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STABLE-HOMOTOPY AND HOMOLOGY INVARIANTS OF BOUNDARY LINKS

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ABSTRACT. An n-dimensional ($n \geq 5$) link in the $(n+2)$-dimensional sphere is *stable* if the ith homotopy group of its complement X vanishes for $2 \leq i \leq (n+1)/3$ and $\pi_1(X)$ is freely generated by meridians. In this paper a classification of stable links in terms of stable homotopy theory is given. For *simple* links this classification gives a complete algebraic description.

We also study Poincaré duality in the space of the free covering of the complement of a boundary link. The explicit computation of the corresponding Ext-functors gives a construction of new homology pairings, generalizing the Blanchfield and the torsion pairings for knots.

INTRODUCTION

The present paper studies new stable-homotopy and homology invariants of boundary links of codimension two. The stable-homotopy invariant, which is called *stable isometry structure*, is a generalization of the corresponding notion for knots (i.e., links with one component) [F2, F3], which in its turn generalizes the Seifert matrix [L1] and the isometry structure [K] of a knot. It is shown here that one may use the Spanier-Whitehead duality theory and Wall's thickening technique [W] to obtain a complete stable-homotopy reduction of the classification problem for *stable links* (cf. §1.4). This reduction gives a general computational scheme which could serve as a basis of applying homotopy theory for algebraic description of different classes of stable links (as was demonstrated in [F3, F4] for stable knots).

The easiest case, where such computations can be conducted without difficulties caused by homotopy theory, is the case of simple odd-dimensional links. This class of links has been classified by Liang [L] (cf. also [Ko1]) in terms of their Seifert matrices. In §4 of this paper another algebraic description of this same class of links is obtained, which is slightly better in some respects; for instance, our equivalence relation is of a more algebraic-invariant nature. Of course, the best possible form in which one could expect to get an algebraic classification of simple odd-dimensional links is the form of the Trotter-Kearton theorem [Tr, K1], which is valid for knots. Presumably a similar theorem is true

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also for links [1] and in this paper we make a step in this direction: we construct invariantly-defined Hermitian pairings on the homology of the free covering corresponding to the link, which in the case of knots represent the Blanchfield form [B] and the torsion (or linking) form [L4, F1]. Another (different) version of the Blanchfield form for boundary links was constructed by Duval [D].

1. Preliminaries

1.1. An *n-dimensional μ-component link* is an oriented smooth submanifold Σ^n of S^{n+2}, where $\Sigma^n = \Sigma_1^n \cup \cdots \cup \Sigma_\mu^n$ is the ordered disjoint union of μ submanifolds of S^{n+2}, each homeomorphic to S^n. Σ is a *boundary link* if there is an oriented smooth submanifold V^{n+1} of S^{n+2}, where $V^{n+1} = V_1^{n+1} \cup \cdots \cup V_\mu^{n+1}$ is the disjoint union of connected submanifolds V_i^{n+1}, such that $\partial V_i = \Sigma_i$ $(i = 1, \ldots, \mu)$. If each V_i is r-connected, we say that V is an *r-simple Seifert manifold* for Σ.

1.2. Let Σ^n be a μ-component link in S^{n+2}, and let $X = S^{n+2} - \Sigma$ be its complement. Fix a base point $* \in X$; for each $i = 1, \ldots, \mu$ the meridian $m_i \in \pi_1(X, *)$ is defined up to conjugation.

A *splitting* [CS] is a homomorphism (which is defined up to conjugation) $\mathscr{S} : \pi_1(X, *) \to F_\mu$ onto the free group with μ generators $t_1, \ldots, t_\mu$ that has the following property: the image of the conjugacy class $[m_i]$ of ith meridian coincides with the conjugacy class $[t_i]$ of $t_i \in F_\mu$.

This notion does not depend on the choice of the base point.

Each Seifert manifold V of a boundary link Σ defines an obvious splitting $\mathscr{S}_V$: if α is a loop in X which is in general position with respect to V, then $\mathscr{S}_V([\alpha])$ is a word in $t_1, \ldots, t_\mu$, obtained by writing down $t_i^{\varepsilon_i}$, $\varepsilon_i = \pm 1$, for each intersection point p of α and V (where i is the number such that $p \in V_i \cap \alpha$ and ε_i is the local intersection number of α and V_i at p), and then multiplying these words in the order of their appearance in α.

A theorem of Gutiérrez [G] states that any link admitting a splitting is a boundary link (cf. also [Sm]).

1.3. An *$\mathscr{F}$-link* [CS] (of dimension n and multiplicity μ) is a pair $(\Sigma, \mathscr{S})$, where Σ is a link (of dimension n and multiplicity μ) and $\mathscr{S}$ is a splitting for Σ. Two $\mathscr{F}$-links $(\Sigma_1, \mathscr{S}_1)$ and $(\Sigma_2, \mathscr{S}_2)$ are *equivalent* if there exists a diffeomorphism $h : S^{n+2} \to S^{n+2}$, taking Σ_1 onto Σ_2, preserving orientations of S^{n+2} and Σ_ν, $\nu = 1, 2$, and mapping $\mathscr{S}_2$ onto $\mathscr{S}_1$.

1.4. An $\mathscr{F}$-link $(\Sigma, \mathscr{S})$ will be called *r-simple* (where r is an integer, $r \geq 1$) if (a) $\mathscr{S}$ is an isomorphism $\pi_1(S^{n+2} - \Sigma, *) \to F_\mu$; and (b) $\pi_i(S^{n+2} - \Sigma) = 0$ for all $1 < i \leq r$. We will consider every $\mathscr{F}$-link to be *0-simple.*

Another theorem of Gutiérrez [G] states that any r-simple n-dimensional $\mathscr{F}$-link $(\Sigma, \mathscr{S})$ admits an r-simple Seifert manifold V with $\mathscr{S} = \mathscr{S}_V$, provided $n \geq 4$.

An n-dimensional r-simple $\mathscr{F}$-link will be called *stable* if $3r \geq n + 1 \geq 6$.

1.5. Let V and W be two Seifert manifolds of a boundary link Σ^n in S^{n+2}.

[1] Added in proof (September 1990). The goal of obtaining an algebraic classification of simple odd-dimensional links in terms of the generalized Blanchfield pairing has been accomplished in [F5].

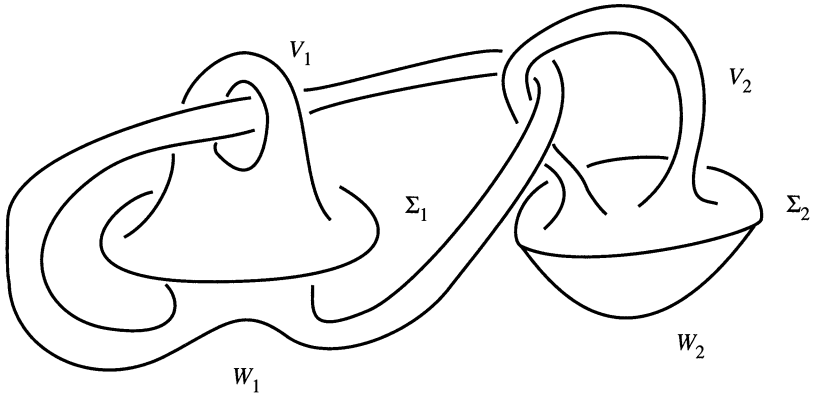

FIGURE 1

We will say that V is *contiguous* to W if

(a) $V_i \cap W_j = \varnothing$ for $i \neq j$, $i, j = 1, \dots, \mu$;
(b) $V_i \cap W_i = \Sigma_i$, $i = 1, \dots, \mu$;
(c) there exist N_i, the arcwise connected component of $S^{n+2} - (V_i \cup W_i)$ with the property that the positive normal to W_i goes inside N_i, such that $\Sigma_j \cap N_i = \varnothing$ for $i, j \in \{1, \dots, \mu\}$.

Here $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_\mu$, $V = V_1 \cup \cdots \cup V_\mu$, and $W = W_1 \cup \cdots \cup W_\mu$. Of course, we assume that V and W are oriented and induce the same orientation on Σ.

Figure 1 shows a typical example of a pair of contiguous Seifert manifolds V and W of a 2-component link.

It is clear that if V is contiguous to W then $-W$ is contiguous to $-V$, where the minus sign means that the orientation is reversed.

Contiguous Seifert manifolds V and W define the same splitting: $\mathscr{S}_V = \mathscr{S}_W$. The contrary is also partially true:

1.6. **Theorem.** *Let V and W be two Seifert manifolds of an n-dimensional link Σ. If $\mathscr{S}_V = \mathscr{S}_W$ then there exists a sequence $U_1, \dots, U_N$ of Seifert manifolds of Σ with*

(a) $U_1 = V$, $U_N = W$;
(b) *for each $i = 1, \dots, N-1$ one of the following three possibilities holds:* (1) *U_i is contiguous to U_{i+1};* (2) *U_{i+1} is contiguous to U_i;* (3) *there is an orientation-preserving diffeomorphism $f : S^{n+2} \to S^{n+2}$ with $f(U_i) = U_{i+1}$.*

Proof. Choose a base point $* \in S^{n+2} - \Sigma$. As explained in §1.2 V and W define epimorphisms

$$f_V, f_W : \pi_1(S^{n+2} - \Sigma) \to F_\mu$$

and the condition $\mathscr{S}_V = \mathscr{S}_W$ means that

$$f_V(x) = a f_W(x) a^{-1}$$

for some element $a \in F_\mu$. As the first step we will describe modifications of W by a chain of contiguities, which give another Seifert manifold W' with $f_V = f_{W'}$.

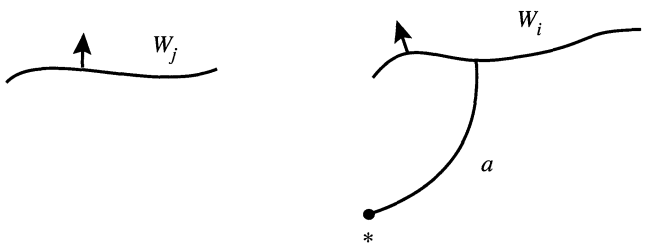

FIGURE 2

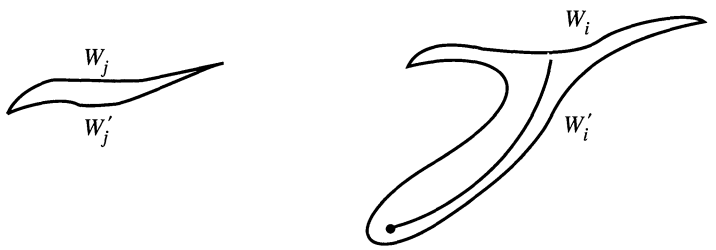

FIGURE 3

Let $W = W_1 \cup \cdots \cup W_\mu$. Join the base point $*$ with W_i by a simple arc a approaching W_i from the negative side and disjoint from W_j for $j \neq i$ (Figure 2). Then construct a new Seifert manifold $W' = W_1' \cup \cdots \cup W_\mu'$ according to Figure 3.

In order to obtain W_j' we shift W_j slightly in the direction of negative normal, and for $j = i$ we then add a boundary of the thickened arc a. It is clear that W' is contiguous to W and

$$f_{W'}(x) = t_i^{-1} f_W(x) t_i$$

for all $x \in \pi_1(S^{n+2} - \Sigma, *)$.

A similar modification with an approaching W_i from the positive side and with shifts in the positive normal direction is shown in Figure 4.

In this case we have

$$f_{W'}(x) = t_i f_W(x) t_i^{-1} ,$$

where $x \in \pi_1(S^{n+2} - \Sigma, *)$.

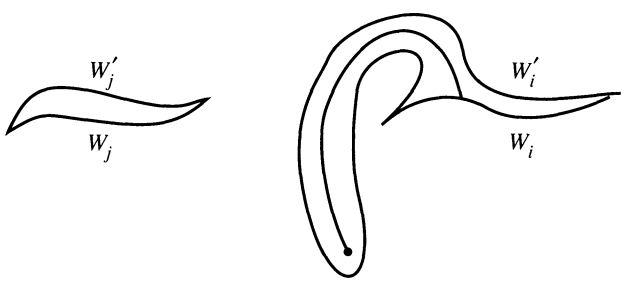

FIGURE 4

By a number of such modifications we change W (by a chain of contiguities) in such a way that a new Seifert manifold (which we also denote by W) has the property

$$f_V(x) = f_W(x)$$

for all $x \in \pi_1(S^{n+2} - \Sigma, *)$. Using the arguments of [F3, 1.9–1.12], we may additionally assume that W_i coincides with V_i in a small tubular neighbourhood of Σ_i, $i = 1, \dots, \mu$.

Now we can apply the Thom-Pontrjagin construction. Using arguments similar to those in [L3, G, CS], we obtain a submanifold $X \subset S^{n+2} \times [0, 1]$ such that

(1) X is compact, $(n+2)$-dimensional, and orientable;
(2) X is a disjoint union of μ arcwise connected components $X_1 \cup \cdots \cup X_\mu$;
(3) $\partial X_i = V_i \times 0 \cup W_i \times 1 \cup \Sigma_i \times [0, 1]$, $i = 1, \dots, \mu$;
(4) V_i and W_i define the same orientation on X_i.

Using general position arguments we may arrange X to have the following additional property:

(5) Let $h : S^{n+2} \times [0, 1] \to [0, 1]$ be the projection; then $h|_X$ is a nondegenerate Morse function, whose critical values are distinct.

We can also modify X in order to cancel critical points of $h|_X$ of index 0 and $n+2$. Let $0 = c_0 < c_1 < \cdots < c_N = 1$ be a small subdivision of the interval $[0, 1]$ such that $[c_\nu, c_{\nu+1}]$ contains at most one critical value of $h|_X$. Denote

$$U_\nu = h^{-1}(c_\nu) \cap X ,$$

$\nu = 0, 1, \dots, N$. Then U_ν is a Seifert manifold of Σ with $U_0 = V$, $U_N = W$. If there is no critical point in $[c_i, c_{i+1}]$ then U_{i+1} may be obtained from U_i by an orientation-preserving diffeomorphism $f_i : S^{n+2} \to S^{n+2}$ (i.e., $U_{i+1} = f_i(U_i)$). If $[c_i, c_{i+1}]$ contains a critical point then U_{i+1} is obtained from U_i by a surgery modification (see Figure 5).

It is clear that any such modification can be decomposed into a contiguity and then diffeomorphism. The only possible trouble can occur when the one-dimensional handles are glued; it is easy to see that a one-dimensional handle

(1) cannot join two different components of U_i and
(2) should approach U_i from one side;

otherwise we get a contradiction with conditions (1), (2).

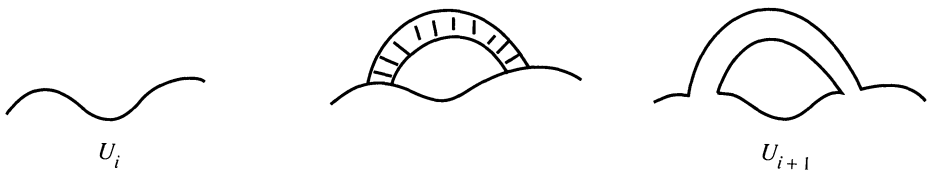

FIGURE 5

A similar proof in the PL case can be obtained by applying the critical level imbedding technique of [KL].

2. Seifert forms and isometry structures

2.1. Let V be a Seifert manifold of a boundary link Σ^n of multiplicity μ. For each $i = 1, \dots, \mu$ fix a base point $x_i \in \Sigma_i = \partial V_i$, where $V = V_1 \cup \dots \cup V_\mu$. Denote by

$$\theta_{ii} : V_i \wedge V_i \to S^{n+1}$$

the homotopy Seifert pairing of V_i (cf. [F2, p. 186]). For $i \neq j$ let θ_{ij} be the canonical map

$$\theta_{ij} : V_i \wedge V_j \to S^{n+1} ,$$

which is well defined since V_i and V_j are disjoint subsets of S^{n+2} (cf., for example, [St, Chapter 3]).

We will use the stable homotopy category Stab_0, described in [F4, p. 509]. Denote by $\widehat{V}$ the sum (in Stab_0)

$$\widehat{V} = V_1 \oplus V_2 \oplus \dots \oplus V_\mu .$$

The pairings θ_{ij} define a pairing

$$\theta : \widehat{V} \otimes \widehat{V} \to S^{n+1}, \qquad \theta = (\theta_{ij}) .$$

We also have a collection of S-maps

$$\pi_i : \widehat{V} \to \widehat{V}, \qquad i = 1, \dots, \mu ,$$

each π_i being the composite $\widehat{V} \to V_i \to \widehat{V}$ of the projection and the inclusion.

The collection $(\widehat{V}, \theta, \pi_1, \dots, \pi_\mu)$ is the *Seifert form* of V.

2.2. **Lemma.** *Any Seifert form $(\widehat{V}, \theta, \pi_1, \dots, \pi_\mu)$ has the following properties*:

(1) $\theta + (-1)^{n+1}\theta' : \widehat{V} \otimes \widehat{V} \to S^{n+1}$ *is a Spanier-Whitehead duality*;
(2) $\theta \circ (\pi_i \otimes \pi_j) = (-1)^n \theta' \circ (\pi_i \otimes \pi_j)$ *for* $i \neq j$;
(3) $\pi_i \circ \pi_j = \delta_{ij}\pi_j$, $\pi_1 + \pi_2 + \dots + \pi_\mu = 1_{\widehat{V}}$.

Here θ' denotes $\theta \circ T$, where $T : \widehat{V} \otimes \widehat{V} \to \widehat{V} \otimes \widehat{V}$ is the interchanging map, and δ_{ij} is Kronecker's delta.

Proof. Statements (2) and (3) are evident. Statement (1) may be deduced from [F1, Theorem 1.4] in the following way. Consider a small $(n+1)$-dimensional disk D^{n+1} imbedded in $S^{n+2} - V$. Connect it by a smooth $(n+1)$-dimensional band B_i with V_i, $i = 1, \dots, \mu$. We assume that $B_i = D_i^n \times I$, $B_i \cap V_i = B_i \cap \partial V_i = D_i^n \times \{0\}$, $B_i \cap D^{n+1} = B_i \cap \partial D^{n+1} = D_i^n \times \{1\}$, and $B_i \cap V_j = \varnothing = B_i \cap B_j$ for $i \neq j$. The union $W = V_1 \cup \dots \cup V_\mu \cup B_1 \cup \dots \cup B_\mu \cup D^{n+1}$ is a Seifert manifold of a spherical knot in S^{n+2} and it is clear that there is a homotopy equivalence $\varphi : W \to \widehat{V}$ such that $\theta \circ (\varphi \otimes \varphi)$ is the homotopy Seifert pairing of the knot. Statement (1) now follows from [F2, Theorem 1.4].

2.3. An *abstract Seifert form* of dimension n and multiplicity μ is a collection $(X, \theta, \pi_1, \dots, \pi_\mu)$, where X is a stable (or virtual) complex (i.e., an object

of Stab_0), $\theta : X \otimes X \to S^{n+1}$ is an S-map with $\theta + (-1)^{n+1}\theta'$ a duality, and $\pi_1, \pi_2, \dots, \pi_\mu$ are S-maps $X \to X$ satisfying

$$\pi_i \circ \pi_j = \delta_{ij}\pi_j, \qquad \pi_1 + \cdots + \pi_\mu = 1_X,$$

and

$$\theta \circ (\pi_i \otimes \pi_j) = (-1)^n \theta' \circ (\pi_i \otimes \pi_j)$$

for $i \neq j$. Two abstract Seifert forms are *isomorphic* if there exists a stable homotopy equivalence between the corresponding spaces, commuting with the π_i's and transforming one pairing θ into the other.

2.4. **Theorem.** *Suppose r and n are integers with $3r \geq n+1 \geq 6$. Then:*

(1) *Any abstract Seifert form $(X, \theta, \pi_1, \dots, \pi_\mu)$ of dimension n with* $\mathrm{length}(X) \leq n - 2r$ *(cf.* [F4, *p.* 510]) *is isomorphic to the Seifert form of an r-simple Seifert manifold V of a boundary μ-component link Σ^n.*

(2) *Assume that Σ^n and Σ'^n are two r-simple μ-component links and, V and V' are the corresponding r-simple Seifert manifolds. If the Seifert forms of V and V' are isomorphic then there exists an isotopy of S^{n+2} taking each V_i onto V_i', $i = 1, \dots, \mu$, and preserving the orientations.*

The proof is given in §2.9 below.

2.5. Another stable homotopy object (which is in fact equivalent to the abstract Seifert form but will be more convenient later) is the *stable isometry structure.* A *stable isometry structure* of dimension n and multiplicity μ is a collection

$$(X, u, z, \pi_1, \dots, \pi_\mu),$$

where X is a virtual complex and $u : X \otimes X \to S^{n+1}$, $z : X \to X$, and $\pi_i : X \to X$, $i = 1, \dots, \mu$ are S-maps, satisfying:

(1) u is a Spanier-Whitehead duality;
(2) $u' = (-1)^{n+1}u$;
(3) $u \circ (z \otimes 1_X) = u \circ (1_X \otimes \bar{z})$, where $\bar{z} = 1_X - z$;
(4) $u \circ (\pi_i \otimes \pi_j) = 0$ for $i \neq j$;
(5) $\pi_i \circ \pi_j = \delta_{ij}\pi$;
(6) $\pi_1 + \cdots + \pi_\mu = 1_X$.

The notion of isomorphism of stable isometry structures can be introduced in an obvious way.

Any stable isometry structure $(X, u, z, \pi_1, \dots, \pi_\mu)$ defines an abstract Seifert form $(X, \theta, \pi_1, \dots, \pi_\mu)$, where $\theta = u \circ (1 \otimes z)$. Conversely, given an abstract Seifert form $(X, \theta, \pi_1, \dots, \pi_\mu)$ one can define $u : X \otimes X \to S^{n+1}$ by $u = \theta + (-1)^{n+1}\theta'$; this is a Spanier-Whitehead duality and so there exists a unique S-map $z : X \to X$ such that $\theta = u \circ (1 \otimes z)$. It is easy to see that the collection $(X, u, z, \pi_1, \dots, \pi_\mu)$ is a stable isometry structure.

The following statement is just another version of Theorem 2.4.

2.6. **Theorem.** *Suppose r and n are integers with $3r \geq n+1 \geq 6$. There is a one-to-one correspondence between the set of isotopy types of embeddings into S^{n+2} of r-simple μ-component Seifert manifolds and the set of isomorphism classes of stable isometry structures $(X, u, z, \pi_1, \dots, \pi_\mu)$ of dimension n with* $\mathrm{length}(X) \leq n - 2r$.

2.7. A *configuration* (of dimension n and multiplicity μ) is a collection $(X_i; v_{ij})$, $i, j = 1, \dots, \mu$, where X_i is a virtual complex (i.e., an object of Stab_0), $v_{ij} : X_i \otimes X_j \to S^{n+1}$, defined for $i \neq j$, $i, j \in \{1, \dots, \mu\}$, such that

$$v_{ij} = (-1)^n v'_{ji},$$

where v'_{ji} denotes $v_{ji} \circ T$, $T : X_i \otimes X_j \to X_j \otimes X_i$ being the "interchanging" map. Two configurations (X_i, v_{ij}) and (Y_i, w_{ij}) of the same dimension and multiplicity are *isomorphic* if there exists a family of S-equivalences $f_i : X_i \to Y_i$, $i = 1, \dots, \mu$, with the property

$$w_{ij} \circ (f_i \otimes f_j) = v_{ij}$$

for all $i, j = 1, \dots, \mu$, $i \neq j$.

Let (X_i, v_{ij}) be a configuration of dimension n and multiplicity μ. Its *geometric realization* is a sequence of compact $(n+2)$-dimensional smooth submanifolds $N_1, \dots, N_\mu$ of S^{n+2} supplied with a family of S-equivalences

$$f_i : X_i \to N_i, \qquad i = 1, \dots, \mu,$$

such that

$$v_{ij} = w_{ij} \circ (f_i \otimes f_j)$$

for all $i \neq j$, $i, j = 1, \dots, \mu$, where $w_{ij} : N_i \otimes N_j \to S^{n+1}$ is the canonical pairing (arising from the embedding of N_i and N_j as disjoint subsets of S^{n+2}). Two geometric realizations (N_i, f_i) and (N'_i, f'_i) will be called *homeomorphic* if there exists a diffeomorphism $h : S^{n+2} \to S^{n+2}$ (which is isotopic to the identity) such that

$$h(N_i) = N'_i, \qquad i = 1, \dots, \mu,$$

and the diagram

$$\begin{array}{ccc} & X_i & \\ {\scriptstyle f_i}\swarrow & & \searrow{\scriptstyle f'_i} \\ N_i & \xrightarrow[h|_{N_i}]{} & N'_i \end{array}$$

commutes for all $i = 1, \dots, \mu$ in stable homotopy.

2.8. **Proposition.** *Let r and n be integers with $3r \geq n+1 \geq 6$. Any configuration (X_i, v_{ij}) of dimension n, with each X_i being r-connected and $(n-r)$-dimensional, admits a geometric realization which is unique up to homeomorphism.*

Proof. For $\mu = 1$ this statement follows immediately from Wall's theory [W]. For $\mu > 1$ one may use the following arguments. Suppose we have a configuration (X_i, v_{ij}) and we have constructed $N_1, \dots, N_{k-1}$ and $f_i : X_i \to N_i$, $i = 1, \dots, k-1$, with the desired properties. Pick an $(n+2)$-dimensional ball $B \subset S^{n+2}$, $B \cap N_i = \varnothing$, $i = 1, \dots, k-1$, and join each N_i with B by nonintersecting 1-handles. Let N denote the union of $N_1, \dots, N_{k-1}$ with these 1-handles and with B. If Y denotes $S^{n+2} - \operatorname{int} N$, then according to the Spanier-Whitehead duality theory, stable homotopy classes $X_k \to Y$ correspond bijectively to systems of pairings

$$\{v_{ki} : X_k \otimes X_i \to S^{k+1};\ i = 1, \dots, k-1\}.$$

Thus if we are given $\{v_{ki}\}$ for $i < k$ we can find a unique stable homotopy class $f : X_k \to Y$ corresponding to it. Using our dimension and connectivity

assumptions we may realize f by a map (which is unique due to the suspension theorem). Now at the last stage we can apply the thickening theory of Wall [W], similarly to the case of knots [F2, F4].

Uniqueness of realization follows similarly.

2.9. *Proof of Theorem* 2.4(1). If we are given an abstract Seifert form $(X, \theta, \pi_1, \dots, \pi_\mu)$ then we can apply the Splitting of Homotopy Idempotents Theorem of [Fr] to conclude that X can be identified with the direct sum $X_1 \oplus X_2 \oplus \cdots \oplus X_\mu$ in such a way that $\pi_i = q_i \circ p_i$, where $p_i : X \to X_i$ and $q_i : X_i \to X$ are the projection and inclusion maps, respectively.

Consider the configuration $\{X_i, v_{ij}\}$, where

$$v_{ij} = \theta(q_i \otimes q_j), \qquad i, j \in \{1, \dots, \mu\}, i \neq j .$$

By Proposition 2.8 there exists a geometric realization $N_1, \dots, N_\mu$ of $\{X_i, v_{ij}\}$. For each $i = 1, \dots, \mu$ we have an $(n+2)$-dimensional submanifold $N_i \subset S^{n+2}$ and a stable homotopy equivalence $f_i : N_i \to X_i$. Our initial data also give us a pairing

$$\theta_i = \theta \circ (q_i \otimes q_i) : X_i \otimes X_i \to S^{n+1} .$$

Using the arguments of the proof of Theorem 1.3 of [F2] (cf. 1.10 of [F2]), one gets an r-connected $(n+1)$-dimensional manifold $V_i \subset \partial N_i$ with ∂V_i a homotopy sphere and with the property that the inclusion $b_i : V_i \to N_i$ is an equivalence and $\theta_i \circ (f_i \circ b_i \otimes f_i \circ b_i)$ is the homotopy Seifert pairing of V_i. It is now clear that the union

$$V = V_1 \cup \cdots \cup V_\mu$$

is the desired Seifert manifold of a boundary link $\Sigma = \partial V$.

Statement (2) can be proved similarly (using the arguments of the proof of Theorem 1.2 in [F2]).

3. Almost closed Seifert surfaces

The aim of this and the next sections is to describe in terms of the stable homotopy theory the precise relation between stable homotopy structures of different Seifert manifolds of a link. On the level of homology this has been done by J. Levine [L3] in terms of S-equivalence of Seifert matrices. A more invariant approach was suggested in [F2, F3] under the name of R-equivalence. The present exposition will be a generalization of [F2, F3, F4] to the case of links.

Almost closed Seifert manifolds, which will be studied in the present section, are used in the next section to measure the difference between different Seifert manifolds of a link.

3.1. **Proposition.** *There is a canonical construction which assigns to any configuration* (X_i, v_{ij}) *a stable isometry structure* $S(X_i, v_{ij})$ *(of the same dimension and multiplicity) with the following properties*:

(1) *Let* (N_i, f_i) *be a geometric realization of* (X_i, v_{ij}), *let* V_i *be obtained from* ∂N_i *by removing a small* $(n+1)$*-dimensional disk in* ∂N_i, *and orient* V_i *by directing the positive normal outside* N_i. *Then* $V = V_1 \cup \cdots \cup V_\mu$ *is a Seifert manifold (of the trivial link) and its stable isometry structure is isomorphic to* $S(X_i, v_{ij})$.

(2) *Stable isometry structures* $S(X_i, v_{ij})$ *and* $S(Y_i, w_{ij})$ *are isomorphic if and only if the configurations* (X_i, v_{ij}) *and* (Y_i, w_{ij}) *are isomorphic.*

(3) *A stable isometry structure* $(X, u, z, \pi_1, \dots, \pi_\mu)$ *can be obtained as* $S(X_i, v_{ij})$ *for some configuration* (X_i, v_{ij}) *if and only if* $\bar{z}\pi_k z = 0$ *for all* $k = 1, \dots, \mu$.

Any Seifert manifold $V = V_1 \cup \cdots \cup V_\mu$ which can be obtained by the construction described in Proposition 3.1(1) will be called *almost closed*.

Proof. The construction of $S(X_i, v_{ij})$ proceeds as follows. For each $i = 1, \dots, \mu$ let $u_i : X_i \otimes Z_i \to S^{n+1}$ be a Spanier-Whitehead duality. For $i \neq j$ there is a unique S-map $\varphi_{ji} : X_j \to Z_i$ with the property that

$$u_i \circ (1_{X_i} \otimes \varphi_{ji}) = v_{ij} .$$

Denote by X the direct sum

$$X = X_1 \oplus \cdots \oplus X_\mu \oplus Z_1 \oplus \cdots \oplus Z_\mu$$

and define $u : X \otimes X \to S^{n+1}$ by

$$u|_{X_i \otimes X_j} = 0, \qquad u|_{X_i \otimes Z_j} = \delta_{ij} u_i ,$$

$$u|_{Z_i \otimes X_j} = (-1)^{n+1} \delta_{ij} u_i' , \qquad u|_{Z_i \otimes Z_j} = 0 ,$$

where δ_{ij} is the Kronecker symbol. The next object $z : X \to X$ will be defined by

$$z|_{X_i} = 0, \qquad z|_{Z_i} = \mathrm{id}_{Z_i} : Z_i \to Z_i ,$$

where id_{Z_i} is the identity map, $i = 1, \dots, \mu$. We define the S-map $\pi_k : X \to X$ by the formulae

$$\pi_k|_{X_i} = \begin{cases} \mathrm{id}_{X_k} + \sum_{j \neq k} \varphi_{kj} & \text{for } i = k , \\ -\varphi_{ik} & \text{for } i \neq k , \end{cases}$$

$$\pi_k | Z_i = \begin{cases} 0 & \text{for } i \neq k , \\ \mathrm{id}_{Z_k} & \text{for } i = k . \end{cases}$$

A trivial check (which is omitted) shows that the collection

$$(X, u, z, \pi_1, \dots, \pi_\mu)$$

is a stable isometry structure, and satisfies $\bar{z}\pi_k z = 0$ for all $k = 1, \dots, \mu$.

To prove the last statement of the proposition, assume that the collection $(X, u, z, \pi_1, \dots, \pi_\mu)$ is an arbitrary stable isometry structure with $\bar{z}\pi_k z = 0$ for all $k = 1, \dots, \mu$. Consider S-maps

$$P_i : X \to X , \qquad i = 1, \dots, 2\mu ,$$

defined by

$$P_i = \begin{cases} \bar{z}\pi_i & \text{for } i = 1, \dots, \mu , \\ \pi_{i-\mu} z & \text{for } i = \mu + 1, \dots, 2\mu . \end{cases}$$

It is clear that

$$P_i P_j = \delta_{ij} P_j , \qquad \sum_{i=1}^{2\mu} P_i = 1_X .$$

By the Freyd theorem [Fr] there is a direct sum decomposition

$$X \underset{q_i}{\overset{p_i}{\rightleftarrows}} X_i$$

with

$$q_i \circ p_i = P_i \ , \quad p_i \circ q_i = \mathrm{id}_{X_i} , \qquad i = 1, 2, \dots , 2\mu .$$

Define $v_{ij} : X_i \otimes X_j \to S^{n+1}$ by

$$v_{ij} = u \circ (\pi_i q_i \otimes z\pi_j q_j)$$

for $i \neq j$, $i, j = 1, \dots , \mu$.

We thus obtain a configuration (X_i, v_{ij}) and we want to prove that the original stable isometry structure $(X, u, z, \pi_1, \dots , \pi_\mu)$ is isomorphic to $S(X_i, v_{ij})$. The identity

$$\begin{aligned} u(q_i \otimes q_j) &= u(q_i \otimes q_j \circ p_j \circ q_j) = u(q_i \otimes P_j \circ q_j) \\ &= u(P_{\bar{j}} \circ q_i \otimes q_j) = \delta_{i\bar{j}} u(q_i \otimes q_j) \ , \end{aligned}$$

where

$$\bar{j} = \begin{cases} j + \mu & \text{for } j \leq \mu, \\ j - \mu & \text{for } j > \mu , \end{cases}$$

shows that $u(q_i \otimes q_j) = 0$ unless $|i - j| = \mu$. It follows that

$$u_i = u \circ (q_i \otimes q_{i+\mu}) : X_i \otimes X_{i+\mu} \to S^{n+1} , \qquad i = 1, \dots , \mu ,$$

is a duality map, and thus we can use it in the construction of $S(X_i, v_{ij})$ instead of u_i: $X_i \otimes Z_i \to S^{n+1}$. It is easy to show that the maps

$$\varphi_{ji} : X_j \to X_{i+\mu} , \qquad i \neq j , \ i, j = 1, \dots , \mu ,$$

which appear in the construction of $S(X_i, v_{ij})$, are given by the formula

$$\varphi_{ji} = P_{i+\mu} \circ \pi_j \circ q_j \ ,$$

and the map $Q : X_1 \oplus \cdots \oplus X_{2\mu} \to X$, $Q = \sum_{i=1}^{2\mu} q_i$, gives an isomorphism between $S(X_i, v_{ij})$ and $(X, u, z, \pi_1, \dots , \pi_\mu)$.

This completes the proof of statement (3).

Statement (2) follows from the arguments used in the proof of (3): we have shown that all objects in $(X, u, z, \pi_1, \dots , \pi_\mu)$ could be uniquely determined in terms of X_i, v_{ij} and vice versa.

Now we have to prove statement (1).

Let $N_1, \dots , N_\mu \subset S^{n+2}$ be a set of compact smooth $(n+2)$-dimensional pairwise disjoint submanifolds and let

$$v_{ij} : N_i \otimes N_j \to S^{n+1} \ , \qquad i \neq j ,$$

be the corresponding pairing, $i, j = 1, \dots , \mu$. We construct V_i by removing from ∂N_i a small $(n+1)$-dimensional disk and we have to show that the stable isometry structure

$$(\widehat{V}, u, z, \pi_1, \dots , \pi_\mu)$$

corresponding to $V = V_1 \cup V_2 \cup \cdots \cup V_\mu$ is isomorphic to $S(N_i, v_{ij})$.

Let B^{n+2} be a small $(n+2)$-dimensional ball in S^{n+2} disjoint from N_i, $i = 1, \dots , \mu$. Join each N_i to B by a 1-handle H_i; in other words, $H_i = D^{n+1} \times$

$[0, 1]$, $H_i \cap N_i = H_i \cap \partial N_i = D^{n+1} \times 0$, $H_i \cap B = H_i \cup \partial B = D^{n+1} \times 1$, $H_i \cap H_j = \varnothing$ for $i \neq j$. Denote by N the union $N_1 \cup \cdots \cup N_\mu \cup H_1 \cup \cdots \cup H_\mu \cup B$. Denote by Y the complement $S^{n+2} - \operatorname{int} N$. The space $\widehat{V}$ can be naturally identified with the result of removing from ∂N a small $(n+1)$-dimensional disk.

By Proposition 2.1 of [F2] there exists a direct sum decomposition

$$Y \underset{\alpha}{\overset{j}{\leftrightarrows}} \widehat{V} \underset{\beta}{\overset{i}{\leftrightarrows}} N$$

(where $i : \widehat{V} \to N$ and $j : \widehat{V} \to Y$ are the inclusions) with the property

$$z = \alpha \circ j \ , \qquad \bar{z} = \beta \circ i$$

(cf. also Corollary 2.2 in [F2] and Lemma 2.9 in [F4]). Identifying N with the wedge of $N_1 \vee \cdots \vee N_\mu$, we have another direct sum decomposition

$$N \underset{q_k}{\overset{p_k}{\rightleftarrows}} N_k , \qquad k = 1, \ldots, \mu .$$

It is clear from the geometry that

$$i \circ \pi_k = q_k \circ p_k \circ i \ , \qquad k = 1, \ldots, \mu ,$$

and thus

$$\beta \circ q_k \circ p_k \circ i = \beta i \pi_k = \bar{z} \pi_k .$$

This shows that the projector $P_k : \widehat{V} \to \widehat{V}$ corresponding to the summand

$$\widehat{V} \underset{\beta q_k}{\overset{p_k i}{\rightleftarrows}} N_k$$

is given by the formula $P_k = \bar{z} \pi_k$ which agrees with the construction of $S(N_i, v_{ij})$. We also have

$$\bar{z} \pi_k z = P_k z = \beta q_k p_k i \alpha j = 0$$

since $i\alpha = 0$. To complete the proof of statement (1) we only have to show that

$$v_{kl} = u \circ (\pi_k \beta q_k \otimes z \pi_l \beta q_l)$$

for $k \neq l$, $k, l = 1, \ldots, \mu$. According to the definition $u \circ (1 \otimes z) = \theta$, where $\theta : \widehat{V} \otimes \widehat{V} \to S^{n+1}$ is the Seifert form (cf. §2.5) and

$$\theta \circ (\pi_k \otimes \pi_l) = v_{kl} \circ (p_k \circ i \otimes p_l \circ i)$$

(obvious). Thus

$$\begin{aligned} u \circ (\pi_k \beta q_k \otimes z \pi_l \beta q_l) &= \theta \circ (\pi_k \otimes \pi_l) \circ (\beta q_k \otimes \beta q_l) \\ &= v_{kl} \circ (p_k i \otimes p_l i) \circ (\beta q_k \otimes \beta q_l) \\ &= v_{kl} \circ (p_k i \beta q_k \otimes p_l i \beta q_l) = v_{kl} \end{aligned}$$

since

$$p_k i \beta q_k = \mathrm{id}_{N_k} , \qquad p_l i \beta q_l = \mathrm{id}_{N_l} .$$

This completes the proof.

3.2. **Corollary.** (1) *If* $(\widehat{V}, u, z, \pi_1, \dots, \pi_\mu)$ *is the stable isometry structure of an almost closed Seifert manifold, then* $\bar{z}\pi_k z = 0$ *for all* $k = 1, 2, \dots, \mu$.

(2) *Let* V *be an* r*-simple Siefert manifold of an* n*-dimensional link, where* $3r \geq n+1 \geq 6$. *If* $\bar{z}\pi_k z = 0$ *for all* $k \in \{1, \dots, \mu\}$, *then* V *is almost closed.*

Proof. (1) follows from statements (1) and (3) of Proposition 3.1. (2) follows from Propositions 2.8 and 3.1 and Theorem 2.6.

4. The classification of $\mathscr{F}$-links

In this section we will describe the relation between stable isometry structures of contiguous Seifert manifolds. Under stability assumptions it will be a complete description. Thus, combining this result with Theorem 1.6 we obtain a stable-homotopy classification of stable $\mathscr{F}$-links. In the case of $(q-1)$-simple $(2q-1)$-dimensional links it gives an algebraic classification.

4.1. Let $\mathfrak{q}_1 = (X_1, u_1, z_1, \pi_1, \dots, \pi_\mu)$ and $\mathfrak{q}_2 = (X_2, u_2, z_2, \widetilde{\pi}_1, \dots, \widetilde{\pi}_\mu)$ be two stable isometry structures of dimension n. We will say that $\mathfrak{q}_1$ is contiguous to $\mathfrak{q}_2$ if there exist S-maps

$$\varphi : X_1 \to X_2, \qquad \psi : X_2 \to X_1$$

with

(a) $u_2 \circ (1_{X_2} \otimes \varphi) = u_1 \circ (\psi \otimes 1_{X_1})$;
(b) $\varphi \circ \pi_i \circ z_1 = z_2 \circ \widetilde{\pi}_i \circ \varphi$;
(c) $\psi \circ \widetilde{\pi}_i \circ \bar{z}_2 = \bar{z}_1 \circ \pi_i \circ \psi$;
(d) $\varphi \circ \pi_i \circ \psi = z_2 \circ \widetilde{\pi}_i \circ \bar{z}_2$;
(e) $\psi \circ \widetilde{\pi}_i \circ \varphi = \bar{z}_1 \circ \pi_i \circ z_1$, for all $i = 1, 2, \dots, \mu$.

This definition generalizes the corresponding notion for knots (cf. [F2, F4]).

Each stable isometry structure is contiguous to itself (one may take $\varphi = z$, $\psi = \bar{z}$).

If $\mathfrak{q} = (X, u, z, \pi_1, \dots, \pi_\mu)$ is an isometry structure, we may define $-\mathfrak{q}$ as $(X, -u, \bar{z}, \pi_1, \dots, \pi_\mu)$. If $\mathfrak{q}_1$ is contiguous to $\mathfrak{q}_2$, then $-\mathfrak{q}_2$ is contiguous to $-\mathfrak{q}_1$.

The following theorem is the main result of this section.

4.2. **Theorem.** (1) *Let* V *and* W *be two Seifert manifolds of a boundary link* Σ^n *in* S^{n+2} *with* V *contiguous to* W. *Then the stable isometry structure* $\mathfrak{q}_V = (\widehat{V}, u_V, z_V, \pi_1^V, \dots, \pi_\mu^V)$, *corresponding to* V, *is contiguous to the stable isometry structure* $\mathfrak{q}_W = (\widehat{W}, u_W, z_W, \pi_1^W, \dots, \pi_\mu^W)$ *corresponding to* W.

(2) *Let* r, n *be integers with* $3r \geq n+1 \geq 6$. *Let* V *and* W *be* r*-simple Seifert manifolds of boundary links* $\Sigma_V = \partial V$ *and* $\Sigma_W = \partial W$, *and let* $\mathfrak{q}_V$ *and* $\mathfrak{q}_W$ *be the corresponding stable isometry structures. If* $\mathfrak{q}_V$ *is contiguous to* $\mathfrak{q}_W$, *then there exists an isotopy of* S^{n+2} *taking* W *onto another Siefert manifold* W' *such that* V *is contiguous to* W'; *in particular, the links* Σ_V *and* Σ_W *are equivalent.*

Proof. (1) Assume that V is contiguous to W. Denote by N_i the closure of the arcwise connected component of $S^{n+2} - (V_i \cup W_i)$ with the positive normal to W_i pointing inside N_i. We have compact connected disjoint $(n+2)$-dimensional submanifolds $N_1, \dots, N_\mu$ in S^{n+2} and we can form an almost closed Seifert

manifold $U = U_1 \cup \cdots \cup U_\mu$, where U_i is obtained from ∂N_i by removing a small $(n+1)$-dimensional disk (cf. the previous section). Orient U according to V.

Let $\mathfrak{q}_U = (\widehat{U}, u_U, z_U, \pi_1^U, \dots, \pi_\mu^U)$ be the stable isometry structure of U. It is clear that there is natural identification of $\widehat{U}$ with $\widehat{V} \oplus \widehat{W}$, and the Seifert form θ_U of U has the matrix presentation

$$\begin{bmatrix} \theta_V & \alpha \\ (-1)^n \alpha' & (-1)^n \theta'_W \end{bmatrix},$$

where $\alpha: \widehat{V} \otimes \widehat{W} \to S^{n+1}$ is the canonical pairing and the prime denotes transposition (cf. [F2, F4]). Therefore, we have

$$u_U = \theta_U + (-1)^{n+1}\theta'_U = \begin{bmatrix} u_V & 0 \\ 0 & -u_W \end{bmatrix},$$

$$z_U = \begin{bmatrix} z_V & \varphi \\ \psi & \bar{z}_W \end{bmatrix}, \qquad \pi_i^U = \begin{bmatrix} \pi_i^V & 0 \\ 0 & \pi_i^W \end{bmatrix},$$

where $\varphi : \widehat{V} \to \widehat{W}$ and $\psi : \widehat{W} \to \widehat{V}$ are S-maps with

$$\alpha = u_V \circ (1_V \otimes \psi) = u_W \circ (\varphi \otimes 1_W).$$

From Corollary 2.6 we know that

$$\bar{z}_U \pi_i^U z_U = 0, \qquad i = 1, \dots, \mu.$$

Thus we have

$$\begin{bmatrix} \bar{z}_V & -\varphi \\ \psi & z_W \end{bmatrix} \circ \begin{bmatrix} \pi_i^V & 0 \\ 0 & \pi_i^W \end{bmatrix} \circ \begin{bmatrix} z_V & \varphi \\ \psi & \bar{z}_W \end{bmatrix} = 0$$

and this matrix equation gives the following four equalities:

$$\varphi \pi_i^V z_V = z_W \pi_i^W \varphi, \qquad \psi \pi_i^W \bar{z}_W = \bar{z}_V \pi_i^V \varphi,$$
$$\psi \pi_i^2 \varphi = \bar{z}_V \pi_i^V z_V, \qquad \varphi \pi_i^V \psi = z_W \pi_i^W \bar{z}_W,$$

which proves (1).

Let us now prove the second statement of the theorem. Assume that $\mathfrak{q}_V$ is contiguous to $\mathfrak{q}_W$ and let $\varphi : \widehat{V} \to \widehat{W}$, $\psi : \widehat{W} \to \widehat{V}$ be the maps satisfying conditions (a)–(e) of 4.1. Define a new stable isometry structure $\mathfrak{q}_U = (\widehat{U}, u_U, z_U, \pi_1^U, \dots, \pi_\mu^U)$, where

$$\widehat{U} = \widehat{V} \oplus \widehat{W}, \qquad u_U = \begin{bmatrix} u_V & 0 \\ 0 & -u_W \end{bmatrix}, \quad z_U = \begin{bmatrix} z_V & \varphi \\ \psi & \bar{z}_W \end{bmatrix},$$

and

$$\pi_i^U = \begin{bmatrix} \pi_i^V & 0 \\ 0 & \pi_i^W \end{bmatrix}.$$

Conditions (a)–(e) of 4.1 are equivalent to the statement that the collection $\mathfrak{q}_U$ is a stable isometry structure with the property $\bar{z}_U \pi_i^U z_U = 0$ for all $i = 1, \dots, \mu$.

Let us define S-maps $P_i : \widehat{U} \to \widehat{U}$, $i = 1, \dots, 2\mu$, by

$$P_i : \widehat{U} \to V_i \to \widehat{U}, \qquad i = 1, \dots, \mu,$$
$$P_i : \widehat{U} \to W_i \to \widehat{U}, \qquad i = \mu + 1, \dots, 2\mu,$$

where all maps are the natural projections and inclusions. The collection

$$\mathfrak{q} = (\widehat{U}, u_V, z_V, P_1, \dots, P_{2\mu})$$

is a stable isometry structure of multiplicity 2μ. By Theorem 1.7 there exists a 2μ-component r-connected Seifert manifold $U = U_1 \cup \cdots \cup U_{2\mu}$ realizing $\mathfrak{q}$ (i.e., having $\mathfrak{q}$ as its stable isometry structure). Now, the union $U_1 \cup \cdots \cup U_\mu$ is a μ-component Seifert manifold whose stable isometry structure is isomorphic to $\mathfrak{q}_V$ and $U_{\mu+1} \cup \cdots \cup U_{2\mu}$ has stable isometry structure isomorphic to $\mathfrak{q}_W$. Therefore, by Theorem 1.6, there exists an isotopy of S^{n+2} taking V onto $U_1 \cup \cdots \cup U_\mu$ with respect to the orientations; similarly, there exists an isotopy of S^{n+2} taking W onto $U_{\mu+1} \cup \cdots \cup U_{2\mu}$ with orientation reversed.

Join each U_i with $U_{i+\mu}$ $(i = 1, \dots, \mu)$ by a 1-handle $H_i = D^n \times [0, 1]$ with $H_i \cap U_i = H_i \cap \partial U_i = D^n \times 0$, $H_i \cap U_{i+\mu} = H_i \cap \partial U_{i+\mu} = D^n \times 1$, $H_i \cap H_j = \varnothing$ for $i \neq j$, and $H_i \cap V_j = \varnothing$ for $j \neq i$, $i + \mu$. We also assume that the orientations on H_i induced by that of U_i and $U_{i+\mu}$, agree. Denote $U_i \cup H_i \cup U_{i+\mu}$ by S_i.

The set $S_1 \cup S_2 \cup \cdots \cup S_\mu = S$ is a μ-component Seifert manifold with the corresponding stable isometry structure

$$\mathfrak{q}_S = (\widehat{U}, u_V, z_U, P_1 + P_{\mu+1}, P_2 + P_{\mu+2}, \dots, P_\mu + P_{2\mu}).$$

It follows from our construction and from Corollary 2.6 that S is almost closed. This means that by a small isotopy we can translate $U_{\mu+1}, U_{\mu+2}, \dots, U_{2\mu}$ into new positions $U'_{\mu+1}, \dots, U'_{2\mu}$ such that

$$U_i \cap U'_{i+\mu} = \partial U_i = \partial U'_{i+\mu}$$

and

$$U_i \cap U'_j = \varnothing$$

for $j \neq i + \mu$. Thus $U_1 \cup \cdots \cup U_\mu$ and $U'_{\mu+1} \cup \cdots \cup U'_{2\mu}$ are contiguous. This proves the theorem.

4.3. Two stable isometry structures $\mathfrak{q}_0$ and $\mathfrak{q}_1$ will be called *R-equivalent* if there exists a sequence $\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_N$ of stable isometry structures with $\mathfrak{p}_0 = \mathfrak{q}_0$ and $\mathfrak{p}_N = \mathfrak{q}_1$, and for each $i \in \{0, 1, \dots, N-1\}$ either $\mathfrak{p}_i$ is contiguous to $\mathfrak{p}_{i+1}$ or $\mathfrak{p}_{i+1}$ is contiguous to $\mathfrak{p}_i$. R-equivalence is an equivalence relation.

From Theorems 1.6 and 4.2 we obtain

4.4. **Corollary.** *The map*

$$\left\{\begin{array}{l}\text{types of } \mathscr{F}\text{-links} \\ \text{of dimension } n \\ \text{and multiplicity } \mu\end{array}\right\} \to \left\{\begin{array}{l}R\text{-equivalence classes of stable isometry} \\ \text{structures of dimension } n \text{ and multiplicity } \mu\end{array}\right\}$$

which sends a link type into the R-equivalence class of the stable isometry structure of an arbitrary Seifert manifold of the link, is correctly defined. This map is a bijection from the subset of stable $\mathscr{F}$-links onto the set of R-equivalence classes of stable isometry structures on $\left[\frac{n+1}{3}\right]$-connected complexes.

4.5. Now we consider simple odd-dimensional links. A *simple link* is an $\mathscr{F}$-link of dimension $(2q-1)$, which is $(q-1)$-simple. We will assume that $q \geq 3$; in this case the link will be stable.

Any simple link of dimension $2q-1$ admits a $(q-1)$-connected Seifert manifold $V = V_1 \cup \cdots \cup V_\mu$ (cf. [G]). From Poincaré duality it follows that $\widehat{V}$ has the homotopy type of a wedge of spheres of dimension q, and so the homotopy type of $\widehat{V}$ is completely determined by the free abelian group $H_q(\widehat{V})$. Moreover, the S-maps $u, z, \pi_1, \dots, \pi_\mu$ from the stable isometry structure of $\widehat{V}$ are completely determined by the induced maps on $H_q(\widehat{V})$; thus we come to the following known algebraic notion of isometry structure (cf. [K, Mio, Ko]):

4.6. An *ε-symmetric* ($\varepsilon = \pm 1$) *isometry structure* of multiplicity μ is a tuple

$$(M, \langle , \rangle, z, \pi_1, \dots, \pi_\mu),$$

where M is a finitely generated free abelian group, $\langle , \rangle : M \otimes M \to \mathbb{Z}$ is an ε-symmetric bilinear form, and $z, \pi_1, \dots, \pi_\mu : M \to M$ are endomorphisms of M, such that

(i) $\langle , \rangle$ is unimodular;
(ii) $\langle zx, y\rangle = \langle x, (1-z)y\rangle \quad \forall x, y \in M$;
(iii) $\langle \pi_i x, \pi_j y\rangle = 0$ for $i \neq j$,
(iv) $\pi_i \circ \pi_j = \delta_{ij}\pi_j$;
(v) $\pi_1 + \pi_2 + \cdots + \pi_\mu = 1_M$.

We will denote an isometry structure simply by M, and consider the other objects (the bilinear form and endomorphisms) to define additional structures on M.

Let M_1 and M_2 be two ε-symmetric isometry structures of multiplicity μ. We will say that M_1 is contiguous to M_2 iff there are homomorphisms

$$\varphi : M_1 \to M_2, \qquad \psi : M_2 \to M_1$$

such that for each $x \in \{1, \dots, \mu\}$, $a \in M_1$, and $b \in M_2$ the following hold:

(a) $\langle a, \psi(b)\rangle = \langle \varphi(a), b\rangle$;
(b) $\varphi\pi_k z(a) = z\pi_k\varphi(a)$;
(c) $\psi\pi_k \bar{z}(b) = \bar{z}\pi_k\psi(b)$;
(d) $\varphi\pi_k\psi(b) = z\pi_k\bar{z}(b)$;
(e) $\psi\pi_k\varphi(a) = \bar{z}\pi_k z(a)$.

The equivalence relation generated by contiguity will be called R-equivalence.

Every Seifert manifold V of a $(2q-1)$-dimensional link determines an isometry structure (on the Betti group $H_q(\widehat{V})/\text{Tors}$) and contiguous Seifert manifolds determine contiguous isometry structures.

4.7. **Theorem.** *For $q \geq 3$ the correspondence described above gives a bijection*

$$\left\{\begin{array}{l}\textit{types of simple}\\ \textit{$\mathcal{F}$-links of dimension}\\ 2q-1 \textit{ and multiplicity } \mu\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\textit{R-equivalence classes of}\\ (-1)^q\textit{-symmetric isometry}\\ \textit{structures of multiplicity } \mu\end{array}\right\}.$$

Proof. The proof follows from Theorems 1.6, 2.6, and 4.2.

5. Homology pairings

In this section we use Poincare duality to study the quadratic structure of the homology of the free covering of a boundary link. We construct two quadratic forms, which generalize well-known forms on the homology of infinite cyclic covering of knots: the Blanchfield form [B] and the torsion form [L4, F1].

Another generalization of the Blanchfield form was suggested by Duval [D].

5.1. Let $(\Sigma, \mathscr{S})$ be an n-dimensional μ-component $\mathscr{F}$-link. Denote by X the complement of an open tubular neighbourhood of Σ in S^{n+2}. The splitting $\mathscr{S}$ determines a homomorphism of $\pi_1(X, *)$ onto F_μ (the free group in $t_1, \dots, t_\mu$), which is defined up to conjugation. The covering $\widetilde{X} \to X$ corresponding to the kernel of the homomorphism $\pi_1(X, *) \to F_\mu$ is called the *free covering*; and has the free group F_μ as the group of covering translations. Homology groups $H_*(\widetilde{X}; \mathbb{Z})$, considered as left modules over $\Lambda = \mathbb{Z}[F_\mu]$, are invariants of $(\Sigma, \mathscr{S})$.

The modules $H_*(\widetilde{X}; \mathbb{Z})$ possess finite presentations (since Λ is coherent) and are of type L (cf. [S]): a module M over Λ is *of type* L if the map

$$\bigoplus_{i=1}^{\mu} M \to M, \qquad (m_1, \dots, m_\mu) \mapsto \sum_{i=1}^{\mu} (t_i - 1)m_i ,$$

where $m_i \in M$, $i = 1, \dots, \mu$, is a $\mathbb{Z}$-isomorphism.

5.2. Fix some notation: Let $^- : \Lambda \to \Lambda$ denote the anti-isomorphism defined by

$$\bar{t}_i = t_i^{-1} , \qquad i = 1, \dots, \mu .$$

If M is a left Λ-module, $\overline{M}$ denotes the right Λ-module structure on M given by

$$m\lambda = \bar{\lambda} m , \qquad m \in M, \ \lambda \in \Lambda .$$

Let $e^i(M)$ denote $\operatorname{Ext}^i_\Lambda(M; \Lambda)$ —the right Λ-module. Let τM denote the $\mathbb{Z}$-torsion part of M, $\tau M = \operatorname{Tors}_{\mathbb{Z}} M$, and let $fM = M/\tau M$ denote the $\mathbb{Z}$-torsion-free part of M.

5.3. **Lemma** [D]. *Let M be a finitely presented module of type L. Then*

(1) $e^0(M) = 0$;
(2) $e^1(M)$ *has no* $\mathbb{Z}$*-torsion*;
(3) $e^1(M) \approx e^1(fM)$;
(4) $e^2(M)$ *is* $\mathbb{Z}$*-torsion*;
(5) $e^2(M) \approx e^2(\tau M)$.

In the next two lemmas we shall compute e^1 and e^2. Let I denote the augmentation ideal of Λ.

5.4. **Lemma.** *Let M be a finitely presented Λ-module of type L and let N be an arbitrary left Λ-module.*

(1) *If* $\bigcap_{n=0}^{\infty} I^n N = 0$, *then* $\operatorname{Hom}_\Lambda(M, N) = 0$.
(2) *If* $\bigcap_{n=0}^{\infty} I^n N = 0$ *and N is complete in the I-adic topology, then*

$$\operatorname{Ext}^1_\Lambda(M, N) = 0 .$$

Proof. The first statement is evident since $I^n M = M$ for any n.

To prove the second statement we have to show that any extension

$$0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to 0$$

splits. We will do this by constructing a sequence of Λ-homomorphisms

$$f_n : X \to N/I^n N, \qquad n = 0, 1, \dots ,$$

with the following properties:

(a) the diagram

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & X \\ {\scriptstyle \pi_n}\searrow & & \swarrow{\scriptstyle f_n} \\ & N/I^nN & \end{array}$$

commutes (where π_n is the projection);

(b) f_{n+1} is a refinement of f_n:

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle f_n}\swarrow & & \searrow{\scriptstyle f_{n+1}} & \\ N/I^nN & \longleftarrow & & & N/I^{n+1}N \end{array}$$

By property (b) the sequence $\{f_n\}$ converges to a homomorphism $f : X \to N$ and by (a) this f will be a splitting of the extension. Thus, the existence of such $\{f_n\}$ will prove (2).

We will construct $f_0, f_1, \dots$ inductively.

Put $f_0 = 0$ and suppose we have constructed $f_0, f_1, \dots, f_n$ with the desired properties. To define f_{n+1}, consider arbitrary $x \in X$; then $\beta(x) \in M$ can be uniquely represented in the form

$$\beta(x) = \sum_{i=1}^{\mu}(t_i - 1)m_i \ ,$$

where $m_i \in M$, $i = 1, \dots, \mu$. Now, $m_i = \beta(x_i)$ for some $x_i \in X$ and

$$x = \sum_{i=1}^{\mu}(t_i - 1)x_i + \alpha(\gamma),$$

where $\gamma \in N$. Let us define $f_{n+1}(x)$ by

$$f_{n+1}(x) = \sum_{i=1}^{\mu}(t_i - 1)f_n(x_i) + \gamma \pmod{I^{n+1}N} .$$

If we choose other elements $x_i' \in X$ with $\beta(x_i') = m_i$, then

$$x_i' = x_i + \alpha(\delta_i), \qquad \delta_i \in N, \ i = 1, \dots, \mu,$$

and we will have another representation

$$x = \sum_{i=1}^{\mu}(t_i - 1)x_i' + \alpha(\gamma')$$

with $\gamma' = \gamma - \sum_{i=1}^{\mu}(t_i - 1)\delta_i$. Hence, the result of computation of $f_{n+1}(x)$ according to the second formula will be

$$\begin{aligned} &\gamma' + \sum_{i=1}^{\mu}(t_i - 1)f_n(x_i') \\ &\quad = \gamma - \sum_{i=1}^{\mu}(t_i - 1)\delta_i + \sum_{i=1}^{\mu}(t_i - 1)\,[f_n(x_i) + f_n(\alpha(\delta_i))] \\ &\quad = \gamma + \sum_{i=1}^{\mu}(t_i - 1)f_n(x_i) \pmod{I^{n+1}N} \ , \end{aligned}$$

and, thus, f_{n+1} is correctly defined.

It can be easily checked that f_{n+1} is a Λ-homomorphism and

$$f_{n+1}(x) = f_n(x) \pmod{I^n N}, \qquad f_{n+1}(\alpha(\gamma)) = \gamma \pmod{I^{n+1} N},$$

where $x \in X$, $\gamma \in N$. This proves the lemma.

Let Γ be the ring of formal power series $\mathbb{Z}[[x_1, \dots, x_\mu]]$ of noncommuting variables $x_1, \dots, x_\mu$. We imbed Λ in Γ by the Magnus imbedding:

$$t_i \to 1 + x_i, \qquad t_i^{-1} \to 1 - x_i + x_i^2 - x_i^3 + \cdots.$$

Let $\mathbb{Q}\Gamma$ be $\mathbb{Q}[[x_1, \dots, x_\mu]]$, $\mathbb{Q}\Lambda = \mathbb{Q}[F_\mu]$, and $\mathbb{Q}\Lambda \to \mathbb{Q}\Gamma$ be the similar imbedding.

5.5. **Proposition.** *If M is a finitely presented module of type L, then*

(1) $e^1(M) \approx \mathrm{Hom}_\Lambda(M; \Gamma/\Lambda)$;

(2) $e^2(M) \approx \mathrm{Hom}_\Lambda(\tau M; \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda))$,

where the sign $\approx$ means "naturally isomorphic as right Λ-modules."

Proof. (1) Lemma 5.4 says that

$$\mathrm{Hom}_\Lambda(M; \Gamma) = 0 = \mathrm{Ext}^1_\Lambda(M, \Gamma),$$

so the exact sequence

$$0 = \mathrm{Hom}_\Lambda(M; \Gamma) \to \mathrm{Hom}_\Lambda(M; \Gamma/\Lambda) \to e^1(M) \to \mathrm{Ext}^1_\Lambda(M; \Gamma) = 0,$$

derived from $0 \to \Lambda \to \Gamma \to \Gamma/\Lambda \to 0$, gives the isomorphism (1).

(2) By Lemma 5.3 we can assume that M is $\mathbb{Z}$-torsion, $M = \tau M$. The exact sequence

$$\mathrm{Ext}^1_\Lambda(M; \mathbb{Q}\Lambda) \to \mathrm{Ext}^1_\Lambda(M; \mathbb{Q}\Lambda/\Lambda) \to e^2(M) \to \mathrm{Ext}^2_\Lambda(M; \mathbb{Q}\Lambda),$$

derived from $0 \to \Lambda \to \mathbb{Q}\Lambda \to \mathbb{Q}\Lambda/\Lambda \to 0$, gives an isomorphism

$$e^2(M) \approx \mathrm{Ext}^1_\Lambda(M; \mathbb{Q}\Lambda/\Lambda),$$

since $\mathrm{Ext}^*_\Lambda(M; \mathbb{Q}\Lambda) = 0$ (M is $\mathbb{Z}$-torsion).

From Lemma 5.4 it follows that $\mathrm{Hom}_\Lambda(M; \mathbb{Q}\Gamma/\Gamma) = 0 = \mathrm{Ext}^1_\Lambda(M; \mathbb{Q}\Gamma/\Gamma)$. The exact sequence

$$\begin{aligned}\mathrm{Hom}_\Lambda(M; \mathbb{Q}\Gamma/\Gamma) &\to \mathrm{Hom}_\Lambda(M; \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)) \\ &\to \mathrm{Ext}^1_\Lambda(M; \mathbb{Q}\Lambda/\Lambda) \to \mathrm{Ext}^1_\Lambda(M; \mathbb{Q}\Gamma/\Gamma),\end{aligned}$$

derived from $0 \to \mathbb{Q}\Lambda/\Lambda \to \mathbb{Q}\Gamma/\Gamma \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}/\Lambda) \to 0$, gives

$$\mathrm{Ext}^1_\Lambda(M; \mathbb{Q}\Lambda/\Lambda) \approx \mathrm{Hom}_\Lambda(M; \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)),$$

which proves the lemma.

5.6. Let $(\Sigma, \mathscr{S})$ be an n-dimensional μ-component $\mathscr{F}$-link and let $\tilde{X} \to X$ be the corresponding free covering. Consider the chain complex C of $\tilde{X}$ constructed by a cell-decomposition of X. The universal coefficient spectral sequence [EC] gives an exact sequence

$$0 \to e^2(H_{i-2}(C)) \to H^i(C; \Lambda) \to e^1(H_{i-1}(C)) \to 0. \tag{1}$$

According to Lemma 5.3, $e^2(H_{i-2}(C))$ coincides with the $\mathbb{Z}$-torsion subgroup of $H^i(C;\Lambda)$.

The Poincaré duality isomorphism [M1]

$$H_{n+2-i}(\widetilde{X}, \partial\widetilde{X}) \to \overline{H^i(C, \Lambda)}$$

gives two families of isomorphisms

$$\tau H_{n+2-i}(\widetilde{X}, \partial\widetilde{X}) \to \overline{e^2(H_{i-2}(C))}, \tag{2}$$

$$B_{n+2-i}(\widetilde{X}, \partial\widetilde{X}) \to \overline{e^1(H_{i-1}(C))}, \tag{3}$$

where $B_j(Y) = fH_j(Y;\mathbb{Z})$ denotes the Betti groups. It is clear that $\tau H_j(\widetilde{X}, \partial\widetilde{X}) = \tau H_j(\widetilde{X})$ for all j and $B_j(\widetilde{X}, \partial\widetilde{X}) = B_j(\widetilde{X})$ for $1 \le j \le n$. These identifications and Proposition 5.5 allow us to rewrite as

$$T_j(\widetilde{X}) \underset{\approx}{\longrightarrow} \overline{\mathrm{Hom}_\Lambda(T_{n-j}(\widetilde{X}); \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda))}, \tag{4}$$

$$B_j(\widetilde{X}) \underset{\approx}{\longrightarrow} \overline{\mathrm{Hom}_\Lambda(B_{n+1-j}(\widetilde{X}); \Gamma/\Lambda)}, \qquad 1 \le j \le n, \tag{5}$$

where T_j denotes τH_j. From this we obtain the following theorem.

5.7. **Theorem.** *In the notation of* §5.6, *there exist two families of forms*:

$$[\ ,\ ]: T_i(\widetilde{X}) \times T_{n-i}(\widetilde{X}) \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda),$$

$$\langle\ ,\ \rangle: B_i(\widetilde{X}) \times B_{n+1-i}(\widetilde{X}) \to \Gamma/\Lambda, \qquad 1 \le i \le n.$$

These forms are conjugate-linear and nonsingular.

In the special case $\mu = 1$ (knots) the pairing $\langle\ ,\ \rangle$ was constructed by Blanchfield [B], and the pairing $[\ ,\ ]$ by Levine [L4] and the author [F1]. Another version of $\langle\ ,\ \rangle$ was constructed by Duval [D].

5.8. In order to obtain explicit computational formulas for $\langle\ ,\ \rangle$ and $[\ ,\ ]$, we have to find the homomorphisms

$$\kappa: H^i(C;\Lambda) \to \mathrm{Hom}_\Lambda(H_{i-1}(C); \Gamma/\Lambda),$$

$$\chi: \mathrm{Tors}_\mathbb{Z} H^i(C;\Lambda) \to \mathrm{Hom}_\Lambda(T_{i-2}(C); \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda))$$

forming the exact sequence (1) in §5.6.

Let $f: C_i \to \Lambda$ be a cocycle. Since $f(Z_i) = 0$, f defines a homomorphism $f_1: B_{i-1} \to \Lambda$ with $f_1 \circ \partial = f$. By Lemma 5.4 f_1 can be extended to a homomorphism $f_2: Z_{i-1} \to \Gamma$ and this extension is unique. Now f_1 and f_2 give a homomorphism

$$\varphi: Z_{i-1}/B_{i-1} = H_{i-1}(C) \longrightarrow \Gamma/\Lambda$$

and hence we have a homomorphism

$$\kappa: H^i(C;\Lambda) \to \mathrm{Hom}(H_{i-1}(C); \Gamma/\Lambda), \qquad \kappa([f]) = \varphi.$$

In order to see that κ is an epimorphism, we have to show that every homomorphism $Z_{i-1} \to \Gamma/\Lambda$ can be lifted to a homomorphism $Z_{i-1} \to \Gamma$. The obstruction to this is an element of $e^1(Z_{i-1}) \approx e^2(B_{i-2}) = 0$.

Let us assume now that $[f] \in H^i(C; \Lambda)$ is of finite order. Then f_2 assumes its values in Λ. We want to construct the following commutative diagram

$$\begin{array}{ccccccccc} Z_{i-1} & \longrightarrow & C_{i-1} & \xrightarrow{\partial} & B_{i-2} & \longrightarrow & Z_{i-2} & \longrightarrow & H_{i-2}(C) \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} & & \downarrow{\scriptstyle f_6} \\ \Lambda & \longrightarrow & \mathbb{Q}\Lambda & \longrightarrow & \mathbb{Q}\Lambda/\Lambda & \longrightarrow & \mathbb{Q}\Gamma/\Gamma & \longrightarrow & \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)\ , \end{array}$$

where all horizontal maps are "natural." The obstruction to constructing f_3 lies in $\operatorname{Ext}^1_\Lambda(B_{i-2}, \mathbb{Q}\Lambda) = \operatorname{Ext}^1_{\mathbb{Q}\Lambda}(\mathbb{Q} \otimes_{\mathbb{Z}} B_{i-2}; \mathbb{Q}\Lambda)$ and the last group is trivial since $\mathbb{Q} \otimes_{\mathbb{Z}} B_{i-2}$ is a submodule of a free $\mathbb{Q}\Lambda$-module $\mathbb{Q} \otimes_{\mathbb{Z}} C_{i-2}$; so the result now follows from the fact that $\mathbb{Q}\Lambda$ is a free-ideal ring [C]. f_3 obviously defines f_4. The existence of f_5 is guaranteed by Lemma 5.4, and f_6 is determined obviously. We have an ambiguity in the construction of f_3 and f_5. It is clear that f_6 is correctly defined modulo $\beta_* \operatorname{Hom}(H_{i-2}(C); \mathbb{Q}\Gamma/\mathbb{Q}\Lambda)$, where $\beta : \mathbb{Q}\Gamma/\mathbb{Q}\Lambda \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$ is the projection. We will show later that

$$\begin{aligned} \sigma : &\operatorname{Hom}(H_{i-2}(C); \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda))/\beta_* \operatorname{Hom}(H_{i-2}(C); \mathbb{Q}\Gamma/\mathbb{Q}\Lambda) \\ &\to \operatorname{Hom}(\tau H_{i-2}(C); \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda))\ , \end{aligned} \tag{6}$$

which is given by taking the restriction to $\tau H_{i-2}(C)$, is an isomorphism; and

(7) any homomorphism $Z_{i-2} \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$ can be lifted to $Z_{i-2} \to \mathbb{Q}\Gamma/\Gamma$.

Assuming that (6), (7) are true, we see that the above arguments give the map

$$\kappa : \tau H^i(C; \Lambda) \to \operatorname{Hom}(\tau H_{i-2}(C); \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda))\ ,$$

$$\chi([f]) = f_6|_{\tau H_{i-2}(C)}\ .$$

The fact that χ is an epimorphism follows immediately from (6) and (7). If $\chi[f] = 0$, then we can construct a diagram as above with $f_6 = 0$; hence f_3 assumes its values in Λ and is a cochain with $\delta f_3 = f$. Thus χ is an isomorphism.

Let us now prove (7). By virtue of the exact sequence

$$0 \to \mathbb{Q}\Lambda/\Lambda \to \mathbb{Q}\Gamma/\Gamma \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda) \to 0$$

the obstruction to lifting $Z_{i-2} \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$ lies in $\operatorname{Ext}^1_\Lambda(Z_{i-2}, \mathbb{Q}\Lambda/\Lambda)$ which is trivial.

Statement (6) follows from the following lemma.

5.9. **Lemma.** *Let M be a finitely generated Λ-module of type L. Then*

(a) *any homomorphism $\tau M \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$ can be extended to $M \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$;*

(b) *if $\tau M = 0$ then any homomorphism $M \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$ can be lifted to $\mathbb{Q}\Gamma/\mathbb{Q}\Lambda$.*

Proof. (a) Let $fM = M/\tau M$. The obstruction to constructing the extension lies in $\operatorname{Ext}^1_\Lambda(fM; \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda))$. The exact sequence

$$0 \to \mathbb{Q}\Lambda/\Lambda \to \mathbb{Q}\Gamma/\Gamma \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda) \to 0$$

gives an exact sequence

$$\mathrm{Ext}^1_\Lambda(fM; \mathbb{Q}\Gamma/\Gamma) \to \mathrm{Ext}^1_\Lambda(fM; \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)) \to \mathrm{Ext}^2_\Lambda(fM; \mathbb{Q}\Lambda/\Lambda),$$

where both side groups are zero (by virtue of Lemma 5.4 and Proposition 4.1 of [D]).

Let us now prove (b). The obstruction to constructing the lifting lies in $\mathrm{Ext}^1_\Lambda(M; \Gamma/\Lambda)$. The exact sequence $0 \to \Lambda \to \Gamma \to \Gamma/\Lambda \to 0$ gives an exact sequence

$$\mathrm{Ext}^1_\Lambda(M; \Gamma) \to \mathrm{Ext}^1_\Lambda(M; \Gamma/\Lambda) \to e^2(M),$$

where both side groups are zero (by Lemmas 5.3 and 5.4).

The above arguments give the following recipes for calculation of κ and χ in §5.8:

5.10. **Proposition.** *Let $f : C_i \to \Lambda$ be a cocycle. Then the homomorphism $\kappa[f] : H_{i-1}(C) \to \Gamma/\Lambda$ acts as follows: For any cycle $c \in Z_{i-1}$ there is a sequence of i-dimensional chains α_m with ∂a_m converging to c in the I-adic topology, and*

$$\kappa([f])([c]) = \lim_{m\to\infty} f(\alpha_m) \mod \Lambda .$$

Here $f(a_m) \in \Lambda$ and the limit is an element of Γ. Its class modulo Λ does not depend on the sequence $\{\alpha_m\}$.

5.11. **Proposition.** *Assume that $f : C_i \to \Lambda$ is a cocyle representing a $\mathbb{Z}$-torsion cohomology class. It defines a homomorphism*

$$\chi[f] : \tau H_{i-2}(C) \to \mathbb{Q}\Gamma/(\Gamma + \mathbb{Q}\Lambda)$$

acting as follows. Let $c \in Z_{i-2}$ represent a $\mathbb{Z}$-torsion homology class. There exist sequences $\alpha_m \in C_{i-1}$ and $\beta_m \in C_i$, and elements $\gamma \in C_{i-1}$ and $N \in \mathbb{Z}$, $N \neq 0$, such that

(a) $Nc = \partial\gamma$;
(b) $\partial\alpha_m \to c$ *in* C_{i-2}; *and*
(c) $N\alpha_m - \partial\beta_m - \gamma \to 0$ *in* C_{i-1}.

Then

$$\chi([f])([c]) = \frac{1}{N} \lim_{m\to\infty} f(\beta_m) ,$$

where the limit is taken in $\mathbb{Q}\Gamma$ and should be considered modulo $\Gamma + \mathbb{Q}\Lambda$. The result does not depend on the choice.

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