrightarrow{(\alpha_2)_0} (B_2)_0,$$
$$(\alpha_3)_+\colon\ (A_3 \otimes \mathbf{Z}_2)_+ \to B_3,$$
$$(\alpha_3)_-\colon\ (A_3 \otimes \mathbf{Z}_2)_- \to B_3.$$

Here $i\colon A_3 \to A_2$ and $(\alpha_3)_+$, $(\alpha_3)_-$ are the embeddings. We define the form $\psi_3\colon B_3 \otimes B_3 \to \mathbf{Z}_4$ by the matrix

$$\begin{bmatrix}
\varkappa & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \varepsilon\delta\\
0 & 0 & 0 & \varepsilon\sigma & 0\\
0 & 0 & \sigma' & 0 & 0\\
0 & \delta' & 0 & 0 & 0
\end{bmatrix}$$

where $\varkappa(x \otimes y) = \psi_2(x \otimes (z\bar{z})^{1-m}y)$, $x, y \in (B_2)_0$, and σ and δ are the restrictions of the canonical evaluation pairing $(A_3 \otimes \mathbf{Z}_2) \otimes \mathrm{Hom}(A_3;\ \mathbf{Z}_2) \to \to \mathbf{Z}_2 \subset \mathbf{Z}_4$. We omit the verification that $\mathfrak{G}_3 = (A_3, B_3, \alpha_3, l_3, \psi_3)$ is a P-quintet.

To prove that $\mathfrak{G}_3$ 1-adjoins $\mathfrak{G}_1$, we construct a morphism $(\eta_1, \xi_1, \hat{\eta}_1, \hat{\xi}_1)$, defining $\eta_1\colon A_1 \to A_3$ and $\hat{\eta}_1\colon A_3 \to A_1$ by the formulae $\eta_1(a) = \eta(a)$, $a \in A_1$, and $\hat{\eta}_1(b) = \eta^{-1}(z\bar{z}b)$, $b \in A_3$, and taking as $\xi_1\colon B_1 \to B_3$ and $\hat{\xi}_1\colon B_3 \to B_1$ any suitable extensions of the homomorphisms $(\xi)_0\colon\ (B_1)_0 \to (B_3)_0 = (B_2)_0$ and $(B_3)_0 \to (B_1)_0$, $x \mapsto \xi^{-1}(z\bar{z}x)$, respectively (these can be constructed using the fact that quintets can be splitting.

We now define a morphism $(\eta_2, \xi_2, \hat{\eta}_2, \hat{\xi}_2)$ from $\mathfrak{G}_3$ to $\mathfrak{G}_2$. Let $\eta_2\colon A_3 \to A_2$ be the embedding, and let $\hat{\eta}_2\colon A_2 \to A_3$ act as follows: $\hat{\eta}_2(a) = (z\bar{z})^{m-1}a$, $a \in A_2$. Let $(\xi_2)_0\colon (B_3)_0 \to (B_2)_0$ be the identity map, and let $(\hat{\xi}_2)_0\colon (B_2)_0 \to (B_3)_0$ be multiplication by $(z\bar{z})^{m-1}$. As above, using the splitting property of the quintets, we can construct extensions ξ_2 and $\hat{\xi}_2$ such that the collection $(\eta_2, \xi_2, \hat{\eta}_2, \hat{\xi}_2)$ is a P-homomorphism from $\mathfrak{G}_3$ to $\mathfrak{G}_2$, and all four homomorphisms $\eta_2 \circ \hat{\eta}_2$, $\hat{\eta}_2 \circ \eta_2$, $\xi_2 \circ \hat{\xi}_2$, and $\hat{\xi}_2 \circ \xi_2$ are multiplication by $(z\bar{z})^{m-1}$.

The proof of the theorem is now completed by induction. ■

10.9. Let $\Lambda = \mathbf{Z}[t,\ t^{-1}]$ be the group ring of the infinite cyclic group. A Λ-*quintet* is a $\mathbf{Z}$-quintet (A, B, α, l, ψ) in which the groups A and B are additionally equipped with Λ-module structures such that (a) A and B are finitely generated over Λ and the multiplications by $1 - t \in \Lambda$ are automorphisms of the modules A and B; (b) α is a Λ-homomorphism; (c) the automorphisms of A and B defined by multiplication by $t \in \Lambda$ are isometries of the forms l and ψ. Two Λ-quintets $(A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu)$ ($\nu = 1$ or 2) are called *isomorphic* if there are Λ-isomorphisms $\eta: A_1 \to A_2$ and $\xi: B_1 \to B_2$ such that the collection $(\eta, \xi, \eta^{-1}, \xi^{-1})$ is a morphism of **Z**-quintets (see 9.1).

10.10. With each P-quintet $\mathfrak{S} = (A, B, \alpha, l, \psi)$ we now associate a Λ-quintet $\widetilde{\mathfrak{S}} = (\widetilde{A}, \widetilde{B}, \widetilde{\alpha}, \widetilde{l}, \widetilde{\psi})$. We put $\widetilde{A} = L \otimes_P A$ and $\widetilde{B} = L \otimes_P B$ (we recall that $L = \mathbf{Z}[t, t^{-1}, (1-t)^{-1}]$and that P is embedded in L by identifying z with $(1-t)^{-1}$; Λ is naturally embedded in L, and so $\widetilde{A}$ and $\widetilde{B}$ are Λ-modules). We define $\widetilde{\alpha}$ as the composite

$$\widetilde{A} \otimes \mathbf{Z}_2 = (L \otimes_P A) \otimes_{\mathbf{Z}} \mathbf{Z}_2 \approx L \otimes_P (A \otimes \mathbf{Z}_2) \xrightarrow{1 \otimes \alpha} L \otimes_P B = \widetilde{B}.$$

By Lemma 10.5, there are natural isomorphisms $(T(A))_0 \to T(\widetilde{A})$ and $(B)_0 \to \widetilde{B}$, which we denote by $\varkappa$ and λ, respectively. By means of them we can define $\widetilde{l}: T(\widetilde{A}) \otimes T(\widetilde{A}) \to \mathbf{Q}/\mathbf{Z}$ and $\widetilde{\psi}: \widetilde{B} \otimes \widetilde{B} \to \mathbf{Z}_4$ as follows: $\widetilde{l}(x \otimes y) = l(\varkappa^{-1}(x) \otimes \varkappa^{-1}(y)), x, y \in T(\widetilde{A})$ and $\widetilde{\psi}(a \otimes b) = \psi(\lambda^{-1}(a) \otimes \lambda^{-1}(b))$, $a, b \in \widetilde{B}$. An easy verification shows that the collection $(\widetilde{A}, \widetilde{B}, \widetilde{\alpha}, \widetilde{l}, \widetilde{\psi})$ is indeed a Λ-quintet. We denote by $\widetilde{\mathfrak{S}} = L \otimes_P \mathfrak{S}$ the fact that the Λ-quintet $\widetilde{\mathfrak{S}}$ is obtained by this construction from the P-quintet $\mathfrak{S}$.

Now we can state the second criterion for R-equivalence of P-quintets.

10.11. *Theorem.* *Two P-quintets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are R-equivalent if and only if the Λ-quintets $\widetilde{\mathfrak{S}}_1 = L \otimes_P \mathfrak{S}_1$ and $\widetilde{\mathfrak{S}}_2 = L \otimes_P \mathfrak{S}_2$ are isomorphic.*

■ Let $(\eta, \xi, \hat{\eta}, \hat{\xi})$ be a morphism from a P-quintet $\mathfrak{S}_1 = (A_1, B_1, \alpha_1, l_1, \psi_1)$ to a P-quintet $\mathfrak{S}_2 = (A_2, B_2, \alpha_2, l_2, \psi_2)$, satisfying the condition for 1-adjoining (see 9.1), that is, all four composites $\eta \circ \hat{\eta}$, $\hat{\eta} \circ \eta$, $\xi \circ \hat{\xi}$, $\hat{\xi} \circ \xi$ are multiplication by $z\bar{z} \in P$. We define Λ-homomorphisms $\Phi: \widetilde{A}_1 \to \widetilde{A}_2$ and $\Xi: \widetilde{B}_1 \to \widetilde{B}_2$ by the formulae

$$\Phi(1 \otimes a) = z^{-1} \otimes \eta(a), \quad \Xi(1 \otimes b) = z^{-1} \otimes \xi(b),$$

where $a \in A_1$, $b \in B_1$. Similarly, we define $\hat{\Phi}: \widetilde{A}_2 \to \widetilde{A}_1$ and $\hat{\Xi}: \widetilde{B}_2 \to \widetilde{B}_1$ by the formulae

$$\hat{\Phi}(1 \otimes a) = \bar{z}^{-1} \otimes \hat{\eta}(a), \quad \hat{\Xi}(1 \otimes b) = \bar{z}^{-1} \otimes \hat{\xi}(b),$$

where $a \in A_2$, $b \in B_2$. Then Φ and $\hat{\Phi}$ are mutually inverse Λ-homomorphisms, and similarly for Ξ and $\hat{\Xi}$. It is easy to see that Φ and Ξ determine an isomorphism between the Λ-quintets $\widetilde{\mathfrak{S}}_1$ and $\widetilde{\mathfrak{S}}_2$. Consequently, the Λ-quintets corresponding to R-equivalent P-quintets are isomorphic.

We now prove the converse. Suppose that $\widetilde{\mathfrak{S}}_1$ and $\widetilde{\mathfrak{S}}_2$ are isomorphic and $\Phi: \widetilde{A}_1 \to \widetilde{A}_2$ and $\Xi: \widetilde{B}_1 \to \widetilde{B}_2$ are the respective isomorphisms. By Proposition 10.2 and what has just been proved, we may assume that A_1 and A_2 are minimal. In this case, by Lemma 2.5, the homomorphisms $\mu_\nu: A_\nu \to \widetilde{A}_\nu = L \otimes_P A_\nu$ taking $a \in A_\nu$ to $1 \otimes a$ are monomorphisms ($\nu = 1$ or 2). As in the proof of Theorem 6.6, there is an integer $n \geqslant 0$ such that

$$\Phi((z\bar{z})^n \operatorname{im} \mu_1) \subset \operatorname{im} \mu_2, \quad (z\bar{z})^n \operatorname{im} \mu_2 \subset \Phi(\operatorname{im} \mu_1).$$

We may assume, in addition, that n is large enough so that

$$z^n(B_\nu)_+ = 0 = \bar{z}^n(B_\nu)_-,$$

$$z^n(A_\nu \otimes \mathbf{Z}_2)_+ = 0 = \bar{z}^n(A_\nu \otimes \mathbf{Z}_2)_-,$$

where $\nu = 1$ or 2. We define P-homomorphisms $\eta: A_1 \to A_2$ and $\hat{\eta}: A_2 \to A_1$ by the formulae

$$\eta(a) = \mu_2^{-1} \circ \Phi \circ \mu_1((z\bar{z})^n a), \qquad a \in A_1,$$

$$\hat{\eta}(a) = \mu_1^{-1} \circ \Phi^{-1} \circ \mu_2((z\bar{z})^n a), \qquad a \in A_2.$$

Similarly, let λ_ν: $(B_\nu)_0 \to \widetilde{B}_\nu$ be the isomorphism given by Lemma 10.5. We denote by $\bar{\lambda}_\nu$: $B_\nu \to \widetilde{B}_\nu$ the homomorphism that is equal to λ_ν on $(B_\nu)_0$ and to 0 on $(B_\nu)_+$ and on $(B_\nu)_-$. We define $\xi: B_1 \to B_2$ and $\hat{\xi}: B_2 \to B_1$ by putting

$$\xi(b) = (z\bar{z})^n \lambda_2^{-1} \circ \Xi \circ \bar{\lambda}_1(b), \qquad b \in B_1,$$

$$\hat{\xi}(b) = (z\bar{z})^n \lambda_1^{-1} \circ \Xi^{-1} \circ \bar{\lambda}_2(b), \qquad b \in B_2.$$

Then the collection $(\eta, \xi, \hat{\eta}, \hat{\xi})$ makes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ $2n$-adjoining. By Theorem 10.1, $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are R-equivalent. ■

As a supplement to Theorem 10.11 we prove the following theorem.

10.12. *Theorem.* *Every Λ-quintet is isomorphic to $L \otimes_P \mathfrak{S}$ for some P-quintet $\mathfrak{S}$.*

■ Let $(\widetilde{A}, \widetilde{B}, \widetilde{\alpha}, \widetilde{l}, \widetilde{\psi})$ be a Λ-quintet. By Theorem 2.7 (1), there exists a P-module A that is finitely generated over **Z** and such that $L \otimes_P A \approx \widetilde{A}$. Without loss of generality we may suppose also that A is minimal. Then $T(A)$ is naturally identified with $T(\widetilde{A})$, and this allows us to define in the obvious way a form l: $T(A) \otimes T(A) \to \mathbf{Q}/\mathbf{Z}$, using $\widetilde{l}$. Let B be the direct sum

$$\widetilde{B} \oplus (A \otimes \mathbf{Z}_2)_+ \oplus (A \otimes \mathbf{Z}_2)_- \oplus (\mathrm{Hom}(A; \mathbf{Z}_2))_+ \oplus (\mathrm{Hom}(A; \mathbf{Z}_2))_-$$

and let ψ: $B \otimes B \to \mathbf{Z}_4$ be given by the matrix

$$\begin{bmatrix} \widetilde{\psi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon\delta \\ 0 & 0 & 0 & \varepsilon\sigma & 0 \\ 0 & 0 & \sigma' & 0 & 0 \\ 0 & \delta' & 0 & 0 & 0 \end{bmatrix},$$

where the dash denotes the transpose, ε the parity, and δ: $(A \otimes \mathbf{Z}_2)_+ \otimes (\mathrm{Hom}(A; \mathbf{Z}_2))_- \to \mathbf{Z}_2 \subset \mathbf{Z}_4$and σ: $(A \otimes \mathbf{Z}_2)_- \otimes (\mathrm{Hom}(A; \mathbf{Z}_2))_+ \to \mathbf{Z}_2 \subset \subset \mathbf{Z}_4$ are the restrictions of the canonical form

$$(A \otimes \mathbf{Z}_2) \otimes \mathrm{Hom}(A; \mathbf{Z}_2) \to \mathbf{Z}_2, \quad x \otimes f \mapsto f(x).$$

It remains for us to construct a homomorphism α: $A \otimes \mathbf{Z}_2 \to B$, which we define as the direct sum of the homomorphism α_0: $(A \otimes \mathbf{Z}_2)_0 \approx \widetilde{A} \otimes \mathbf{Z}_2 \xrightarrow{\widetilde{\alpha}} \widetilde{B} \subset B$ and the natural embeddings α_+: $(A \otimes \mathbf{Z}_2)_+ \to B$ and

$\alpha_-: (A \otimes \mathbf{Z}_2)_- \to B$. It is easy to see that $\mathfrak{S} = (A, B, \alpha, l, \psi)$ is a P-quintet and that the Λ-quintet $L \otimes_P \mathfrak{S}$ is isomorphic to the original one. ∎

Theorems 9.3, 10.11, and 10.12 have the following corollary.

10.13. *Corollary.* *For $q > 3$ the map that assigns to a knot the Λ-quintet $L \otimes_P \mathfrak{S}$, where $\mathfrak{S}$ is the P-quintet of some Seifert manifold, is a bijection from the set of simple $2q$-dimensional knots in S^{2q+2} onto the set of isomorphism classes of Λ-quintets of parity $(-1)^{q+1}$.* ∎

10.14. Suppose that (S^{2q+2}, k^{2q}) is a simple knot. How can one describe in terms of invariants the Λ-quintet (A, B, α, l, ψ) corresponding to it by Corollary 10.13? From Theorem 2.6 and the definitions in 9.2 and 10.10 it follows that the Λ-module A is isomorphic to $H_q(\tilde{X})$, where $\tilde{X}$ is the infinite cyclic covering of the complement of the knot in question. Similarly, the Λ-module B is isomorphic to $\sigma_{q+2}(\tilde{X})$ (this follows from Definitions 9.2 and 10.10 and the remark that the statement and proof of Theorem 2.6 remain valid for any generalized homology theory, in particular, for the theory σ_* of stable homotopy groups).

We claim that the homomorphism $\alpha: H_q(\tilde{X}) \otimes \mathbf{Z}_2 \to \sigma_{q+2}(\tilde{X})$ in the Λ-quintet under consideration acts according to the formula

$$\alpha(x \otimes 1) = h^{-1}(x) \circ \eta,$$

where $x \in H_q\tilde{X}$, $h: \sigma_q(\tilde{X}) \to H_q\tilde{X}$ is the Hurewicz homomorphism (which in our case is an isomorphism, since $\tilde{X}$ is $(q-1)$-connected), and $\eta \in \sigma_{q+2}(S^q)$ is the unique non-zero element. For let $V^{2q+1} \subset S^{2q+2}$ be an arbitrary $(q-1)$-connected Seifert manifold of the knot in question and let int $V \to \tilde{X}$ be an arbitrary lifting. By Definitions 9.2 and 10.10 and the construction of the isomorphisms given by Theorem 2.6, there is a commutative diagram

$$\begin{array}{ccc} H_q(\tilde{X}) \otimes \mathbf{Z}_2 & \xrightarrow{\alpha} & \sigma_{q+2}(\tilde{X}) \\ \uparrow{\scriptstyle f_* \otimes 1} & & \uparrow{\scriptstyle f_{**}} \\ H_q(V) \otimes \mathbf{Z}_2 & \xrightarrow[\alpha_V]{} & \sigma_{q+2}(V), \end{array}$$

where f_* and f_{**} denote the homomorphisms induced by the lifting, and α_V is the homomorphism in the P-quintet of V whose action is defined in 9.2. By Lemma 10.5, the vertical maps in this diagram are epimorphisms. Therefore, if $x \in H_q\tilde{X}$, then there is a $y \in H_qV$ with $(f_* \otimes 1)(y \otimes 1) = x \otimes 1$ and we have

$$\alpha(x \otimes 1) = f_{**}(\alpha_V(y \otimes 1)) = f_{**}(h^{-1}(y) \circ \eta) = h^{-1}(f_*(y)) \circ \eta = h^{-1}(x) \circ \eta,$$

which is what was to be proved.

The forms l and ψ with the above interpretation of A and B, become

$$l: T_q(\tilde{X}) \otimes T_q(\tilde{X}) \to \mathbf{Q}/\mathbf{Z} \text{ and } \psi: \sigma_{q+2}(\tilde{X}) \otimes \sigma_{q+2}(\tilde{X}) \to \mathbf{Z}_4,$$

where $T_q(\tilde{X})$ denotes $\mathrm{Tors}_{\mathbf{Z}} H_q(\tilde{X})$. It follows from Theorems 9.3 (2) and 10.11 that *these forms are invariants of the knot in question.*

In the next theorem we summarize the computations of §§9 and 10, restating Corollary 10.13 in the light of the remarks in 10.14.

10.15. *Theorem.* (1) *The collection $(H_q(\tilde{X}), \sigma_{q+2}(\tilde{X}), \alpha, l, \psi)$ associated with any simple knot (S^{2q+2}, k^{2q}), where $\tilde{X}$ is the infinite cyclic covering of the complement, and $\alpha: H_q(\tilde{X}) \otimes \mathbf{Z}_2 \to \sigma_{q+2}(\tilde{X})$, $l: T_q(\tilde{X}) \otimes T_q(\tilde{X}) \to \mathbf{Q}/\mathbf{Z}$, $\psi: \sigma_{q+2}(\tilde{X}) \otimes \sigma_{q+2}(\tilde{X}) \to \mathbf{Z}_4$ are the homomorphisms defined above, is a Λ-quintet of parity $(-1)^{q+1}$;* (2) *the Λ-quintets corresponding to equivalent knots are isomorphic;* (3) *for $q > 3$ simple $2q$-dimensional knots with isomorphic Λ-quintets are equivalent;* (4) *for any Λ-quintet of parity $\varepsilon (= \pm 1)$ and any $q > 3$ with $(-1)^{q+1} = \varepsilon$ there exists a simple $2q$-dimensional knot whose Λ-quintet is isomorphic to the given one.*

We recall the meaning of Theorem 10.15, (1). The phrase "*the collection (A, B, α, l, ψ) is a Λ-quintet of parity ε*" means (see 9.1 and 10.9) that (a) A and B are finitely generated Λ-modules and multiplication by $(1-t) \in \Lambda$ defines automorphisms of them; (b) $\alpha: A \otimes \mathbf{Z}_2 \to B$ is a Λ-homomorphism; (c) the forms $l: T(A) \otimes T(A) \to \mathbf{Q}/\mathbf{Z}$ and $\psi: B \otimes B \to \mathbf{Z}_4$ are non-degenerate and ε-symmetric; (d) the automorphisms of $T(A)$ and B defined by multiplication by $t \in \Lambda$ are isometries of the forms l and ψ, respectively; (e) there is an exact sequence

$$0 \to A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B \xrightarrow{\beta} \mathrm{Hom}_{\mathbf{Z}}(A; \mathbf{Z}_2) \to 0,$$

in which the homomorphism β adjoins α relative to ψ, that is, $\beta(b)(a) = \psi(b \otimes \alpha(\pi(a)))$ for $b \in B$ and $a \in A$, where $\pi: A \to A \otimes \mathbf{Z}_2$ is the projection; (f) the composite

$$B \xrightarrow{\gamma} A \xrightarrow{\pi} A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B$$

is multiplication by 2, where $\gamma(b)$ for $b \in B$ is defined as an element of $T(A)$ such that $\psi(b \otimes \alpha(\pi(a))) = l(\gamma(b) \otimes a)$ for all $a \in T(A)$.

In certain cases some of the objects in the Λ-quintet (A, B, α, l, ψ) vanish, or can be expressed in terms of the others. Suppose, for example, that A is 2-divisible. Then $A \otimes \mathbf{Z}_2 = 0$, $\mathrm{Hom}(A; \mathbf{Z}_2) = 0$, and as follows from (e) above, $B = 0$, so that $\alpha = 0$ and $\psi = 0$. Thus, by Theorem 10.15, *for $q > 3$ the type of a simple $2q$-dimensional knot with 2-divisible Alexander module $H_q(\tilde{X})$ is completely determined by the Λ-module $H_q(\tilde{X})$ and the form $l: T_q(\tilde{X}) \otimes T_q(\tilde{X}) \to \mathbf{Q}/\mathbf{Z}$.*

Now we consider a Λ-quintet (A, B, α, l, ψ) in which the module A is finite of exponent 2. Each element $x \in A$ determines a homomorphism $\nu: A \to \mathbf{Z}_2$, where $\nu(a) = l(x \otimes a)$ for $a \in A$. By (e) there exists a $b \in B$ with $\beta(b) = \nu$. Then $\gamma(b) = x$, and so the homomorphism $\gamma: B \to A$ defined in (f) is an epimorphism. Hence, by (e) and (f), α maps A onto the subgroup $2B$ of B, and B as an Abelian group is isomorphic to a direct sum of copies of $\mathbf{Z}_4$ whose number is equal to the rank of A over the field $\mathbf{Z}_2$. If $b, b' \in B$, then it follows from (f) that

$$2\psi(b \otimes b') = \psi(b \otimes 2b') = \psi(b \otimes \alpha(\pi(\gamma(b')))) = l(\gamma(b) \otimes \gamma(b')).$$

Thus, ψ determines l and a complete system of invariants is the pair (B, ψ) consisting of the Λ-module B and the pairing ψ: $B \otimes B \to \mathbf{Z}_4$. It now follows from Theorem 10.15 that *the type of a simple 2q-dimensional knot with finite Alexander module of exponent 2 is determined for $q > 3$ by the Λ-module $\sigma_{q+2}(\widetilde{X})$ and the form ψ: $\sigma_{q+2}(\widetilde{X}) \otimes \sigma_{q+2}(\widetilde{X}) \to \mathbf{Z}_4$.*

Notes to §10. Theorem 10.15 is the central result of this section. It was announced in the author's paper [13], 8.13, where there is a detailed proof of a similar theorem for fibred knots see also [12], Theorem 10.

If (A, B, α, l, ψ) is an arbitrary Λ-quintet, then the modules $T(A)$ and B are finite, and it follows from Theorem 10.15 that the Alexander module $H_q(\widetilde{X})$ determines for $q > 3$ the type of a simple $2q$-dimensional knot up to finitely many possibilities. A more general assertion of this kind is contained in the author's lecture [60]: *the middle-dimensional Alexander module and the Blanchfield pairing (the latter in the odd-dimensional case) determine the type of a stable knot up to finitely many possibilities.* There is a proof of an analogous theorem for fibred knots in [13].

The form l: $T_q(\widetilde{X}) \otimes T_q(\widetilde{X}) \to \mathbf{Q}/\mathbf{Z}$ in Theorem 10.15 was discovered independently by Levine [41], [42] and the author [10], [11] (the full text of my paper had been deposited at VINITI in 1974, see Ref. Zh. Matematika, 1974, 9A640). Later Kobel'skii [61] constructed an analogous form for links. The new construction of l presented in this paper follows the ideas of the author's paper "Knots and stable homotopy" (Proc. Leningrad Internat. Topology Conf.), in which a general construction is proposed for forms on extraordinary homology groups of the infinite cyclic covering of the complement of a knot. More precisely, it is shown there that for any n-dimensional knot there is an invariantly defined form

$$\sigma_i(\widetilde{X}) \otimes \sigma_j(\widetilde{X}) \to \sigma_{i+j}(S^{n+1}),$$

provided that $i+j > n+1$. For $n = 2q$, $i = j = q+2$ there arises in this way the form

$$\sigma_{q+2}(\widetilde{X}) \otimes \sigma_{q+2}(\widetilde{X}) \to \mathbf{Z}_{24} = \sigma_{2q+4}(S^{2q+1}),$$

which for simple knots takes values in $\mathbf{Z}_4 \subset \mathbf{Z}_{24}$ and coincides with the form ψ in Theorem 10.15.

The fact that the form l constructed in the present paper coincides with the form of the same name in [10] and [11] is easily deduced from the results of [11], §7.

The simple knots of a fixed dimension form an Abelian semigroup under the connected sum. The elements of this semigroup do not always split uniquely into a sum of primes, that is, not further decomposable, elements. Various results on the uniqueness and non-uniqueness of such decompositions are presented in [19] and in earlier papers cited in [19].

For other questions not touched upon in the present paper we refer the reader to the surveys [23], [35], [51].

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Translated by J. Howie

Institute of Cybernetics,
Academy of Sciences
Azerbaidzhan SSR

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