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Hermitian forms on periodic modules and even-dimensional links

*M. Farber**

In this paper we will study a class of Hermitian forms on $\mathbb{Z}$-torsion modules over the group ring of a free group F_μ. These forms emerge geometrically as higher Blanchfield forms on homology modules associated to even-dimensional links.

The first part of the paper (sections 1-3) is devoted to an algebraic study of these forms, using the general treatment of link modules developed in [F]. It is shown that the structure of the form is determined completely by a finite algebraic object consisting of a minimal lattice plus a scalar form on that lattice. In the case of knots ($\mu = 1$), but only in this case, the minimal lattice coincides with the whole module and the scalar form constructed here reduces to the linking form found by J. Levine [L1] and the author [F1]. This shows a sense in which the properties of knots ($\mu = 1$) and of links ($\mu > 1$) are quite different.

In the second part of the paper these algebraic results are applied to study torsion links. These are simple even-dimensional links with the middle-dimensional Alexander module being odd-$\mathbb{Z}$-torsion (the precise definition is given in §4). Any torsion link determines a form (a "secondary" Blanchfield form) which is precisely of the type studied in the first (algebraic) part of the paper. It is proved here that this gives a one-to-one correspondence between the isotopy classes of torsion links and the isomorphism classes of Hermitian forms. This result is in the spirit of the well-known Trotter-Kearton theorem [T],[K1] about simple odd-dimensional knots; formally it is a generalization of [Ko]; cf. also [F2].

The results of Browder-Levine [BL] and Browder [Br] show that any torsion knot (= torsion link of one component) admits a unique minimal Seifert manifold which is the fiber of the fibering of the knot complement over the circle. We will prove here that torsion links share this uniqueness property: the minimal Seifert manifold is unique up to ambient isotopy. It seems to be an interesting

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open question to understand the structure of the link complement in terms of this minimal Seifert manifold.

We show also that the homology of the minimal Seifert manifold provides a complete finite algebraic invariant of a torsion link.

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§1. Lattices in link modules

This section gives a brief review of some definitions and results of [F] that will be used in the present paper.

Fix an integer $\mu > 0$ and a subring $k \subset \mathbb{Q}$. Let F_μ denote the free group on μ generators $t_1, \dots, t_\mu$ and let $\Lambda = k[F_\mu]$ be the group ring.

1.1 A left Λ-module M has the *Sato property* if $\mathrm{Tors}_q^\Lambda(k, M) = 0$ for all q, where k is regarded as a right Λ-module with trivial action via the augmentation map. As was shown by Sato [S], this condition is equivalent to the following: the map

$$M^\mu = \underbrace{M \times \cdots \times M}_{\mu \text{ times}} \to M\ ,$$

given by $(m_1, \dots, m_\mu) \mapsto \sum_{i=1}^\mu (t_i - 1)m_i$ is a bijection. In other words, each $m \in M$ has unique representation in the form

$$m = \sum_{i=1}^{\mu} (t_i - 1)m_i\ .$$

Let us define "*derivations*" $\partial_i : M \to M$, $i = 1, \dots, \mu$, by

$$\partial_i(m) = m_i\ ,$$

where $m_i \in M$ is the element appearing in the above decomposition. Thus,

$$m = \sum_{i=1}^{\mu} (t_i - 1)\partial_i(m)\ , \qquad m \in M\ .$$

If $\lambda \in \Lambda$, then

$$\partial_i(\lambda m) = \partial_i(\lambda)m + \varepsilon(\lambda)\partial_i(m)\ ,$$

where $\partial_i(\lambda) \in \Lambda$ is the Fox derivative with respect to t_i [CF], and $\varepsilon(\lambda) \in k$ is the augmentation.

We can think of M as also having a left module structure over the ring $D = k[\partial_1, \dots, \partial_\mu]$ of polynomials in the non-commuting variables $\partial_1, \dots, \partial_\mu$. Any Λ-homomorphism $f : M_1 \to M_2$ between modules having the Sato property is also a D-homomorphism. The converse is also true. Thus

$$\mathrm{Hom}_\Lambda(M_1, M_2) = \mathrm{Hom}_D(M_1, M_2)\ .$$

1.2 The most important example of a module with the Sato property is the following:

Let $\Gamma = k\,[[x_1, \dots, x_\mu]]$ be the ring of formal power series of non-commuting variables $x_1, \dots, x_\mu$. The ring Λ may be embedded in Γ via the Magnus embedding $t_i \mapsto 1 + x_i,\ \ t_i^{-1} \mapsto 1 - x_i + x_i^2 - x_i^3 + \cdots$. Then Γ/Λ is a left Λ-module with the Sato property. The derivation $\partial_i : \Gamma/\Lambda \to \Gamma/\Lambda$ acts as cancellation of x_i from the left on monomials containing x_i on the leftmost position, and sends all other monomials to zero.

In fact, the above-mentioned rule defines an additive map $\partial_i : \Gamma \to \Gamma$ with the property

$$\gamma = \varepsilon(\gamma) + \sum_{i=1}^{\mu} x_i \partial_i(\gamma)\ ,$$

where $\varepsilon(\gamma) \in k$ is the augmentation. ∂_i maps Λ into itself and the restriction $\partial_i|_\Lambda$ coincides with the Fox derivative $\partial/\partial t_i$ [CF].

These remarks allow us to introduce a left D-module structure on Γ and Λ, which will be used later.

1.3 A *module of type* L is a left finitely generated Λ-module with the Sato property.

Γ/Λ is *not* a module of type L.

Modules of type L appear as homology modules of free coverings of boundary links [S], cf. also §4.

We shall now introduce some more operations in modules M having the Sato property. If $m \in M$ then the equation

$$m = \sum_{i=1}^{\mu} (t_i - 1)\partial_i(m)$$

is equivalent to

$$m = \sum_{i=1}^{\mu} (t_i^{-1} - 1)\,\overline{\partial}_i(m)\ ,$$

where $\overline{\partial}_i : M \to M,\ i = 1, \dots, \mu$ is defined by $\overline{\partial}_i(m) = -t_i\partial_i(m)$. Define

$$\pi_i(m) = -\partial_i(m) - \overline{\partial}_i(m) = (t_i - 1)\partial_i(m)\ ,$$

which will be called the i-th *component* of m. Then

$$\begin{aligned}
&m = \pi_1(m) + \cdots + \pi_\mu(m), \quad m \in M,\\
&\pi_i \circ \pi_i = \pi_i,\\
&\pi_i \circ \pi_j = 0 \quad \text{for} \quad i \neq j,\\
&\partial_i = \partial_i \circ \pi_i,\\
&\overline{\partial}_i = \overline{\partial}_i \circ \pi_i\ .
\end{aligned}$$

Let us also introduce an operator $z : M \to M$ by

$$z = -\partial_1 - \cdots - \partial_\mu\ .$$

One can express ∂_i and $\overline{\partial}_i$ in terms of z and π_i:

$$\begin{aligned}
\partial_i &= -z\pi_i\ ,\\
\overline{\partial}_i &= -\overline{z}\pi_i\ ,
\end{aligned}$$

where

$$\overline{z} = 1 - z : M \to M\ .$$

Thus, the whole structure is given by a decomposition of unity $\{\pi_i\}_{i=\overline{1,\mu}}$, which gives a splitting of M into a direct sum (over k)

$$M \approx X_1 \oplus \cdots \oplus X_\mu\ ,$$

and an endomorphism

$$z : M \to M\ .$$

1.4 Let M be a Λ-module of type L. A *lattice* in M is a k-submodule $A \subset M$ which:
(a) is invariant under ∂_i, $\overline{\partial}_i$, $i = 1, \dots, \mu$;
(b) generates M over Λ;
(c) is finitely generated over k.

Condition (a) is equivalent to each of the following conditions (a′),(a″),(a‴):
(a′) A is invariant under z and π_i, $i = 1, \dots, \mu$;
(a″) A is invariant under ∂_i and $\pi_i, i = 1, \dots, \mu$;
(a‴) A is invariant under $\overline{\partial}_i$ and π_i, $i = 1, \dots \mu$.

1.5 Lemma. *(1) Each Λ-module M of type L contains a lattice; (2) If A_1 and A_2 are two lattices in M then $A_1 + A_2$ and $A_1 \cap A_2$ are also lattices; (3) Let $A \subset M$ be a lattice and B be a Λ-module with the Sato property. Then any D-homomorphism $A \to B$ can be uniquely extended to a Λ-homomorphism $M \to B$. Thus,* $\mathrm{Hom}_\Lambda(M, B) = \mathrm{Hom}_D(A; B)$. *In particular, two modules of type L are isomorphic if and only if they admit lattices which are isomorphic as D-modules.*

For the proof cf. [F], Lemmas 1.5 and 2.6.

1.6 Assume that $k = \mathbb{Z}$ and M is a module of type L. We will say that M is *periodic* if there is an integer $N \in \mathbb{Z}$, $N \neq 0$ with $NM = 0$. As follows, from Lemma 1.5.(1), this is equivalent to $M = \mathrm{Tors}_\mathbb{Z} M$.

Any lattice of a periodic module of type L is finite and conversely, any module of type L admitting a finite lattice is periodic.

1.7 Theorem. *Let M be a module of type L. Assume that either (i) $k = \mathbb{Q}$ or (ii) $k = \mathbb{Z}$ and M is periodic. Then M contains a minimal lattice $A \subset M$, which is the intersection of all lattices in M. A lattice $A \subset M$ is the minimal lattice if and only if for any $k = 1, \dots, \mu$*

$$\pi_k z A = \pi_k A \quad \text{and} \quad \pi_k \overline{z} A = \pi_k A$$

(where $\overline{z} = 1 - z$).

Proof. Cf. [F], §1.

§2. The dual of a periodic module

2.1 Let S denote $\mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$; it is Λ-Λ-bimodule. If M is a periodic Λ-module of type L define the dual module $\widehat{M}$ as the set of all left Λ-homomorphisms $M \to S$:

$$\widehat{M} = \mathrm{Hom}_\Lambda(M; S)\ .$$

$\widehat{M}$ has a natural right Λ-module structure. We shall transform it into a left Λ-module structure by using the standard involution $t_i \to t_i^{-1}$ of Λ. In other words, we consider $\widehat{M}$ with the following left Λ-module structure

$$(t_i f)(m) = f(m) t_i^{-1}$$

for

$$f \in \widehat{M}\ , \quad m \in M\ , \quad i = 1, \dots, \mu\ .$$

2.2 Proposition. *$\widehat{M}$ is a periodic Λ-module of type L.*

Proof. First of all, arguments similar to those of 2.3, 2.4, 2.5 of [F] show that $\widehat{M}$ has the Sato property.

Secondly, $\widehat{M}$ is periodic. Indeed, let A be the minimal lattice in M. There exists an integer $N \neq 0$ with $NA = 0$. Thus, for any $f \in \widehat{M}$, Nf vanishes on A and thus $Nf = 0$.

We only have left to show that $\widehat{M}$ is finitely generated over Λ. To do this we consider the following map

$$\psi : \mathrm{Hom}_D(A; \mathbb{Q}\Gamma/\Gamma) \to \mathrm{Hom}_D(A; S) .$$

We will establish the following facts:
(1) $\mathrm{Hom}_D(A; S)$ is naturally isomorphic to $\widehat{M}$;
(2) the image of ψ generates $\mathrm{Hom}_D(A; S)$ over Λ;
(3) the module $\mathrm{Hom}_D(A; \mathbb{Q}\Gamma/\Gamma)$ is isomorphic to $A^* = \mathrm{Hom}_{\mathbb{Z}}(A; \mathbb{Q}/\mathbb{Z})$ and so it is finite.

(1) follows directly from Lemma 1.5.
Let us prove (3). If $F : A \to \mathbb{Q}\Gamma/\Gamma$ is a D-homomorphism then

$$F(a) = \sum_\alpha x^\alpha f_\alpha(a) \quad \text{for} \quad a \in A ,$$

where α runs over all tuples $(i_1, \ldots, i_s)$ with $i_1, \ldots, i_s \in \{1, \ldots, \mu\}$ and x^α denotes the monomial

$$x_{i_1} x_{i_2} \ldots x_{i_s} ,$$

with the convention

$$x^\phi = 1 .$$

For each multi-index α, $f_\alpha : A \to \mathbb{Q}/\mathbb{Z}$ is a $\mathbb{Z}$-homomorphism. Since F is a D-homomorphism, we get

$$f_\alpha(\partial_i a) = f_{i\alpha}(a) ,$$

and so

$$f_\alpha(a) = f_\phi(\partial_{i_s} \ldots \partial_{i_1} a) , \qquad a \in A ,$$

for

$$\alpha = (i_1, \ldots, i_s) .$$

Thus, the whole D-homomorphism F is determined by $f_\phi : A \to \mathbb{Q}/\mathbb{Z}$. Conversely, given a $\mathbb{Z}$-homomorphism $f_\phi : A \to \mathbb{Q}/\mathbb{Z}$ one can define a map $F : A \to \mathbb{Q}\Gamma/\Gamma$ by

$$F(a) = \sum_\alpha x^\alpha f_\phi(\partial^\alpha a) ,$$

where

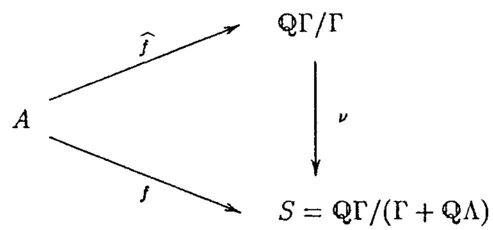
$$\partial^\alpha = \partial_{i_s} \ldots \partial_{i_1} ,$$

for

$$\alpha = (i_1, \ldots, i_s) .$$

It is clear that F is a D-homomorphism. This proves (3).

To prove (2), consider a D-homomorphism $f : A \to S$. It is clear that f admits a $\mathbb{Z}$-lifting $\widehat{f} : A \to \mathbb{Q}\Gamma/\Gamma$



For $i = 1, \ldots, \mu$ consider the map $g_i : A \to \mathbb{Q}\Lambda/\Lambda$,

$$g_i(a) = \partial_i \widehat{f}(a) - \widehat{f}(\partial_i a) , \qquad a \in A .$$

They measure the obstructions for $\widehat{f}$ to be a D-homomorophism. Let us write

$$g_i(a) = \sum_{\pi \in F_\mu} \pi \cdot g^i_\pi(a) ,$$

where $g^i_\pi : A \to \mathbb{Q}/\mathbb{Z}$ is a $\mathbb{Z}$-homomorphism and g^i_π is nonzero only for a finite number of pairs (i, π), $i = 1, \ldots, \mu$, $\pi \in F_\mu$.
Let $\widehat{g}^i_\pi : A \to \mathbb{Q}\Gamma/\Gamma$ be the D-homomorphism

$$\widehat{g}^i_\pi(a) = \sum_\alpha x^\alpha g^i_\pi(\partial^\alpha a) .$$

Define

$$\widehat{g} : A \to \mathbb{Q}\Gamma/\Gamma$$

by

$$\widehat{g}(a) = \sum_{i,\pi} \widehat{g}^i_\pi(a) x_i \pi ,$$

the sum being in fact finite.
Let us show that $\widehat{h} = \widehat{g} - \widehat{f} : A \to \mathbb{Q}\Gamma/\Gamma$ is a D-homomorphism:

$$\begin{aligned} \partial_j \widehat{g}(a) - \widehat{g}(\partial_j a) &= \sum_{i,\pi,\alpha} x^\alpha g^i_\pi(\partial^\alpha \partial_j a) x_i \pi + \sum_\pi g^j_\pi(a)\pi - \sum_{i,\pi,\alpha} x^\alpha g^i_\pi(\partial^\alpha \partial_j a) x_i \pi \\ &= \sum_\pi g^j_\pi(a)\pi = g_j(a) \\ &= \partial_j \widehat{f}(a) - \widehat{f}(\partial_j a) . \end{aligned}$$

Denote, $h = \nu \circ \widehat{h}$, $h : A \to S$. It is a D-homomorphism. Obviously,

$$h \in \mathrm{im}(\psi) .$$

Denote, $g = \nu \circ \widehat{g}$, $g : A \to S$. This is also a D-homomorphism. The formula above for $\widehat{g}$ shows that

$$g \in \Lambda(\operatorname{im}(\psi)) .$$

Thus

$$f = g - h \in \Lambda(\ker(\psi)) .$$

This proves Proposition 2.2.

2.3 Theorem. *Let M be a periodic Λ-module of type L and let $A \subset M$ be its minimal lattice. Consider the following homomorphism*

$$\varphi : A^* = \operatorname{Hom}_{\mathbb{Z}}(A; \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}_D(A; S) = \widehat{M} ,$$

$$\varphi(f)(a) = \sum_{i=1}^{\mu} \sum_{\alpha} x^\alpha f(\pi_i \partial^\alpha a) x_i \in S ,$$

where $f \in A^*$, $a \in A$ and α runs over all multi-indices $\alpha = (i_1, \dots, i_s)$ with $i_1, \dots, i_s \in \{1, \dots, \mu\}$ and x^α denotes $x_{i_1} x_{i_2} \dots x_{i_s}$ while ∂^α denotes $\partial_{i_s} \partial_{i_{s-1}} \cdots \partial_{i_1}$. Then φ is a monomorphism and its image coincides with the minimal lattice of $\widehat{M}$.

The proof makes use of Lemma 2.5 below and will therefore be postponed until after that lemma has been presented..

2.4 Let C be a D-module. We will say that C is a *D-module of type* 0 if $\partial_\kappa C = 0$ for all $\kappa = 1, \dots, \mu$. We will say that C is a *D-module of type i* (where $i \in \{1, 2, \dots, \mu\}$) if $\partial_\kappa C = 0$ for $\kappa \neq i$, $\kappa \in \{1, \dots, \mu\}$ and $(1 + \partial_i)C = 0$.

A D-module Y will be called *primitive* if it has a filtration

$$0 = Y_0 \subset Y_1 \subset Y_2 \subset \dots , \qquad \bigcup Y_i = Y$$

with a property that for each $j = 1, 2, \dots$ there exists a number $i = i(j) \in \{0, 1, \dots, \mu\}$ such that Y_j / Y_{j-1} is a D-module of type i.

Any submodule and any factor-module of a primitive D-module is also primitive. The direct sum of two primitive modules is also primitive.

A basic example of a primitive D-module is provided by $Y = \mathbb{Q}\Lambda$ with the D-module structure given by Fox derivatives. To show this, one can proceed as follows. Let

$$e_0, e_1, e_2, \dots$$

be all elements of the free group F_μ in a linear ordering such that each e_n is a reduced word and can be written as

$$e_n = x_i^\varepsilon e_\kappa ,$$

where $\kappa < n$, $i \in \{1, \dots, \mu\}$ and $\varepsilon = \pm 1$. Let Y_n be the linear hull of $e_0, e_1, \dots, e_n$. Then Y_n / Y_{n-1} is a D-module of type 0 (if $\varepsilon = 1$) or of type i, $i \in \{1, \dots, \mu\}$ (if $\varepsilon = -1$).

2.5 Lemma. *Let M be a Λ-module of type L and let $X \subset M$ be a D-submodule which is finitely generated over $\mathbb{Z}$ and generates M over Λ. Let Y be a primitive D-module. Then the kernel of any D-homomorphism*

$$g : X \to Y$$

generates M over Λ.

Proof. See the proof of Lemma 4.4 in [F].

2.6 Proof of Theorem 2.3. Let us show that φ is a monomorphism. Assume that $f \in A^*$ and $\varphi(f) = 0$. This means that for each $a \in A$ the power series

$$\sum_{i=1}^{\mu} \sum_{\alpha} x^\alpha f(\pi_i \partial^\alpha a) x_i$$

represents an element of $\mathbb{Q}\Lambda/\Lambda$. For each $i = 1, \dots, \mu$ define

$$F_i : A \to \mathbb{Q}\Lambda/\Lambda$$

by

$$F_i(a) = \sum_{\alpha} x^\alpha f(\pi_i \partial^\alpha a) .$$

F_i is a D-homomorphism and, on the other hand, $\mathbb{Q}\Lambda/\Lambda$ is a primitive D-module. From Lemma 2.5 it follows that $K_i = \ker F_i$ generates M over Λ. If $a \in K_i$ then $\partial_\kappa a \in K_i$ and $\pi_\kappa a \in K_i$ for each $\kappa \in \{1, \dots, \mu\}$. Thus K_i is a lattice and $K_i \subset A$, which implies $K_i = A$. Because this is true for any i it follows that $f = 0$.

Our next step will be to show that $\operatorname{im}(\varphi)$ is invariant under $\overline{z}$ and $\pi_1, \dots, \pi_\mu$. To do this we will introduce operations

$$\overline{z}, \pi_1, \dots, \pi_\mu : A^* \to A^*$$

by

$$(\overline{z} f)(a) = f(za)$$
$$(\pi_i f)(a) = f(\pi_i a)$$

for

$$f \in A^* , \quad a \in A , \quad i = 1, \dots, \mu .$$

To show that φ commutes with $\bar{z}, \pi_1, \ldots, \pi_\mu$ we compute:

$$\begin{aligned}\varphi(\bar{z}f)(a) &= \sum_{i=1}^{\mu}\sum_{\alpha} x^\alpha f(z\pi_i\partial^\alpha a)x_i \\ &= -\sum_{i=1}^{\mu}\sum_{\alpha} x^\alpha f(\partial_i\partial^\alpha a)x_i \\ &= -\sum_{|\alpha|\geq 1} x^\alpha f(\partial^\alpha a) \\ &= (\bar{z}\varphi(f))(a) \qquad (\mathrm{mod}\, \mathbb{Q}\Lambda)\end{aligned}$$

and

$$\begin{aligned}\varphi(\pi_j f)(a) &= \sum_{i=1}^{\mu}\sum_{\alpha} x^\alpha f(\pi_j\pi_i\partial^\alpha a)x_i \\ &= \sum_{\alpha} x^\alpha f(\pi_j\partial^\alpha a)x_j \\ &= (\pi_j\varphi(f))(a)\ .\end{aligned}$$

To prove that $\mathrm{im}(\varphi)$ is a lattice there remains to show that $\mathrm{im}(\varphi)$ generates $\mathrm{Hom}_D(A; S) = \widehat{M}$ over Λ.

Consider the homomorphism

$$\psi : A^* = \mathrm{Hom}_{\mathbb{Z}}(A; \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_D(A, S) = \widehat{M}\ ,$$

$$\psi(f)(a) = \sum_{\alpha} x^\alpha f(\partial^\alpha a)$$

for $f \in A^*$ and $a \in A$, where α runs over all multi-indices. Note that ψ is essentially the homomorphism which appeared in the proof of Proposition 2.2; it was proved there that $\mathrm{im}(\psi)$ generates $\widehat{M}$. Now, it is easy to check that

$$\psi(f) = \varphi(g)\ ,$$

where $g \in A^*$, $g(a) = -f(za)$ for $a \in A$. Thus, $\mathrm{im}(\varphi) \supset \mathrm{im}(\psi)$ and so $\mathrm{im}(\varphi)$ generates $\widehat{M}$ over Λ.

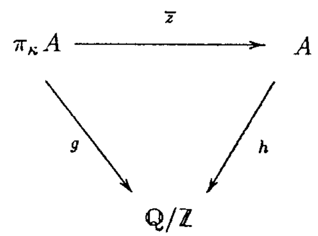
In order to show that $\mathrm{im}(\varphi)$ is the *minimal* lattice we can check the condition of Theorem 1.7 for A^*. Since

$$(\pi_\kappa z f)(a) = f(\bar{z}\pi_\kappa a)$$

for $f \in A^*$, $a \in A$, the identity

$$\pi_\kappa z A^* = \pi_\kappa A^*$$

is equivalent to the following statement: for each $\mathbb{Z}$-homomorphism $g : \pi_\kappa A \to \mathbb{Q}/\mathbb{Z}$ there exists a $\mathbb{Z}$-homomorphism $h : A \to \mathbb{Q}/\mathbb{Z}$ such that the diagram



commutes. The last statement is equivalent to the fact that $\bar{z}|_{\pi_\kappa A}$ is a monomorphism, which is in fact true: if $a \in \pi_\kappa A$ and $\bar{z}a = 0$ then $\bar{\partial}_i a = 0$ for all $i = 1, \ldots, \mu$ and $a = \sum_{i=1}^{\mu} \bar{x}_i\bar{\partial}_i(a) = 0$. The identity

$$\pi_\kappa \bar{z} A^* = \pi_\kappa A^*$$

follows similarly.

This proves the theorem.

§3. Hermitian forms on periodic modules

3.1 Let M_1, M_2 be two periodic Λ-modules of type L. Consider a $\mathbb{Z}$-bilinear pairing

$$[\ ,\ ] : M_1 \times M_2 \to S = \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$$

with the properties:

(a) $[\lambda a, b] = \lambda[a, b]$ for $\lambda \in \Lambda$, $a \in M_1$, $b \in M_2$;

(b) $[a, \lambda b] = [a, b]\bar{\lambda}$;

(c) $[\ ,\ ]$ is non-degenerate in the following sense: for $b \in M_2$ let $\varphi_b : M_1 \to S$ be the Λ-homomorphism defined by $\varphi_b(a) = [a, b]$, then the map

$$M_2 \to \widehat{M_1} = \mathrm{Hom}_\Lambda(M_1; S)\ , \qquad b \mapsto \varphi_b\ ,$$

is an isomorphism.

In the case $M_1 = M_2 = M$ we will consider an additional property:

(d) $[a, b] = \varepsilon\overline{[b, a]}$ for $a, b \in M$, $\varepsilon = \pm 1$.

3.2 Theorem. *Let M_1, M_2 be two periodic modules of type L supplied with a pairing*

$$[\ ,\ ] : M_1 \times M_2 \to S$$

satisfying (a),(b),(c) *of subsection* 3.1. *Then there exists a unique $\mathbb{Z}$-bilinear form*

$$\langle\ ,\ \rangle : A_1 \times A_2 \to \mathbb{Q}/\mathbb{Z}$$

(the scalar form) defined on the minimal lattices $A_1 \subset M_1$, $A_2 \subset M_2$ *such that*
(1) *for* $a \in A_1$, $b \in A_2$

$$[a,b] = \sum_{i=1}^{\mu} \sum_{\alpha} x^{\alpha} \langle \partial^{\alpha} a, \pi_i b \rangle \, x_i \qquad (\operatorname{mod} \mathbb{Q}\Lambda) \, ,$$

where α *runs over all multi-indices* $\alpha = (i_1, \ldots, i_s)$ *with* $i_j \in \{1, \ldots, \mu\}$, $x^{\alpha} = x_{i_1} x_{i_2} \ldots x_{i_s}$, *and* $\partial^{\alpha} = \partial_{i_s} \partial_{i_{s-1}} \ldots \partial_{i_1}$;
(2) $\langle \pi_i a, b \rangle = \langle a, \pi_i b \rangle$ *for all* $a \in A_1$, $b \in A_2$, $i = 1, \ldots, \mu$;
(3) $\langle za, b \rangle = \langle a, \bar{z} b \rangle$, *where* $\bar{z} = 1 - z$;
(4) $\langle \; , \; \rangle$ *is non-degenerate, i.e. the associated map* $A_2 \to A_1^* = \operatorname{Hom}_{\mathbb{Z}}(A_1; \mathbb{Q}/\mathbb{Z})$ *is an isomorphism.*

Conversely, given a scalar form $\langle \; , \; \rangle$ *with the above properties, the formula in (1) defines a pairing* $A_1 \times A_2 \to S$ *which can be uniquely extended to a pairing* $M_1 \times M_2 \to S$ *satisfying* (a),(b),(c) *of 3.1.*

Proof. Defining a pairing $M_1 \times M_2 \to S$ satisfying (a),(b),(c) of 3.1 is equivalent to specifying a Λ-isomorphism

$$M_2 \to \operatorname{Hom}_{\Lambda}(M_1; S) = \widehat{M_1} \, ,$$

and by Lemma 2.6 of [F] and Theorem 2.3 this is equivalent to specifying a D-isomorphism

$$A_2 \to A_1^*$$

which is the restriction of the above homomorphism to the minimal lattices and which represents the scalar form.

3.3 Theorem. *Let* M *be a periodic module of type* L *and* $[\; , \;] : M \times M \to S$ *be a pairing satisfying* (a),(b),(c) *of* 3.1. *The pairing* $[\; , \;]$ *satisfies* (d) *of* 3.3 *if and only if the scalar form* $\langle \; , \; \rangle$ *is* $(-\varepsilon)$*-symmetric:* $\langle a, b \rangle = -\varepsilon \langle b, a \rangle$ *for* $a, b \in A$; *here* A *is the minimal lattice of* M.

The proof is similar to that of Theorem 3.3 of [F].

§4. Blanchfield forms

4.1 $\mathcal{F}$-links and Seifert surfaces. An *n-dimensional μ-component* link is an oriented smooth submanifold Σ^n of S^{n+2}, where $\Sigma^n = \Sigma_1^n \cup \ldots \cup \Sigma_{\mu}^n$ is the ordered disjoint union of μ submanifolds of S^{n+2}, each homeomorphic to S^n. Let $X = S^{n+1} - T(\Sigma)$ be the complement of a tubular neighbourhood $T(\Sigma)$ of Σ in

S^{n+2}. Fix a base point $* \in X$; for each $i = 1, \ldots, \mu$ the meridian $m_i \in \pi_1(X, *)$ is defined up to conjugation.

A *splitting* [CS] for Σ^n is a homomorphism (which is defined up to conjugation) $s : \pi_1(X, *) \to F_{\mu}$ onto the free group with μ generators $t_1, \ldots, t_{\mu}$ having the following property: the image of the ith meridian m_i is conjugate to $t_i \in F_{\mu}$, $i = 1, \ldots, \mu$.

An *$\mathcal{F}$-link* [CS] is a pair (Σ^n, s), where Σ^n is a link and s is a splitting for Σ. Two $\mathcal{F}$-links (Σ_1, s_1) and (Σ_2, s_2) of the same dimension and multiplicity are *equivalent* if there exists a diffeomorphism $h : S^{n+2} \to S^{n+2}$, taking Σ_1^n onto Σ_2^n, preserving the orientations of S^{n+2} and Σ_{ν}, $\nu = 1, 2$ and mapping s_2 onto s_1.

A link Σ^n is a *boundary link* if there is an oriented smooth submanifold V^{n+1} of S^{n+2} such that $V^{n+1} = V_1^{n+1} \cup \ldots \cup V_{\mu}^{n+1}$ is the disjoint union of submanifolds V_i^{n+1} satisfying $\partial V_i = \Sigma_i$ $(i = 1, \ldots, \mu)$. If each V_i is connected, we say that V is a Seifert manifold for Σ.

Any Seifert manifold V of a boundary link Σ defines an obvious splitting s_V: if α is a loop in X, then $s_V([\alpha])$ is the word in $t_1, \ldots, t_{\mu}$ obtained by writing down $t_i^{\varepsilon_i}$ $(\varepsilon_i = \pm 1)$ for each intersection point p of α and V_i (where ε_i is the local intersection number) and then multiplying these words in order of the appearance of the corresponding intersection points in α.

Conversely, any link admitting a splitting is a boundary link, cf. [G]. Choosing a splitting is equivalent to a choice of a Seifert surface up to embedded concordance, c.f. [CS],[F5]. All possible transformations of the Seifert surface which preserve the $\mathcal{F}$-structure were described in [F5].

4.2 Torsion links. In this paper we will consider a particular class of even-dimensional $\mathcal{F}$-links. An $\mathcal{F}$-link (Σ^{2q}, s) is called *torsion* if
(a) the splitting $s : \pi_1(S^{2q+2} - \Sigma^{2q}) \to F_{\mu}$ is an isomorphism;
(b) $\pi_i(S^{2q+2} - \Sigma) = 0$ for $1 < i < q$;
(c) the group $\pi_q(S^{2q+2} - \Sigma)$ is $\mathbb{Z}$-torsion and has no elements of order 2.

It is the goal of this paper to classify torsion links in purely algebraic terms for the case $q \geq 4$.

Let V be a Seifert manifold for Σ with $s_V = s$. We will say that V is *minimal* if each component V_j of V is $(q-1)$-connected and the maps $i_+, i_- : V_j \to S^{n+2} - V$ (defined as small shifts in directions of the positive and negative normal to V_j, respectively) induce monomorphisms in q-dimensional homology. Gutierrez [G] has shown that such minimal Seifert manifolds always exist.

4.3 Poincaré duality. Let (Σ^n, s) be an $\mathcal{F}$-link in S^{n+2} and let $X = S^{n+2} - T(\Sigma^n)$ be the complement of a tubular neighbourhood $T(\Sigma)$ of Σ in S^{n+2}. Fix a particular epimorphism $s_0 : \pi_1(X, *) \to F_{\mu}$ onto the free group F_{μ} in $t_1, \ldots, t_{\mu}$.

Consider the covering

$$\widetilde{X} \to X$$

corresponding to the kernel of s_0; the group F_μ acts on $\widetilde{X}$ as the group of covering transformations. The homology groups

$$H_\kappa(\widetilde{X};\mathbb{Z}) , \qquad \kappa = 1,2,\ldots$$

are modules over $\Lambda = \mathbb{Z}[F_\mu]$. Sato [S] has shown that *these modules are of type* L.

Fix a triangulation of X and consider the corresponding equivariant triangulation of $\widetilde{X}$ and the simplicial chain complex $C_*(\widetilde{X})$. Let X^1 denote the dual triangulation of X and let $C_*(\widetilde{X}^1)$ denote the similar chain complex. $C_*(\widetilde{X})$ and $C_*(\widetilde{X}^1)$ are complexes of free finitely generated left Λ-modules. There is an intersection pairing (cf. Milnor [M1])

$$C_i(\widetilde{X}^1) \times C_j(\widetilde{X}, \partial\widetilde{X}) \to \Lambda , \qquad (\alpha,\beta) \mapsto \alpha\cdot\beta$$

for $i+j = n+2$ with the properties:

(i) it is bilinear over $\mathbb{Z}$;

(ii) $(g\alpha)\cdot\beta = g(\alpha\cdot\beta)$, $\alpha\cdot(g\beta) = (\alpha\cdot\beta)g^{-1}$ for $g \in F_\mu$, $\alpha \in C_i(\widetilde{X}^1)$, $\beta \in C_j(\widetilde{X}, \partial\widetilde{X})$.

This pairing defines a chain map

$$C_j(\widetilde{X}, \partial\widetilde{X}) \to \overline{\mathrm{Hom}_\Lambda\,(C_i(\widetilde{X}^1);\Lambda)}$$

inducing the Poincaré duality isomorphism

$$H_j(\widetilde{X}, \partial\widetilde{X}) \to \overline{H^i(C(\widetilde{X}^1);\Lambda)} ,$$

where the bar means that the module under it, which is naturally a right Λ-module, should be converted into a left Λ-module using the standard involution $t_i \mapsto t_i^{-1}$ of Λ.

4.4 Ext-functors. The cohomology module $H^i(C(\widetilde{X}^1);\Lambda)$ could be computed using the universal coefficient spectral sequence [EC]. As shown in §5 of [F5], there is an exact sequence

$$0 \to e^2(H_{i-2}(\widetilde{X}^1)) \to H^i(C(\widetilde{X}^1);\Lambda) \to e^1(H_{i-1}(\widetilde{X}^1)) \to 0$$

and the image of $e^2(H_{i-2}(\widetilde{X}^1))$ coincides with the $\mathbb{Z}$-torsion part of $H^i(C;\Lambda)$. Here $e^\nu(M)$ denotes

$$\mathrm{Ext}^\nu_\Lambda(M;\Lambda) .$$

It was also shown in [F5] that for any module M of type L there are natural isomorphisms

$$e^1(M) \approx \mathrm{Hom}_\Lambda(M;\Gamma/\Lambda) ,$$
$$e^2(M) \approx \mathrm{Hom}_\Lambda\,(\tau M;\mathbb{Q}\Gamma/(\Gamma+\mathbb{Q}\Lambda)) ,$$

where Γ denotes the ring of formal power series in non-commuting variables $x_1,\ldots,x_\mu$ with integral coefficients, $\mathbb{Q}\Gamma$ is a similar ring with rational coefficients, Λ is embedded in Γ and $\mathbb{Q}\Lambda$ is embedded in $\mathbb{Q}\Gamma$ via the Magnus embedding.

4.5 The Poincaré duality isomorphism together with the above mentioned formulas for e^1 and e^2 produce two families of non-degenerate Hermitian forms:

$$B_{i-1}(\widetilde{X}) \times B_j(\widetilde{X}) \to \Gamma/\Lambda ,$$
$$T_{i-2}(\widetilde{X}) \times T_j(\widetilde{X}) \to \mathbb{Q}\Gamma/(\Gamma+\mathbb{Q}\Lambda) ,$$

where $i+j = n+2$, and $T_j(\widetilde{X}) = \mathrm{Tors}_\mathbb{Z}\, H_j(\widetilde{X})$, $B_j(\widetilde{X}) = H_j(\widetilde{X})/T_j(\widetilde{X})$. We refer to [F5] for a more detailed description of these forms.

Note that from the existence of these forms it follows that all Alexander modules $H_\kappa(\widetilde{X})$, $1 < \kappa \le 2q$ of a torsion link are zero, except for the module $H_q(\widetilde{X})$.

4.6 Consider also the completed chain complex

$$C'_*(\widetilde{X}^1) = \Gamma \otimes_\Lambda C_*(\widetilde{X}^1)$$

(the completion of $C_*(\widetilde{X}^1)$ with respect to powers of the augmentation ideal of Λ). There is obviously an intersection pairing

$$C'_i(\widetilde{X}^1) \times C_j(\widetilde{X}, \partial\widetilde{X}) \to \Gamma$$

with properties similar to those of 4.3. The completed complex is acyclic [F5]. Using this fact one can find an explicit description of the Hermitian forms constructed in subsection 4.5. We will do this now for the case of the pairing

$$[\ ,\ ] : T_q(\widetilde{X}) \times T_q(\widetilde{X}) \to \mathbb{Q}\Gamma/(\Gamma+\mathbb{Q}\Lambda) ,$$

under the assumption that $n = 2q$.

Let the cycles $\alpha \in C_q(\widetilde{X}^1)$ and $\beta \in C_q(\widetilde{X})$ represent classes $[\alpha], [\beta] \in T_q(\widetilde{X})$. Then there is an "infinite" chain c in $C'_{q+1}(\widetilde{X}^1)$ with $\partial c = \alpha$. On the other hand there is an integer $N \neq 0$ and a chain $d \in C_{q+1}(\widetilde{X})$ with $N\beta = \partial d$. Then,

$$[a,b] = \frac{1}{N} c\cdot d \in \mathbb{Q}\Gamma/(\Gamma+\mathbb{Q}\Lambda) ,$$

where $a = [\alpha]$, $b = [\beta] \in T_q(\widetilde{X})$. The last formula is essentially contained in the arguments of [F5], §5.

§5. Seifert manifold and Alexander modules

5.1 Let us first introduce an algebraic notion which describes the homology structure of a Seifert manifold of an even-dimensional link.

An *ε-symmetric torsion isometry structure of multiplicity* μ is a tuple

$$(A, \langle\ ,\ \rangle, z, \pi_1, \dots, \pi_\mu)$$

where A is a finite abelian group, $\langle\ ,\ \rangle : A \otimes A \to \mathbb{Q}/\mathbb{Z}$ is an ε-symmetric non-degenerate bilinear form and $z, \pi_1, \dots, \pi_\mu : A \to A$ are endomorphisms satisfying
(i) $\langle za, b\rangle = \langle a, \bar{z}b\rangle$, where $a, b \in A$ and $\bar{z}$ denotes $1 - z : A \to A$'
(ii) $\langle \pi_i a, b\rangle = \langle a, \pi_i b\rangle$;
(iii) $\pi_1 + \pi_2 + \dots + \pi_\mu = 1_A$;
(iv) $\pi_i \cdot \pi_j = \delta_{ij}\pi_j$.

Two torsion isometry structures are *isomorphic* iff there exists an isomorphism between the corresponding groups, commuting with $z, \pi_1, \dots, \pi_\mu$, and bringing one form $\langle\ ,\ \rangle$ onto the other.

5.2 Non-degenerate Hermitian forms

$$[\ ,\ ] : M \times M \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$$

on periodic modules, which were studied in §3, provide an algebraic source of examples of torsion isometry structures: if $A \subset M$ is the minimal lattice of M and $\langle\ ,\ \rangle : A \times A \to \mathbb{Q}/\mathbb{Z}$ is the scalar form, then the tuple

$$(A, \langle\ ,\ \rangle, z, \pi_1, \dots, \pi_\mu)$$

is a torsion isometry structure (here $z, \pi_1, \dots, \pi_\mu$ denote the restrictions to A of the corresponding operators on M). If the original form $[\ ,\ ]$ is ε-Hermitian, then $\langle\ ,\ \rangle$ is $(-\varepsilon)$-symmetric, cf. 3.3.

We will say that an abstract torsion isometry structure A *admits an embedding* in a periodic module M of type L supplied with a non-degenerate Hermitian form $[\ ,\ ] : M \times M \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$ if A is isomorphic (as a torsion isometry structure) to the minimal lattice of M.

If a torsion isometry structure admits embeddings in two periodic Λ-modules $(M_\nu, [\ ,\ ]_\nu)$, $\nu = 1, 2$, then there exists an isomorphism $f : M_1 \to M_2$ preserving the forms; this follows from Lemma 1.5 and Theorem 3.2.

A torsion isometry structure A will be called *minimal* if for every $a \in A$, $\kappa = 1, \dots, \mu$, either of the conditions $z\pi_\kappa a = 0$ or $\bar{z}\pi_\kappa a = 0$ implies $\pi_\kappa a = 0$

Every torsion isometry structure, admitting an embedding in the periodic Λ-module, is minimal.

5.3 Geometrically, torsion isometry structures appear as the middle dimensional homology of Seifert manifolds of even-dimensional links. Namely, let (Σ^{2q}, s) be

a μ-component $\mathcal{F}$-link and $V^{2q+1} \subset S^{2q+2}$ be any Seifert surface for (Σ^{2q}, s). Denote $A = T_q(V) = \mathrm{Tors}_{\mathbb{Z}} H_q(V)$ and let

$$\langle\ ,\ \rangle : A \times A \to \mathbb{Q}/\mathbb{Z}$$

be the classical linking form (note, that orientation of V is specified by the orientation of Σ). Let $\pi_i : A \to A$, $i = 1, \dots, \mu$, be the restriction on the torsion subgroup of the composition $H_q(V) \to H_q(V_i) \to H_q(V)$. An operation $z : A \to A$ is defined as follows. Let Y denote the result of cutting the sphere S^{2q+2} along V. Let $i_+, i_- : V \to Y$ be small shifts of V in the directions of the positive and negative normals to V, respectively. Then the map $i_{+*} - i_{-*} : H_*(V) \to H_*(Y)$ is an isomorphism (cf. [F4], §1.1) and we define $z(v) \in H_q(V)$ for $v \in H_q(V)$ by

$$(i_{+*} - i_{-*})(z(v)) = i_{+*}(v)\ .$$

It is easy to see that the tuple $(A, \langle\ ,\ \rangle, z, \pi_1, \dots, \pi_\mu)$ is a $(-1)^{q+1}$-symmetric torsion isometry structure.

5.4 From now on we will be dealing with torsion links, cf. 4.2. A Seifert manifold V^{2q+1} of a torsion link is *minimal* iff each component is $(q-1)$-connected and the corresponding torsion isometry structure (described in 5.3) is minimal. In this and the subsequent sections we will show that the torsion isometry structure of the minimal Seifert manifold admits an embedding in the Alexander module supplied with the Blanchfield form. This result combined with the algebraic considerations of §3 will give a proof of the fact that the isomorphism type of the torsion isometry structure of a minimal Seifert manifold is uniquely determined by the link.

5.5 Let (Σ^{2q}, s) be a torsion link and V^{2q+1} be its minimal Seifert manifold. Consider the $(2q+2)$-dimensional manifold Y obtained by cutting $S^{2q+2} - \Sigma$ along V. The boundary of Y is the disjoint union of

$$\partial_1^+ Y \cup \partial_1^- Y \cap \partial_2^+ Y \cup \partial_2^- Y \cup \dots \cup \partial_\mu^+ Y \cup \partial_\mu^- Y\ ,$$

where each $\partial_i^\varepsilon Y$, $\varepsilon = \pm$, is homeomorphic to V_i. There is an identification map

$$\psi : Y \to S^{2q+2} - \Sigma$$

which is a homeomorphism on $\operatorname{int} Y$ and maps $\partial_i^+ Y$ and $\partial_i^- Y$ onto V_i. The internal normal on $\partial_i^+ Y$ corresponds under ψ to the positive normal on V_i.

The map $\psi : Y \to S^{2q+2} - \Sigma = X$ can be lifted into $\widetilde{X}$, where $\widetilde{X} \to X$ is the universal covering. One can find a lifting $\widetilde{\psi} : Y \to \widetilde{X}$ such that

$$\widetilde{\psi}(Y) \cap g\widetilde{\psi}(Y) = \begin{cases} \partial_i^- Y & \text{for } g = t_i\ , \\ \partial_i^+ Y & \text{for } g = t_i^{-1}\ , \\ \emptyset & \text{for other } g \in F_\mu,\ g \neq 1. \end{cases}$$

Identify Y with its image in $\widetilde{X}$ under $\widetilde{\psi}$. It is clear that

$$\bigcup_{g \in F_\mu} gY = \widetilde{X} .$$

5.6 Let $f : H_q(V) \to H_q(\widetilde{X})$ be the composition of $i_{+*} - i_{-*} : H_q(V) \to H_q(Y)$ and $i_* : H_q(Y) \to H_q\widetilde{X})$, where $i : Y \to \widetilde{X}$ is the inclusion.

We claim that the map

$$f : H_q(V) \to H_q(\widetilde{X})$$

has the following properties:

(i) it is a monomorphism;

(ii) $f(\pi_i a) = \pi_i f(a)$ for $a \in H_q(V)$, $i = 1, \dots, \mu$;

(iii) $f(za) = zf(a)$;

(iv) the image of f generates $H_q(\widetilde{X})$ over Λ.

The proof is almost identical to those of [F], 6.18-6.25, where odd-dimensional links where considered, and will therefore, be omitted.

Note that from (i) it follows that the group $H_q(V)$ is in fact finite.

5.7 Corollary. $\mathrm{im}(f)$ *is the minimal lattice in* $H_q(\widetilde{X})$.

Proof. Conditions (ii),(iii),(iv) mean that $A = \mathrm{im}(f)$ is a lattice. Let us show that it is minimal. Assuming the contrary, one may conclude that by Theorem 1.7 there exists an integer $\kappa \in \{1, \dots, \mu\}$ with

$$\pi_\kappa z A \subsetneqq \pi_\kappa A$$

or

$$\pi_\kappa \bar{z} A \subsetneqq \pi_\kappa A .$$

If $\pi_\kappa z A \subsetneqq \pi_\kappa A$ then there is $x \in \pi_\kappa A$ with $x \neq 0$ and

$$\langle \pi_\kappa z A, x \rangle = 0 .$$

From this it follows that

$$\langle A, \bar{z} x \rangle = 0$$

and $\bar{z}x = 0$. Now $x = 0$ follows from the minimality of A; this gives a contradiction. The assumption

$$\pi_\kappa \bar{z} A \subsetneqq \pi_\kappa A$$

might be considered similarly.

§6. Computation of the Blanchfield form

6.1 Let (Σ^{2q}, s) be a torsion link, $X = S^{2q+2} - T(Z)$ be the complement of a tubular neighbourhood of Σ, and $\widetilde{X} \to X$ be the free covering. Further, let V be a minimal Seifert manifold for (Σ, s), and let $f : H_q(V) \to H_q(\widetilde{X})$ be the map constructed in 5.6. From §4 we know that there is a Blanchfield form

$$[\ ,\ ] : H_q(\widetilde{X}) \times H_q(\widetilde{X}) \to \mathbb{Q}\Gamma(\Gamma + \mathbb{Q}\Lambda)$$

and our aim is to compute this form on the image of f. We want to express the result in terms of the torsion isometry structure of the Seifert surface. The answer is as follows.

6.2 Theorem. *For* $a, b \in H_q(V)$

$$[f(a), f(b)] = \sum_{i=1}^{\mu} \sum_{\alpha} x^\alpha \langle \partial^\alpha a, \pi_i b \rangle x_i \quad \in \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda) ,$$

where α *runs over all multi-indices* $\alpha = (i_1, \dots, i_s)$, *with* $i_1, i_2, \dots, i_s \in \{1, \dots, \mu\}$, x^α *denotes the monomial*

$$x_{i_1} x_{i_2} \dots x_{i_s}$$

and ∂^α *denotes*

$$\partial_{i_s} \partial_{i_{s-1}} \dots \partial_{i_1}$$

with $\partial_i = -z\pi_i : H_q(V) \to H_q(V)$. *The brackets* $\langle\ ,\ \rangle$ *denote the linking form on* V.

Comparing the formula of Theorem 6.2 with Theorem 3.2 and Corollary 5.7 we obtain

6.3 Corollary. *The torsion isometry structure of any minimal Seifert manifold of a torsion link admits an embedding (cf. 5.2) in the Alexander module of the link supplied with the Blanchfield form.*

In fact the map $f : H_q(V) \to H_q(\widetilde{X})$ provides such an embedding.

6.4 Corollary. *The isomorphism type of torsion isometry structure of any minimal Seifert manifold of the torsion link is uniquely determined by the link.*

This follows from results of §3: a minimal lattice with its scalar form is uniquely determined by the Hermitian form.

6.5 *Proof of Theorem 6.2.* According to 4.6, in order to compute the value of the Blanchfield form $[f(a), f(b)]$ we have, to find an "infinite" chain $c \in C'_{q+1}(\widetilde{X}^1)$

in the completed complex such that ∂c is finite and represents $f(a)$, and also a chain $d \in C_{q+1}(\widetilde{X})$ with $\partial d = Nx$, where x is a cycle representing $f(b)$ and N is a nonzero integer. Then

$$[f(a), f(b)] = \frac{1}{N} c \cdot d \in \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda) .$$

Thus, our first aim is to construct this infinite chain c.

But before doing this we must discuss some general constructions of $(q+1)$-dimensional chains in $\widetilde{X}$.

6.6 Let Y be the manifold obtained from $S^{2q+2} - \Sigma$ by cutting along V. We will identify Y with its embedding in $\widetilde{X}$ constructed in 5.5, and will identify V with $\partial^+ Y \subset Y \subset \widetilde{X}$.

Let v be a q-dimensional cycle in V and let $\pi_i zv$ be a cycle representing $\pi_i z[v] \in H_q(V)$, $i = 1, \dots, \mu$. By the definition of z (cf. 5.3), there exists a $(q-1)$-dimensional chain c_v in Y with $\partial c_v \in \partial Y$ such that

$$\partial c_v = i_+(v) - (i_+ - i_-)(zv) .$$

Identifying $i_+(v)$ with v and $i_+(zv)$ with zv, we should identify $i_-(zv)$ with

$$\sum_{i=1}^{\mu} t_i(\pi_i zv)$$

and we have

$$\partial c_v = v + \sum_{i=1}^{\mu} x_i(\pi_i zv) ,$$

where $x_i = t_i - 1 \in \Lambda$.

6.7 Lemma. *Let $[c_v]$ be the homology class in $H_{q+1}(Y, \partial Y)$ represented by the cycle c_v constructed above. Let*

$$\ell^Y : H_{q+1}(Y, \partial Y) \times H_q(Y) \to \mathbb{Q}/\mathbb{Z}$$

denote the linking pairing in Y and

$$\langle\, ,\rangle : H_q(V) \times H_q(V) \to \mathbb{Q}/\mathbb{Z}$$

denote the linking pairing in Y. Then for any class $x \in H_q(V)$ the following formula holds

$$\ell^Y([c_v], (i_{+*} - i_{-*})(x)) = -\langle [v], x\rangle ,$$

where $[v] \in H_q(V)$ denotes the homology class of v.

Note first that $H_{q+1}(Y, \partial Y)$ as well as $H_q(V)$ are in fact $\mathbb{Z}$-torsion and thus the linking pairings are correctly defined.

The proof of Lemma 6.7 is identical to the proof of Lemma 6.22 in [F] and is therefore omitted.

6.8 We now proceed with the construction of the "infinite" chain c such that $\partial c = f(a)$.

We are given a homology class $a \in H_q(V)$. For each multi-index $\alpha = (i_1, \dots, i_s)$ and each number $i \in \{1, \dots, \mu\}$ define

$$a_\alpha^i = \pi_{i_s} z \pi_{i_{s-1}} z \dots \pi_{i_1} z \pi_i a \in H_q(V) .$$

Let a cycle v_α^i realize a_α^i. By 6.6, there is a $(q+1)$-dimensional chain c_α^i in Y with $\partial c_\alpha^i \subset \partial Y$ and

$$\partial c_\alpha^i = v_\alpha^i + \sum_{j=1}^{\mu} x_j v_{\alpha j}^i ,$$

where $\alpha j = (i_1, \dots, i_s, j)$ for $\alpha = (i_1, \dots, i_s)$. Put

$$c = \sum_\alpha \sum_{i=1}^{\mu} (-1)^{|\alpha|+1} x_i x^\alpha c_\alpha^i ,$$

where α runs over all multi-indices and $|\alpha|$ denotes s for $\alpha = (i_1, \dots, i_s)$. This is a convergent power series in $C'_{q+1}(\widetilde{X}^1)$ and a short computation identical with that given in 6.25 of [F] shows that ∂c is finite and represents $f(a)$.

6.9 Going back to the proof of Theorem 6.2, assume that we have two homology classes $a, b \in H_q(V)$. In the previous subsection we have found an infinite chain c with ∂c representing $f(a)$. Let d be a chain in V with $\partial d = N\beta$, where β is a cycle representing b. Then

$$\begin{aligned} [f(a), f(b)] &= \frac{1}{N} c \cdot (i_+ - i_-)(d) \\ &= \sum_\alpha \sum_{i=1}^{\mu} (-1)^{|\alpha|+1} x_i x^\alpha \ell^Y([c_\alpha^i], (i_+ - i_-)(b)) \\ &= \sum_\alpha \sum_{i=1}^{\mu} (-1)^{|\alpha|} x_i x^\alpha \langle a_\alpha^i, b\rangle \end{aligned}$$

(by virtue of Lemma 6.7), and then computations identical with those of 6.25 in [F] complete the proof of Theorem 6.2.

Combining Theorem 6.2 with 5.7 we obtain:

6.10 Theorem. *Let V^{2q+1} be a minimal Seifert manifold of a torsion link (Σ^{2q}, s). Then the torsion isometry structure determined by V (cf. 5.3) is isomorphic to the torsion isometry structure of the minimal lattice in $H_q\widetilde{X}$ supplied with the*

Blanchfield form

$$H_q(\widetilde{X}) \times H_q(\widetilde{X}) \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$$

(cf. 5.2). Thus, the isomorphism type of the torsion isometry structure of a minimal Seifert manifold is determined uniquely by the link.

6.11 Corollary. *The Blanchfield form*

$$H_q(\widetilde{X}) \times H_q(\widetilde{X}) \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$$

of a torsion link of dimension $2q$ *is* $(-1)^q$*-Hermitian.*

Proof. The scalar form on the minimal lattice is isomorphic to the torsion isometry structure of any minimal Seifert manifold, which is obviously $(-1)^{q+1}$-symmetric. The result now follows from Theorem 3.3.

§7. The main results

7.1 Theorem. *Let* (Σ^{2q}, s), $q \geq 4$, *be a torsion link. Then any two minimal Seifert manifolds* V *and* W *of* Σ *(with* $s_V = s = s_W$*) are ambient isotopic (i.e. there exists a smooth isotopy* $h_t : S^{2q+2} \to S^{2q+2}$, $t \in [0,1]$, $h_0 = id$, *with* $h_1(V_j) = W_j$ *for each component* V_j *of* V *and the corresponding component* W_j *of* W, $j = 1, \ldots, \mu$. *Moreover,* $h_1|_{V_j} : V_j \to W_j$ *preserves the orientations).*

Proof. We will use the stable-homotopy reduction of the classification problem established in [F5]. Let $(\widehat{V}, u_V, z_V, \pi_1^V, \ldots, \pi_\mu^V)$ and $(\widehat{W}, u_W, z_W, \pi_1^W, \ldots, \pi_\mu^W)$ be stable isometry structures of the Seifert manifolds V and W, respectively ($\widehat{V}$ denotes the bouquet $\bigvee_{i=1}^{\mu} V_i$ and similarly for W). By Theorem 2.6 of [F5] in order to show that V and W are ambient isotopic it is enough to construct an S-equivalence

$$f : \widehat{V} \to \widehat{W}$$

with

$$f \circ z_V = z_W \circ f\,,$$
$$f \circ \pi_i^V = \pi_i^W \circ f\,, \quad i = 1, \ldots, \mu$$
$$u^W \circ (f \wedge f) = u^V$$

(the sign "=" means "stably homotopic" here). Now, both $\widehat{V}$ and $\widehat{W}$ are $(q-1)$-connected CW-complexes. By Poincaré duality, we may assume that $\dim \widehat{V} =$

$\dim \widehat{W} = q + 1$. We also know that $H_{q+1}(\widehat{V}) = H_{q+1}(\widehat{W}) = 0$ and that $H_q(\widehat{V})$ and $H_q(\widehat{W})$ are finite groups with no element of order two. We can apply (an easy part of) the classification Theorems 6.2 and 7.2 of [F3] to show that such an S-equivalence f exists if and only if the torsion isometry structures of V and W are isomorphic. But this last fact follows from Theorem 6.10.

This completes the proof.

7.2 Theorem. *Two torsion links* (Σ_ν^{2q}, s_ν), $\nu = 1, 2$, *with* $q \geq 4$, *are equivalent if and only if the corresponding Alexander modules* $H_q\widetilde{X}_\nu$, $\nu = s$, *together with their Blanchfield forms*

$$H_q(\widetilde{X}_\nu) \times H_q(\widetilde{X}_\nu) \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$$

are isomorphic.

Proof. Assume that there is a Λ-isomorphism

$$f : H_q(\widetilde{X}_1) \to H_q(\widetilde{X}_2)$$

preserving the Blanchfield forms. Let V_ν, $\nu = 1, 2$, be a minimal Seifert manifold of Σ_ν. The restriction of f to a minimal lattice of $H_q(\widetilde{X}_1)$ is an isomorphism between minimal lattices of $H_q(\widetilde{X}_1)$ and $H_q(\widetilde{X}_2)$. This restriction preserves the scalar forms (this follows from Theorem 3.2). By Theorem 6.10 we obtain that the torsion isometry structures of V_1 and V_2 are isomorphic. Then, using the results of [F5] we obtain by arguments similar to those of the proof of Theorem 7.1 that V_1 and V_2 are ambient isotopic; in particular Σ_1 and Σ_2 are equivalent (as $\mathcal{F}$-links).

7.3 Theorem. *Two torsion links* (Σ_ν^{2q}, s_ν), $\nu = 1, 2$, *with* $q \geq 4$, *are equivalent if and only if the torsion isometry structures of any pair of their respective minimal Seifert manifolds are isomorphic.*

Proof. This follows from Theorem 7.2 plus the remark in 5.2.

7.4 Theorem. *Given a minimal* ε*-symmetric torsion isometry structure* A *of multiplicity* μ *with no 2-torsion and an integer* $q \geq 4$ *with* $(-1)^{q+1} = \varepsilon$, *there exists a torsion link* (Σ^{2q}, s) *of* μ *components and its minimal Seifert manifold* V^{2q+1} *such that the torsion isometry structure of* V *is isomorphic to* A.

Proof. This follows from Theorem 2.6 of [F5] plus the homotopy classification of maps between A_q^1-spaces given in [F3].

7.5 Theorem. *Given a periodic* Λ*-module* M *of type* L *with no elements of order 2 and an* ε*-Hermitian non-degenerate form*

$$[\,,\,] : M \times M \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)\,,$$

for each $q \geq 4$, *with* $(-1)^q = \varepsilon$, *there exists a torsion link* (Σ^{2q}, s) *in* S^{2q+2} *such that the Alexander module* $H_q(\widetilde{X})$ *of* (Σ^{2q}, s), *considered together with its Blanchfield form*

$$H_q(\widetilde{X}) \times H_q(\widetilde{x}) \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$$

is isomorphic to $(M, [\ ,\ ])$.

Proof. Consider the minimal lattice of M together with its scalar form (cf. §3). By Theorem 7.4, one may realize the corresponding torsion isometry structure by a minimal Seifert manifold. The result now follows from 5.2.

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