KNOTS AND STABLE HOMOTOPY

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An n-dimensional k n o t is a pair (S^{n+2}, k^n), where $k \subset S^{n+2}$ is a smooth oriented submanifold homeomorphic to the n-sphere. A knot is called s t a b l e if its complement has the homotopy $[(n+3)/3]$ -type of S^1 and $n \geqslant 5$. Two knots are i s o t o p i c if there exists an isotopy of the ambient sphere sending one knot onto another with preserved orientations.

The first section of this paper provides a classification of stable knots in terms of the stable homotopy theory. Our main invariant is a homotopy generalization of isometry structure introduced by Kervaire [10]. Detailed analysis of all possible modifications of Seifert manifold enables us to formulate an equivalence relation on the set of stable isometry structures, factor set being exactly the set of all stable knot types. This found equivalence relation is new for the algebraic situation of [10] as well; it gives there an effective algebraic description of isotopy types (instead of cobordism classes as in [10]) of simple odd-dimensional knots.

As it was shown in [7], the classical homology invariants fail to form a complete system even for simple even-dimensional knots. The results of §1 allow us, however, to expect that more or less extensive algebraic classification of stable knots might have been constructed by applying generalized homology theories. §2, where we study extaordinary Alexander modules expressing them through modules of Seifert manifolds, gives a few steps in this direction. Our techique of covering functors makes it possible to manage difficulties caused by non-compactness of the infinite cyclic covering and by lack of suitable duality theorems for generalized homology. This technique suggests a general construction of various forms on extraordinary Alexander modules among which there are, on the one hand, all known forms and, on the other hand, a number of new.

1. A STABLE-HOMOTOPY CLASSIFICATION OF KNOTS

1.1. THE CARVING MAP. Let $V^{n+1} \subset S^{n+2}$ be a smooth compact connected oriented submanifold with boundary being a homology sphere.

Let $i_+, i_- : V \to S^{n+2} - V$ be small translations along positive and negative normal fields, respectively. It is easy to show that for $k > 0$ the homomorphism $H_k V \to H_k(S^{n+2} - V)$, sending $a \in H_k V$ to $i_{+*}(a) - i_{-*}(a)$, is an isomorphism. In fact, if $a = \{\alpha\}$ is in its kernel, then there is a chain β in $S^{n+2} - V$ with $\partial\beta = i_{+*}\alpha - i_{-*}\alpha$; if we add to β the cylinder over α we obtain a cycle γ which intersects V along α. This implies $a = 0$, because γ is the boundary of some chain δ lying in $S^{n+2} - \partial V$ (since $H_{k+1}(S^{n+2} - \partial V) = 0$ for $k > 0$) and so α is the boundary of the intersection of δ with V. The fact that $i_{+*} - i_{-*}$ is onto may be proved similarly.

Consider the map $h : SV \to S(S^{n+2} - V)$, where

$$h[v,t] = \begin{cases} [i_+(v), 2t] & \text{for } 0 \leqslant t \leqslant 1/2 \\ [i_-(v), 2-2t] & \text{for } 1/2 \leqslant t \leqslant 1, \end{cases}$$

S denoting non-reduced suspension. By the statement in the previous paragraph h induces isomorphism in integer homology. Besides, both spaces in question are simply connected and so h is a homotopy equivalence. Therefore there exists map $z : SV \to SV$ with $h \circ z$ homotopic to Si_+ and z is unique up to homotopy. This map z will be called c a r v i n g m a p.

The map z acts on homology of V and this action is easy to describe: if $a = \{\alpha\} \in H_k V$ and β is a $(k+1)$-chain in $S^{n+2} - \partial V$ with boundary $i_{+*}(\alpha)$ then $za \in H_k V$ is the homology class of $\beta \cap \operatorname{int} V$. For classical knots the matrix representing the action of z on the one-dimensional homology appeared in [17] and was denoted by Γ.

1.2. THE INTERSECTION FORM. Let Y be the complement of an open tubular neighbourhood of V in S^{n+2}. Fix base-points in V and in Y and consider the canonical Spanier-Whitehead duality $v : V \wedge Y \to S^{n+1}$. Regarding $i_+, i_- : V \to Y$ as S-maps we may form the S-map $u : V \wedge V \to S^{n+1}$ by $u = v \circ (1 \wedge (i_+ - i_-))$. This map u will be called i n t e r s e c t i o n f o r m of V. It may be shown that u does not depend on the imbedding $V \subset S^{n+2}$, and is determined by the topology type of V. But we shall not use this fact and so omit its proof.

To formulate properties of the S-maps u and z we shall use the following notion: A s t a b l e i s o m e t r y s t r u c t u r e of dimension n is a triplet (X, u, z), where X is a finite pointed CW-complex and $u : X \wedge X \to S^{n+1}$, $z : X \to X$ are two S-maps satisfying: (a) u is a duality; (b) $u' = (-1)^{n+1} u$;

c) $u\circ(1\wedge z)=u\circ(\bar z\wedge 1)$. Here in (b) u' denotes $u\circ\gamma$, where $\gamma: X\wedge X\to X\wedge X$ is the interchanging map. In (c) $\bar z$ denotes the S-map $1-z: X\to X$. These notations will be also used below.

Two stable isometry structures (X_ν, u_ν, z_ν), $\nu=1,2$, of the same dimension will be called isomorphic if there exists an S-equivalence $f: X_1\to X_2$ such that $f\circ z_1=z_2\circ f$ and $u_2\circ(f\wedge f)=u_1$.

1.3. THEOREM. (1) Let $V^{n+1}\subset S^{n+2}$ be a smooth compact connected oriented submanifold with boundary being a homology sphere. Then the triplet (V,u,z), where u is the intersection from and z is the stable class of the carving map, is a stable isometry structure. (2) For $n\geqslant 4$ any stable isometry structure (X,u,z) of dimension n with $[(n+2)/3]$ -connected X is isomorphic to the stable isometry structure of some smooth compact simply-connected oriented submanifold $V^{n+1}\subset S^{n+2}$ with ∂V being a homotopy sphere.

PROOF. (1). From the fact that $i_+-i_-: V\to V$ (see 1.1) is an S-equivalence and from the definition it follows that the intersection form is a duality. This proves (a) of 1.2. To prove (b) use the following relations (see [4], p. 188).

$$v\circ(1\wedge i_+)=(-1)^n v'\circ(i_-\wedge 1),\ v\circ(1\wedge i_-)=(-1)^n v'\circ(i_+\wedge 1),$$

where $=$ means the equality of S-maps. We have

$$u'=v'\circ((i_+-i_-)\wedge 1)=(-1)^n v\circ(1\wedge i_-)-(-1)^n v\circ(1\wedge i_+)=(-1)^{n+1}u,$$

To prove (c) note that the definition of carving map in 1.1 implies the following equality of S-maps $(i_+-i_-)\circ z=i_+$. Thus $u\circ(1\wedge z)=v\circ(1\wedge(i_+-i_-)\circ z=v\circ(1\wedge i_+)$. Similarly $(i_+-i_-)\circ\bar z=-i_-$ and $u\circ(\bar z\wedge 1)=(-1)^{n+1}u'\circ(\bar z\wedge 1)=(-1)^n v'\circ(i_-\wedge 1)=v\circ(1\wedge i_+)$, which proves (c). □

PROOF (2). Let (X,u,z) be a stable isometry structure of dimension $n\geqslant 4$ with $[(n+2)/3]$ -connected X. Since u is a duality $H^i(X)\approx H_{n+1-i}(X)=0$ for $i>n-\tau$, where $\tau=[(n+2)/3]$ and so we may suppose X to be $(n-\tau)$-dimensional. Denote $u\circ(1\wedge z)$ by $\xi: X\wedge X\to S^{n+1}$. By virtue of Theorem 1.3 from [4] there is a compact oriented submanifold $V^{n+1}\subset S^{n+1}$ with simplyconnected boundary ∂V and a homotopy equivalence $g: V\to X$ such that $\xi\circ(g\wedge g)$ is homotopic to the homotopy Seifert pairing of V. Since $[\xi+(-1)^{n+1}\xi']\circ(g\wedge g)=u\circ(g\wedge g)$ is a duality and by Theorem 1.4 of [4] ∂V is a homology sphere. It remains only to show that g yields an isomorphism between the stable isometry structure of V and (X,u,z). But this follows from the fact that homotopy Seifert pairing θ and stable isometry structure (V,u_V,z_V) determine each other. In fact, if θ is given then $u_V=\theta+(-1)^{n+1}\theta'$ and S-map z_V is uniquely determined by the relation $u_V\circ(1\wedge z_V)=$ $=\theta$. The last relation allows to find θ if u_V and z_V are given. □

These arguments and Theorem 1.2 of [4] imply

1.4. THEOREM. Let $n\geqslant 5$ and V^{n+1}, $W^{n+1}\subset S^{n+2}$ be smooth compact $[(n+3)/3]$ -connected oriented submanifolds with boundaries being homotopy spheres. If the stable isometry structures of V and W are isomorphic then there exists an isotopy of S^{n+2} sending V onto W with respect to the orientations. □

A knot bounds many different Seifert manifolds. If there is given one of them then any other may be obtained by surgery along imbedded handles. It is a known fact. We shall use its following more precise version.

1.5. THEOREM. Let V^{n+1}, $W^{n+1}\subset S^{n+2}$ be smooth compact τ-connected oriented submanifolds bounding homotopy spheres. If the oriented knots $(S^{n+2},\partial V)$, $(S^{n+2},\partial W)$ are isotopic and $\tau\geqslant 2$, $n\geqslant 4$, then there exists a finite sequence $U_0, U_1,\ldots, U_N$ of smooth compact τ-connected oriented submanifolds such that (a) $U_0=V$; (b) for $i=1,\ldots,N-1$ int $U_{i-1}\cap$ int $U_i=\varnothing$, $\partial U_{i-1}=\partial U_i$ and the orientations of U_{i-1} and U_i agree on ∂U_i; (c) U_N is ambient isotopic to W with respect to the orientations.

For the proof see [4], pp. 201-107, although this Theorem was not formulated there explicitly.

The novelty of this Theorem is in the τ-connectness of all U_i. The condition (b) means that $U_{i-1}\cup U_i$ is a closed $(n+1)$-dimensional manifold in S^{n+2}, bounding a solid N_i. This N_i may be considered as a cobordism from U_{i-1} to U_i; any its handle decomposition provides a sequence of modifications of U_{i-1} resulting in U_i. Because of the τ-connectness of all U_i we may construct a decomposition including only handles of indices j with $\tau<j<n-\tau+2$.

1.6. R-EQUIVALENCE RELATION. Let $m\geqslant 0$ be an integer and let (X_ν,u_ν,z_ν), $\nu=1,2$, be two stable isometry structures of the same dimension. They will be called m-contiguous if there exist S-maps $\varphi: X_1\to X_2$ and $\psi: X_2\to X_1$ such that

$$\varphi\circ z_1=z_2\circ\varphi,\quad \psi\circ z_2=z_1\circ\psi,\quad u_1\circ(1_{X_1}\wedge\psi)=u_2\circ(\varphi\wedge 1_{X_2}),$$
$$\varphi\circ\psi=(z_2\circ\bar z_2)^m,\quad \psi\circ\varphi=(z_1\circ\bar z_1)^m,$$

where $\bar z_1=1_{X_1}-z_1$, $\bar z_2=1_{X_2}-z_2$. The first two equalities mean that φ commute with the carving map, the third equality expresses the conjugateness of φ and ψ (and so φ determines ψ and vice versa).

Evidently, 0-contiguity coicides with the isomorphism. If $\mathcal{A},\mathcal{B},\mathcal{C}$ are three stable isometry structures with $\mathcal{A}$ and $\mathcal{B}$ m-contiguous and $\mathcal{B}$ and $\mathcal{C}$ ℓ-contiguous, then $\mathcal{A}$ and $\mathcal{C}$ are $(m+\ell)$-contiguous.

Two stable isometry structures $\mathcal{A}$ and $\mathcal{B}$ of the same dimension we shall call R-equivalent if there is a finite sequence of stable isometry structures $C_0, C_1, \dots, C_N$ such that $\mathcal{A}=C_0$, $\mathcal{B}=C_N$ and besides C_i and C_{i+1} are 1-contiguous for all $i=0$, $1,\dots,N-1$.

1.7. THEOREM. Suppose V^{n+1}, $W^{n+1} \subset S^{n+2}$ are smooth oriented submanifolds bounded by homotopy spheres. (1) If $\operatorname{int} V \cap \operatorname{int} W = \emptyset$, $\partial V = \partial W$ and the orientations of V and W agree on ∂V then the stable isometry structures of V and W are 1-contiguous. (2) Conversely, if the stable isometry structures of V and W are 1-contiguous, V and W are $[(n+3)/3]$ -connected and $n \geqslant 5$ then there exists an isotopy of S^{n+2} sending V onto a submanifold $U \subset S^{n+2}$ with $\operatorname{int} U \cap \operatorname{int} W = \emptyset$, $\partial U = \partial W$ and the orientations of U and W agree on ∂W.

We shall deduce this Theorem from Theorem 2.4, 2.5 of [4] using the following lemma.

1.8. LEMMA. Let (X, u, z) be a stable isometry structures of dimension n and let $\theta = u \circ (1 \wedge z);\ X \wedge X \longrightarrow S^{n+1}$ be the corresponding homotopy Seifert pairing, The condition $z^2 = z$ is equivalent to the existence of an S-equivalence $f: K \vee L \longrightarrow X$ and of a duality $v: L \wedge K \longrightarrow S^{n+1}$ such that $\theta \circ (f \wedge f)$ is given by the matrix.

$$\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}$$

PROOF. Suppose $z^2 = z$. By the Freyd Theorem [9] the idempotent z splits, i.e. there is a complex K and S-maps $i_1: K \longrightarrow X$, $\pi_1: X \longrightarrow K$ with $\pi_1 \circ i_1 = 1_K$, $i_1 \circ \pi_1 = z$. Similarly there is a complex L and S-maps $i_2: L \longrightarrow X$, $\pi_2: X \longrightarrow L$ with $\pi_2 \circ i_2 = 1_L$, $i_2 \circ \pi_2 = \bar{z}$. Let $f: K \vee L \longrightarrow X$ and $g: X \longrightarrow K \vee L$ be given by i_1, i_2 and π_1, π_2 respectively. Then f and g are mutually inverse S-equivalences. The relations $z \circ i_1 = i_1$, $z \circ i_2 = 0$, $u \circ (i_1 \wedge i_1) = 0$, $u \circ (i_2 \wedge i_2) = 0$ imply that $\theta \circ (f \wedge f)$ is given by the matrix as required with $v = u \circ (i_2 \wedge i_1)$. Now it only remains to show that v is a duality. But this follows from the fact that $u \circ (f \wedge f)$, being a duality, is given by the matrix

$$\begin{pmatrix} 0 & (-1)^{n+1} v' \\ v & 0 \end{pmatrix}.$$

The inverse statement is evident. □

1.9. PROOF OF THEOREM 1.7. Let (V, u_V, z_V), (W, u_W, z_W) be the stable isometry structures and let $\theta_V = u_V \circ (1_V \wedge z_V)$, $\theta_W = u_W \circ (1_W \wedge z_W)$ be the homotopy Seifert pairings corresponding to V and W respectively. By the Theorem 2.4 of [4] under the assumptions of the statement (1) there exists a pairing $d: V \wedge W \longrightarrow S^{n+1}$ such that the pairing $\xi: (V \vee W) \wedge (V \vee W) \longrightarrow S^{n+1}$, given by the matrix

$$\begin{pmatrix} \theta_V & d \\ (-1)^n d' & (-1)^n \theta_W' \end{pmatrix},$$

is congruent to the pairing of the form considered in the Lemma 1.8. Let $(V \vee W, U, Z)$ be the stable isometry structure corresponding to ξ. Then U and Z are given by

$$\begin{pmatrix} u_V & 0 \\ 0 & -u_W \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_V & \varphi \\ \psi & \bar{z}_W \end{pmatrix}$$

respectively, where S-maps $\varphi: V \longrightarrow W$ and $\psi: W \longrightarrow V$ have to be determined by $d = u_V \circ (1_V \wedge \psi) = u_W \circ (\varphi \wedge 1_W)$. By Lemma 1.8, $Z^2 = Z$ and so we obtain the following four equalities for the corresponding entries

$$\varphi \circ z_V = z_W \circ \varphi,\quad \psi \circ z_W = z_V \circ \psi,\quad \psi \circ \varphi = z_V \circ \bar{z}_V,\quad \varphi \circ \psi = z_W \circ \bar{z}_W.$$

This means that (V, u_V, z_V) and (W, u_W, z_W) are 1-contiguous.

To prove (2) one should bring the same arguments in the inverse order and use the Theorem 2.5 of [4]. If (V, u_V, z_V) and (W, u_W, z_W) are 1-contiguous and $\varphi: V \longrightarrow W$, $\psi: W \longrightarrow V$ are the corresponding S-maps, then we may construct the stable isometry structure $(V \vee W, U, Z)$ defining U and Z by the matrices as above. Then $Z^2 = Z$ and the Lemma 1.8 is applicable. Finally the Theorem 2.5 of [4] gives us the necessary isotopy. □

Theorems 1.5 and 1.7 imply:

1.10. COROLLARY. The R-equivalence class of the stable isometry structure of a Seifert manifold is a knot invariant. □

1.11. COROLLARY. Stable knots are isotopic if and only if the stable isometry structures of some their Seifert manifolds are R-equivalent.

PROOF. This follows from the Theorems 1.3, 1.7 and from the Levine Theorem [11], which states that any stable n-dimensional knot has an $[(n+3)/3]$ -connected Seifert manifold. □

The results of this section may also be interpreted as a classification of all knots up to the stable equivalence (see [8]).

2. ALGEBRAIC INVARIANTS

2.1. EXTRAORDINARY ALEXANDER MODULES. Let h_* be a homology theory on the category of finite cell complexes with base points. Let (S^{n+2}, k^n) be a knot and let $p: \tilde{X} \to X$ be the infinite cyclic covering of the complement of a tubular neighbourhood of k in S^{n+2}. Denote the space $\tilde{X}/\tilde{m}$ by $\hat{X}$, where $\tilde{m} = p^{-1}(m)$, $m \subset \partial X$ being a meridian. Define $h_i(\hat{X})$ as the direct limit of $h_i(K)$ where K runs over all finite subcomplexes of $\hat{X}$ containing the base-point. $\hat{X}$ has the natural action of the infinite cyclic group with generator t (defined by the orientation of the knot k), the base-point being invariant under this action. So $h_i(\hat{X})$ is a module over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$.

Let $V^{n+1} \subset S^{n+2}$ be a Seifert manifold of the knot. The carving map defines on $h_i(V)$ a module structure over the ring $P = \mathbb{Z}[z]$. Denote the localized ring $\mathbb{Z}[z, (z\bar{z})^{-1}]$ by L, where $\bar{z} = 1 - z$. We shall consider Λ as a subring in L supposing that $t = 1 - z^{-1}$, $t^{-1} = 1 - \bar{z}^{-1}$. Thus $L \otimes_P h_i(V)$ is a Λ-module.

2.2. THEOREM. Λ-module $h_i(\hat{X})$ and $L \otimes_P h_i(V)$ are isomorphic.

PROOF (sketch). Denote $W = V \cap X$. We may suppose V-int W to be a collar of $k = \partial V$ and so there is a retraction $r: V \to W$. The inclusion $W \to X$ may be lifted to the covering p and thus we obtain a map $j: W \to \hat{X}$. Consider the homomorphism $f: h_i(V) \to h_i(\hat{X})$, $f(v) = j_* r_*(v)$ for $v \in h_i(V)$. The Theorem follows easily from the assertions: (1) for any $v \in h_i(V)$, $(1-t)f(zv) = f(v)$; (2) for $v \in h_i(V)$ $f(v) = 0$ if and only if $(\bar{z}z)^m v = 0$ for some $m > 0$; (3) for any $x \in h_i(\hat{X})$ there are integers $l \geq 0$ and $m \geq 0$ such that $x = (1-t)^l t^{-m} f(v)$ for some $v \in h_i(V)$. We omit further details. □

2.3. FUNCTORS OF KNOT MODULES. Recall that module of type K is a finitely generated Λ-module for which the multiplication by $1-t \in \Lambda$ is an isomorphism. Since $L = \Lambda[(1-t)^{-1}]$, each module of type K is an L-module. Denote by $\mathcal{K}$ the full subcategory of L-mod generated by modules of type K.

As it was shown in [5], an L-module is of type K if and only if it is isomorphic to $L \otimes_P X$ for some finitely generated over $\mathbb{Z}$ P-module X. Denote by P-mod$_f$ the full subcategory of P-mod consisting of finitely generated over $\mathbb{Z}$ P-modules. Let $M: P$-mod$_f \to \mathcal{K}$ be the functor $L \otimes_P$.

Let $F: \mathbb{Z}$-mod $\to \mathbb{Z}$-mod be an additive functor (covariant or contravariant) and let R be a commutative ring. If X is an R-module then for any $\lambda \in R$ the homomorphism $X \to X$ of multiplication by λ is defined. Applying F we obtain homomorphism $F(X) \to F(X)$. It may be considered as the multiplication by λ in $F(X)$ and so $F(X)$ has the natural R-module structure. Thus the functor R-mod $\to R$-mod is defined and we shall denote it by RF. Further we shall take P or L for R.

2.4. THEOREM. ([6]). For any additive functor $F: \mathbb{Z}$-mod $\to \mathbb{Z}$-mod there exists unique functor $\hat{F}: \mathcal{K} \to L$-mod such that the functors $M \circ PF$ and $\hat{F} \circ M$ are naturally equivalent on P-mod$_f$. □

In other words this Theorem states that $L \otimes_P PF(X)$ depends only on $L \otimes_P X$ and this dependence is functorial. The functor $\hat{F}$ will be called *c o v e r i n g f u n c t o r* for F.

A functor F as above will be called *f i n i t e* if $F(X)$ is a finite group for any finitely generated abelian group X.

2.5. PROPOSITION. The covering functor $\hat{F}$ is naturally isomorphic to the restriction of $LF: L$-mod $\to L$-mod if one of the following conditions holds: (1) F is covariant and commutes with direct limits; (2) F is finite and $F = G \circ H$, where H is an additive covariant finite functor, commuting with direct limits, G being additive and contravariant.

PROOF. The ring L as a P-module is isomorphic to the limit of the system $P \to P \to P \to$, where all maps are the multiplications by $z\bar{z} \in P$. So $L \otimes_P PF(X)$ is naturally isomorphic to the limit $PF(X) \to PF(X) \to PF(X) \to \cdots$. Under the assumptions (1) this limit equals to $PF(\lim(X \to X \to)) \approx LF(L \otimes_P X)$. In the case (2) arguments are similar.

Functors $X \otimes G$, $X * G$ satisfy the condition (1); functor $Ext(X; G)$ and, if G is finite also $Hom(X; G)$, satisfy (2). In general $\hat{F}$ is not isomorphic to the restriction of LF:

EXAMPLE. Suppose $F(X) = Hom(X; \mathbb{Z})$. Then $\hat{F}(A) = Hom_L(A; Q(L)/L)$, where $Q(L)$ is the quotient field of L. For the proof see [6].

2.6. FORMS ON THE ALEXANDER MODULES. Adopt the following convention. If R is a commutative ring with involution and A is an R-module, then $\bar{A}$ will denote the R-module on the same group A where for $a \in \bar{A}$, $r \in R$ the product ra is equal to $\bar{r}a$, computed in A. We shall take below L (with involution given by $z \mapsto \bar{z} = 1 - z$) or its subring P for R.

Let h_* be a multiplicative homology theory on the category of finite based CW-complexes. We shall suppose that $h_i(X)$ is finitely generated over $\mathbb{Z}$ for all $i \in \mathbb{Z}$. Let (S^{n+2}, k^n) be a knot and let V be some its Seifert manifold. The intersection form u (see 1.2) defines a ρ-homomorphism

$$h_i(V) \longrightarrow \rho\, \mathrm{Hom}_{\mathbb{Z}}(\overline{h_j(V)}; h_{i+j}(S^{n+1})) \qquad (*)$$

by the rule $a \mapsto (b \mapsto u_*(a \wedge b) \in h_{i+j}(S^{n+1}))$, where $a \in h_i(V)$, $b \in h_j(V)$. Apply the functor M to it. If $G = h_{i+j}(S^{n+1})$ is finite then using Theorems 2.2, 2.4, 2.5.(2) we obtain L-homomorphism

$$h_i(\hat{X}) \longrightarrow L\, \mathrm{Hom}_{\mathbb{Z}}(\overline{h_j(\hat{X})}; G)$$

or $\mathbb{Z}$-homomorphism

$$h_i(\hat{X}) \otimes_L \overline{h_j(\hat{X})} \longrightarrow G,$$

where $\hat{X}$ is defined in 2.1. The forms of this kind appear in the algebraic classification of simple even-demensional knots (see [7]) where h_* is the theory of stable homotopy groups, $G = \mathbb{Z}_{24}$, $i = j = q+2$, $n = 2q$).

If G is infinite then by the same way we obtain another form. For instance, if $G = \mathbb{Z}$ then the example in the subsection 2.5 leads us to the Hermitian form

$$h_i(\hat{X}) \otimes_L \overline{h_j(\hat{X})} \longrightarrow Q(L)/L$$

generalizing the Blanchfield form.

The forms just constructed have been obtained by applying the functor M to $(*)$ with subsequent interpretation of the resulted homomorphism in terms of the Alexander modules. However, the intersection form u determines side by side with $(*)$ a number of other homomorphisms and any of them may make an origin for the similar construction. For example, u determines the well-known ρ-homomorphism

$$\rho\, \mathrm{Tor}_{\mathbb{Z}}(H_i(V); Q/\mathbb{Z}) \longrightarrow \rho\, \mathrm{Ext}_{\mathbb{Z}}(\overline{H_{n-i}(V)}; \mathbb{Z})$$

corresponding to the form of linking coefficients in V. Since the covering functors for $\mathrm{Tor}_{\mathbb{Z}}(\ ; G)$ and $\mathrm{Ext}_{\mathbb{Z}}(\ ; G)$ are $L\mathrm{Tor}_{\mathbb{Z}}(\ ; G)$ and $L\mathrm{Ext}_{\mathbb{Z}}(\ ; G)$ respectively (by virtue of Proposition 2.5), applying Theorems 2.2 and 2.4 we get L-homomorphism

$$L\mathrm{Tor}_{\mathbb{Z}}(H_i(\hat{X}); Q/\mathbb{Z}) \longrightarrow L\mathrm{Ext}_{\mathbb{Z}}(\overline{H_{n-i}(\hat{X})}; \mathbb{Z})$$

with the associated form being just the linking form

$$T_i(\hat{X}) \otimes_L \overline{T_{n-i}(\hat{X})} \longrightarrow Q/\mathbb{Z}$$

(where $T_j(\hat{X})$ denotes the $\mathbb{Z}$-torsion subgroup of $H_j(\hat{X})$). This form was originally constructed by Levine [14], [15] and by the author [2], [3] independently, using two different lines of arguments. The construction presented here differs from both of them.

The forms of Milnor [16] and of Erle [1] may be also constructed by a slight modification of the described technique.

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ON THE HOMOTOPICAL STRUCTURE AND APPLICATIONS OF MORAVA'S EXTRAORDINARY K-THEORIES

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INTRODUCTION

Let p be a prime and BP be the corresponding Brown-Peterson spectrum. Let $\{x_1, \ldots, x_m, \ldots\}$ be a set of polynomial generators of $\pi_*(BP) = \mathbb{Z}_{(p)}[x_1, \ldots, x_m, \ldots]$, $\deg x_m = 2(p^m - 1)$. An extraordinary Morava's k-theory (connected version) may be obtained from BP-theory à la Bass-Sullivan by the killing of the elements p, $x_1, \ldots, x_{n-1}, x_{n+1}, \ldots$ of $\pi_*(BP)$ (see [8]). (In general this theory depends on the choice of the generators x_i). It is well known that $k(n)$ is multiplicative, and commutative for $p > 2$ cf. [11] . Further, $\pi_*(k(n)) \cong F_p[t]$, $\deg t = 2(p^n - 1)$ and $H^*(k(n)) \cong A/AQ_n$, see [4] (Here and elsewhere $H^*(\cdot)$ denotes cohomology $\mathrm{mod}\, p$, and H denotes the corresponding Eilenberg-MacLane spectrum, A denotes the Steenrod algebra $\mathrm{mod}\, p$, and Q_n denotes Milnor's operation, $\deg Q_n = 2p^n - 1$.

Now we consider the theory $k^{\tau}(n)$, obtained from $k(n)$ by killing of the $t^{\tau+1}$. One can show that $k^{\tau}(n)$ is multiplicative, and its coefficient ring is $F_p[t]/t^{\tau+1}$. There is an obvious $(2\tau(p^n-1)-1)$ - equivalence $k^{\tau}(n) \longrightarrow k^{\tau-1}(n)$. So we get a tower

$$\ldots \to k^{\tau+1}(n) \to k^{\tau}(n) \to k^{\tau-1}(n) \to \ldots$$

THEOREM 1.1. (see [2]). The tower constructed above is the Postnikov tower for the spectrum $k(n)$. The Postnikov's k-invariants of $k(n)$ are higher cohomology operations $Q_n^{(\tau)}$, $\deg Q_n^{(\tau)} = 2\tau(p^n-1)+1$, where $Q_n^{(1)} = \lambda Q_n$ $\lambda \not\equiv 0 (\mathrm{mod}\, p)$ and $Q_n^{(\tau+1)}$ corresponds to the relation $Q_n Q_n^{(\tau)} = 0$. All operations $Q_n^{(\tau)}$ are non-trivial, and $H^*(k^{\tau}(n)) \cong (A/AQ_n) \oplus (A/AQ_n)(Q_n^{(\tau)})$.

Let E be a commutative ring spectrum with $\pi_*(E) \cong F_p[t]$ $p > 2$ (for example, any $k(n)$-theory). Obviously, E is an MU-module spectrum, and hence it defines a formal group over $\pi_*(E) \cong$ $\cong F_p[t]$. The following theorem gives a multiplicative classification of such theories. (In particular, we get multiplicative classi-