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Poincaré-Reidemeister metric, Euler structures, and torsion

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Abstract. In this paper we define a *Poincaré-Reidemeister scalar product* on the determinant line of the cohomology of any flat vector bundle over a closed orientable odd-dimensional manifold. It is a combinatorial "torsion-type" invariant which refines the PR-*metric* introduced in [Fa] and contains an additional sign or phase information. We compute the PR-scalar product in terms of the torsions of *Euler structures*, introduced in [T1], [T2]. We show that the sign of our PR-scalar product is determined by the Stiefel-Whitney classes and the semi-characteristic of the manifold. As an application, we compute the Ray-Singer analytic torsion via the torsions of Euler structures. Another application: a computation of the twisted semi-characteristic in terms of the Stiefel-Whitney classes.

§1. Introduction

Let F be a flat real vector bundle over a closed odd-dimensional smooth manifold X. Ray and Singer [RS] used the Laplace operators and their zeta-function regularized determinants to define a norm on the determinant line of the cohomology $\det H^*(X;F)$. Ray and Singer showed that their norm is topologically invariant. They conjectured that for bundles with orthogonal structure group, this norm coincides with the Reidemeiter norm on $\det H^*(X;F)$, defined using a piecewise linear triangulation of X and the classical Reidemeister-Franz torsion. This conjecture was proven by J. Cheeger and W. Müller in their celebrated papers [C] and [Mu].

Although the topologically invariant Ray-Singer norm is defined for an *arbitrary* flat real vector bundle F over X, the combinatorial counterpart, the Reidemeister norm, was known only for bundles with unimodular structure group. In 1994 W. Müller [Mu1] extended the result of [C], [Mu] to all unimodular F.

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In [Fa], it was shown how to construct combinatorially a norm on $\det H^*(X;F)$ for an arbitrary flat real vector bundle F without the unimodularity assumption. The construction of [Fa] uses a combination of the Reidemeister torsion with the Poincaré duality; the resulting norm on $\det H^*(X;F)$ is called the *Poincaré-Reidemeister norm*. It was proven in [Fa], that this norm coincides with the Ray-Singer norm for any F. The proof of this theorem uses fundamental results of J.-M. Bismut and W. Zhang [BZ].

A different approach to the Reidemeister torsions was introduced in [T1]–[T3]. It is observed in these papers that the indeterminacy of the Reidemeister torsion is controlled by additional structures on the manifold X, the *homology orientations* and the *Euler structures*. A homology orientation is an orientation of the determinant line of real homologies $\det H_*(X;\mathbb{R})$. An Euler structure on X can be described in terms of an Euler chain on a PL-triangulation of X; it may also be described via vector fields or via Spin^c-structures (for 3-manifolds), see loc. cit. The constructions of [T1], [T2] yield torsions of Euler structures on X which refine the usual Reidemeister torsions.

The initial goal of this research was to find a relation between the approaches of [Fa] and [T1], [T2].

In this paper we define the *Poincaré-Reidemeister scalar product* on $\det H^*(X;F)$, which determines the PR-norm defined in [Fa] and contains an additional sign or phase information. We show that the sign of the PR-scalar product is determined by the Stiefel-Whitney classes of F and X and the semi-characteristic of X. The main result of this paper computes the Poincaré-Reidemeister scalar product in terms of the torsions of Euler structures on X. More precisely, in the case of even-dimensional F, we give a formula expressing the PR-scalar product applied to the torsion of an Euler structure ξ on X in terms of a characteristic homology class $c(\xi) \in H_1(X)$ associated to ξ. For odd-dimensional F, we establish a similar formula with the only difference that the torsion of ξ depends also on a choice of a homology orientation of X. Using these formulas and the main result of [Fa] we compute the analytic Ray-Singer torsion in terms of the Euler structures.

As an application, we compute the residue $\bmod 2$ of the twisted semi-characteristic of X with coefficients in a flat vector bundle with orthogonal structure group. We give a formula for this residue in terms of the Stiefel-Whitney classes. (For related formulas, see [LMP].)

In order to prove our results we develop general algebraic tools, allowing to treat the sign anomalies, which appear in the formalism of the determinant lines. In [T1], the canonical isomorphism between the determinant lines of a chain complex and its homology was modified by introducing an additional sign factor. In this paper we introduce more sign factors in the natural maps between the determinant lines and we show that these sign choices are compatible.

§2. Determinant lines of chain complexes

In this section we recall the canonical isomorphism relating the determinant line of a chain complex and the determinant line of its homology. Our formula (cf. (2.2)) contains a sign refinement, suggested in [T1], of the standard formula [M2]. We will introduce also some sign involving factors in the natural commutativity and duality maps between the determinant lines. We will establish a few technical results concerning the compatibility of these sign involving choices.

2.1. Determinant lines. We shall denote by $\boldsymbol{k}$ a fixed ground field of characteristic zero. The most important special cases are $\boldsymbol{k} = \mathbb{R}$ and $\boldsymbol{k} = \mathbb{C}$.

If V is a finite dimensional vector space over $\boldsymbol{k}$, *the determinant line of* V is denoted by $\det V$ and is defined as the top exterior power of V, i.e., $\Lambda^n V$, where $n = \dim V$. The dual line $\mathrm{Hom}_{\boldsymbol{k}}(\Lambda^n V, \boldsymbol{k})$ is denoted by $(\det V)^{-1}$. This notation is justified by the obvious equality $\det V \otimes (\det V)^{-1} = \boldsymbol{k}$. If $V = 0$ then by definition $(\det V)^{-1} = \det V = \boldsymbol{k}$.

For a finite dimensional graded vector space $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$, its *determinant line* $\det V$ is defined as the tensor product

$$\det V = \det V_0 \otimes (\det V_1)^{-1} \otimes \det V_2 \otimes \cdots \otimes (\det V_m)^{(-1)^m}.$$

2.2. Torsion of a chain complex. Let C be a finite dimensional chain complex

$$0 \to C_m \xrightarrow{d} C_{m-1} \xrightarrow{d} \cdots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \to 0$$

over $\boldsymbol{k}$. In the theory of torsions a crucial role is played by a canonical isomorphism

$$\varphi_C : \det C \to \det H_*(C), \tag{2.1}$$

where both C and $H_*(C)$ are considered as graded vector spaces. The definition of the mapping φ_C is as follows. Choose for each $q = 0, \ldots, m$ non-zero elements $c_q \in \det C_q$ and $h_q \in \det H_q(C)$. Set $c = c_0 \otimes c_1^{-1} \otimes c_2 \otimes \cdots \otimes c_m^{(-1)^m} \in \det C$ and

$$h = h_0 \otimes h_1^{-1} \otimes h_2 \otimes \cdots \otimes h_m^{(-1)^m} \in \det H_*(C),$$

where -1 in the exponent denotes the dual functional; for example, c_1^{-1} is a $\boldsymbol{k}$-linear mapping $\det C_1 \to \boldsymbol{k}$ such that $c_1^{-1}(c_1) = 1$. We define φ_C by

$$\varphi_C(c) = (-1)^{N(C)}[c : h]h, \tag{2.2}$$

where $N(C)$ is a residue modulo 2 defined below and $[c : h]$ is a nonzero element of $\boldsymbol{k}$, defined by

$$[c : h] = \prod_{q=0}^{m} [d(b_{q+1})\hat{h}_q b_q / \hat{c}_q]^{(-1)^{q+1}}. \tag{2.3}$$

Here b_q is a sequence of vectors of C_q whose image $d(b_q)$ under the boundary homomorphism $d : C_q \to C_{q-1}$ is a basis of $\mathrm{Im}\, d$; the symbol $\hat{h}_q$ denotes a sequence of cycles in C_q such that the wedge product of their homology classes equals h_q; the symbol $\hat{c}_q$ denotes a basis of C_q whose wedge product equals c_q; the number $[d(b_{q+1})\hat{h}_q b_q / \hat{c}_q]$ is the determinant of the matrix transforming $\hat{c}_q$ into the basis $d(b_{q+1})\hat{h}_q b_q$ of C_q. The residue $N(C)$ is defined by

$$N(C) = \sum_{q=0}^{m} \alpha_q(C)\beta_q(C) \pmod 2, \tag{2.4}$$

where

$$\alpha_q(C) = \sum_{j=0}^{q} \dim C_j \pmod 2, \quad \beta_q(C) = \sum_{j=0}^{q} \dim H_j(C) \pmod 2. \tag{2.5}$$

We shall deal with chain complexes with zero Euler characteristic so that the residues (2.5) vanish for big q.

It is clear that $[c : h]$ is independent of the choice of b_q's and also that the isomorphism φ_C is independent of the choice of h_q's and c_q's.

Formula (2.2) involves the sign refinement of the standard formula suggested in [T1]. In the next subsections we introduce similar signs in other natural maps arising in this setting. We shall show that these signs are compatible with isomorphism (2.1) and with each other. For more information on torsions, see [M2], [BGS], and [Fr].

2.3. The fusion homomorphism. For two finite-dimensional graded vector spaces $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ and $W = W_0 \oplus W_1 \oplus \cdots \oplus W_m$, we define a canonical isomorphism

$$\mu_{V,W} : \det V \otimes \det W \to \det(V \oplus W), \tag{2.6}$$

by

$$\mu_{V,W} = (-1)^{M(V,W)} \bigotimes_q \mu_q^{(-1)^q}, \tag{2.7}$$

where

$$\mu_q^{+1} = \mu_q : \det V_q \otimes \det W_q \to \det(V_q \oplus W_q)$$

is the isomorphism defined by

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge w_2 \wedge \cdots \wedge w_l) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_l \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_k,$$

with $k = \dim V_q, l = \dim W_q$, the isomorphism

$$\mu_q^{-1} : (\det V_q)^{-1} \otimes (\det W_q)^{-1} \to \det(V_q \oplus W_q)^{-1}$$

is the transpose of the inverse of μ_q,

$$M(V, W) = \sum_{q=1}^{m} \alpha_{q-1}(V)\alpha_q(W) \in \mathbb{Z}/2\mathbb{Z}, \tag{2.8}$$

with

$$\alpha_q(V) = \sum_{j=0}^{q} \dim V_j \ (\mathrm{mod}\, 2) \in \mathbb{Z}/2\mathbb{Z}, \quad q = 0, 1, \ldots, m,$$

and $\alpha_q(W) \in \mathbb{Z}/2\mathbb{Z}$ defined similarly.

We will call (2.6) *the fusion homomorphism*.

2.4. Lemma. *Let*

$$= (0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to 0)$$

and

$$C' = (0 \to C'_m \to C'_{m-1} \to \cdots \to C'_0 \to 0)$$

be two finite dimensional chain complexes over $\boldsymbol{k}$. *Then the following diagram, involving the canonical isomorphisms* (2.1) *and* (2.6), *is commutative*:

$$\begin{array}{ccc} \det C \otimes \det C' & \xrightarrow{\varphi_C \otimes \varphi_{C'}} & \det H_*(C) \otimes \det H_*(C') \\ \Big\downarrow \mu_{C,C'} & & \Big\downarrow \mu_{H_*(C), H_*(C')} \\ \det(C \oplus C') & \xrightarrow{\varphi_{C \oplus C'}} & \det H_*(C \oplus C') = \det\big(H_*(C) \oplus H_*(C')\big). \end{array} \tag{2.9}$$

Proof. Fix non-zero $c_q \in \det C_q$, $h_q \in \det H_q$ and $c'_q \in \det C'_q$, $h'_q \in \det H'_q$, where $H_q = H_q(C)$ and $H'_q = H_q(C')$. We obtain non-zero elements

$$c = c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_m^{(-1)^m} \in \det C \quad \text{and} \quad h = h_0 \otimes h_1^{-1} \otimes \cdots \otimes h_m^{(-1)^m} \in \det H_*,$$

and similarly $c' \in \det C'$ and $h' \in \det H'_*$. Set

$$cc' = (c_0 \wedge c'_0) \otimes (c_1 \wedge c'_1)^{-1} \otimes \cdots \otimes (c_m \wedge c'_m)^{(-1)^m} \in \det(C \oplus C')$$

and

$$hh' = (h_0 \wedge h'_0) \otimes (h_1 \wedge h'_1)^{-1} \otimes \cdots \otimes (h_m \wedge h'_m)^{(-1)^m} \in \det(H_* \oplus H'_*).$$

According to definitions,

$$\begin{aligned} \mu_{H_*, H'_*}\big((\varphi_C \otimes \varphi_{C'})(c \otimes c')\big) &= (-1)^{N(C)+N(C')}[c : h][c' : h']\mu_{H_*(C), H*(C')}(h \otimes h') \\ &= (-1)^{N(C)+N(C')+M(H_*, H'_*)}[c : h][c' : h']hh'. \end{aligned}$$

Similarly,

$$(\varphi_{C \oplus C'}\mu_{C,C'})(c \otimes c') = (-1)^{M(C,C')}\varphi_{C \oplus C'}(cc') = (-1)^{N(C \oplus C')+M(C,C')}[cc' : hh']hh'.$$

To prove the lemma we should show that

$$(2.10)\quad (-1)^{N(C)+N(C')+M(H_*,H'_*)}[c:h][c':h'] = (-1)^{N(C\oplus C')+M(C,C')}[cc':hh'].$$

Let b_q be a sequence of vectors of C_q whose image $d(b_q)$ under the boundary homomorphism $d: C_q \to C_{q-1}$ is a basis of $\operatorname{Im} d$. Similarly choose a sequence $b'_q \subset C'_q$ for each q. By definition,

$$\frac{[cc':hh']}{[c:h][c':h']} = \prod_{q=0}^{m}\left(\frac{[(db_{q+1}db'_{q+1}\hat{h}_q\hat{h}'_q b_q b'_q)/c_q]}{[(db_{q+1}\hat{h}_q b_q)/c_q][(db'_{q+1}\hat{h}'_q b'_q)/c_q]}\right)^{(-1)^{q+1}}.$$

The q-th factor on the right-hand side is equal to

$$(2.11)\qquad (-1)^{\operatorname{card}(b'_{q+1})\dim H_q + \operatorname{card}(b_q)\dim H'_q + \operatorname{card}(b_q)\operatorname{card}(b'_{q+1})}.$$

Since

$$\operatorname{card}(b_{q+1}) \equiv \alpha_q(C) + \beta_q(C) \pmod 2, \quad \dim H_q \equiv \beta_q(C) + \beta_{q-1}(C) \pmod 2$$

and similarly for C', we obtain that the product of the signs (2.11) equals $(-1)^y$, where

$$y = \beta_m(C)\beta_m(C') + \sum_{q=0}^{m}\{\beta_q(C)\alpha_q(C') + \beta_{q-1}(C)\beta_q(C') + \alpha_{q-1}(C)\alpha_q(C') + \alpha_{q-1}(C)\beta_{q-1}(C')\}.$$

It is easy to check that

$$y \equiv N(C) + N(C') + M(C,C') - N(C\oplus C') - M\big(H_*(C), H_*(C')\big) \pmod 2.$$

This implies (2.10) and the lemma. □

2.5. Duality operator *D*. Let $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ be a finite dimensional graded vector space over $\boldsymbol{k}$ with odd m. We define the *dual graded vector space* over $\boldsymbol{k}$ by $V' = V'_0 \oplus V'_1 \oplus \cdots \oplus V'_m$ where $V'_q = (V_{m-q})^* = \operatorname{Hom}_{\boldsymbol{k}}(V_{m-q}, \boldsymbol{k})$. We define a *duality operator*

$$D = D_V : \det V \to \det V'$$

as follows. Let $v_q \in \det V_q$ be a volume element determined by a basis of V_q and let $v'_{m-q} \in \det V'_{m-q}$ be the volume element determined by the dual basis of V'_{m-q}, for $q = 0, 1, \ldots, m$. Then

$$D(v_0 \otimes v_1^{-1} \otimes v_2 \otimes \cdots \otimes v_m^{-1}) = (-1)^{s(V)} v'_0 \otimes (v'_1)^{-1} \otimes v'_2 \otimes \cdots \otimes (v'_m)^{-1},$$

where the residue $s(V) \in \mathbb{Z}/2\mathbb{Z}$ is given by

$$s(V) = \sum_{q=1}^{m} \alpha_{q-1}(V)\alpha_q(V) + \sum_{q=0}^{(m-1)/2} \alpha_{2q}(V).$$

Recall that $\alpha_q(V) = \sum_{j=0}^{q} \dim V_j \pmod 2$. It is easy to check that $D = D_V$ does not depend on the choice of $v_q \in \det V_q$ $(q = 0, 1, \dots, m)$.

In the next lemma we shall use the notion of a dual chain complex. For a chain complex $C = (0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to 0)$ over $\boldsymbol{k}$ the *dual chain complex* $C' = (0 \to C'_m \to C'_{m-1} \to \cdots \to C'_0 \to 0)$ is defined by $C'_q = (C_{m-q})^*$. The boundary homomorphism $C'_{q+1} \to C'_q$ is defined to be $(-1)^{m-q} d^*_{m-q-1}$ where d_{m-q-1} is the boundary homomorphism $C_{m-q} \to C_{m-q-1}$. For odd m, the construction above yields a duality operator $D_C : \det C \to \det C'$.

2.6. Lemma. *Let $C = (0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to 0)$ be a finite dimensional chain complex with odd m and with $\chi(C) \equiv 0 \bmod 2$, and let*

$$C' = (0 \to C'_m \to C'_{m-1} \to \cdots \to C'_0 \to 0)$$

be the dual chain complex. Then the following diagram, involving the canonical isomorphisms (2.1), *is commutative*:

$$\begin{array}{ccc} \det C & \xrightarrow{D_C} & \det C' \\ \varphi_C \Big\downarrow & & \Big\downarrow \varphi_{C'} \\ \det H_*(C) & \xrightarrow{D_{H_*(C)}} & \det H_*(C'). \end{array} \tag{2.12}$$

Note that the duality between C and C' induces a duality between the graded vector spaces $H_*(C)$ and $H_*(C')$ so that we can consider the duality operator $D_{H_*(C)}$.

Proof. Lemma 2.6 is a sign-refined version of the standard duality for torsions of chain complexes (see [M1]). For the computation of signs, see Lemma 7 in the Appendix to [T1]. □

2.7. Lemma. (1) *Let $V = V_0 \oplus \cdots \oplus V_m$ and $W = W_0 \oplus \cdots \oplus W_m$ be finite-dimensional graded $\boldsymbol{k}$-vector spaces such that $\alpha_m(V) = \alpha_m(W) = 0 \in \mathbb{Z}/2\mathbb{Z}$. Then the following diagram is commutative*:

$$\begin{array}{ccc} \det V \otimes \det W & \xrightarrow{\mu_{V,W}} & \det(V \oplus W) \\ S \Big\downarrow & & \Big\downarrow \det(s) \\ \det W \otimes \det V & \xrightarrow{\mu_{W,V}} & \det(W \oplus V). \end{array} \tag{2.13}$$

Here s denotes the natural map $V \oplus W \to W \oplus V$ interchanging the summands and S interchanges the factors $v \otimes w \mapsto w \otimes v$.

(2) *For V and W as in* (1) *with odd m, the following diagram involving the dual graded vector spaces V' and W' and the canonical isomorphisms D and μ is commutative*:

$$\begin{array}{ccc} \det V \otimes \det W & \xrightarrow{D_V \otimes D_W} & \det V' \otimes \det W' \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ \det(V \oplus W) & \xrightarrow{D_{V \oplus W}} & \det(V' \oplus W'). \end{array} \tag{2.14}$$

(3) *For any triple of finite-dimensional graded vector spaces U, V, W, the diagram*

$$\begin{array}{ccc} \det U \otimes \det V \otimes \det W & \xrightarrow{\mu_{U,V} \otimes 1} & \det(U \oplus V) \otimes \det(W) \\ {\scriptstyle 1 \otimes \mu_{V,W}}\downarrow & & \downarrow{\scriptstyle \mu_{U \oplus V, W}} \\ \det U \otimes \det(V \oplus W) & \xrightarrow{\mu_{U, V \oplus W}} & \det(U \oplus V \oplus W) \end{array} \tag{2.15}$$

is commutative.

Proof. Statement (1) is equivalent to

$$M(V, W) + M(W, V) + \sum_q (\dim V_q)(\dim W_q) \equiv 0 \pmod 2,$$

which follows easily.

Statement (2) follows from

$$M(V', W') \equiv M(W, V) \pmod 2$$

(using $\alpha_m(V) = \alpha_m(W) = 0$) and then

$$M(V, W) + M(W, V) = s(V \oplus W) + s(V) + s(W) \pmod 2.$$

Statement (3) follows from the easy equality

$$M(U, V) + M(U \oplus V, W) = M(V, W) + M(U, V \oplus W). \quad \square$$

§3. The Reidemeister torsion

In this section we discuss the classical construction of the Reidemeister torsion of a flat unimodular bundle. We view this torsion as an element of the determinant line of the homology of the bundle. We show that the torsion has no indeterminacy in the case of an even-dimensional bundle and has a sign indeterminacy in the case of an odd-dimensional bundle.

3.1. Torsion of a unimodular flat vector bundle. Let F be a flat $\boldsymbol{k}$-vector bundle over a finite connected CW-space X. Recall a definition of the homology of X with coefficients in F. Orient all cells of X. For a cell a of X, denote by $\Gamma(a, F)$ the vector space of flat sections of F over a. (Clearly, $\dim \Gamma(a, F) = \dim F$.) The vector space of q-chains in X with values in F is defined by

$$C_q(X;F) = \bigoplus_{\dim a = q} \Gamma(a,F). \tag{3.1}$$

The boundary homomorphism $C_q(X;F) \to C_{q-1}(X;F)$ is defined by restricting the flat sections to the faces with the signs determined in the usual way by the orientations of the cells. Denote the resulting chain complex by $C = C_*(X;F)$ and set $H_*(X;F) = H_*(C)$. The graded vector space $H_*(X;F)$ is a homotopy invariant of the pair (X,F).

Recall the Reidemeister-Franz construction of the torsion of (X,F). We consider here only the case of unimodular F, for the general case, see Remark 3.4 and Section 6. The bundle F is called unimodular, if its top exterior power $\Lambda^{\dim F} F$ is a trivial flat vector bundle. The bundle F is unimodular iff it has a flat volume form, i.e., a linear volume form on each fiber $F_x, x \in X$ invariant under the parallel transport along any path in X. Fix such a form ω. For every cell a of X choose a basis of $\Gamma(a,F)$ of ω-volume 1. The concatenation of these bases over all q-dimensional cells gives a basis in $C_q(X;F)$ via (3.1). The wedge product of the elements of this basis yields a non-zero element $c_q \in \det C_q(X;F)$. Set

$$\tau(X;F) = \varphi_C(c_0 \otimes c_1^{-1} \otimes c_2 \otimes \cdots \otimes c_m^{(-1)^m}) \in \det H_*(X;F)$$

where $m = \dim X$ and φ_C is the isomorphism $\det C \to \det H_*(X;F)$ constructed in Section 2.2. In particular, if $H_*(X;F) = 0$, then $\det H_*(X;F) = \boldsymbol{k}$ and $\tau(X;F) \in \boldsymbol{k}$ is the Reidemeister-Franz torsion of the pair (X,F).

The definition of $\tau(X;F)$ involves certain choices. Note first that $\tau(X;F)$ does not depend on the choice of ω-volume 1 bases in $\{\Gamma(a,F)\}_a$. If we replace ω with $k\omega$ for a non-zero $k \in \boldsymbol{k}$, then the torsion $\tau(X;F)$ is multiplied by $k^{-\chi(X)}$ where χ is the Euler characteristic. Another indeterminacy comes from orders and orientations of the cells. To apply (3.1), we need to order the q-cells of X; a permutation in this order leads to multiplication of $\tau(X;F)$ by $(-1)^{\dim F}$. Finally, when we invert the orientation of a cell of X (used in the definition of the boundary homomorphisms), the torsion $\tau(X;F)$ is also multiplied by $(-1)^{\dim F}$. We sum up this discussion in the following lemma.

3.2. Lemma. *Let F be a unimodular flat vector bundle over a finite connected* CW-*space X with $\chi(X) = 0$. The element $\tau(X;F) \in \det H_*(X;F)$ is well defined up to multiplication by $(-1)^{\dim F}$. In particular, if F is even-dimensional, then $\tau(X;F)$ is a well defined element of* $\det H_*(X;F)$.

A fundamental property of the torsion is its combinatorial invariance which allows to consider the torsions of flat vector bundles over PL-manifolds. We have the following version of the combinatorial invariance.

3.3. Lemma. *Under the conditions of Lemma* 3.2, *the torsion $\tau(X;F)$ with indeterminacy given in Lemma* 3.2 *is invariant under cell subdivisions of X.*

Proof. The standard arguments imply the combinatorial invariance of $\tau(X;F)$ modulo ± 1. This yields the lemma in the case of odd-dimensional F. Let us prove the lemma for even-dimensional F. (We follow the argument given in [T1], Section 3.2.1.) Since a cellular subdivision is a simple homotopy equivalence, it is enough to prove that $\tau(X;F)$ is invariant under simple homotopy equivalences. It is well known that any simple homotopy equivalence may be presented as a composition of elementary cellular expansions and contractions. Therefore it suffices to consider one such transformation. Assume that a CW-space X' is obtained from X by attaching a closed j-dimensional ball D along a cellular mapping of a closed $(j-1)$-dimensional ball $D' \subset \partial D$ into X. The cellular structure in X' is obtained from the one in X by adding two open cells $a = \operatorname{Int} D$ and $b = \partial D \backslash D'$. The flat vector bundle F over X extends to a flat vector bundle F' over X'. Clearly, $H_*(X;F) = H_*(X';F')$ and we should prove that $\tau(X;F) = \tau(X';F')$.

We orient and numerate the cells of X' (in each dimension) so that the newly attached cells a, b appear at the very end. Denote the chain complex $C_*(X';F')$ and its subcomplex $C_*(X;F)$ by C' and C, respectively. It is clear that $C'_q = C_q$ for $q \neq j, j-1$ and $C'_j = C_j \oplus \Gamma(a,F')$, $C'_{j-1} = C_{j-1} \oplus \Gamma(b,F')$. We choose a flat volume form on F' and volume 1 bases in $\Gamma(a,F')$, $\Gamma(b,F')$, and C_q, as in Section 3.1. Denote these bases by A, B, and $\hat{c}_q$, respectively. Note that $\operatorname{card} A = \operatorname{card} B = \dim F$. Choose for each q a non-zero element $h_q \in \det H_q(C) = \det H_q(C')$. Choose a sequence of vectors b_q in C_q whose image under the boundary homomorphism $d_{q-1} : C_q \to C_{q-1}$ is a basis of $\operatorname{Im} d_{q-1}$. It is easy to see that the image of the boundary homomorphism $d'_{q-1} : C'_q \to C'_{q-1}$ equals to $\operatorname{Im} d_{q-1}$ for $q \neq j$ and that $d_{j-1}(b_j), d'_{j-1}(A)$ is a basis of $\operatorname{Im} d'_{j-1}$. Note that the residues $N(C)$, $N(C')$ introduced in Section 2.2 are both equal to 0. Now, it follows from definitions that

$$\frac{\tau(X';F')}{\tau(X;F)} = \left(\frac{[d_j(b_{j+1})\hat{h}_j b_j A/\hat{c}_j A]}{[d_j(b_{j+1})\hat{h}_j b_j/\hat{c}_j]}\right)^{(-1)^j} \times \left(\frac{[d_{j-1}(b_j)d'_{j-1}(A)\hat{h}_{j-1}b_{j-1}/\hat{c}_{j-1}B]}{[d_{j-1}(b_j)\hat{h}_{j-1}b_{j-1}/\hat{c}_{j-1}]}\right)^{(-1)^{j-1}}.$$

It is obvious that the first factor on the right-hand side equals 1. The second factor on the right-hand side equals $(-1)^{rs}\varepsilon^r$ where $r = \operatorname{card} A = \dim F$, $s = \operatorname{card} \hat{h}_{j-1} + \operatorname{card} b_{j-1}$, and ε is the incidence sign of the oriented cells a, b. Since r is even, we obtain $\tau(X';F') = \tau(X;F)$. $\square$

3.4. Remark. It is easy to generalize the definition of $\tau(X;F)$ to the case of a non-unimodular flat vector bundle F over a finite connected CW-space X with $\chi(X) = 0$. This gives an element $\tau(X;F) \in \det H_*(X;F)$ defined up to multiplication by $(-1)^{\dim F}$ and $\det_F\big(H_1(X)\big) \subset \boldsymbol{k}^*$ where $\det_F : H_1(X) \to \boldsymbol{k}^*$ is the determinant of the monodromy of F. We shall consider a more subtle torsion in Section 6.

§4. The Poincaré-Reidemeister scalar product

In this section we introduce the Poincaré-Reidemeister scalar product on the determinant line of the homology of a flat vector bundle over a closed orientable odd-dimensional PL-manifold. It determines the Poincaré-Reidemeister metric, introduced in [Fa], and carries an additional information in the form of a phase (if $\boldsymbol{k} = \mathbb{C}$) or in the form of a sign (if $\boldsymbol{k} = \mathbb{R}$).

4.1. The dual flat vector bundle. Let F be a flat $\boldsymbol{k}$-vector bundle over a finite connected CW-space X with $\chi(X) = 0$. Recall the dual flat vector bundle F^*. The fiber of F^* over a point $x \in X$ is the dual vector space $F^*_x = \operatorname{Hom}_{\boldsymbol{k}}(F_x, \boldsymbol{k})$. For a path $\gamma : [0,1] \to X$, the parallel transport $F_x^* \to F_y^*$ along γ is the transpose of the parallel transport $F_y \to F_x$ along the inverse path γ^{-1}.

It is clear that for any loop γ in X we have $\det_F(\gamma) \cdot \det_{F^*}(\gamma) = 1$ and therefore $F \oplus F^*$ is a unimodular flat vector bundle. Since it is also even-dimensional, we can apply the construction of Section 3 to obtain a well defined non-zero element $\tau(X; F \oplus F^*) \in \det H_*(X; F \oplus F^*)$.

4.2. The duality operator. Let X be a *closed connected oriented piecewise linear manifold of odd dimension m*. Let F be a flat $\boldsymbol{k}$-vector bundle over X. The standard homological intersection pairing

$$H_q(X; F^*) \otimes H_{m-q}(X; F) \to \boldsymbol{k} \tag{4.1}$$

allows us to identify the dual of $H_{m-q}(X; F)$ with $H_q(X; F^*)$. Applying the construction of Section 2.5 to the graded vector space $\bigoplus_{q=0}^{m} H_q(X; F)$ we obtain a canonical isomorphism

$$D : \det H_*(X; F) \to \det H_*(X; F^*). \tag{4.2}$$

By definition, $D = (-1)^{s(F)} \bigotimes_{q=0}^{m} \psi_q$ where the residue $s(F) \in \mathbb{Z}/2\mathbb{Z}$ is given by

$$s(F) = \sum_{q=0}^{m} \beta_{q-1}\beta_q + \sum_{q=0}^{(m-1)/2} \beta_{2q} \pmod 2, \quad \beta_q = \sum_{q=0}^{q} \dim H_q(X; F)$$

and ψ_q with even q denotes the isomorphism

$$\det H_q(X; F) \to \left(\det H_{m-q}(X; F^*)\right)^{-1} \tag{4.3}$$

induced by the intersection form, while ψ_q with odd q denotes the isomorphism

$$\psi_q : \left(\det H_q(X; F)\right)^{-1} \to \det H_{m-q}(X; F^*)$$

inverse to the transpose of (4.3).

It is easy to check that D does not depend on the choice of the orientation of X and therefore can be considered for *orientable* manifolds. (Hint: $\beta_m \equiv \chi(X) = 0 \pmod 2$.) As an exercise, the reader may check that $s(F) = s(F^*)$ (we shall not use it).

4.3. The Poincaré-Reidemeister pairing. Let F be a flat $\boldsymbol{k}$-vector bundle over a closed connected orientable odd-dimensional PL-manifold X. Denote by μ the canonical fusion isomorphism

$$\det H_*(X; F) \otimes \det H_*(X; F^*) \to \det\big(H_*(X; F) \oplus H_*(X; F^*)\big) = \det H_*(X; F \oplus F^*)$$

defined in Section 2.3. Consider the bilinear pairing

$$\langle\ ,\ \rangle_{\mathrm{PR}} : \det H_*(X; F) \times \det H_*(X, F) \to \boldsymbol{k}, \tag{4.4}$$

given by

$$\langle a, b \rangle_{\mathrm{PR}} = \mu\big(a \otimes D(b)\big)/\tau(X; F \oplus F^*) \in \boldsymbol{k},$$

where $a, b \in \det H_*(X; F)$ and D is the isomorphism (4.2). In other words, $\langle a, b \rangle_{\mathrm{PR}}$ is an element of $\boldsymbol{k}$ such that

$$\mu\big(a \otimes D(b)\big) = \langle a, b \rangle_{\mathrm{PR}} \tau(X; F \oplus F^*).$$

The pairing (4.4) is called the *Poincaré-Reidemeister scalar product*.

The Poincaré-Reidemeister scalar product determines *the Poincaré-Reidemeister metric* (or norm) on the determinant line $\det H_*(X; F)$, which was introduced in [Fa]. It is given by

$$a \mapsto \sqrt{|\langle a, a \rangle_{\mathrm{PR}}|}, \quad a \in \det H_*(X; F)$$

(the positive square root of the absolute value of $\langle a, a \rangle_{\mathrm{PR}}$). The PR-scalar product contains an additional phase or sign information.

In the sequel we shall compute the Poincaré-Reidemeister scalar product in terms of Euler structures and their torsions. As an application, we describe when this scalar product is positive definite in terms of the Stiefel-Whitney classes $w_1(F) \in H^1(X, \mathbb{Z}/2\mathbb{Z})$ and $w_{m-1}(X) \in H^{m-1}(X, \mathbb{Z}/2\mathbb{Z})$. Namely, we shall prove the following theorem.

4.4. Theorem. *Let F be a flat $\mathbb{R}$-vector bundle over a closed connected orientable* PL*-manifold X of odd dimension m. If $m \equiv 3 \pmod 4$ then the Poincaré-Reidemeister scalar product on* $\det H_*(X; F)$ *is positive definite. If $m \equiv 1 \pmod 4$, then the Poincaré-Reidemeister scalar product on* $\det H_*(X; F)$ *is positive definite if and only if*

$$\langle w_1(F) \cup w_{m-1}(X), [X] \rangle = s\chi(X) \cdot \dim F \pmod 2, \tag{4.5}$$

where $s\chi(X)$ is the semi-characteristic of X, defined by

$$s\chi(X) = \sum_{i=0}^{(m-1)/2} \dim H_{2i}(X; \mathbb{R}).$$

Theorem 4.4 implies that the Poincaré-Reidemeister scalar product is negative definite if and only if $m \equiv 1 (\mathrm{mod}\, 4)$ and

$$\langle w_1(F) \cup w_{m-1}(X), [X] \rangle = s\chi(X) \cdot \dim F + 1 \pmod 2.$$

Theorem 4.4 will be proven in Section 6.

§5. Combinatorial Euler structures

In this section we recall combinatorial Euler structures on CW-spaces and PL-manifolds following [T2].

5.1. Euler structures on CW-spaces. Let X be a finite connected CW-space with $\chi(X) = 0$. An *Euler chain in* X is a singular 1-chain ξ in X such that

$$d\xi = \sum_a (-1)^{|a|} p_a \tag{5.1}$$

where a runs over all cells of X and p_a is a point in a; the symbol $|a|$ denotes the dimension of a. The vanishing of the Euler characteristic guarantees the existence of Euler chains. An *Euler structure on* X is an equivalence class of Euler chains with respect to an equivalence relation which we now describe.

Suppose that ξ and η are two Euler chains in X. Additionally to (5.1) we have

$$d\eta = \sum_a (-1)^{|a|} q_a, \quad \text{where} \quad q_a \in a.$$

For each cell a choose a path γ_a in a joining p_a to q_a. Then the chain

$$\xi - \eta + \sum_a (-1)^{|a|} \gamma_a$$

is a 1-cycle; we denote by $d(\xi, \eta)$ its homology class in $H_1(X) = H_1(X; \mathbb{Z})$. The class $d(\xi, \eta)$ is clearly independent of the choice of the paths $\{\gamma_a\}_a$. We say that the Euler chains ξ and η are equivalent if $d(\xi, \eta) = 0$. The set of equivalence classes (i.e., the set of Euler structures on X) is denoted by $\mathrm{Eul}(X)$. Sometimes we shall denote an Euler structure and a representing it Euler chain by the same letter.

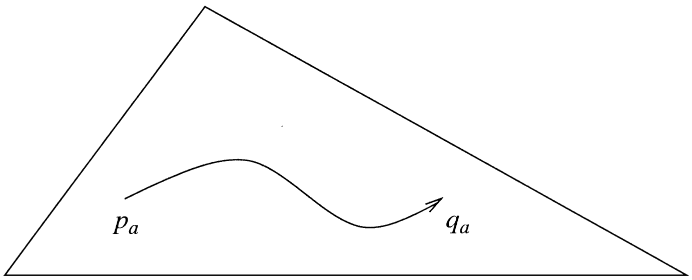

Figure 1. Path γ_a

It is clear that $H_1(X)$ acts on the set $\mathrm{Eul}(X)$: a 1-cycle h acts on a Euler chain ξ giving another Euler chain $h + \xi$. This action of $H_1(X)$ on $\mathrm{Eul}(X)$ is free and transitive. We shall use multiplicative notation both for this action and for the group operation in $H_1(X)$.

Suppose now that X' is a cellular subdivision of X. Then there is a canonical bijection

$$\sigma_{X,X'} : \mathrm{Eul}(X) \to \mathrm{Eul}(X').$$

It is defined as follows. Let ξ be an Euler chain in X so that (5.1) holds. Every cell b of X' is contained in a unique cell a of X. Choose a path γ_b in a leading from the point p_a to a certain point in b. Set

$$\xi' = \xi + \sum_b (-1)^{|b|} \gamma_b$$

where b runs over all cells of X'. It is easy to check that ξ' is an Euler chain in X'. The correspondence $\xi \mapsto \xi'$ determines a map $\sigma_{X,X'} : \mathrm{Eul}(X) \to \mathrm{Eul}(X')$. It is $H_1(X)$-equivariant and therefore bijective.

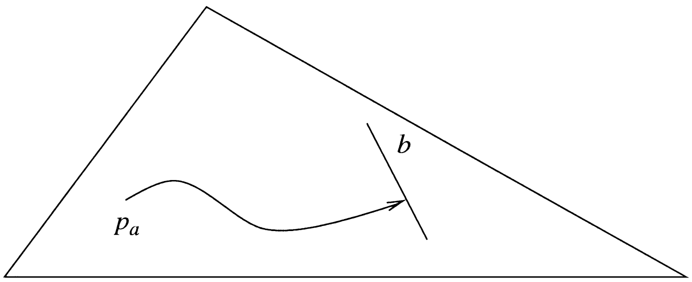

Figure 2. a, b and γ_b

5.2. Euler structures on PL-manifolds. Let X be a closed connected PL-manifold with $\chi(X) = 0$. Each piecewise linear triangulation ρ of X makes X a CW-space and allows us to consider the $H_1(X)$-set $\mathrm{Eul}(X, \rho)$. For a smaller triangulation ρ' we have the equivariant bijection

$$\sigma_{\rho,\rho'} : \mathrm{Eul}(X, \rho) \to \mathrm{Eul}(X, \rho'). \tag{5.2}$$

These sets and bijections form an inductive system whose inductive limit

$$\mathrm{Eul}(X) = \lim_{\rho} \mathrm{Eul}(X, \rho)$$

is the set of *Euler structures on X*. The group $H_1(X)$ acts on $\mathrm{Eul}(X)$ freely and transitively.

For each Euler structure ξ on X we define its *characteristic class* $c(\xi) \in H_1(X)$ following [T2], Section 5.3 and Appendix B. Choose a PL-triangulation ρ of X. Let W be the 1-chain in X defined by

$$W = \sum_{a_0 < a_1 \in \rho} (-1)^{|a_0|+|a_1|} \langle \underline{a}_0, \underline{a}_1 \rangle,$$

where a_1 runs over all simplices of ρ, a_0 runs over all proper faces of a_1, and $\langle \underline{a}_0, \underline{a}_1 \rangle$ is a path in a_1 going from the barycenter $\underline{a}_0$ of a_0 to the barycenter $\underline{a}_1$ of a_1. It is easy to check (see [HT]) that

$$\partial W = \big(1 - (-1)^m\big) \sum_{a \in \rho} (-1)^{|a|} \underline{a}$$

where $m = \dim X$. Now, any Euler structure on X can be presented by an Euler chain ξ in (X, ρ) such that $\partial\xi = \sum_a (-1)^{|a|}\underline{a}$. It is clear that $\big(1 - (-1)^m\big)\xi - W$ is a 1-cycle. Denote its homology class in $H_1(X)$ by $c(\xi)$. It follows from [T2], Lemma B.2.1, that the mapping $c : \mathrm{Eul}(X, \rho) \to H_1(X)$ commutes with the subdivision isomorphisms (5.2), i.e., $c \circ \sigma_{\rho,\rho'} = c$. In this way, we obtain a mapping $c : \mathrm{Eul}(X) \to H_1(X)$.

Note a few easy properties of the characteristic class c. If $m = \dim X$ is even, then $c(\xi)$ does not depend on ξ. If m is odd, then (in multiplicative notation)

$$c(h\xi) = h^2 c(\xi) \tag{5.3}$$

for any $\xi \in \mathrm{Eul}(X), h \in H_1(X)$. For odd m, the mod 2 reduction of $c(\xi)$ is independent of ξ and equals to the dual of the Stiefel-Whitney class $w_{m-1}(X) \in H^{m-1}(X, \mathbb{Z}/2\mathbb{Z})$. This follows from the fact that $W (\mathrm{mod}\, 2)$ represents the dual of $w_{m-1}(X)$, see [HT].

Using the characteristic class c we define a mapping $\xi \mapsto \xi^* : \mathrm{Eul}(X) \to \mathrm{Eul}(X)$ by

$$\xi^* = \big(c(\xi)\big)^{-1}\xi. \tag{5.4}$$

This mapping is an involution. It is easy to see this for odd m. Indeed, set $h = c(\xi)$ and observe that

$$\xi^{**} = \big(c(\xi^*)\big)^{-1}\xi^* = \big(c(h^{-1}\xi)\big)^{-1}h^{-1}\xi = (h^{-2}h)^{-1}h^{-1}\xi = \xi.$$

For even m, the involutivity of $*$ follows from the fact that the 1-cycle $2W$ is a boundary, see [HT].

The involution $*$ admits a simple geometric interpretation. Let ρ be a PL-triangulation of X and let ρ^* be the dual cellular decomposition of X. Let us represent $\xi \in \mathrm{Eul}(X)$ by an Euler chain in (X, ρ) denoted by the same letter ξ. We can choose this chain so that $\partial\xi = \sum_{a\in\rho} (-1)^{|a|}\underline{a}$. Since the barycenter $\underline{a}$ of a belongs to the dual $(m - |a|)$-dimensional cell a^*, the 1-chain $(-1)^m\xi$ is an Euler chain in (X, ρ^*). It represents the Euler structure $\xi^* \in \mathrm{Eul}(X) = \mathrm{Eul}(X, \rho^*)$ (for a proof, see [T2], Lemma B.2.3).

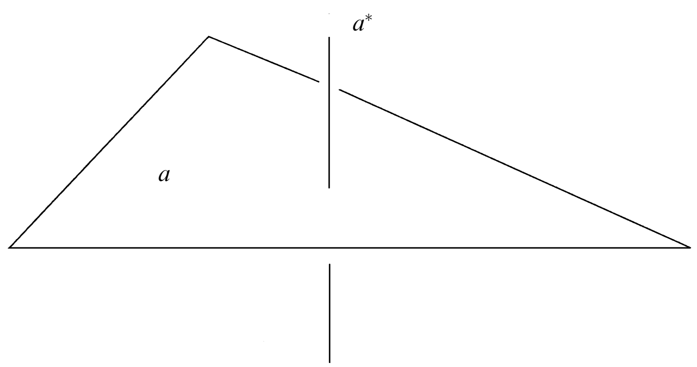

Figure 3. Simplex a and the dual cell a^*

§6. Refined torsions. Main theorem

In this section we recall the torsions of Euler structures essentially following [T1], [T2]. The novelty here (compared to [T1], [T2]) is that we view the torsion as an element of the determinant line of homology and also in a systematic treatment of the torsions of flat vector bundles. In this section we state our main theorem relating the torsion to the Poincaré-Reidemeister scalar product.

We shall first define the torsions of Euler structures modulo $(-1)^{\dim F}$; in particular this gives well-defined torsions, for even-dimensional F. For odd-dimensional F, we need to involve additional data (a homology orientation of the base) to make the torsions of Euler structures well-defined.

6.1. Torsion of Euler structures. Let F be a flat vector bundle over a finite connected CW-space X with $\chi(X) = 0$. For each Euler structure ξ on X we define a torsion $\tau(X, \xi; F)$ which is an element of the determinant line $\det H_*(X; F)$ defined up to multiplication by $(-1)^{\dim F}$.

As in Section 3 we consider the chain complex $C = C_*(X; F)$ and the associated torsion isomorphism $\varphi_C : \det C \to \det H_*(X; F)$. Set

$$\tau(X, \xi; F) = \varphi_C(c_0 \otimes c_1^{-1} \otimes c_2 \otimes \cdots \otimes c_m^{(-1)^m}) \in \det H_*(X; F) \tag{6.1}$$

where $m = \dim X$ and $c_q \in \det C_q(X; F)(q = 0, 1, \ldots, m)$ are non-zero elements defined as follows. Fix a point $x \in X$ and a basis e_x in the fiber F_x. Let $\beta_a : [0, 1] \to X$ be a path connecting $x = \beta_a(0)$ to a point $\beta_a(1) \in a$. The assumption $\chi(X) = 0$ implies that the 1-chain $\sum_a (-1)^{|a|}\beta_a$ (where a runs over all cells of X) is an Euler chain with boundary $\sum_a (-1)^{|a|}\beta_a(1)$. We choose the paths $\{\beta_a\}_a$ so that this chain represents ξ. We apply the parallel transport to e_x along β_a to obtain a basis in the fiber $F_{\beta_a(1)}$ and we extend it to a basis of flat sections over a. The concatenation of these bases over all q-dimensional cells gives a basis in $C_q(X; F)$ via (3.1). The wedge product of the elements of this basis yields $c_q \in \det C_q(X; F)$.

Let us check the indeterminacy in the definition of $\tau(X, \xi; F)$. A different choice of e_x transforms the bases in $\{\Gamma(a, F)\}_a$ via one and the same invertible matrix, A. The torsion $\tau(X, \xi; F)$ is multiplied by $(\det A)^{\chi(X)} = 1$ and therefore does not depend on the choice of e_x. We can replace the path β_a by its composition with a path in a beginning in the point $\beta_a(1)$. This does not change the basis of $\Gamma(a, F)$ constructed above and therefore does not change $\tau(X, \xi; F)$. We can also multiply each β_a by a loop $\gamma_a : ([0, 1], 0, 1) \to (X, x, x)$ such that the product $\prod_a (-1)^{|a|}\gamma_a$ is homologically trivial. When we replace β_a by $\beta_a\gamma_a$, the element $c_{|a|} \in \det C_{|a|}(X; F)$ is multiplied by $\det_F([\gamma_a])$ where $[\gamma_a] \in H_1(X)$ is the homological class of γ_a and $\det_F : H_1(X) \to \boldsymbol{k}^*$ is the determinant of the monodromy of F. The torsion $\tau(X, \xi; F)$ is multiplied by $\prod_a \det_F(\gamma_a)^{(-1)^{|a|}} = 1$ and therefore is not changed. We can also simultaneously replace the paths $\{\beta_a\}$ by their compositions $\{\beta_a\gamma\}$ where γ is a path in X leading from a point $y \in X$ to x. Choosing as e_y the basis in F_y obtained from e_x by the parallel transport along γ^{-1} we observe that the data $y, e_y, \{\beta_a\gamma\}$ give rise to the same $c_q \in \det C_q(X;F)$ $(q = 0, 1, \dots, m)$. Therefore $\tau(X, \xi; F)$ does not depend on the choice of the point x. Finally, as in Section 3.1, there is a sign indeterminacy $(-1)^{\dim F}$ coming from orders and orientations of the cells. We conclude that $\tau(X, \xi; F)$ is defined up to multiplication by $(-1)^{\dim F}$. In particular, for even-dimensional F, the torsion $\tau(X, \xi; F)$ is a well defined element of $\det H_*(X; F)$.

It follows directly from definitions that

$$\tau(X, h\xi; F) = \det_F(h) \cdot \tau(X, \xi; F),$$

for any $h \in H_1(X)$ and $\xi \in \mathrm{Eul}(X)$. For unimodular F, we have $\tau(X, \xi; F) = \tau(X; F)$ where $\tau(X; F)$ is the torsion defined in Section 3.1.

It follows from [T2], Lemma 3.2.3 that the torsion $\tau(X, \xi; F)$ is invariant under cellular subdivisions of X. More precisely, if X' is a cellular subdivision of X then

$$\tau(X, \xi; F) = \tau\big(X', \sigma_{X,X'}(\xi); F\big) \tag{6.2}$$

where $\sigma_{X,X'} : \mathrm{Eul}(X) \to \mathrm{Eul}(X')$ is the canonical bijection constructed in Section 5.1. (Note that both parts of (6.2) are defined up to multiplication by $(-1)^{\dim F}$.) This fact allows us to consider torsions of Euler structures on PL-manifolds.

6.2. Main Theorem (even-dimensional case). *Let F be an even-dimensional flat* ***k****-vector bundle over a closed connected orientable* PL*-manifold X of odd dimension. Then for any Euler structure $\xi \in \mathrm{Eul}(X)$, we have*

$$\langle \tau(X, \xi; F), \tau(X, \xi; F) \rangle_{\mathrm{PR}} = \det_F\big(c(\xi)\big). \tag{6.3}$$

Since the mod 2 reduction of the characteristic class $c(\xi)$ is dual to the Stiefel-Whitney class $w_{m-1}(X)$, Theorem 6.2 implies Theorem 4.4 in the case of even-dimensional F.

Using the equality $\xi = c(\xi)\xi^*$ we can reformulate formula (6.3) as follows:

$$\langle \tau(X, \xi; F), \tau(X, \xi^*; F) \rangle_{\mathrm{PR}} = 1.$$

To give similar formulas for odd-dimensional bundles, we need a sign-determined version of $\tau(X, \xi; F)$ discussed in the next subsection.

6.3. Sign-refined torsion. Let F be an odd-dimensional flat vector bundle over a finite connected CW-space X with $\chi(X) = 0$. Assume that X is endowed with an orientation η of the determinant line of real homologies $\det H_*(X; \mathbb{R})$. (Such X is said to be homology oriented.) Following [T1], we introduce for each $\xi \in \mathrm{Eul}(X)$ a torsion $\tau(X, \eta, \xi; F) \in \det H_*(X; F)$ which has no indeterminacy.

Let us orient and order the cells of X. Set

$$\tau_0 = \varphi_C(c_0 \otimes c_1^{-1} \otimes c_2 \otimes \cdots \otimes c_m^{(-1)^m}) \in \det H_*(X; F)$$

where $m = \dim X$ and $c_q \in \det C_q(X;F)$ $(q = 0, 1, \dots, m)$ are non-zero elements determined by ξ as in Section 6.1. Consider the cellular chain complex $C_{\mathbb{R}} = C_*(X;\mathbb{R})$ determined by the trivial line bundle over X. Clearly, $H_*(C) = H_*(X;\mathbb{R})$. The orientation and order of the cells of X yield a basis of $C_{\mathbb{R}}$ which determines an element $c \in \det C_{\mathbb{R}}$. Recall the torsion isomorphism $\varphi_{C_{\mathbb{R}}} : \det C_{\mathbb{R}} \to \det H_*(X;\mathbb{R})$. Set $\tau(X,\eta,\xi;F) = \tau_0 \in \det H_*(X;F)$ if the element $\varphi_{C_{\mathbb{R}}}(c) \in \det H_*(X;\mathbb{R})$ defines the orientation η. In the opposite case set $\tau(X,\eta,\xi;F) = -\tau_0 \in \det H_*(X;F)$. It is easy to check that $\tau(X,\eta,\xi;F)$ has no indeterminacy. In particular, when we change the orientation or order of the cells of X the signs $(-1)^{\dim F} = -1$ appear simultaneously in $\varphi_{C_{\mathbb{R}}}(c)$ and τ_0 and cancel each other (cf. Section 3.1).

Clearly, $\tau(X,\xi;F) = \pm\tau(X,\eta,\xi;F)$ is the torsion discussed in Section 6.1. Note that $\tau(X,-\eta,\xi;F) = -\tau(X,\eta,\xi;F)$ and $\tau(X,\eta,h\xi;F) = \det_F(h)\tau(X,\eta,\xi;F)$ for any $\xi \in \mathrm{Eul}(X)$ and $h \in H_1(X)$.

The torsion $\tau(X,\eta,\xi;F)$ is invariant under cell subdivisions of X, see [T1], Theorem 3.2.1. (It is to ensure this that we need the signs $(-1)^{N(C)}$ and $(-1)^{N(C_{\mathbb{R}})}$ in the definition of the torsion isomorphisms $\varphi_C, \varphi_{C_{\mathbb{R}}}$.) The invariance of $\tau(X,\eta,\xi;F)$ under cell subdivisions allows us to apply this torsion to PL-manifolds.

6.4. Main Theorem (odd-dimensional case)**.** *Let F be an odd-dimensional flat* ***k****-vector bundle over a closed connected orientable* PL*-manifold X of odd dimension m. Then for any Euler structure $\xi \in \mathrm{Eul}(X)$ and any homology orientation η of X, we have*

$$\langle \tau(X,\eta,\xi;F), \tau(X,\eta,\xi;F)\rangle_{\mathrm{PR}} = (-1)^z \det_F\big(c(\xi)\big) \tag{6.4}$$

where z is the residue given by

$$z = \begin{cases} 0, & \text{if } \dim F \text{ is even or } m \equiv 3 \pmod 4, \\ s\chi(X) \pmod 2, & \text{if } \dim F \text{ is odd and } m \equiv 1 \pmod 4. \end{cases} \tag{6.5}$$

Theorem 6.4 implies the identity

$$\langle \tau(X,\eta,\xi;F), \tau(X,\eta,\xi^*;F)\rangle_{\mathrm{PR}} = (-1)^z.$$

Theorems 6.2 and 6.4 are the main results of this paper. They compute the Poincaré-Reidemeister scalar product in terms of Euler structures and their characteristic classes and torsions. A proof of Theorems 6.2 and 6.4 is given in Section 8 using the results of Section 7.

6.5. Proof of Theorem 4.4. It follows from Theorems 6.2 and 6.4, that the Poincaré-Reidemeister scalar product on $\det H_*(X;F)$ is positive definite if and only if the real number $(-1)^z \det_F(h)$ is positive, where $h \in H_1(X)$ is a class whose mod 2 reduction is dual to $w_{m-1}(X)$ and $z \in \mathbb{Z}/2\mathbb{Z}$ is the residue given by (6.5). The sign of the non-zero real number $\det_F(h)$ is equal to $(-1)^{w_1(F)(h)}$ where

$$w_1(F)(h) = \langle w_1(F), w_{m-1}(X) \cap [X]\rangle = \langle w_1(F) \cup w_{m-1}(X), [X]\rangle.$$

This proves Theorem 4.4 for $m \equiv 1 \pmod 4$.

It is a theorem of W. Massey [Ma], Theorem III, that $w_{m-1}(X) = 0$ for any closed orientable smooth manifold X of dimension $m \equiv 3 \pmod 4$. This together with the previous argument gives the claim of Theorem 4.4 for $m \equiv 3 \pmod 4$ assuming that M is smoothable. Vanishing of the class $w_{m-1}(X)$ for any closed orientable PL-manifold X of dimension $m \equiv 3 \pmod 4$ can be obtained similarly to [Ma]. It also follows from our arguments used in the proof of Theorem 11.2 (cf. formula (11.2) and Remark 11.4). This gives our statement for $m \equiv 3 \pmod 4$ in the PL case. □

6.6. Remarks. 1. Any closed oriented manifold X of odd dimension m has a canonical homology orientation determined by any basis in $\bigoplus_{i<m/2} H_i(X;\mathbb{R})$ followed by the Poincaré dual basis in $\bigoplus_{i>m/2} H_i(X;\mathbb{R})$.

2. We could formulate a version of Theorem 6.4 without involving the sign-refined torsions. Namely, for odd-dimensional F, we have

$$\langle \tau(X,\xi;F), \tau(X,\xi;F)\rangle_{\mathrm{PR}} = (-1)^z \det{}_F\big(c(\xi)\big) \tag{6.6}$$

where z is the number defined by (6.5). This formula makes sense: although the torsion $\tau(X,\xi;F)$ is defined up to sign, the scalar product on the left hand side of (6.6) is well defined. Formula (6.6) directly follows from (6.4).

§7. Properties of the torsion: multiplicativity and duality

In this section we establish two important properties of the torsion of Euler structures: multiplicativity with respect to direct sums and compatibility with the duality operator. These properties will be used in the proof of Theorems 6.2 and 6.4 in Section 8.

7.1. Theorem. *Let F, F' be flat vector bundles over a finite connected* CW*-space X with $\chi(X) = 0$. Let*

$$\mu = \mu_{H_*(X;F),\,H_*(X;F')}$$

be the canonical fusion isomorphism

$$\det H_*(X;F) \otimes \det H_*(X;F') \to \det\big(H_*(X;F) \oplus H_*(X;F')\big) = \det H_*(X;F \oplus F')$$

defined in Section 2.3. *If both F and F' are even-dimensional then for any $\xi \in \mathrm{Eul}(X)$*

$$\tau(X,\xi;F \oplus F') = \mu\big(\tau(X,\xi;F) \otimes \tau(X,\xi;F')\big). \tag{7.1}$$

If both F and F' are odd-dimensional then for any $\xi \in \mathrm{Eul}(X)$ and any homology orientation η of X,

$$\tau(X,\xi;F \oplus F') = \mu\big(\tau(X,\eta,\xi;F) \otimes \tau(X,\eta,\xi;F')\big). \tag{7.2}$$

Proof. Denote by α_q the number of cells of X of dimension $\leqq q$ and by $r_q = \alpha_q - \alpha_{q-1}$ the number of q-dimensional cells of X. Consider the chain complexes $C = C_*(X;F)$, $C' = C_*(X;F')$, and $\tilde{C} = C_*(X;F \oplus F')$. It is clear that $\tilde{C} = C \oplus C'$.

Let us orient and order the cells of X and fix a spider-like Euler chain representing ξ as in Section 6.1. The constructions of Section 6.1 provide bases in C_q, C'_q, and $\tilde{C}_q$. The basis in C_q is formed by a sequence $D_1, \dots, D_{r_q}$ where D_s is a flat basis of F over the s-th q-dimensional cell of X. The basis in C'_q is formed by a sequence $D'_1, \dots, D'_{r_q}$ where D'_s is a flat basis of F' over the s-th q-dimensional cell of X. The basis in $\tilde{C}_q$ is formed by a sequence $D_1, D'_1, D_2, D'_2, \dots, D_{r_q}, D'_{r_q}$. Consider the corresponding wedge products $c_q \in \det C_q$, $c'_q \in \det C'_q$, and $\tilde{c}_q \in \det \tilde{C}_q$. Using the canonical identification $\det \tilde{C}_q = \det C_q \otimes \det C'_q$ we obtain

$$\tilde{c}_q = (-1)^{\frac{(\alpha_q - \alpha_{q-1} - 1)(\alpha_q - \alpha_{q-1})}{2} dd'} (c_q \otimes c'_q)$$

where $d = \dim F = \operatorname{card} D_s$ and $d' = \dim F' = \operatorname{card} D_s$ for all s.

Consider the case where both d and d' are even. In this case $\tilde{c}_q = c_q \otimes c'_q$ for all q. By definition,

$$\tau(X, \xi; F) = \varphi_C(c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_m^{(-1)^m}),$$

and

$$\tau(X, \xi; F') = \varphi_{C'}\big(c'_0 \otimes (c'_1)^{-1} \otimes \cdots \otimes (c'_m)^{(-1)^m}\big).$$

Lemma 3.3 implies that

$$\begin{aligned} &\mu\big(\tau(X, \xi; F) \otimes \tau(X, \xi; F')\big) \\ &\quad = (\varphi_{\tilde{C}} \mu_{C,C'})\big(c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_m^{(-1)^m} \otimes c'_0 \otimes (c'_1)^{-1} \otimes \cdots \otimes (c'_m)^{(-1)^m}\big). \end{aligned}$$

By definition of $\mu_{C,C'}$ and by $\tilde{c}_q = c_q \otimes c'_q$, the right-hand side equals

$$(-1)^{M(C,C')} \varphi_{\tilde{C}}\big(\tilde{c}_0 \otimes (\tilde{c}_1)^{-1} \otimes \cdots \otimes (\tilde{c}_m)^{(-1)^m}\big) = \tau(X, \xi; F \oplus F').$$

Here we use the fact that $\alpha_q(C) = d \cdot \alpha_q$ is even so that $M(C, C') = 0$.

Assume that both d and d' are odd. By definition, $\tau(X, \eta, \xi; F) = \varepsilon \tau_0$, and $\tau(X, \eta, \xi; F') = \varepsilon \tau'_0$, where

$$\tau_0 = \varepsilon \varphi_C(c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_m^{(-1)^m}), \quad \tau'_0 = \varepsilon \varphi_{C'}\big(c'_0 \otimes (c'_1)^{-1} \otimes \cdots \otimes (c'_m)^{(-1)^m}\big),$$

and $\varepsilon = \pm 1$ is a sign determined by η and the chosen orientations and order of the cells of X. It is important that one and the same sign ε appears in the expressions for $\tau(X, \eta, \xi; F)$ and $\tau(X, \eta, \xi; F')$. The same argument as above shows that

$$\mu\big(\tau(X, \eta, \xi; F) \otimes \tau(X, \eta, \xi; F')\big) = \mu(\tau_0 \otimes \tau'_0) = (-1)^{M(C,C') + R} \tau(X, \xi; F \oplus F')$$

where

$$R = \sum_{q=0}^{m} \frac{(\alpha_q - \alpha_{q-1} - 1)(\alpha_q - \alpha_{q-1})}{2}.$$

It remains to show that $M(C,C')+R$ is even. By definition,

$$M(C,C') = \sum_{q=0}^{m} dd'\alpha_{q-1}\alpha_q \equiv \sum_{q=0}^{m} \alpha_{q-1}\alpha_q \pmod 2.$$

A direct computation yields

$$R \equiv \sum_{q=0}^{m} \alpha_q - \sum_{q=0}^{m} \alpha_{q-1}\alpha_q - (\alpha_m + \alpha_m^2)/2 \pmod 2.$$

If P (resp. Q) is the number of even-dimensional (resp. odd-dimensional) cells of X then $P-Q=\chi(X)=0$, $\alpha_m = P+Q = 2P$, and $\sum_{q=0}^{m} \alpha_q \equiv P \equiv Q \pmod 2$. This implies that $M(C,C')+R$ is even and completes the proof of the lemma. □

7.2. Theorem. *Let F be a flat vector bundle over a closed connected orientable* PL-*manifold X of odd dimension m. Let $\xi \in \mathrm{Eul}(X)$ and let $D : \det H_*(X;F) \to \det H_*(X;F^*)$ be the isomorphism* (4.2). *If* $\dim F$ *is even then*

$$D\big(\tau(X,\xi;F)\big) = \tau(X,\xi^*;F^*). \tag{7.3}$$

If $\dim F$ *is odd then for any homology orientation η of X*

$$D\big(\tau(X,\eta,\xi;F)\big) = (-1)^z \tau(X,\eta,\xi^*;F^*), \tag{7.4}$$

where z is the number given by (6.5).

Theorem 7.2 is a refined version of the classical duality for torsions due to Franz and Milnor, see also [T1], [T2].

Proof. Fix an orientation of X. Consider first the case of even-dimensional F. Fix a piecewise linear triangulation ρ of X. We orient and order the simplices of ρ in an arbitrary way. Fix a point $x \in X$. For each simplex a of ρ, choose a path $\beta_a : [0,1] \to X$ connecting $x = \beta_a(0)$ to the barycenter of a so that the 1-chain $\sum_a (-1)^{|a|}\beta_a$ represents ξ in $\mathrm{Eul}(X,\rho)$. As in Section 6.1, this chain and a basis e_x of the fiber F_x determine an ordered basis of the simplicial chain complex $C = C_*\big((X,\rho);F\big)$ and a distinguished element, $c \in \det C$. By definition, $\tau(X,\xi;F) = \varphi_C(c)$.

To compute the torsion $\tau(X,\xi^*;F^*)$ we shall use the dual cellular subdivision ρ^* of X. It is well known that the simplicial chain complex $C = C_*\big((X,\rho);F\big)$ and the cellular chain complex $C' = C_*\big((X,\rho^*);F^*\big)$ are dual to each other. Let us provide the cells of ρ^* with the order and orientation induced by the order and orientation of the simplices of ρ. (To define the induced orientation in the dual cells we use the orientation of X.) According to the last remark of Section 5.2, the chain $-\sum_a (-1)^{|a|}\beta_a$ represents ξ^* in $\mathrm{Eul}(X,\rho^*)$. As in Section 6.1, this chain and a basis of the fiber F_x^* determine an ordered basis of $C' = C_*\big((X,\rho^*);F^*\big)$ and a distinguished element, $c' \in \det C'$. By definition, $\tau(X,\xi^*;F^*) = \varphi_{C'}(c')$. Observe, that if in the role of the basis in F_x^* we take the dual basis e_x^*, then the basis in C' constructed in this way is dual to the basis in C constructed above. Note that all vector spaces C'_q are even-dimensional so that $\alpha_q(C') = 0$ for all q. Therefore, in this case $c' = D_C(c)$. It remains to apply Lemma 2.6 to the complex C (cf. 2.5 and 4.2). This gives

$$D\big(\tau(X, \xi; F)\big) = D_{H_*(C)}\big(\varphi_C(c)\big) = \varphi_{C'}\big(D_C(c)\big) = \varphi_{C'}(c') = \tau(X, \xi^*; F^*).$$

Assume now that F is odd-dimensional. As above, we construct distinguished elements $c \in \det C, c' \in \det C'$ and observe that $c' = (-1)^s D_C(c)$ where $s \in \mathbb{Z}/2\mathbb{Z}$ is given by

$$s = s(C) = \sum_{q=1}^{m} \alpha_{q-1}\alpha_q + \sum_{q=0}^{(m-1)/2} \alpha_{2q} \pmod 2 \tag{7.5}$$

where α_q is the number of simplices of ρ of dimension $\leqq q$. Consider the simplicial chain complex $C_{\mathbb{R}} = C_*\big((X, \rho); \mathbb{R}\big)$ and the volume element $c_{\mathbb{R}} \in \det C_{\mathbb{R}}$ determined by the orientation and order of the simplices of ρ. Similarly, consider the cellular chain complex $C'_{\mathbb{R}} = C_*\big((X, \rho^*); \mathbb{R}\big)$ and the volume element $c'_{\mathbb{R}} \in \det C'_{\mathbb{R}}$ determined by the orientation and order of the cells of ρ^*. Recall the torsion isomorphisms $\varphi_{C_{\mathbb{R}}} : \det C_{\mathbb{R}} \to \det H_*(X; \mathbb{R})$ and $\varphi_{C'_{\mathbb{R}}} : \det C'_{\mathbb{R}} \to \det H_*(X; \mathbb{R})$. By definition, $\tau(X, \eta, \xi; F) = \varepsilon\varphi_C(c)$ where $\varepsilon = +1$ if the volume element $\varphi_{C_{\mathbb{R}}}(c_{\mathbb{R}}) \in \det H_*(X; \mathbb{R})$ defines the given homology orientation η and $\varepsilon = -1$ otherwise. Similarly, $\tau(X, \eta, \xi^*; F^*) = \varepsilon'\varphi_{C'}(c')$ where $\varepsilon' = +1$ if

$$\varphi_{C'_{\mathbb{R}}}(c'_{\mathbb{R}}) \in \det\, H_*(X; \mathbb{R})$$

defines η and $\varepsilon' = -1$ otherwise. As above, $c'_{\mathbb{R}} = (-1)^s D_{C_{\mathbb{R}}}(c_{\mathbb{R}})$ where

$$s = s(C_{\mathbb{R}}) = s(C) \in \mathbb{Z}/2\mathbb{Z}$$

is the residue (7.5). By Lemma 2.6,

$$\varphi_{C'_{\mathbb{R}}}(c'_{\mathbb{R}}) = \varphi_{C'_{\mathbb{R}}}\big((-1)^s D_{C_{\mathbb{R}}}(c_{\mathbb{R}})\big) = (-1)^s D_{H_*(X;\mathbb{R})}\big(\varphi_{C_{\mathbb{R}}}(c_{\mathbb{R}})\big).$$

We can conclude that $\varepsilon\varepsilon' = (-1)^s \nu$ where $\nu = +1$ if the linear mapping

$$D_{H_*(X;\mathbb{R})} : \det H_*(X; \mathbb{R}) \to \det H_*(X; \mathbb{R})$$

preserves the orientation of the line $\det H_*(X; \mathbb{R})$ and $\nu = -1$ otherwise. A computation in [T1], pp. 178–179 (see also Section 11) shows that $\nu = (-1)^z$ where z is the number given by (6.5). As in the even dimensional case, we apply Lemma 2.6 to the complex C and to the duality operator $D_{H_*(C)} = D : \det H_*(C) \to \det H_*(C')$ (cf. 2.5 and 4.2). This gives

$$\begin{aligned} D\big(\tau(X, \eta, \xi; F)\big) &= D_{H_*(C)}\big(\varepsilon\varphi_C(c)\big) = \varepsilon\varphi_{C'}\big(D_C(c)\big) \\ &= \varepsilon\varphi_{C'}\big((-1)^s c'\big) = (-1)^s \varepsilon\varepsilon'\tau(X, \eta, \xi^*; F^*) = (-1)^z \tau(X, \eta, \xi^*; F^*). \quad \square \end{aligned}$$

§8. Proof of Theorems 6.2 and 6.4

8.1. Proof of Theorem 6.2. Set $T = \tau(X, \xi; F)$. We should prove that $\langle T, T\rangle_{\mathrm{PR}} = \det_F\big(c(\xi)\big)$.

Since the bundle $F \oplus F^*$ is even-dimensional and unimodular, the torsion $\tau(X; F \oplus F^*)$ is well defined and equals $\tau(X, \xi; F \oplus F^*)$, for any $\xi \in \mathrm{Eul}(X)$. By Theorem 7.1,

$$\tau(X; F \oplus F^*) = \mu\big(\tau(X, \xi; F) \otimes \tau(X, \xi; F^*)\big) = \mu\big(T \otimes \tau(X, \xi; F^*)\big)$$

where μ is the canonical fusion isomorphism

$$\det H_*(X; F) \otimes \det H_*(X; F^*) \to \det\big(H_*(X; F) \oplus H_*(X; F^*)\big) = \det H_*(X; F \oplus F^*)$$

defined in Section 2.3. By Theorem 7.2,

$$\begin{aligned} D(T) &= \tau(X, \xi^*; F^*) = \tau\big(X, (c(\xi))^{-1}\xi; F^*\big) \\ &= \det_{F^*}\big((c(\xi))^{-1}\big)\tau(X, \xi; F^*) = \det_F\big(c(\xi)\big)\tau(X, \xi; F^*). \end{aligned}$$

By definition,

$$\begin{aligned} \langle T, T \rangle_{\mathrm{PR}} &= \mu\big(T \otimes D(T)\big)/\tau(X; F \oplus F^*) \\ &= \mu\big(T \otimes \det_F\big(c(\xi)\big)\tau(X, \xi; F^*)\big)/\mu\big(T \otimes \tau(X, \xi; F^*)\big) = \det_F\big(c(\xi)\big). \quad \square \end{aligned}$$

8.2. Proof of Theorem 6.4. Set $T = \tau(X, \eta, \xi; F)$. By Theorem 7.1,

$$\tau(X; F \oplus F^*) = \tau(X, \xi; F \oplus F^*) = \mu\big(T \otimes \tau(X, \eta, \xi; F^*)\big).$$

By Theorem 7.2,

$$D(T) = (-1)^z \tau(X, \eta, \xi^*; F^*) = (-1)^z \det_F\big(c(\xi)\big)\tau(X, \xi; F^*).$$

Thus,

$$\begin{aligned} \langle T, T \rangle_{\mathrm{PR}} &= \mu\big(T \otimes D(T)\big)/\tau(X; F \oplus F^*) \\ &= \mu\big(T \otimes (-1)^z \det_F\big(c(\xi)\big)\tau(X, \eta, \xi; F^*)\big)/\mu\big(T \otimes \tau(X, \eta, \xi; F^*)\big) \\ &= (-1)^z \det_F\big(c(\xi)\big). \quad \square \end{aligned}$$

§9. Cohomological torsions and the PR-pairing

In this section we give cohomological versions of both the Poincaré-Reidemeister scalar products and the torsions of Euler structures. This cohomological formulation is better suited for a comparison with the analytical approach, see Section 10.

9.1. Cohomology of a flat vector bundle. Let F be a flat $\boldsymbol{k}$-vector bundle over a finite connected CW-space X. Recall a definition of the cohomology of X with coefficients in F. Orient all cells of X. As in Section 3.1, for a cell a of X, denote by $\Gamma(a, F)$ the vector space of flat sections of F over a. The vector space of q-cochains in X with values in F coincides with the vector space of q-chains and is defined by

$$C^q(X;F) = \bigoplus_{\dim a = q} \Gamma(a, F). \tag{9.1}$$

The boundary homomorphism $\delta_q : C^q(X;F) \to C^{q+1}(X;F)$ is defined as follows. Let s_a be a flat section of F over a q-cell a. We set

$$\delta_q(s_a) = \sum_b \varepsilon(a,b) s_a^b$$

where the sum runs over all $(q+1)$-cells b incident to a, the sign $\varepsilon(a,b) = \pm 1$ is determined in the usual way by the orientations of a and b, and s_a^b denotes the unique flat section over b extending s_a. (It is understood that each b enters this sum with multiplicity equal to the number of appearances of a in ∂b.) Denote the resulting cochain complex by $C^*(X;F)$ and set $H^*(X;F) = H^*(C^*(X;F))$. The graded vector space $H^*(X;F)$ is a homotopy invariant of the pair (X, F).

It is clear that the vector space $C^q(X;F)$ is dual to $C_q(X;F^*)$, i.e.,

$$C^q(X;F) = \mathrm{Hom}_{\boldsymbol{k}}(C_q(X;F^*), \boldsymbol{k})$$

and the boundary homomorphism δ_q introduced above is dual to the boundary homomorphism $C_{q+1}(X;F^*) \to C_q(X;F^*)$. Therefore for each q, we have a non-singular evaluation pairing

$$H^q(X;F) \otimes H_q(X;F^*) \to \boldsymbol{k}.$$

These pairings for $q = 0, \dots, \dim X$ induce a non-singular pairing

$$[\,,\,] : \det H^*(X;F) \otimes \det H_*(X;F^*) \to \boldsymbol{k}. \tag{9.2}$$

9.2. Cohomological torsion. Let F be a flat $\boldsymbol{k}$-vector bundle over a finite connected CW-space X with $\chi(X) = 0$. If $\dim F$ is odd, then we additionally assume that X is provided with a homology orientation (which we suppress in the notation). For every $\xi \in \mathrm{Eul}(X)$, we define the cohomological torsion $\tau^\bullet(X, \xi; F)$ as the unique element of $\det H^*(X;F)$ such that

$$[\tau^\bullet(X, \xi, F), \tau(X, \xi, F^*)] = 1 \tag{9.3}$$

where $\tau(X, \xi; F^*) \in \det H_*(X;F^*)$ is the torsion defined in Section 6. The cohomological torsion $\tau^\bullet(X, \xi; F) \in \det H^*(X;F)$ satisfies properties similar to those of the homological torsion. In particular, it is invariant under cell subdivisions and has no indeterminacy. For any $h \in H_1(X)$, we have

$$\tau^\bullet(X, h\xi; F) = \det_F(h) \cdot \tau^\bullet(X, \xi; F). \tag{9.4}$$

9.3. Cohomological Poincaré-Reidemeister scalar product. We define a cohomological version of the Poincaré-Reidemeister scalar product. The norm determined by this scalar product was originally defined in [Fa].

Let F be a flat $\boldsymbol{k}$-vector bundle over a closed connected orientable PL manifold X of odd dimension m. Given $\alpha, \beta \in \det H^*(X;F)$, we define a number $\langle\alpha,\beta\rangle_{\mathrm{PR}} \in \boldsymbol{k}$ by

$$\langle\alpha,\beta\rangle_{\mathrm{PR}} = \frac{[\alpha,a]\cdot[\beta,b]}{\langle a,b\rangle_{\mathrm{PR}}} \tag{9.5}$$

for any nonzero $a, b \in \det H_*(X;F^*)$. Here $\langle a,b\rangle_{\mathrm{PR}}$ denotes the homological Poincaré-Reidemeister scalar product (defined in Section 4) and the square brackets denote the pairing (9.2). Formula (9.5) yields a well-defined bilinear form on $\det H^*(X;F)$ called the cohomological Poincaré-Reidemeister scalar product.

Let us show that the norm,

$$\alpha \mapsto |\langle\alpha,\alpha\rangle_{\mathrm{PR}}|^{1/2}, \quad \alpha \in \det H^*(X;F),$$

determined by scalar product (9.5), coincides with the Poincaré-Reidemeister norm on $\det H^*(X;F)$, introduced in [Fa], section 4.7. This fact will be used in Section 10.

Denote by

$$\mu^\bullet : \det H^*(X;F) \otimes \det H^*(X;F^*) \to \det H^*(X;F\oplus F^*)$$

the canonical isomorphism, defined similarly to (2.6) (ignoring the signs). Let

$$D^\bullet : \det H^*(X;F) \to \det H^*(X;F^*),$$

be the Poincaré duality isomorphism (a cohomological version of (4.2)). In our present notation, the Poincaré-Reidemeister metric on $\det H^*(X;F)$, which defined in Section 4.7 of [Fa], is given by

$$\alpha \mapsto |\mu^\bullet(\alpha\otimes D^\bullet\alpha)/\tau^\bullet(X;F\oplus F^*)|^{1/2},$$

where $\alpha \in \det H^*(X;F)$. In order to prove compatibility with (9.5) it is enough to show that for any $a, b \in \det H_*(X;F^*)$, and $\alpha, \beta \in \det H^*(X;F)$ holds

$$\big(\mu^\bullet(\alpha\otimes D^\bullet\beta)/\tau^\bullet(X;F\oplus F^*)\big)\cdot\big(\mu(a\otimes Db)/\tau(X;F^*\oplus F)\big) = \pm[\alpha,a]\cdot[\beta,b].$$

The left hand side may be rewritten as

$$\frac{[\mu^\bullet(\alpha\otimes D^\bullet\beta), \mu(\alpha\otimes D\beta)]}{[\tau^\bullet(X;F\oplus F^*), \tau(X;F^*\oplus F)]}$$

where the brackets [,] denote the pairing (9.2) for the flat vector bundle $F\oplus F^*$. By (9.3), the denominator of the last expression is equal to ± 1. It remains to check that

$$[\mu^\bullet(\alpha\otimes D^\bullet\beta), \mu(\alpha\otimes D\beta)] = \pm[\alpha,a]\cdot[\beta,b]$$

where the brackets [,] denote the pairings (9.2) for F, F^*, and $F\oplus F^*$. The last equality follows from

$$[\mu^{\bullet}(\alpha \otimes D^{\bullet}\beta), \mu(\alpha \otimes D\beta)] = \pm[\alpha, a] \cdot [D^{\bullet}\beta, Db] = \pm[\alpha, a] \cdot [\beta, b]$$

(cf. [Fa], Section 3.4).

9.4. Main Theorem (cohomological version). *Let F be a flat $\boldsymbol{k}$-vector bundle over a closed connected orientable* PL-*manifold X of odd dimension m. If $\dim F$ is odd, then we additionally assume that X is provided with a homology orientation. Then for any $\xi \in \mathrm{Eul}(X)$,*

$$\langle \tau^{\bullet}(X, \xi; F), \tau^{\bullet}(X, \xi; F) \rangle_{\mathrm{PR}} = (-1)^{z} \det{}_{F}\big(c(\xi)\big), \tag{9.6}$$

where $\langle\ ,\ \rangle_{\mathrm{PR}}$ is the cohomological Poincaré-Reidemeister scalar product and z is the number given by (6.5).

Proof. Applying (9.5) to

$$\alpha = \beta = \tau^{\bullet}(X, \xi; F) \in \det H^{*}(X; F)$$

and

$$a = b = \tau(X, \xi; F^{*}) \in \det H_{*}(X; F^{*})$$

we obtain

$$\begin{aligned}&\langle \tau^{\bullet}(X, \xi; F), \tau^{\bullet}(X, \xi; F) \rangle_{\mathrm{PR}} \\ &\qquad = \langle \tau(X, \xi; F^{*}), \tau(X, \xi; F^{*}) \rangle_{\mathrm{PR}}^{-1} \cdot [\tau^{\bullet}(X, \xi, F), \tau(X, \xi, F^{*})]^{2}.\end{aligned}$$

By (9.3), $[\tau^{\bullet}(X, \xi, F), \tau(X, \xi, F^{*})] = 1$. By Theorems 6.2 and 6.4,

$$\langle \tau(X, \xi; F^{*}), \tau(X, \xi; F^{*}) \rangle_{\mathrm{PR}}^{-1} = (-1)^{z} \det{}_{F^{*}}\big(c(\xi)\big)^{-1} = (-1)^{z} \det{}_{F}\big(c(\xi)\big).$$

This implies the claim of the theorem. □

§10. Analytic torsion via Euler structures

In this section we describe a relationship between the analytic torsion of Ray and Singer [RS] and the combinatorial torsion of Euler structures. The analytic torsion of a flat vector bundle F over a closed odd-dimensional manifold X can be viewed as a norm (*the Ray-Singer norm*) on the determinant line $\det H^{*}(X; F)$. The main result of this section expresses the Ray-Singer norm of the cohomological torsion $\tau^{\bullet}(X, \xi; F)$ of any Euler structure $\xi \in \mathit{Eul}(X)$ in terms of the monodromy of F along the characteristic class $c(\xi)$.

10.1. Ray-Singer norm. We recall the construction of the Ray-Singer norm. Let X be a closed smooth manifold, and let F be a flat real vector bundle over X. (Here the ground field $\boldsymbol{k}$ is $\mathbb{R}$.) Choose an arbitrary Riemannian metric on X and a smooth metric on F. Then the space $\Omega^{*}(X; F)$ of differential forms on X with values in F has a scalar product. The flat structure on F determines a flat connection $\nabla : \Omega^{q}(X; F) \to \Omega^{q+1}(X; F)$, so that $\nabla^{2} = 0$. We have

$$H^*(X;F) = \ker(\nabla)/\operatorname{im}(\nabla)$$

(the cohomology of the twisted de Rham complex). Using the Hodge decomposition, the cohomology can be embedded into $\Omega^*(X;F)$ as the space of harmonic forms; this embedding induces a norm $|\cdot|^{\mathrm{RS}}$ on the determinant line $\det H^*(X;F)$. The Ray-Singer norm $\|\cdot\|^{\mathrm{RS}}$ on $\det H^*(X;F)$ is defined by

$$\|\cdot\|^{\mathrm{RS}} = |\cdot|^{\mathrm{RS}} \prod_{q=0}^{\dim X} (\mathfrak{Det}\,\Delta_q')^{(-1)^q q/2}, \tag{10.1}$$

where $\mathfrak{Det}\,\Delta_q'$ denotes the zeta-function regularized determinant of the Laplacian Δ_q' acting on the space of q-forms orthogonal to the harmonic forms. Recall the definition of $\mathfrak{Det}\,\Delta_q'$ following [RS]. Consider the positive eigenvalues of the Laplacian $\Delta_q : \Omega^q(X;F) \to \Omega^q(X;F)$

$$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \cdots, \quad \lambda_k \to \infty$$

and form the ζ-function

$$\zeta_q(s) = \sum_{k=1}^{\infty} \lambda_k^{-s}, \quad \mathrm{Re}(s) \text{ is large.}$$

It is a meromorphic function holomorphic at $s = 0$. Now,

$$\mathfrak{Det}\,\Delta_q' = \exp\left(-\frac{d}{ds}\,\zeta_q(s)|_{s=0}\right).$$

The fundamental property of the Ray-Singer norm (10.1) for *odd-dimensional X* is its topological invariance: it does not depend on the choice of metrics on X and F, used in the construction. For even-dimensional X this is not the case, see [BZ] for a detailed description of the dependence of the Ray-Singer norm on the metrics.

10.2. Theorem (analytic torsion and Euler structures). *Let X be a closed connected orientable smooth manifold of odd dimension and let F be a flat $\mathbb{R}$-vector bundle over X. If* $\dim F$ *is odd, then we additionally assume that X is provided with a homology orientation. For any Euler structure* $\xi \in \mathrm{Eul}(X)$, *the Ray-Singer norm of its cohomological torsion* (*cf.* 9.2) $\tau^\bullet(X,\xi;F) \in \det H^*(X;F)$ *is equal to the positive square root of the absolute value of the monodromy of F along the characteristic class* $c(\xi) \in H_1(X)$:

$$\|\tau^\bullet(X,\xi;F)\|^{\mathrm{RS}} = |\det_F c(\xi)|^{1/2}. \tag{10.2}$$

In the special case, where the flat bundle F is acyclic, i.e., $H^*(X;F) = 0$, the torsion $\tau^\bullet(X,\xi;F)$ is a real number and Theorem 10.2 yields

$$\prod_{q=0}^{\dim X} (\mathfrak{Det}\,\Delta_q')^{(-1)^{q+1} q} = \frac{(\tau^\bullet(X,\xi;F))^2}{|\det_F c(\xi)|}. \tag{10.3}$$

Note that the RHS of this formula does not depend on the choice of ξ, this follows directly from (6.3) and the properties of the torsion.

Theorem 10.2 generalizes the classical Cheeger-Müller theorem [C], [Mu1] concerning the orthogonal flat real bundles F and the (more general) theorem of Müller [Mu2] concerning the unimodular flat real bundles F. Note that if F is unimodular then $|\det_F c(\xi)| = 1$ and the torsion $\tau^\bullet(X, \xi; F)$ does not depend on the choice of ξ.

Proof of Theorem 10.2. The main theorem of [Fa], Theorem 3.2, states that the norm on $\det H^*(X; F)$ associated with the Poincaré-Reidemeister scalar product coincides with the Ray-Singer norm. More precisely, for any $\alpha \in \det H^*(X; F)$,

$$\langle \alpha, \alpha \rangle_{\mathrm{PR}} = \pm (\|\alpha\|^{\mathrm{RS}})^2. \tag{10.4}$$

Substituting here $\alpha = \tau^\bullet(X, \xi, F)$ and using (9.6) we obtain (10.2). □

Note that the sign in (10.4) is completely described in Theorem 4.4.

10.3. Questions. Formula (10.2) computes $\pm\tau^\bullet(X, \xi; F)$ in analytical terms. Is there a way to compute $\tau^\bullet(X, \xi; F)$ (without the sign indeterminacy) using the analytic tools? One may expect that the η-invariant of Atiyah, Patodi and Singer will be relevant for this purpose.

There is a similar question. Suppose that F is a flat complex bundle. Then we have the *complex torsion* $\tau^\bullet(X, \xi; F)$ lying in the complex determinant line $\det H^*(X; F)$. Now, one may also consider F as the real flat bundle $F_{\mathbb{R}}$ and consider the *real torsion* $\tau^\bullet(X, \xi; F_{\mathbb{R}}) \in \det H^*(X; F_{\mathbb{R}})$. It can be shown that the real torsion $\tau^\bullet(X, \xi; F_{\mathbb{R}})$ may be considered as an "*absolute value*" of the complex torsion $\tau^\bullet(X, \xi; F)$; it can be expressed in terms of the analytic torsion of Ray and Singer and the information contained in the characteristic class $c(\xi)$, using our Theorem 10.2. One may ask how to recover *the* "*phase information*" of the complex torsion $\tau^\bullet(X, \xi; F)$ using the analytic tools?

§11. Semi-characteristics of manifolds

In this section we will apply the results obtained above to compute the residue mod 2 of the twisted semi-characterictic of a closed orientable smooth manifold of dimension $\equiv 1 \pmod 4$.

11.1. Twisted semi-characteristics. Let F be a flat vector bundle over a manifold X of odd dimension m. By the *twisted semi-characteristic* of X (with coefficients in F) we mean the integer

$$s\chi_F(X) = \sum_{i=0}^{(m-1)/2} \dim H_{2i}(X; F).$$

In the case of the trivial real line bundle we recover the semi-characteristic $s\chi(X) \in \mathbb{Z}$ which appeared in Section 4.4. The next theorem computes $s\chi_F(X) \pmod 2$ as a function of an orthogonal real vector bundle F in the case $m \equiv 1 \pmod 4$. We refer to [LMP], [K], for other properties of the twisted semi-characteristics.

11.2. Theorem. *Let X be a closed connected orientable smooth manifold of dimension* $m \equiv 1 \pmod 4$. *Let F be a flat $\mathbb{R}$-vector bundle over X with orthogonal structure group. Then*

$$s\chi_F(X) \equiv \langle w_1(F) \cup w_{m-1}(X), [X] \rangle + s\chi(X) \cdot \dim F \pmod 2.$$

Proof. When we add to F the trivial line bundle, both sides of the formula increase by $s\chi(X)$. Therefore it is enough to prove the theorem for even-dimensional F. Set $x = \langle w_1(F) \cup w_{m-1}(X), [X] \rangle \in \mathbb{Z}/2\mathbb{Z}$. We should prove that $s\chi_F(X) \equiv x \pmod 2$.

Fix an Euler structure $\xi \in \mathrm{Eul}(X)$ and consider the torsion $\tau(X, \xi; F)$, which is an element of $\det H_*(X; F)$ (see Section 6.1). By Theorem 7.2 and remarks in Section 4.4,

$$\begin{aligned}(11.1)\qquad D\big(\tau(X, \xi; F)\big) &= \tau(X, \xi^*; F^*) = \big(\det_{F^*} c(\xi)\big)^{-1} \tau(X, \xi; F^*)\\ &= \det_F\big(c(\xi)\big)\tau(X, \xi; F^*) = (-1)^x \tau(X, \xi; F^*)\end{aligned}$$

where $D : \det H_*(X; F) \to \det H_*(X; F^*)$ is the duality operator (4.2).

The flat scalar product on F gives an isomorphism of flat vector bundles $\phi : F^* \to F$ which induces an isomorphism $\phi_* : \det H_*(X; F^*) \to \det H_*(X; F)$. By formula (11.1),

$$\phi_*\Big(D\big(\tau(X, \xi; F)\big)\Big) = (-1)^x \phi_*\big(\tau(X, \xi; F^*)\big) = (-1)^x \tau(X, \xi; F).$$

Therefore the number $(-1)^x$ equals the degree of the linear endomorphism $\phi_* \circ D$ of the real line $\det H_*(X; F)$. We shall show below that the degree of $\phi_* \circ D$ equals $(-1)^{s\chi_F(X)}$. This would imply $s\chi_F(X) = x \pmod 2$ and complete the proof of the theorem.

We have $H_*(X; F) = \bigoplus_p V_p$, where $V_p = H_p(X; F) \oplus H_{m-p}(X; F)$ and

$$p = 0, 1, \dots, (m-1)/2.$$

We consider each V_p as a graded vector space with zero entries in degrees $\neq p$, $m - p$. Using the isomorphism $\phi : F^* \to F$, we can identify V_p with its dual. In this way we obtain duality operators

$$D_p : \det V_p \to \det V_p, \quad p = 0, 1, \dots, (m-1)/2.$$

Using the fusion isomorphism constructed in Section 2.3 and Lemma 2.7.3 we obtain a natural isomorphism $\mu : \det H_*(X; F) \to \bigotimes_p \det V_p$. Lemma 2.7.2 implies that the conjugation by μ transforms $\phi_* \circ D$ into the tensor product $\bigotimes_p D_p$. Therefore

$$\deg(\phi_* D) = \prod_{p=0}^{(m-1)/2} \deg D_p.$$

We shall show that $\deg D_p = (-1)^{\dim H_p(X;F)}$. This will imply that $\deg(\phi_* D) = (-1)^{s\chi_F(X)}$.

It is clear (from the definitions introduced in Section 2.5) that $\deg D_p = (-1)^{s(V_p)}$. Obviously, we have

$$\alpha_q(V_p) = \begin{cases} \dim H_p(X; F), & \text{if } p \leqq q < m - p, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, one easily verifies that $\sum_{q=1}^{m} \alpha_{q-1}(V_p)\alpha_q(V_p) = 0 \in \mathbb{Z}/2\mathbb{Z}$ and

$$(11.2) \qquad s(V_p) = \sum_{q=0}^{(m-1)/2} \alpha_{2q}(V_p) = \big((m+1)/2\big) \cdot \dim H_p(X; F) \pmod 2.$$

Therefore we obtain $s(V_p) \equiv \dim H_p(X; F) \pmod 2$ assuming that $m \equiv 1 \pmod 4$. □

11.3. Example. Let $X = S^1$. Then Theorem 11.2 reduces to the following simple statement: *For $A \in O(n)$, the codimension of the linear space of fixed points of A is even if $A \in SO(n)$, and is odd otherwise.*

11.4. Remark. One may use formula (11.2) in the case $m \equiv 3 \pmod 4$ to conclude that $s(V_p) \equiv 0 \bmod 2$. Together with the arguments used in the proof of Theorem 11.2, this gives an independent proof (which works also for PL manifolds) of the theorem of Massey [Ma] about vanishing of the Stiefel-Whitney class $w_{m-1}(X)$ for any closed orientable smooth manifold X of dimension $m \equiv 3 \pmod 4$.

References

[BGS] *J.-M. Bismut*, *H. Gillet*, *C. Soulé*, Analytic torsion and holomorphic determinant bundles, I, Comm. Math. Phys. **115** (1988), 49–78.

[BZ] *J.-M. Bismut*, *W. Zhang*, An extension of a theorem by Cheeger and Müller, Astérisque **205** (1992).

[BFK] *D. Burghelea*, *L. Friedlander*, *T. Kappeler*, Asymptotic expansion of the Witten deformation of the analytic torsion, preprint 1994.

[C] *J. Cheeger*, Analytic torsion and the heat equation, Ann. Math. **109** (1979), 259–322.

[Fa] *M. Farber*, Combinatorial invariants computing the Ray-Singer analytic torsion, Diff. geom. appl. **6** (1996), 351–366.

[Fr] *D. S. Freed*, Reidemeister torsion, spectral sequences, and Brieskorn spheres, J. reine angew. Math. **429** (1992), 75–89.

[HT] *S. Halperin*, *D. Toledo*, Stiefel-Whitney homology classes, Ann. Math. **96** (1972), 511–525.

[K] *G. Kempf*, Deformations of semi-Euler characteristics, Amer. J. Math. **114** (1992), 973–978.

[LMP] *G. Luszig*, *J. Milnor* and *F. Peterson*, Semi-characteristic and cobordism, Topology **8** (1969), 357–359.

[Ma] *W. Massey*, On the Shtiefel-Whitney classes of a manifold, Amer. J. Math. **82** (1960), 92–102.

[M1] *J. Milnor*, A duality theorem for Reidemeister torsion, Ann. Math. **76** (1962), 137–147.

[M2] *J. Milnor*, Whitehead torsion, Bull. Amer. Math. Soc. **72** (1966), 358–426.

[Mu1] *W. Müller*, Analytic torsion and *R*-torsion for Riemannian manifolds, Adv. Math. **28** (1978), 233–305.

[Mu2] *W. Müller*, Analytic torsion and *R*-torsion for unimodular representations, J. Amer. Math. Soc. **6** (1993), 721–743.

[RS] *D. B. Ray*, *I. M. Singer*, *R*-torsion and the Laplacian on Riemannian manifolds, Adv. Math. **7** (1971), 145–210.

[RS1] *C. P. Rourke*, *B. J. Sanderson*, Introduction to piecewise-linear topology, Springer-Verlag, Berlin-Heidelberg-New York 1972.

[T1] *V. G. Turaev*, Reidemeister torsion in knot theory, Uspekhi Mat. Nauk **41 : 1** (1986), 97–147; Engl. transl.: Russian Math. Surv. **41 : 1** (1986), 119–182.
[T2] *V. G. Turaev*, Euler structures, nonsingular vector fields, and torsion of Reidemeister type, Izv. Acad. Sci. USSR **53 : 3** (1989), 130–146; Engl. transl.: Math. USSR Izv. **34 : 3** (1990), 627–662.
[T3] *V. G. Turaev*, Torsion invariants of Spin^c-structures on 3-manifolds, Math. Res. Letters **4 : 5** (1997), 679–695.

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