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Noncommutative rational functions and boundary links

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The study of finite automata and their associated formal languages led Kleene [Kl] and Schutzenberger [Sc] to the notion of a *rational* formal power series, generalizing the usual notion of a rational function. Given a finite alphabet $X = \{x_1, \ldots, x_\mu\}$, a *language* L is a set of words w in X. L is called *regular* (or *recognizable*) if there exists a finite automaton answering the following question: given a word w in the alphabet X, does w belong to L? With each language L one associates a formal power series

$$\chi_L = \sum_w a(w)w \ ,$$

the *characteristic series of* L; here w runs over all words in X and $a(w)$ is 1 or 0 according to w belongs to L or not. Theorem of Kleene [Kl] and Schutzenberger [Sc] states that L is regular if and only if χ_L is *rational*. There are several equivalent characterizations of rational power series; one of them is taken in Sect. 1 as the definition. A recent exposition of the theory of rational power series and its applications to languages and automata can be found in [BR]. Here I would like to mention some easy facts which are important for the understanding of the sequel. Although it is an *infinite* formal power series, a rational function in non-commuting variables is in fact a *finite* object: due to the number of linear recurrence relations just a finite number of coefficients determine the other ones. Moreover, any rational series can be represented by a finite analytic formula.

It turns out that there is a surprisingly tight relation between noncommutative rational functions and algebraic objects appearing in topology of manifolds in the study of boundary links. One aspect of this connection was established in [FV, Sect. 3]: it was shown that any rational function generates a *link module*, i.e. a module which can be realized as homology module of a boundary link, cf. Sato

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[S]. (Note that any semi-simple link module can be obtained in this way, cf. Sect. 4 of the present paper.) In [FV] it was also proved that the ring of rational functions coincides with some universal ring (the Cohn localization of the free group ring).

The present paper establishes a relation between link modules and rational power series in the other directions:

$$\{\text{link}\} \to \{\text{rational function}\} .$$

It is shown here that each link module M defines a rational formal power series χ_M, containing all *semi-simple* information about M. In the case of knots χ_M can be expressed through the classical Alexander polynomial [L]. Examples show that for links of $\mu > 1$ components the rational function χ_M is richer than the link Alexander polynomial derived by studying the homology of the free abelian covering [B, H].

Note that there exists another path connecting logic, languages, automata (and rational functions) with topology: it is the study of automatic and hyperbolic groups, cf. [Gr, E].

More precisely, the aim of the present paper is to associate a sequence of rational formal power series

$$\chi_1, \ldots, \chi_n$$

with any n-dimensional boundary link, providing rather strong link invariants. To compute χ_i one has to calculate a *finite* number of integers (the traces of certain linear maps acting on the homology of a Seifert surface).

Among the other results of the paper let us mention the symmetry property (cf. Theorem 8.2)

$$\chi_q + \bar{\chi}_q = 0$$

(similar to the well-known symmetry property of the Alexander polynomial) and also

$$\chi_q = \eta - \bar{\eta}$$

for a rational function η with integral coefficients if the link is null concordant (it is a generalization of a theorem of Fox and Milnor [FM]).

The construction of χ_M uses the fact (established in [F3]) that any link module M contains a unique minimal lattice $A \subset M$ which is a finite dimensional vector space supplied with a number of endomorphisms. The minimal lattice A determines M uniquely. Then we apply ideas of Procesi [P] who showed the role of trace-type invariants in the semi-simple classification.

1 Noncommutative rational functions

Here we review basic definitions of the theory of noncommutative rational functions, refering to [BR] for a complete systematic study.

1.1. Let k be a field. A formal power series in noncommuting variables $x_1, \ldots, x_\mu$ with coefficients in k is an expression of the form

$$\gamma = \sum_\alpha a(\alpha) x^\alpha$$

where α runs over all multi-indices $\alpha = (i_1, \ldots, i_s)$ with $i_j \in \{1, \ldots, \mu\}$ (the empty sequence $\varnothing$ is also allowed), the symbol x^α stands for the monomial

$$x_{i_1} x_{i_2} \cdots x_{i_s}$$

(with the convention $x^\varnothing = 1$); a is a function on the set of multi-indices with values in k, the *coefficient function* of the power series γ.

A power series γ is *rational* if there is a finite-dimensional linear "machine" which produces the coefficient function of γ. More precisely, γ is called *rational* if there exists a finite dimensional vector space V over k, linear operators

$$T_1, T_2, \ldots, T_\mu : V \to V$$

and two elements $v \in V$, $f \in V^*$ such that for any multi-index $\alpha = (i_1, \ldots, i_s)$

$$a(\alpha) = f(T_\alpha v)$$

where

$$T_\alpha = T_{i_s} \circ T_{i_{s-1}} \circ \cdots \circ T_{i_1} : V \to V$$

(the composition).

The minimal possible dimension of V in the above representation is called the *rank* of γ; it is denoted by $\mathrm{rk}(\gamma)$.

1.2. The set of all formal power series in $x_1, \ldots, x_\mu$ with coefficients in k is denoted by

$$\Gamma = k\langle\langle x_1, \ldots, x_\mu \rangle\rangle .$$

It is a ring (with respect to usual addition and multiplication). Rational formal power series form a subset $\mathscr{R} \subset \Gamma$. It is a subring [FV]. The ring of rational functions $\mathscr{R}$ was characterized in [FV] as the Cohn localization of the free group ring with respect to the augmentation.

Note that for $\mu = 1$ any rational formal power series can be represented as

$$\gamma = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and $q(0) \neq 0$, cf, [BR, FV]. Thus, the notion of a rational power series generalizes the classical notion of a rational function.

The following two statements show that a finite number of coefficients of a rational power series determine the whole coefficient function.

1.3. Proposition (cf. [BR, p. 33]). *Let*

$$\gamma = \sum_\alpha a(\alpha) x^\alpha$$

be a rational function with $rk(\gamma) \leqq n$. Suppose that $a(\alpha) = 0$ for all multi-indices α of length $\leqq n - 1$. Then $\gamma = 0$.

1.4. Corollary. *Let γ_1, γ_2 be two rational power series of rank $\leqq n$ and let $a(\alpha)$, $b(\alpha)$ be their coefficient functions. If $a(\alpha) = b(\alpha)$ for every multi-index α of length $\leqq 2n - 1$ then $\gamma_1 = \gamma_2$.*

2 *P*-modules and their rational functions

Our main goal in this paper is to study link modules which are modules over the group ring of the free group satisfying certain additional properties, cf. Sect. 7. It turns out that the structure of a link module is determined by a simpler finite-dimensional object, the *minimal lattice*. It is a P-module, P being the ring defined below. The present section gives a construction associating a rational function to a P-module.

2.1. Let k be a field and $\mu > 0$ be a fixed integer. Let P denote the k-algebra with generators $z, \pi_1, \ldots, \pi_\mu$ subject to the following relations

$$\pi_i^2 = \pi_i, \quad i = 1, \ldots, \mu ,$$

$$\pi_i \circ \pi_j = 0 \quad \text{for } i \neq j ,$$

$$\pi_1 + \pi_2 + \cdots + \pi_\mu = 1 .$$

P has an involution $\bar{\ } : P \to P$ defined by

$$z \mapsto \bar{z} = 1 - z, \qquad \pi_i \mapsto \pi_i, \quad i = 1, \ldots, \mu .$$

We will consider also another k-algebra

$$D = k\langle \partial_1, \ldots, \partial_\mu \rangle ,$$

the free k-algebra of polynomials in non-commuting $\partial_1, \ldots, \partial_\mu$. There is a ring homomorphism

$$D \to P, \qquad \partial_i \mapsto -z\pi_i, \quad i = 1, \ldots, \mu .$$

This homomorphism converts any P-module into a D-module.

2.2. The ring Γ (defined in 1.2) is a D-module where $\partial_i : \Gamma \to \Gamma$ is the map given by

$$\partial_i \gamma = \sum_\alpha a(i\alpha) x^\alpha$$

for

$$\gamma = \sum_\alpha a(\alpha) x^\alpha .$$

Let $\Lambda = k[F_\mu]$ be the group ring of the free group F_μ with generators $t_1, \ldots, t_\mu$. The ring Λ is embedded in Γ via the Magnus embedding

$$t_i \mapsto 1 + x_i, \qquad t_i^{-1} \mapsto 1 - x_i + x_i^2 - x_i^3 + \cdots$$

It is easy to see that Λ is a D-submodule of Γ and the restriction $\partial_i|_\Lambda : \Lambda \mapsto \Lambda$ coincides with the Fox derivative with respect to t_i.

The ring of rational functions $\mathscr{R} \subset \Gamma$ is also a D-submodule.

An example of P-module is provided by Γ/Λ. The P-module structure is given by defining $\pi_i\gamma$ to be the sum of all monomials in γ which start with x_i on the left. Thus

$$\pi_i\gamma = \sum_\alpha a(i\alpha)x_i x^\alpha \mod(\Lambda)$$

for

$$\gamma = \sum_\alpha a(\alpha)x^\alpha \mod(\Lambda)\,;$$

the multiplication by $z \in P$ is given by

$$z\gamma = (-\partial_1 - \partial_2 - \cdots - \partial_\mu)\gamma\,.$$

The factor-module $\mathscr{R}/\Lambda$ is a P-submodule in Γ/Λ. It has the following finiteness property:

2.3. Lemma. *Let γ be an element of $\mathscr{R}/\Lambda$ and let $A \subset \mathscr{R}/\Lambda$ be the P-submodule generated by γ. Then* $\dim_k A < \infty$.

Proof. It is well known that any rational function generates a D-submodule of Γ which is finite-dimensional over k, (cf. [BR]). Thus, the D-submodule $B \subset \mathscr{R}/\Lambda$ generated by γ has finite dimension over k. But $A = \pi_1 B + \pi_2 B + \cdots + \pi_2 B$ and the lemma follows.

We will see later that $\mathscr{R}/\Lambda$ is a "universal" P-module: it contains (almost) every simple P-module A with $\dim_k A < \infty$.

2.4. The relation between P-modules and rational functions is twofold. We have seen that any rational function generates a finitely dimensional P-submodule of $\mathscr{R}/\Lambda$. Now we will show that one may invariantly assign a rational function to a P-module.

Let A be a P-module with $\dim_k A < \infty$.

Choose a k-basis $a_1, \ldots, a_n$ of A and let $b_1, \ldots, b_n$ be the dual basis of A^*, the dual vector space. Define

$$\chi_A = \sum_{i=1}^{n}\sum_{k=1}^{\mu}\sum_\alpha x^\alpha \langle b_i, \pi_k \partial_\alpha a_i\rangle x_k\,.$$

Here α runs over all multi-indices and for $\alpha = (i_1, \ldots, i_s)$ ∂_α denotes $\partial_{i_s} \circ \partial_{i_s - 1} \circ \cdots \circ \partial_{i_1}$ (which agrees with the conventions of Sect. 1). The brackets $\langle b, a\rangle$ denote the value of the functional $b \in A^*$ on $a \in A$. χ_A is a formal power series, an element of Γ.

2.5. Proposition. (a) *The element $\chi_A \in \Gamma$ does not depend on the choice of the basis* $a_1, \ldots, a_n \in A$;

(b) *χ_A is a rational power series, i.e. $\chi_A \in \mathscr{R}$.*

(c) *If*

$$0 \to A' \to A \to A'' \to 0$$

is an exact sequence of P-modules then

$$\chi_A = \chi_{A'} + \chi_{A''}\,.$$

(d) *Let* $0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$ *be a filtration of* A *by* P*-submodules. Then*

$$\chi_A = \sum_{i=1}^{n} \chi_{A_i/A_{i-1}} .$$

In particular, χ_A *depends only on the composition factors of* A.

Proof. (a) Note that

$$\sum_{i=1}^{n} \langle b_i, \pi_k \partial_\alpha a_i \rangle = \begin{cases} \dim \pi_k A & \text{if } \alpha = \varnothing \\ \operatorname{Tr}[\pi_k \partial_\alpha : A \to A] & \text{if } |\alpha| \geqq 1 \end{cases} .$$

Thus, one may write χ_A in the form

$$\chi_A = \sum_{k=1}^{\mu} (\dim \pi_k A) x_k + \sum_{k=1}^{\mu} \sum_{\alpha} (\operatorname{Tr}[\pi_k \partial_\alpha \partial_k : A \to A]) x_k x^\alpha x_k .$$

It is now clear χ_A does not depend on the choice of the basis.

(b) It is enough to show that for each $i = 1, \ldots, n$ and $k = 1, \ldots, \mu$ the power series

$$\sum_{\alpha} x^\alpha \langle b_i, \pi_k \partial_\alpha a_i \rangle$$

is rational. But this obviously follows from the definition of 1.1: one may take $V = A$, $T_i = \partial_i$, $v = a_i$ and $f \in A^*$ to be given by $f(a) = \langle b_i, \pi_k a \rangle$.

(c) and (d) are now obvious.

This completes the proof.

2.6. *Example.* Suppose that $\mu = 1$ and the field k is algebraically closed. Then $P = k[z]$ and any simple P-module is one dimensional. The isomorphism type of a simple module is determined by an element $\lambda \in k$ (the eigenvalue of z). If A_λ denotes such a module then

$$\chi_{A_\lambda} = \frac{x}{1 + \lambda x} .$$

From 2.4(d) we get that in the general case

$$\chi_A = \sum_{i=1}^{N} \frac{x}{1 + \lambda_i x}$$

where A is a P-module with $\mu = 1$ and $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of z, each repeated as many times as its multiplicity.

2.7. Let us estimate the rank of rational function χ_A.

Since

$$\operatorname{rk}(\gamma_1 + \gamma_2) \leqq \operatorname{rk}(\gamma_1) + \operatorname{rk}(\gamma_2)$$

$$\operatorname{rk}(\gamma_1 \gamma_2) \leqq (\operatorname{rk}(\gamma_1) + 1)\operatorname{rk}(\gamma_2)$$

(cf. Proposition 1.6 of [FV]) and the rank of

$$\sum_{\alpha} x^{\alpha} \langle f_i, \pi_k \partial_\alpha a_i \rangle$$

is less or equal to $\dim_k A$, while $\mathrm{rk}(x_k) = 2$, we obtain

$$\mathrm{rk}(\chi_A) \leqq 2\mu \cdot \dim_k A \cdot (\dim_k A + 1) .$$

2.8. Given a P-module A with $\dim_k A < \infty$ one may consider the P-submodule $B \subset \mathcal{R}/\Lambda$ generated by χ_A. The following theorem establishes the relationship between A and B.

2.9. Theorem. *Suppose that the field k is algebraically closed. Let A be a simple non-primitive P-module with* $\dim_k A < \infty$. *Then the P-submodule of* $\mathcal{R}/\Lambda$ *generated by the image of* χ_A *is isomorphic (as a P-module) to the direct sum*

$$\underbrace{A \oplus \cdots \oplus A}_{n \text{ times}}$$

where $n = \dim_k A$.

The notion "non-primitive" will be explained in the next section.
The proof of Theorem 2.9 is given in 4.3.

3 Primitive modules

Primitive modules provide an exceptional class of modules which need a separate study.

3.1. Let C be a D-module. We will say that C is a *D-module of type* 0 if $\partial_k C = 0$ for all $k = 1, \dots, \mu$. We will say that C is a *D-module of type i* (where $i \in \{1, 2, \dots, \mu\}$) if $\partial_k C = 0$ for $k \neq i$, $k \in \{1, \dots, \mu\}$ and $(1 + \partial_i)C = 0$.

A D-module Y will be called *primitive* if it has a filtration

$$0 = Y_0 \subset Y_1 \subset \dots \subset Y, \quad \bigcup Y = Y$$

with the property that for each $j = 1, 2, \dots$ there exists a number $i = i(j) \in \{0, \dots \mu\}$ such that Y_j / Y_{j-1} is a D-module of type i.

Any submodule and any factor-module of a primitive D-module is also primitive. An important example of a primitive D-module is provided by $Y = \Lambda$ with D-module structure given by the Fox derivatives, cf. Sect. 2.

3.2. Now we will define primitive P-modules. For any integer $j \in \{1, \dots, \mu\}$ let A_j denote the following P-module: A_j is one dimensional over k, π_i acts as zero on A_j for $j \neq i$ and $\pi_j = 1$. The multiplication by $z \in P$ is zero.

Let also B_j for $j \in \{1, \dots, \mu\}$ denote the P-module with $\dim_k B_j = 1$, $\pi_i = 0$ for $i \neq j$, $\pi_j = 1$ and $z: B_j \to B_j$ acts as the identity.

A P-module A with $\dim_k A < \infty$ will be said to be *primitive* if each of its composition factors is isomorphic to one of $A_1, \dots, A_\mu, B_1, \dots, B_\mu$.

A will be called to be *primitive-free* if it has no composition factors among $A_1, \dots, A_\mu, B_1, \dots, B_\mu$.

Because of the homomorphism $D \to P$ (cf. 2.1) any P-module has an associated D-module structure. A primitive P-module is also primitive as D-module. A partial converse to this statement is given in the following Lemma.

3.3. Lemma. *A primitive-free P-module A with* $\dim_k A < \infty$ *cannot be primitive as a D-module.*

Proof. Assume the contrary. Then A contains a D-submodule $B \subset A$ of type $i \in \{0, 1, \ldots, \mu\}$. If $i = 0$ there exists an element $a \in A$, $a \neq 0$ with $\partial_k a = 0$, $k = 1, 2, \ldots, \mu$. Since $a = \pi_1 a + \cdots + \pi_\mu a$, one of $\pi_j(a)$ is non-zero and the subspace $X \subset A$ generated by this $\pi_j(a)$ is a P-submodule isomorphic to A_j, a contradiction.

Assume now that $i \in \{1, \ldots \mu\}$. Then we obtain an element $a \in A$, $a \neq 0$ with $\partial_k a = 0$ for $k \neq i$, $k \in \{1, \ldots, \mu\}$ and $(1 + \partial_i)a = 0$. Write $a = \pi_1(a) + \cdots + \pi_\mu a$. If one of $\pi_j a$ with $j \neq i$ is non-zero then we can proceed as in the previous paragraph to get a contradiction. Assuming now that all $\pi_j a = 0$ for $j \neq i$. We obtain $a = \pi_i a$ and the linear subspace X generated by a is a P-submodule isomorphic to B_i — a contradiction.

4 Proof of Theorem 2.9

4.1. Let A be a P-module with $\dim_k A < \infty$. For any $f \in A^* = \mathrm{Hom}_k(A, k)$ define a map

$$w_f : A \to \mathscr{R}/\Lambda$$

by

$$w_f(a) = \sum_{k=1}^{\mu} \sum_{\alpha} x^\alpha x_k \langle f, \pi_k \partial_\alpha a \rangle \mod \Lambda .$$

Arguments similar to those used in the proof of the statement (b) of Proposition 2.5 show that $w_f(a)$ belongs to $\mathscr{R}/\Lambda$. It is easy to see that w_f is a P-homomorphism.

4.2. Proposition. *Assume that the field k is algebraically closed. Let A be a simple P-module which is not primitive. Then for any linearly independent* $f_1, \ldots, f_s \in A^*$ *the map*

$$F_s : \underbrace{A \oplus \cdots \oplus A}_{s \text{ rimes}} \to \mathscr{R}/\Lambda, \qquad F_s = w_{f_1} + w_{f_2} + \cdots + w_{f_s}$$

is a monomorphism.

Proof. We proceed by induction on s. Consider first the case $s = 1$. We have to show that w_f is a monomorphism for $f \neq 0$. Because A is assumed to be simple, $\ker w_f \neq 0$ implies $w_f = 0$. This means that for any $k = 1, 2, \ldots, \mu$ the function

$$\varphi_k : A \to \Gamma$$

given by

$$\varphi_k(a) = \sum_\alpha x^\alpha \langle f, \pi_k \partial_\alpha a \rangle$$

takes its values in $\Lambda \subset \Gamma$. Consider $\ker \varphi_k$ which coincides with the set of all $a \in A$ such that $\langle f, \pi_k \partial_\alpha a \rangle = 0$ for any multi-index α. It is clear that $a \in \ker \varphi_k$ implies

$\pi_j a \in \ker \varphi_k$ and also $\partial_j(a) \in \ker \varphi_k$ for any $j \in \{1, \ldots, \mu\}$. Thus $\ker \varphi_k$ is a P-submodule of A. Because A is simple and $f \neq 0$ we get that $\ker \varphi_k = 0$ at least for one $k = 1, \ldots, \mu$.

Note that φ_k is a D-homomorphism. Because A is primitive as D-module and $\varphi_k : A \to A$ is a D-embedding we obtain that A is also D-primitive. But this contradicts Lemma 3.3. Thus w_f is an embedding and we have proved our statement for $s = 1$.

Asssume that Proposition 4.2 has been proved for some s. Consider the case of $s + 1$ linearly independent functionals $f_1, \ldots, f_{s+1}$. If the map

$$F_{s+1} : \underbrace{A \oplus \cdots \oplus A}_{s+1 \text{ times}} \to \mathscr{R}/A, \qquad F_{s+1} = w_1 + \cdots + w_{f_s} + w_{f_{s+1}}$$

is not a monomorphism then there is a P-homomorphism $g : A \to \underbrace{A \oplus \cdots \oplus A}_{s \text{ times}}$ such that the diagram

$$\begin{array}{ccc} \underbrace{A \oplus \cdots \oplus A}_{s \text{ times}} & \xrightarrow{F_s} & \mathscr{R}/A \\ & \nwarrow g \qquad \nearrow w_{f_{s+1}} & \\ & A & \end{array}$$

commutes (to show the existence of g note that F_s and $w_{f_{s+1}}$ are monomorphisms whose images have non-trivial intersection and A is simple; thus the image of $w_{f_{s+1}}$ is contained in the image of F_s).

Since A is simple and k is algebraically closed, there exist $\lambda_1, \lambda_2, \ldots, \lambda_s \in k$ such that

$$g(a) = (\lambda_1 a, \ldots, \lambda_s a)$$

for any $a \in A$. Then we get

$$w_{f_{s+1}}(a) = \sum_{j=1}^{s} w_{f_j}(\lambda_j a) = \sum_{j=1}^{s} w_{\lambda_j f_j}(a)$$

and thus

$$w_f = 0$$

where

$$f = f_{s+1} - \sum_{j=1}^{s} \lambda_j f_j .$$

From the case $s = 1$ of the proposition it follows now that $f = 0$ and so $f_1, f_2, \ldots, f_s, f_{s+1}$ are linearly dependent.

This completes the proof.

4.3. *Poof of Theorem 2.9.* Let $a_1, \ldots, a_n \in A$ be a basis for A and let $f_1, \ldots, f_n \in A^*$ be the dual basis. Then

$$\chi_A = \sum_{i=1}^{n} w_{f_i}(a_i)$$

according to the definition of Subsect. 2.4. By Proposition 4.2 the P-submodule of $\mathscr{R}/\Lambda$ generated by χ_A is isomorphic $\left(\text{via } \sum_{i=1}^{n} w_{f_i}\right)$ to the submodule of

$$\underbrace{A \oplus \cdots \oplus A}_{n \text{ times}} = A^n, \quad n = \dim_k A$$

generated by $a = (a_1, \ldots, a_n) \in A^n$.

Since $a_1, \ldots, a_n$ is a basis of A, for any ordered set $(b_1, \ldots, b_n) \in A^n$ there exists a k-linear map $B: A \to A$ such that $Ba_i = b_i$ for any $i = 1, \ldots, n$. By a Theorem of Bernside (cf. [Bu, Chap. VIII, Sect. 4.3]) there exists $p \in P$ such that $pa_i = b_i$ for any $i = 1, \ldots, n$. Thus $a = (a_1, \ldots, a_n) \in A^n$ generates the whole A^n over P.

This completes the proof.

5 Semi-simple *P*-modules

We prove here that χ-function gives a semi-simple classification of P-modules.

5.1. Consider first χ-functions of primitive P-modules. If A_j and B_j are primitive simple modules described in 3.2 then

$$\chi_{A_j} = x_j, \qquad \chi_{B_j} = \sum_{k \geqq 0} (-1)^k x_j^{k+1} = \frac{x_j}{1 + x_j} = -\bar{x}_j .$$

Thus the χ-function of any primitive P-module has the form

$$\chi = \sum_{j=1}^{\mu} (\alpha_j x_j - \beta_j \bar{x}_j) \in \Lambda$$

with $\alpha_j, \beta_j \in \mathbb{Z}$, $\alpha_j \geqq 0$, $\beta_j \geqq 0$. Here α_j and β_j are the multiplicities of A_j and B_j, respectively, appearing as simple composition factors. If the characteristic of k is 0 one may determine α_j and β_j by means of χ (note that $-\beta_j$ is the coefficient of x_j^2). Thus we have proved the following

5.2. Proposition. *Suppose* char$(k) = 0$. *Two semi-simple primitive P-modules A and B are isomorphic if and only if* $\chi_A = \chi_B$.

5.3. Let A be a P-module with $\dim_k P < \infty$. Consider the P-submodule $B \subset \mathscr{R}/\Lambda$ generated by the image of χ_A in $\mathscr{R}/\Lambda$. By Lemma 2.3 $\dim_k B < \infty$ and we may consider the χ-function χ_B of B. We will denote

$$\chi'_A = \chi_B ,$$

χ'_A will be called *the secondary χ-function of A*.

Note that the secondary χ-function of a primitive module is zero.

5.4. Let A be a primitive-free P-module with $\dim_k A < \infty$ and let $C_1, \ldots, C_n$ be its distinct composition factors appearing with multiplicities $m_1, \ldots, m_n \geqq 1$.

Then by 2.5

$$\chi_A = \sum_{j-1}^{n} m_j \chi_{C_j}$$

and by Theorem 2.9 above and by Proposition 9 in Chap. VIII, sect. 3, n 4 of [Bu]

$$\chi'_A = \sum_{j=1}^{n} (\dim_k C_j) \cdot \chi_{C_j} .$$

Thus the secondary χ-function does not depend on the multiplicities m_j.

5.5. Theorem. *Suppose that k is an algebraically closed field with* $\mathrm{Char}(k) = 0$. *Two semi-simple P-modules with* $\dim_k A < \infty$, $\dim_k B < \infty$ *are isomorphic if and only if* $\chi_A = \chi_B$.

Proof. We have to prove that $\chi_A = \chi_B$ implies that A and B are isomorphic. Consider the P-submodule X of $\mathscr{R}/\Lambda$ generated by the image of $\chi_A = \chi_B$ in $\mathscr{R}/\Lambda$. By Theorem 2.9

$$X = C_1^{\dim C_1} \oplus \ldots \oplus C_n^{\dim C_n}$$

where $C_1, \ldots, C_n$ are distinct primitive-free composition factors of A (or B), appearing with positive multiplicities. Thus, A and B have the same primitive-free composition factors and we may write

$$A = C_1^{m_1} \oplus \ldots \oplus C_n^{m_n} \oplus A_1$$

$$B = C_1^{m'_1} \oplus \ldots \oplus C_n^{m'_n} \oplus B_1$$

where A_1 and B_1 are primitive, $m_j, m'_j \in \mathbb{Z}$, $m_j \geqq 1$, $m'_j \geqq 1$, $j \in \{1, \ldots, n\}$. Assume that $m_1 > m'_1$. Then $m_1 = m'_1 + r$, $r \in \mathbb{Z}$, $r \geqq 0$. Consider the P-modules

$$A' = C_1^r \oplus C_2^{m_2} \oplus \ldots \oplus C_n^{m_n} + A_1$$

$$B' = C_2^{m'_2} \oplus \ldots \oplus C_n^{m'_n} \oplus B_1 .$$

We have

$$\chi_{A'} = \chi_A - m'_1 \chi_{C_1} = \chi_{B'}$$

and A', B' are semi-simple. As we have seen before the condition $\chi_{A'} = \chi_{B'}$ implies that A' and B' have the same primitive-free composition factors. But C_1 is a composition factor of A' but not of B'.

This proves that $m'_j = m_j$ for $j = 1, \ldots, n$.

Now we obtain $\chi_{A_1} = \chi_{B_1}$ and applying Proposition 5.2 completes the proof.

6 The dual module

6.1. Since the ring P has an involution (cf. 2.1), the dual vector space A^* of any P-module (with $\dim_k A < \infty$) has a natural P-module structure:

$$\langle zf, a \rangle = \langle f, \bar{z}a \rangle, \qquad \langle \pi_j f, a \rangle = \langle f, \pi_j a \rangle$$

for any $f \in A^*$, $a \in A$.

We want to compute χ_{A^*} in terms of χ_A.
Consider the involution $\bar{}: \Gamma \to \Gamma$ given by

$$x_i \mapsto \bar{x}_i = -x_i + x_i^2 - \cdots = -\frac{x_i}{1+x_i}.$$

The restriction of $\bar{}$ to $\mathscr{R}$ is an involution $\bar{}: \mathscr{R} \to \mathscr{R}$.

6.2. Proposition. $\chi_{A^*} = -\bar{\chi}_A$.

Proof. Let $a, \ldots, a_n \in A, f_1, \ldots, f_n \in A^n$ be a pair of dual bases. The arguments used in proof of Theorem 3.3 of [F3] show that for any $i = 1, \ldots, n$.

$$\sum_{k=1}^{\mu} \sum_{\alpha} x^{\alpha} x_k \langle f_i, \pi_k \partial_\alpha a_i \rangle = - \sum_{\substack{(i_1, i_2, \ldots, i_s) \\ s \geq 1}} \bar{x}_{i_1} \bar{x}_{i_2} \ldots \bar{x}_{i_s} \langle \pi_{i_1} \partial_{i_2} \ldots \partial_{i_{i_s}} f_i, a_i \rangle .$$

Summing up these expressions for $i = 1, \ldots, n$ we obtain

$$\bar{\chi}_A = -\chi_{A^*}.$$

This completes the proof.

7 Link modules and their lattices

The aim of this section is to describe the relation between link modules and P-modules.

7.1. Let F_μ denote the free group on μ generators $t_1, \ldots, t_\mu$ and let $\Lambda = k[F_\mu]$ be the group ring (k is a field).

A finitely generated left Λ-module M is called *link module* (or module of type L) if

$$\mathrm{Tor}_q^\Lambda(k, M) = 0$$

for all q, where k is regarded as a right Λ-module with trivial action via the augmentation map.

7.2. Lemma. *Let f: $M' \to M$ be a Λ-homomorphism between link modules. Then* $\mathrm{im}(f)$ *and* $\ker(f)$ *are also link modules.*

The proof of the lemma will be given later at the end of 7.5.

7.3. A link module M will be said to be *simple* if it has no non-trivial submodules which are link modules.

A link module is *semi-simple* if it is a finite direct sum of simple link modules.

7.4. Theorem. *Any link module is of finite length; that is, any link module M has a filtration*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that $M_0, M_1, \ldots, M_{n-1}$ are link modules and M_i/M_{i-1} is simple for $i = 1, \ldots, n$.

The proof will be given later in 7.8; it will be based on the study of lattices in link modules.

Note that finitely generated Λ-modules are of infinite length in general (look at Λ itself as an example).

7.5. It was shown by Sato [S] that the condition $\mathrm{Tor}_q^\Lambda(k, M) = 0$ for all q is equivalent to the following: each $m \in M$ has a unique representation in the form

$$m = \sum_{i=1}^{\mu} (t_i - 1)m_i, \quad m_i \in M .$$

Thus, one may define "derivations" ∂_i: M, $i = 1, \ldots, \mu$, by $\partial_i(m) = m_i$, where $m_i \in M$ is the element appearing in the above decomposition, cf. [F3].

Let us also define the operations

$$\pi_1, \ldots, \pi_\mu : M \to M \quad \text{by}$$

$$\pi_i(m) = (t_i - 1)\partial_i(m) .$$

Each π_i is a projection $\pi_i^2 = \pi_i$ and $\pi_1 + \cdots + \pi_\mu = \mathrm{id}$. Moreover, $\pi_i \circ \pi_j = 0$ for $i \neq j$.

Define a map $z: M \to M$ by

$$z(m) = -\partial_1(m) - \partial_2(m) - \cdots - \partial_\mu(m) .$$

Then one can express $\partial_1, \ldots, \partial_\mu$ by means of $z, \pi_1, \ldots, \pi_\mu$:

$$\partial_i = -z \circ \pi_i, \quad i = 1, \ldots, \mu .$$

More information can be found in [F3, Sect. 1].

The operations $z, \pi_1, \ldots, \pi_\mu$ define on a link module a P-module structure. The operations $\partial_1, \ldots, \partial_\mu$ define a D-module structure.

Any Λ-homomorphism between link modules f: $M' \to M$ is also a P- and D-homomorphism. Thus, $\mathrm{im}(f)$ is a D-submodule with each $m \in \mathrm{Im}(f)$ having a unique representation of the form

$$m = \sum_{i=1}^{\mu} (t_i - 1)m_i, \quad m_i \in \mathrm{im}(f) .$$

Since $\mathrm{im}(f)$ is finitely generated over Λ it follows that $\mathrm{im}(f)$ is a link module.

Similar arguments show that $\ker(f)$ is a link module; the fact that $\ker(f)$ is finitely generated over Λ follows from the coherence of Λ, cf. [W].

This proves Lemma 7.2.

7.6. A *lattice* in a link module M is a k-submodule $A \subset M$ which

(a) is invariant under ∂_i and π_i, $i = 1, \ldots, \mu$;

(b) generates M over Λ;

(c) is finitely generated over k

It was shown in [F3] that:

(1) any link module contains a lattice;

(2) any lattice $A \subset M$, considered as a finite dimensional vector space with self maps $z, \pi_1, \ldots, \pi_\mu : A \to A$ (i.e. as a P-module) determines the whole module M (cf. [F3, Lemma 2.6]);

(3) There is a *minimal lattice* $A \subset M$ which is the intersection of all lattices in M (cf. [F3, Lemma 2.6]);

7.7. Lemma. *Any Λ-homomorphism $f\colon M_1 \to M_2$ between link modules maps the miminal lattice $A_1 \subset M_1$ into the minimal lattice $A_2 \subset M_2$.*

Proof. Consider the composition

$$A_1 \to M_1 \xrightarrow{f} M_2 \to M_2/A_2 .$$

All these maps are P-homomorphisms and so the kernel K of the composition is a P-submodule of A_1. On the other hand M_2/A_2 is a primitive P-module (cf. [F3, Sect. 1]) and from Lemma 4.4 of [F3] it follows that K generates M_1 over Λ; thus K is a lattice. The minimality of A_1 gives $K \supset A_1$ and so $f(A_1) \subset A_2$.

7.8. *Proof of Theorem 7.4.* Let $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ be a filtration of a link module M by link submodules. Let $A_1, A_2, \ldots, A_n = A$ be the minimal lattices of $M_1, M_2, \ldots, M_n$ respectively. Then by Lemma 2.3 we have

$$0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = A .$$

Each inclusion here is proper (because of 7.6(2)).

Thus $n \leqq \dim_k A$. This completes the proof.

7.9. The minimal lattice in a link module $A \subset M$ can be considered as a P-module. Such P-modules have an additional property: they have no primitive submodules and factor-modules.

In fact, if A has a primitive simple submodule (of type A_i say, cf. 3.2) then there is an element $a \in A$ with $a = \pi_i a$, $a \neq 0$, $\partial_i a = 0$. Then

$$a = (t_i - 1)b$$

for some $b \in M$ and $\partial_i a = b = 0$ implies $a = 0$.

Similar arguments show that A has no primitive submodules of type B_i.

The fact that the minimal lattices have no nontrivial primitive factor-modules follows from Theorem 1.11 of [F3]; in fact this property characterizes minimal lattices.

The converse to the previous statement is also true: any P-module A with $\dim_k A < \infty$ having no primitive submodules and factor-modules is the minimal lattice in a unique link module. This is the content of the following Lemma.

7.10. Lemma. *Let A be a P-module,* $\dim_k A < \infty$, *having no primitive submodules and factor-modules. Let*

$$u\colon \Lambda \otimes_k A \to \Lambda \otimes_k A$$

be a Λ-homomorphism given by

$$u(\lambda \otimes a) = \lambda \otimes a - \sum_{i=1}^{\mu} \lambda(t_i - 1) \otimes \partial_i(a)$$

where $\lambda \in \Lambda$, $a \in A$. Denote $M(A) = \mathrm{coker}(u)$ and $e: \Lambda \otimes_k A \to M(A)$ the projection. Then:

(a) *u is a monomorphism;*

(b) *$M(A)$ is a link module;*

(c) *the map $A \to M(A)$ given by $a \mapsto e(1 \otimes a)$ is a monomorphism;*

(d) *the image of the map $A \to M(A)$ in* (c) *coincides with the minimal lattice in $M(A)$.*

(e) *let A', A'' be two P-modules satisfying conditions of the Lemma and let $f: A' \to A''$ be a P-homomorphism; then there is a unique Λ-homomorphism*

$$\hat{f}: M(A') \to M(A'')$$

whose restriction on minimal lattices coincides with f.

(f) *let $A' \xrightarrow{f} A \xrightarrow{g} A''$ be an exact sequence of P-modules with no primitive submodules and factor-modules then the corresponding seqence of link modules*

$$M(A') \xrightarrow{\hat{f}} M(A) \xrightarrow{\hat{g}} M(A'')$$

is exact.

Proof. (a) Consider the map

$$u': \Gamma \otimes_k A \to \Gamma \otimes_k A$$

given by the same formula as u. Let $\varepsilon: \Gamma \to k$ be the augmentation $\varepsilon(x_i) = 0$, $i = 1, \ldots, \mu$. It is easy to see that the $(\dim A \times \dim A)$-matrix (λ_{ij}), $\lambda_{ij} \in \Lambda \subset \Gamma$ defining u' has the property that

$$\det \varepsilon(\lambda_{ij}) = 1$$

and so u' is an isomorphism. Thus $\ker(u) \subset \ker(u') = 0$.

(b) The sequence

$$0 \to \Lambda \otimes_k A \xrightarrow{u} \Lambda \otimes_k A \xrightarrow{e} M(A) \to 0$$

is a free resolution of $M(A)$. Using it to compute $\mathrm{Tor}_*^\Lambda(k, M)$ one easily gets $\mathrm{Tor}_*^\Lambda(k, M) = 0$.

(c) Consider the graph T of the free group F_μ with respect to the free generators $t_1, \ldots, t_\mu$. Vertices of T are in $1-1$ correspondence with elements $g \in F_\mu$, edges of T are enumerated by pairs (g, i), $g \in F_\mu$, $i \in \{1, \ldots, \mu\}$ with (g, i) joining g and gt_i. The neighborhood of a vertex g looks as shown in the picture

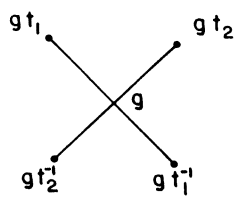

Fig. 1.

It is known that T is a tree. It follows that for any $g \in F_\mu$ there is a unique path α_g joining g to $1 \in F_\mu$. A finite subset $X \subset F_\mu$ will be called *star-convex* if for any $g \in X$ all vertices of the path α_g also belong to X. The minimal star-convex set containing a given set X will be called the *closure* of X and denoted $\mathrm{cl}(X)$.

Let $x \in \Lambda \otimes_k A$. Then x admits a unique representation of the form

$$x = \sum_{g \in F_\mu} g \otimes a_g, \quad a_g \in A$$

with only finite number of a_g non-zero. The support of x will be defined as

$$\mathrm{supp}(x) = \mathrm{cl}\{g; a_g \neq 0\} .$$

The following statement is the key point in the proof of 7.10(c): *if A has no primitive submodules then for any $x \in \Lambda \otimes_k A$ the support of $u(x)$ contains the support of x,* $\mathrm{supp}(x) \subset \mathrm{supp}(u(x))$.

Let us prove this statement. Consider an extremal vertex g of $\mathrm{supp}(x)$. The path α_g can arrive at g in two different ways:

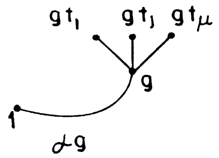

Fig. 2.

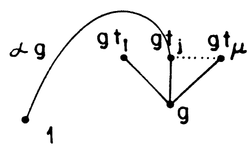

Fig. 3.

The formula

$$u(g \otimes a_g) = g \otimes \bar{z} a_g - \sum_{i=1}^{\mu} g t_i \otimes \partial_i(a_g)$$

where $\bar{z} = 1 - z$, $\bar{z} a_g = a_g + \partial_1(a_g) + \ldots + \partial_\mu(a_g)$, shows that if in the first case (Fig. 2) none of the vertices $gt_1, \ldots, gt_\mu$ belongs to $\mathrm{supp}(u(x))$ then

$$\partial_i(a_g) = 0, \quad i = 1, \ldots, \mu$$

which implies (since A has no primitive submodules) that $a_g = 0$, a contradiction.

Consider now the second case corresponding to Fig. 3. If the vertices g, $gt_1, \ldots, gt_{j-1}, gt_{j+1}, \ldots, gt_\mu$ do not belong to $\mathrm{supp}(u(x))$ then from the formula for u above we obtain

$$\partial_i(a_g) = 0, \quad i \neq j, i \in \{1, 2, \ldots, \mu\}$$

and also

$$\bar{z} a_g = 0 .$$

Write $a_g = \pi_1(a_g) + \ldots + \pi_\mu(a_g)$. If one of $\pi_i(a_g)$ with $i \neq j$ is non-zero then we get a non-trivial primitive submodule of A (of type A_i, cf. 3.2). If all $\pi_i(a_g) = 0$, $i \neq j$, then $a_g = \pi_j a_g$ and $\bar{z} a_g = 0$ imply that A has a primitive submodule (of type B_j, cf. 3.2).

Thus in both cases the vertex g will belong to $\text{supp}(u(x))$ and we obtain $\text{supp}(x) \subset \text{supp}\, u(x)$.

Remark. Note that the above arguments show additionally that $\text{supp}\, x$ is a *proper* subset of $\text{supp}\, u(x)$ if $\text{supp}\, x$ has at least one extremal vertex of the type shown in Fig. 2.

Let us now prove 7.10(c). If $e(1 \otimes a) = 0$ for $a \in A$ then $1 \otimes a = u(x)$ for some $x \in \Lambda \otimes_k A$. By the statement above $\text{supp}(x) \subset \text{supp}(1 \otimes a) = \{1\}$. Thus x is of the form $1 \otimes b$, $b \in A$. But then we may apply the remark above, obtaining a contradiction.

(d) This follows from Theorem 1.11 of [F3] since A has no primitive factor-modules.

(e) This is a consequence of Lemma 2.6 of [F3].

(f) Considering the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \Lambda \otimes A' & \xrightarrow{u} & \Lambda \otimes A' & \to & M(A') & \to & 0 \\ & & \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle \hat{f}} & & \\ 0 & \to & \Lambda \otimes A & \xrightarrow{u} & \Lambda \otimes A & \to & M(A) & \to & 0 \\ & & \downarrow{\scriptstyle 1\otimes g} & & \downarrow{\scriptstyle 1\otimes g} & & \downarrow{\scriptstyle \hat{g}} & & \\ 0 & \to & \Lambda \otimes A'' & \xrightarrow{u} & \Lambda \otimes A'' & \to & M(A'') & \to & 0 \end{array}$$

one easily finds that $\ker(\hat{g})/\text{im}(\hat{f})$ is isomorphic to a submodule of $\text{coker}[1 \otimes g : \Lambda \otimes A \to \Lambda \otimes A''] = \Lambda \otimes (A''/g(A))$. But according to Lemma 7.2, $\ker(\hat{g})/\text{im}(\hat{f})$ is a link module. On the other hand any submodule of a free Λ-module is free [C]; Since a link module cannot be free as a Λ-module, it follows that $\ker(\hat{g})/\text{im}(\hat{f}) = 0$.

This finishes the proof of Lemma 7.10.

7.11. Lemma 7.10 shows that correspondence $\{P\text{-module}\} \to \{\text{link module}\}$ is an exact functor on the sub-category of P-modules with no primitive submodules and factor-modules. In particular there is a $1-1$ correspondence between isomorphism types of such P-modules and isomorphism types of link modules.

There is a functor in the opposite direction $\{\text{link modules}\} \to \{P\text{-modules}\}$ which assigns to a link module its minimal lattice and to a homomorphism of link module its restriction on the minimal lattices, cf. 7.7. But this factor is not exact.

Example. Consider a P-module C with $\dim_k A = 5$, $\mu = 2$, with the k-basis a_1, a_2, b_1, c_1, c_2 of A. The action of z, π_1, π_2 is given by

$$\begin{aligned} za_1 &= a_1 + a_2 & \pi_i a_i &= a_i \\ za_2 &= a_1 & \pi_i b_i &= b_i \\ zb_1 &= a_2 & \pi_i c_i &= c_i, \quad i = 1, 2 \\ zc_1 &= b_1 + c_2 & & \\ zc_2 &= c_1 + c_2\,. & & \end{aligned}$$

This nodule has the composition series

$$A \subset B \subset C$$

where $A = \{a_1, a_2\}$, $B = \{a_1, a_2, b_1\}$, $C = \{a_1, a_2, b_1, c_1, c_2\}$. The composition factors A, B/A, C/B are simple and the "middle" factor B/A is primitive. On the other hand C has no primitive submodules and factor modules. We therefore have a sequence of link modules

$$0 \to M(A) \to M(C) \to M(C/B) \to 0$$

whose exactness might be established by arguments similar to that used in the proof of Lemma 7.10(f). The corresponding sequence of minimal lattices

$$0 \to A \to C \to C/B \to 0$$

is not exact.

7.12. Proposition. *Let* $0 \to M' \overset{f}{\to} M \overset{g}{\to} C \to 0$ *be an exact sequence of link modules and* Λ*-homomorphisms. Let* $A' \subset M'$, $A \subset M$, $A'' = M''$ *be the minimal lattices. Then in the sequence*

$$0 \to A' \overset{f'}{\to} A \overset{g'}{\to} A'' \to 0, \qquad f' = f|_{A'}, g' = g|_A$$

f' *is a monomorphism,* g' *is an epimorphism and the homology in the middle* $\ker(g')/\mathrm{im}(f')$ *is a primitive P-module. Thus the induced sequence* $0 \to A' \to A \to A'' \to 0$ *is exact in each of the following cases*:

(a) *the original sequence* $0 \to M' \to M \to M'' \to 0$ *splits*;

(b) A *is a primitive-free P-module.*

Proof. The assertion about f' is trivial. If g' is not onto consider $g(A) \subset A''$; it is a P-submodule which generates M'' over Λ and so it is a lattice; now $g(A) = A''$ since A'' is the minimal lattice.

Considering the diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & M'/A' & \overset{f''}{\to} & M/A & \overset{g''}{\to} & M''/A'' & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & M' & \overset{f}{\to} & M & \overset{g}{\to} & M & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & A' & \overset{f'}{\to} & A & \overset{g'}{\to} & A'' & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

one finds

$$\ker(g')/\mathrm{im}(f') \simeq \ker(f'')$$

and $\ker(f'')$ is primitive since it is a submodule of a primitive P-module M'/A', cf. [F3, 1.8].

8 Rational functions associated with a link module

8.1. Let M be a link module. Consider its minimal lattice $A \subset M$. It is a P-module with $\dim_k A < \infty$; according to 2.4 we have defined $\chi_A \in R$. Now define

$$\chi_M = \chi_A .$$

We will define also the *rank* of a link module M as

$$\mathrm{rk}(M) = \dim_k A .$$

8.2. Theorem. (a) *The rational formal power series $\chi_M \in \mathscr{R}$ is correctly determined by the link module M;*

(b) $\chi_{M_1 \oplus M_2} = \chi_{M_1} + \chi_{M_2}$;

(c) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of link modules then*

$$\chi_M = \chi_{M'} + \chi_{M''} + \ell$$

where ℓ is a "linear" rational function of the form

$$\ell = \sum_{j=1}^{\mu} (\alpha_j x_j - \beta_j \bar{x}_j) ,$$

where $\alpha_j, \beta_j \in \mathbb{Z}$, $\alpha_j \geqq 0$, $\beta_j \geqq 0$, $\bar{x}_j = -\dfrac{x}{1+x_j}$.

(d) *If $0 = M_0 \subset M_1 \ldots \subset M_n = M$ is a composition series of M with simple factors M_i/M_{i-1} then*

$$\chi_M = \sum_{i=1}^{n} \chi_{M_i/M_{i-1}} + \ell$$

where ℓ is of the form

$$\ell = \sum_{j=1}^{\mu} (\alpha_j x_j - \beta_j \bar{x}_j), \alpha_j, \beta_j \in \mathbb{Z}, \alpha_j \geqq 0, \beta_j \geqq 0 ;$$

here α_j is the number of primitive modules of type A_j (cf. 3.2) *appearing as simple composition factors of the minimal lattice $A \subset M$; β_j is the similar number of simple composition factors of type B_j.*

(e) *If the field k is algebraically closed and* $\mathrm{Char}(k) = 0$ *then any two semi-simple link modulus M_1 and M_2 are isomorphic if and only if $\chi_{M_1} = \chi_{M_2}$;*

(f) *Let $M^* = \mathrm{Hom}_\Lambda(M; \Gamma/\Lambda)$ be the dual link module, cf.* [F3, 2.2]. *Then* $\chi_{M^*} = -\bar{\chi}_M$.

(g) *If M admits a non-singular Hermitian form*

$$M \times M \to \Gamma/\Lambda$$

(cf. [F3, Sect. 3]) *then*

$$\chi_M + \bar{\chi}_M = 0$$

where $\bar{\ }: \mathscr{R} \to \mathscr{R}$ is the involution given by $x_i \mapsto \bar{x}_i = -\dfrac{x_i}{1+x_i}$;

(h) *The rank of the rational function χ_M can be estimated as follows:*

$$\mathrm{rank}\, \chi_M \leqq 2\mu(\mathrm{rk}(M)^2 + \mathrm{rk}(M))$$

and thus the whole rational function χ_M is determined by a finite set of numbers: the rank of M and coefficients of all monomials in χ_M of length $\leqq 4\mu(\mathrm{rk}(M)^2 + \mathrm{rk}(M)) - 1$.

Proof. (a) follows from existence of the minimal lattice (cf. [F3, Theorem 1.11]) and Proposition 2.5;

(b) and (c) follow from Proposition 7.12 and Proposition 2.5;
(d) is a consequence of (c);
(e) follows from Theorem 5.5
(f) follows from Proposition 6.2 and Theorem 2.9 of [F3].
(g) follows from (a) and (f)
(h) follows from Corollary 1.4 and Remark 2.7.

9 Rational functions of boundary links

9.1. An *n-dimensional μ-component link* is an oriented smooth submanifold Σ^n of S^{n+2}, where $\Sigma^n = \Sigma_1^n \cup \ldots \cup \Sigma_\mu^n$ is the ordered disjoint union of μ submanifolds of S^{n+2}, each homeomorphic to S^n. Σ is a *boundary link* if there is an oriented smooth submanifold V^{n+1} of S^{n+2}, $V^{n+1} = V_1^{n+1} \cup \ldots \cup V_\mu^{n+1}$ the disjoint union of the submanifolds V_i^{n+1}, such that $\partial V_i = \Sigma_i (i = 1, \ldots, \mu)$. If each V_i is connected, we say that V is a *Seifert manifold* for Σ.

9.2. Let S^n be a μ-component link in S^{n+2}, and let $X = S^{n+2} - T(\Sigma)$ be the complement of a tubular neighbourhood $T(\Sigma)$ of Σ in S^{n+2}. Fix a base point $* \in X$; for each $i = 1, \ldots, \mu$ the *meridan* $m_i \in \pi_1(X, *)$ (an element represented by a small loop around Σ_i joined by a path to the base point) is defined up to conjugation.

A *splitting* [CS] is a homomorphism (which is defined up to conjugation) $\sigma : \pi_1(X, *) \to F_\mu$ onto the free group with μ generators $t_1, \ldots, t_\mu$ having the following property: the image of the conjugacy class of the i-th meridian m_i coincides with the conjugacy class $[t_i]$ of $t_i \in F_\mu$.

This notion does not depend on the choice of the base point.

If Σ is a boundary link then each Seifert manifold V defines an obvious splitting σ_V: if α is a loop in X which is in general position with respect to V, then $\sigma_V([\alpha])$ is a word in $t_1, \ldots, t_\mu$, obtained by writing down $t_i^{\varepsilon_i} (\varepsilon_i = \pm 1)$ for each intersection point p of α and V (where i is the number with $p \in V_i \cap \alpha$ and ε_i is the local intersection number of α and V at p) and then multiplying these words in order of their appearance in α.

A theorem of Gutiérrez [G] states that any link admitting a splitting is a boundary link; cf. also [Sm].

9.3. An *$\mathscr{F}$-link* [CS] (of dimension n multiplicity μ) is a pair (Σ, σ), where Σ is a link (of dimension n multiplicity μ) and σ is a splitting for Σ. Two $\mathscr{F}$-links (Σ_1, σ_1) and (Σ_2, σ_2) are *equivalent* if there exists a diffeomorphism $h: S^{n+2} \to S^{n+2}$, taking, Σ_1 onto Σ_2, preserving orientations of S^{n+2} and $\Sigma_\nu, \nu = 1, 2$ and mapping σ_2 onto σ_1.

9.4. Let (Σ, σ) be an $\mathscr{F}$-link. Fix a particular epimorphism $\sigma_0 : \pi_1(X, *) \to F_\mu$ conjugate to σ. Consider the covering

$$\tilde{X} \to X$$

corresponding to the kernel of σ_0. The group F_μ acts on $\tilde{X}$ as the group of covering transformations. The homology $H_i(\tilde{X}; k)$ with coefficients in a field k is a $\Lambda = k[F_\mu]$-module. Sato [S] has shown that $H_i(\tilde{X}, k)$ is a link module (cf. 7.1) for $1 \leqq i \leqq n$.

Applying the construction of Sect. 8 to $H_i(\tilde{X}, k)$, $1 \leqq i \leqq n$, we obtain a rational function χ_i. Thus we have a sequence of rational functions

$$\chi_1, \chi_2, \ldots, \chi_n$$

associated to a boundary link (Σ^n, σ).

9.5. Proposition. (1) $\chi_i + \bar{\chi}_{n+1-i} = 0$ *for* $1 \leqq i \leqq n$; *in particular, if* $n = 2q - 1$ *then* $\chi_q + \bar{\chi}_q = 0$ (*the bar* $^-$*stands for the involution defined in* 6.1);

(2) *if the field k is of characteristic zero then all coefficients of* χ_i *are integral,* $1 \leqq i \leqq n$.

Proof of (1). From the duality Theorem 5.7 of [F2] it is known that there is a non-singular Hermitian form

$$H_i(\tilde{X}; k) \otimes H_{n+1-i}(\tilde{X}; k) \to \Gamma/\Lambda$$

and now (1) follows from Theorem 8.2(f).

The proof of (2) will be given in 9.7.

9.6. Now we will discuss the computation of χ_i in terms of the Seifert manifold; this will also give a proof of statement (2) of Proposition 9.5.

Let $V = V_1 \cup V_2 \cup \ldots \cup V_\mu$ be a Seifert manifold of (Σ, σ). Denote by Y the complement $S^{n+2} - V$ and let $i_+, i_-: V \to Y$ be maps given by small shifts in the direction of positive and negative normals to V, respectively. The map

$$i_{+*} - i_{-*}: H_i(V) \to H_i(Y)$$

is an isomorphism (cf. [F, Sect. 1.1]) and we will define a map

$$z: H_i(V) \to H_i(V)$$

by

$$(i_{+*} - i_{-*})(z(v)) = i_{+*}(v), \quad v \in H(V) .$$

This definition works for any coefficient system including $\mathbb{Z}$.

We will also define projections $\pi_1, \ldots, \pi_\mu$; $H_i(V)$ with π_j be the projection corresponding to $V_j \subset V$.

Thus, $H_i(V; k)$ is a P-module.

Seifert manifold V will be called *minimal* if $H_i(V, k)$ has no primitive P-submodules (cf. Sect. 3); this is equivalent to the definition of 6.12 in [F3].

General theorems on the existence of minimal Seifert surfaces for knots were obtained in [F1]; most of them can be generalized to boundary links.

There is a canonical lifting $i: Y \to \tilde{X}$ (cf. [F3, 6.16]); composing it with the isomorphism $i_{+*} - i_{-*}: H_i(V; k) \to H_i(Y; k)$ we get a homomorphism

$$f: H_i(V; k) \to H_i(\tilde{X}; k) .$$

The arguments of 6.16, 6.17, 6.18 of [F3] prove that

(1) f is a monomorphism if V is a minimal Seifert surface;

(2) f establishes a P-isomorphism between the minimal lattice in $H_i(\tilde{X}; t)$ and $H_i(V; k)$.

Thus, in order to compute χ_i one should compute the χ-function of the P-module $H_i(V; k)$ (cf. Sect. 2). An example of such computation is given in Sect. 10.

The results of Sect. 7 prove that the χ-function of $H_i(V; k)$ does not depend on the choice of V if V is minimal.

In general, if W is an abitrary Seifert manifold (which is not supposed to be minimal) then the image of $f^W : H_i(W; k) \to H_i(\tilde{X}; k)$ is a lattice (may be not the minimal one) and the kernel of f^W is a primitive P-module in $H_i(W; k)$. Thus from the general results of Sect. 2 we obtain

$$\chi_i^W = \chi_i^V + \sum_{j=1}^{\mu} (\alpha_j x_j - \beta_j \bar{x}_j)$$

with $\alpha_i, \beta_i \in \mathbb{Z}$, $\alpha_i, \beta_i \geqq 0$. Suppose now that $n = 2q - 1$ and $i = q$ is the middle dimension. Then we have

$$\chi_q^V + \overline{\chi_q^V} = 0$$

and

$$\chi_q^W + \overline{\chi_q^W} = 0 .$$

It follows now that in this case $\beta_j = \alpha_j$ and thus

$$\chi_q^T = \chi_q^V + \sum_{j=1}^{\mu} \alpha_j (x_j - \bar{x}_j), \quad \alpha_j \in \mathbb{Z}^+ .$$

9.7. *Proof of Proposition 9.5(2).* It is enough to show that the χ-function of $H_i(V; k)$ is integral. But this is obvious since $H_i(V; k)$ is $H_i(V; \mathbb{Z}) \otimes k$ and the maps $\pi_1, \ldots, \pi_\mu, z$ are defined on $H_i(V; \mathbb{Z})$ as well.

9.8. An $\mathscr{F}$-link (Σ^n, σ) is called *null-concordant* if there exists oriented disjoint submanifolds $W_1^{n+2}, W_2^{n+2}, \ldots, W_\mu^{n+2}$ of D^{n+3} such that

(i) $\partial W_i = V_i \cup M_i$, where $M_i \cap V_i = \Sigma_i$;

(ii) $W_i \cap \partial D^{n+3} = V_i$;

(iii) M_i is homeomorphic to an $(n + 1)$-dimensional disk.

(iv) the union $V_1 \cup V_2 \cup \ldots \cup V_\mu$ is a Seifert manifold of (Σ, σ) realizing the given splitting σ.

9.9. Proposition. *Let (Σ^{2q-1}, σ) be an $\mathscr{F}$-link. If (Σ, σ) is null-concordant then the corresponding middle-dimensional rational function χ_q has the form*

$$\chi_q = \eta - \bar{\eta}$$

where η is a rational function with integral coefficients.

Proof. Let $W = W_1 \cup W_2 \cup \ldots \cup W_\mu \subset D^{n+3}$ be a submanifold as in 9.8. Consider the isometry structure of Seifert manifold $V = V_i \cup \ldots \cup V_\mu$, $V_i = W_i \cap \partial D^{n+3}$:

$$(H_q(V; k), z, \pi_1, \ldots, \pi_\mu, \langle\, , \rangle) .$$

Here $\langle\, , \rangle : H_q(V) \times H_q(V) \to k$ is the intersection pairing. It was shown in [M] that in this case there exists a P-submodule $A \subset H_q(V; k)$ which coincides with its annihilator with respect to $\langle\, , \rangle$. Thus we obtain an exact sequence of P-modules

$$0 \to A \to H_q(V; k) \to H_q(V; k)/A \to 0$$

and an isomorphism

$$H_q(V; k)/A \simeq A^* .$$

Thus, from 2.5(c) and 6.2 we obtain

$$\chi^V = \chi_A + \chi_{A^*} = \chi_A - \overline{\chi_A} ,$$

where χ^V is the χ-function of $H_q(V; k)$. Now, let χ_q be the χ-function of $H_q(\tilde{X}; k)$. Then as in 9.6 we have

$$\chi^V = \chi_q + \sum_{j=1}^{\mu} \alpha_j(x_j - \bar{x}_j) .$$

It follows that

$$\chi_q = \eta - \bar{\eta}$$

where

$$\eta = \chi_A - \sum_{j=1}^{\mu} \alpha_j x_j .$$

This completes the proof.

10 Examples

10.1. Let $L = k[\mathbb{Z}^\mu]$ be the group ring of the free abelian group $\mathbb{Z}^\mu$ and let $\alpha : \Lambda = k[F_\mu] \to L$ be the natural homomorphism.

As the first example we will construct a link module M which is not trivial (and so has a non-trivial χ-function) with the property that $L \otimes_\Lambda M = 0$.

Let γ be the following rational function

$$\gamma = (1 + x_1 x_2 - x_2 x_1)^{-1} .$$

Let M be the link submodule of $\mathcal{R}/\Lambda$ generated by γ. More precisely, let

$$a = x_1 x_2 \gamma \qquad b_1 = - x_2 x_1 \gamma$$

$$a_2 = x_1 \gamma \qquad b_2 = x_2 \gamma$$

and let M be the left Λ-submodule of $\mathcal{R}/\Lambda$ generated by a_1, a_2, b_1, b_2. M contains a lattice A, the vector space over k generated by a_1, a_2, b_1, b_2. Here the P-structure on A is given by

$$\pi_i a_i = a_i, \qquad \pi_i b_i = b_i, \quad i = 1, 2$$

and the map $z : A \to A$ given by the matrix in the base a_1, a_2, b_1, b_2

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} .$$

The conditions of Proposition 7.10 (i.e. A has no primitive submodules and factor-modules) can be easily checked. Thus A is the minimal lattice in M.

To compute $L \otimes_\Lambda M$ note that this module appears in exact sequence (cf. 7.10)

$$L \otimes_\Lambda A \xrightarrow{w} L \otimes_k A \to L \otimes_\Lambda M \to 0$$

where

$$w(\lambda \otimes a) = \lambda \otimes a + (\tau_1 - 1)\lambda \otimes z\pi_1(a) + (\tau_2 - 1)\lambda \otimes z\pi_2(a)$$

for $\lambda \in L$, $a \in A$. Here $\tau_1, \tau_2 \in \mathbb{Z}^2$ are the generators, cf. 7.10. Thus, the presentation matrix for $L \otimes_\Lambda M$ is

$$\begin{bmatrix} 1 & 0 & 0 & -(\tau_1 - 1) \\ \tau_2 - 1 & 1 & \tau_2 - 1 & 0 \\ 0 & (\tau_1 - 1) & 1 & 0 \\ \tau_2 - 1 & 0 & \tau_2 - 1 & 1 \end{bmatrix}.$$

It has determinant 1. Thus, $L \otimes_\Lambda M = 0$.

10.2. By the realization of Sato [S] one can realize the link module M from 10.1 as $H_i(\tilde{X}; k)$ for some boundary link with $1 < i$. Consider now the free abelian covering of the complement

$$\bar{X}_{\text{ab}} \to X .$$

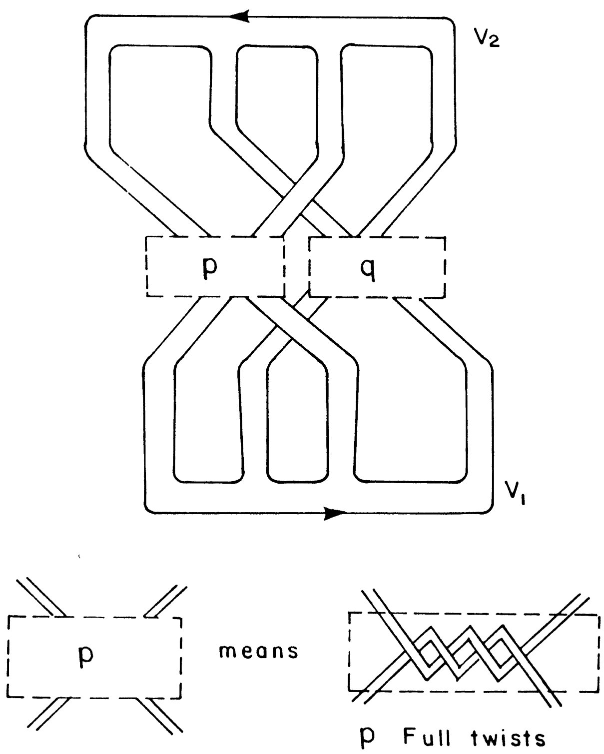

Fig. 4.

Since

$$\mathrm{Tor}_q^L(L, M) = 0$$

for $q \geqq 1$ and for any link module M, we get (assuming $i > 1$)

$$H_i(\bar{X}_{\mathrm{ab}}; k) = L \otimes_\Lambda H_i(\bar{X}; k) = L \otimes_\Lambda M = 0 .$$

Thus all abelian Alexander invariants of this link (in dimension i) are trivial, but the χ-function χ_i is not trivial.

10.3. The next example gives an instance of computing χ-function of a link. Consider the followng 2-component boundary link $L_{p,q}$ in S^3 shown on Fig. 4.

It has the Seifert surface $V = V_1 \cup V_2$ shown in the picture defining the $\mathscr{F}$-structure.

The isometry structure $A = H_1(V; \mathbb{Q}) = H_1(V_1; \mathbb{Q}) \oplus H_1(V_2; \mathbb{Q})$ of the Seifert manifold V has dimension 4. In a natural basis (given by the handles of V) A has 4 generators $\alpha_1, \beta_1 \in H_1(V_1; \mathbb{Q})$ and $\alpha_2, \beta_2 \in H_1(V_2; \mathbb{Q})$. The operator $z: A \to A$ acts as follows:

$$z\alpha_1 = p\alpha_2, \qquad z\beta_1 = \beta_1 + q\beta_2$$

$$z\alpha_2 = -q\alpha_1, \qquad z\beta_2 = \beta_2 - p\beta_1$$

with

$$\pi_i\alpha_i = \alpha_i, \qquad \pi_i\beta_i = \beta_i, \quad i = 1, 2 ,$$

$$\langle \alpha_i, \beta_j \rangle = \delta_{ij} .$$

If $pq \neq 0$ this isometry structure satisfies the conditions of 7.10 and so it is embedded in the corresponding link module $H_1(\tilde{X}, \mathbb{Q})$ and is isomorphic to the minimal lattice.

It is clear that (as P-module) A is a direct sum of two P-submodules, one generated by α_1, α_2 and the second generated by β_1, β_2. Thus, the χ-function (in dimension one) $\chi_{L_{p,q}}$ of the link $L_{p,q}$ is the sum

$$\chi_1 + \chi_2$$

where χ_1 (respectively χ_1) is the χ-function of the submodule generated by α_1, α_2 (respectively β_1, β_2). An easy computation shows that

$$\chi_1 = (1 + pqx_1x_2)^{-1}x_1 + (1 + pqx_2x_1)^{-1}x_2$$

and

$$\chi_2 = -\overline{\chi_1} \cdot$$

10.4. As the last remark we show the relation between the χ-function and the Alexander polynomial for knots ($\mu = 1$).

Let $\Delta_i(t)$ be the Alexander polynomial of $H_i(\tilde{X}; \mathbb{Q})$ where $\tilde{X}$ is infinite cyclic cover of the complement of the knot. One can write

$$\Delta_i(t) = \prod_{j=1}^{n} (t - \mu_j), \quad \mu_j \in \mathbb{C}, \quad \mu_j \neq 0 .$$

Then

$$\chi_i(x) = \sum_{j=1}^{n} \frac{x}{1 + \lambda_j x}$$

where

$$\lambda_j = \frac{1}{1 - \mu_j}, \quad j = 1, \ldots, n, \quad \lambda_j \neq 0, 1 .$$

This follows from 2.6; it is clear now that $\Delta_i(t)$ and $\chi_i(x)$ determine each other.

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