

Jumps of the eta-invariant

(With an Appendix by Shmuel Weinberger:
Rationality of ρ -invariants)

Michael S. Farber^{1,*}, Jerome P. Levine^{2,*,**}

¹ School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel
(e-mail: farber@math.tau.ac.il, Fax: 972-3-6407543)

² Department of Mathematics, Brandeis University, Waltham, MA 02254, USA
(e-mail: levine@binah.cc.brandeis.edu, Fax: 617-736-3085)

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In [2], Atiyah, Patodi and Singer introduced an invariant η_D of any self-adjoint elliptic differential operator D on an odd-dimensional oriented closed manifold M , in order to prove an index theorem for manifolds with boundary. For the germinal case of the “signature operator” the relevant D is $\pm(*d - d*)$, where the Hodge duality operator $*$ is determined by the Riemannian metric on M . They consider, more generally, the signature twisted by a flat connection ∇ on a Hermitian vector bundle \mathcal{E} , or equivalently, a unitary representation α of $\pi_1(M)$. Then D is replaced by $\pm(*\nabla - \nabla*)$ and the corresponding eta-invariant is denoted η_∇ . An important observation is that the “reduced” invariant $\eta_\nabla - k\eta_d$ (where $k = \dim \mathcal{E}$) is a “topological” (more precisely, a C^∞) invariant of (M, α) – the ρ -invariant.

The eta-invariant η_∇ , considered as a real valued function of a flat connection ∇ , demonstrates two different phenomena. First, it has *integral jumps*, known also as the *spectral flow*, which plays a central role in modern low-dimensional topology. And, second, it varies *smoothly* if η_∇ is considered modulo integers, i.e., as a function (called *the reduced eta-invariant*) with values in \mathbb{R}/\mathbb{Z} .

The results of this paper contribute to the understanding of these phenomena. Our main result, Theorem 1.5, states that the jumps (or, equivalently, the infinitesimal spectral flow) can be calculated *homologically* by means of a linking form, constructed directly in terms of *deformations of the monodromy representations of the fundamental group*. More specifically, we show that given an analytic path of unitary representations, one may explicitly construct a local coefficient system over the manifold and an *algebraic* linking form on the homology of this local system such that the signature invariants of the linking form determine the spectral flow completely.

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To prove the above mentioned result we study first a more general situation of analytic families D_t of arbitrary elliptic self-adjoint operators on closed manifolds. We show that any such family can be viewed as a single operator \tilde{D} acting on the space of germs of analytic curves of smooth sections; such an operator \tilde{D} determines, in a canonical way, a linking pairing whose signatures measure the jumps of the eta-invariant of the family. This linking form we call *analytic* in order to distinguish it from the algebraic linking form mentioned above. In order to prove our principal result, Theorem 1.5, we need to find a relation between these two linking forms (they are not isomorphic, although have the same signatures); this is done with the use of parametrized Hodge decomposition, described in Sect. 4, and a version of the De Rham theorem for the germ-complex, cf. Sect. 5. These are the main ingredients of the proof.

The major results of the present paper were announced in [11].

A recent preprint [17] of P. Kirk and E. Klassen also addresses the problem of homological computation of the spectral flow. They proved that the contribution of the "first order terms" to the spectral flow is equal to the signature of a quadratic form given as a cup-product (compare Corollary 3.15). Our Theorem 1.5 gives a more precise and complete answer; it describes the contributions of terms of all orders. In preprint [19] P. Kirk and E. Klassen give an alternative (but equivalent to ours) description of the total spectral flow in terms similar to the spectral sequence of quadratic forms described in the subsections 3.10–3.15 of the present paper. They actually show in [19] that the spectral sequence can be expressed in terms of Massey products (in the sense of V. Retakh). This result of [19] can be compared with the detailed description of the spectral sequence of quadratic forms given in subsections 3.10–3.13 of the present paper; we study a more general case and formulate the answer in terms of the Taylor power series decomposition of the deformation. In yet another very recent preprint [18] P. Kirk and E. Klassen extended the construction of their work [17] to the case of manifolds with boundary.

Theorem 1.5 has some ideological similarity with the results of X. Dai [9], who studied adiabatic limit of the eta-invariant in the case when the Dirac operators along the fibers have nontrivial kernels, forming a vector bundle. Before him the adiabatic limit formula was obtained by J.-M. Bismut and J. Cheeger [3] under the assumption that the kernels along the fibers are trivial. X. Dai proved in [9] that the adiabatic limit formula for the signature operator contains an additional topological invariant τ which has a twofold characterization: it measures nonmultiplicativity of the signature and is equal to the sum of the signatures of pairings determined on the terms of the Leray spectral sequence.

Another relevant reference is the paper of R.R. Mazzeo and R.B. Melrose [25] where the behaviour of the space of harmonic forms under the adiabatic limit of the riemannian metric is studied. R.R. Mazzeo and R.B. Melrose consider spaces of curves of smooth sections and a parametrized Hodge decomposition having some similarity (and some essential distinctions) with our treatment in Sect. 3 and Sect. 4. The distinction between the approach of [25] and the present paper consists in the fact that we consider germs of curves of sections at ordinary points, while R.R. Mazzeo and R.B. Melrose study the

germs at infinity. On the other hand, we focus our attention on the *torsion part* of the germ cohomology, while R.R. Mazzeo and R.B. Melrose discuss essentially the *torsion free* part of the germ cohomology.

We apply our theorem about jumps of the eta-invariant to study the problem of homotopy invariance of the Atiyah–Patodi–Singer ρ -invariant. We give an intrinsic homotopy theoretic definition of the ρ -invariant, up to indeterminacy in the form of a locally constant function on the space of unitary representations, which is zero at the trivial representation. The proof relies on some auxiliary results (mostly known) about variation of the mod \mathbb{Z} -reduction of the eta-invariant. We conjecture that this indeterminacy is rational-valued.

A consequence of our results is that the ρ -invariants of homotopy equivalent manifolds differ by a rational-valued, locally constant function on every component of the representation space containing some representation which factors through a group satisfying the Novikov conjecture. But Shmuel Weinberger shows that such representations are actually dense in the representation space and so the difference is rational valued everywhere. We are grateful to Weinberger for including his proof as an Appendix to our paper.

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1 Deformations of the monodromy representations and jumps of the eta-invariant

1.1 Let M be a compact oriented Riemannian manifold of odd dimension $2l-1$ and \mathcal{E} a *flat Hermitian vector bundle* of rank m over M . This means that (1) a Hermitian metric has been specified on each fibre \mathcal{E}_x which varies smoothly with $x \in M$; (2) there is given a covariant derivative

$$\nabla : A^k(M; \mathcal{E}) \rightarrow A^{k+1}(M; \mathcal{E}), \quad k = 0, 1, \dots \quad (1)$$

acting on the space of C^∞ -forms on M with values in \mathcal{E} ; (3) ∇ is *flat*, i.e., $\nabla^2 = 0$; and (4) the covariant derivative ∇ is *compatible* with the Hermitian structure on \mathcal{E} ; the latter can also be expressed by saying that the Hermitian metric on \mathcal{E} is *flat*.

In this situation Atiyah, Patodi and Singer [2] have defined the following first order differential operator acting on forms of even degree $\phi \in A^{2p}(M, \mathcal{E})$ by

$$B\phi = i^l(-1)^{p+1}(*\nabla - \nabla*)\phi, \quad B : A^{ev}(M; \mathcal{E}) \rightarrow A^{ev}(M; \mathcal{E}) \quad (2)$$

where the star denotes the Hodge duality operator. The operator B is elliptic and self-adjoint. To any such operator Atiyah, Patodi and Singer in [2] assigned a numerical invariant, $\eta(B)$, called the eta-invariant which plays a crucial role in the index theorem for manifolds with boundary. Recall that the eta-invariant

$\eta(B)$ is defined as follows. Consider the eta-function of B :

$$\eta_B(s) = \sum_{\lambda \neq 0} \text{sign}(\lambda) |\lambda|^{-s},$$

where λ runs over all eigenvalues of B . It follows from the general theory of elliptic operators that for large $\Re(s)$ this formula defines a holomorphic function of s which has a meromorphic continuation to the whole complex plane. Atiyah, Patodi and Singer proved in [2] that the eta-function is holomorphic at $0 \in \mathbb{C}$ (cf. also P. Gilkey [14], Theorem 4.3.8, where a more general fact is proven). The eta-invariant of B is then defined as the value of the eta-function at the origin $\eta_B(0)$.

1.2 Suppose now that the covariant derivative ∇ is being *deformed*. By this we understand that there is given an *analytic family* of differential operators

$$\nabla_t : A^k(M; \mathcal{E}) \rightarrow A^{k+1}(M; \mathcal{E}), \quad k = 0, 1, \dots \quad (3)$$

where the parameter t varies in an interval around zero $(-\varepsilon, \varepsilon)$ such that:

(i) for every value of t the operator ∇_t is a covariant derivative on the vector bundle \mathcal{E} having curvature 0 (i.e., $\nabla_t^2 = 0$) and the Hermitian metric on \mathcal{E} is flat with respect to every ∇_t ;

(ii) for $t = 0$ the operator ∇_0 coincides with the original covariant derivative ∇ .

The analyticity of the family of connections ∇_t we understand as follows. Represent $\nabla_t = \nabla + \Omega_t$ where $\Omega_t \in A^1(M; \text{End}(\mathcal{E}))$; then the curve $t \mapsto \Omega_t$ is supposed to be analytic with respect to any Sobolev norm; cf. section 3.3 below.

The corresponding self-adjoint operators B_t , constructed using the connections ∇_t as explained above, will also be functions of the parameter t . Let η_t denote the eta-invariant of B_t . It is shown in [2] that η_t may have only simple discontinuities – integral jumps occurring when some eigenvalues of B_t cross zero. In other words, the limits

$$\lim_{t \rightarrow +0} \eta_t = \eta_+, \quad \lim_{t \rightarrow -0} \eta_t = \eta_- \quad (4)$$

exist and the discontinuities (jumps) $\eta_+ - \eta_0$ and $\eta_- - \eta_0$ are integers; here η_0 denotes the eta-invariant of the operator B_0 . Atiyah, Patodi and Singer [2] show that the integer $\eta_+ - \eta_-$ has the meaning of *infinitesimal spectral flow*; the integers $\eta_+ - \eta_0$ and $\eta_- - \eta_0$ have the meaning of *infinitesimal half flows*.

Our aim in the present paper is to compute these integral jumps $\eta_+ - \eta_0$ and $\eta_- - \eta_0$ in terms of some *homological* invariants constructed by means of the *germ of the deformation* ∇_t . More precisely, we will express these jumps in terms of the germ of the *deformation of the monodromy representation*.

1.3 Fix a base point $x \in M$. Given a flat Hermitian connection ∇ on \mathcal{E} there is the corresponding monodromy representation

$$\rho : \pi = \pi_1(M, x) \rightarrow U(\mathcal{E}_x)$$

where \mathcal{E}_x is the fibre above $x \in M$ and $U(\mathcal{E}_x)$ denotes the unitary group. We can express this by saying that \mathcal{E}_x has the structure of a left $\mathbb{C}[\pi]$ -module, where for $g \in \pi$ and $v \in \mathcal{E}_x$ the product $g \cdot v \in \mathcal{E}_x$ is the value of the flat section obtained from $v \in \mathcal{E}_x$ by parallel displacement along a loop representing g .

When the flat connection is being deformed, the corresponding monodromy representation is deformed as well. Thus, we obtain from the family ∇_t an *analytic one-parameter family of left $\mathbb{C}[\pi]$ -module structures on \mathcal{E}_x* . In other words, we obtain a family of maps

$$\mu_t : \pi \times \mathcal{E}_x \rightarrow \mathcal{E}_x, \quad (g, v) \mapsto g \cdot_t v, \quad g \in \pi, v \in \mathcal{E}_x$$

where $t \in (-\varepsilon, \varepsilon)$, such that for any t the map $(g, v) \mapsto g \cdot_t v$ is a linear unitary action of π on \mathcal{E}_x , depending analytically on t . (The analyticity follows from Lemma 5.2 below.)

Let \mathcal{O} denote the ring of germs of complex valued holomorphic functions $f : (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}$ at the origin. An element of \mathcal{O} can also be represented by a power series

$$f(t) = \sum_{n \geq 0} a_n t^n, \quad a_n \in \mathbb{C}$$

having a non-zero radius of convergence. We will consider \mathcal{O} together with the involution which is induced by complex conjugation on \mathbb{C} ; it has the property that $\bar{t} = t$ (i.e., t is real).

Let $\mathcal{O}\mathcal{E}_x$ be the set of germs of holomorphic curves in $\mathcal{E}_x : \alpha : (-\varepsilon, \varepsilon) \rightarrow \mathcal{E}_x$. It is a left \mathcal{O} -module where

$$(f \cdot \alpha)(t) = f(t) \cdot \alpha(t), \quad f \in \mathcal{O}, \quad \alpha \in \mathcal{O}\mathcal{E}_x.$$

It is clear that $\mathcal{O}\mathcal{E}_x$ is free of rank $m = \dim \mathcal{E}_x$ over \mathcal{O} . The Hermitian metric on the fibre \mathcal{E}_x defines (by pointwise multiplication) the following bilinear map

$$\langle \cdot, \cdot \rangle : \mathcal{O}\mathcal{E}_x \times \mathcal{O}\mathcal{E}_x \rightarrow \mathcal{O} \tag{5}$$

which is Hermitian, non-degenerate and \mathcal{O} -linear with respect to the first variable.

Given a deformation of the monodromy representation as above consider the following map

$$\pi \times \mathcal{O}\mathcal{E}_x \rightarrow \mathcal{O}\mathcal{E}_x, \quad (g, \alpha) \mapsto g \cdot \alpha$$

where $g \in \pi$, $\alpha \in \mathcal{O}\mathcal{E}_x$ and $g \cdot \alpha$ denotes the germ of the following curve

$$t \mapsto (g \cdot \alpha)(t) = g \cdot_t (\alpha(t)) \in \mathcal{E}_x$$

This map defines a left $\mathcal{O}[\pi]$ -module structure on $\mathcal{O}\mathcal{E}_x$.

We arrive at the conclusion that “an analytic one parameter family of left $\mathbb{C}[\pi]$ -module structures on \mathcal{E}_x ” can be understood as a left module $\mathcal{V} = \mathcal{O}\mathcal{E}_x$

over the ring $\mathcal{O}[\pi]$ having the following properties:

(a) \mathcal{V} is free of rank $m = \dim \mathcal{E}_x$ over \mathcal{O} and is supplied with a Hermitian form

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{O};$$

(b) the form $\langle \cdot, \cdot \rangle$ is non-singular and \mathcal{O} -linear in the first variable and anti-linear in the second variable;

(c) $\langle \lambda v, w \rangle = \langle v, \bar{\lambda} w \rangle$ for $\lambda \in \mathcal{O}[\pi], v, w \in \mathcal{V}$. Here the involution acts on $g \in \pi$ as $\bar{g} = g^{-1}$;

(d) If \mathfrak{m} denotes the maximal ideal of \mathcal{O} then there is an isomorphism

$$\mathcal{V}/\mathfrak{m}\mathcal{V} \approx \mathcal{E}_x$$

of $\mathbb{C}[\pi]$ -modules such that the form $\langle \cdot, \cdot \rangle$ reduces under this isomorphism to the original Hermitian form on the fibre \mathcal{E}_x .

The pair consisting of the $\mathcal{O}[\pi]$ -module \mathcal{V} and the form $\langle \cdot, \cdot \rangle$ on it will be referred to as a *deformation of the monodromy representation*. Thus, a deformation ∇_t of a flat Hermitian bundle determines a deformation of the monodromy representation. In 5.3 we will describe \mathcal{V} as a monodromy of a locally flat presheaf over M .

1.4 Now we will show that some standard homological constructions (using Poincaré duality) lead to certain *linking forms* constructed from the deformation of the monodromy representation. The linking form described here we will sometimes call *homological* or *algebraic* in order to distinguish it from another linking form also determined by the deformation which we will call *analytic*; it is constructed in Sect. 3.

Consider the cohomology of the manifold M with local coefficient system defined by \mathcal{V} ; this we understand as

$$H^*(\text{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{M}), \mathcal{V}))$$

and will denote it by $H^*(M; \mathcal{V})$. Here \tilde{M} is the universal cover of M and the group π acts on \tilde{M} from the left by covering translations; $C_*(\tilde{M})$ denotes the singular chain complex of \tilde{M} . Note that $H^*(M; \mathcal{V})$ is a finitely generated \mathcal{O} -module. Since \mathcal{O} is a principal ideal domain the module $H^k(M; \mathcal{V})$ can be represented as the sum

$$H^k(M; \mathcal{V}) = F^k \oplus T^k$$

of its \mathcal{O} -torsion $T^k = T^k(M)$ and the free part $F^k = H^k(M; \mathcal{V})/T^k$ for every $k = 0, 1, \dots$. Note that $T^k(M)$ is finitely generated over \mathbb{C} .

Let us construct now the *homological linking form of the deformation*

$$\{ \cdot, \cdot \} : T^l(M) \times T^l(M) \rightarrow \mathcal{M}/\mathcal{O} \quad (6)$$

where l is the middle dimension of M , $\dim M = 2l - 1$ and \mathcal{M} denotes the field of germs of meromorphic functions at the origin; note that an element of

\mathcal{M} can be represented in the form of a Laurent series

$$f(t) = \sum_{n \geq -N} a_n t^n, \quad a_n \in \mathbb{C}$$

for some non-negative integer N having non-zero radius of convergence.

Let $\mathcal{M}\mathcal{V}$ denote $\mathcal{M} \otimes \mathcal{O}\mathcal{V}$ considered as a left $\mathcal{M}[\pi]$ -module. Since \mathcal{V} is free over \mathcal{O} we have the following exact sequence

$$0 \rightarrow \mathcal{V} \rightarrow \mathcal{M}\mathcal{V} \rightarrow \mathcal{M}\mathcal{V}/\mathcal{V} \rightarrow 0$$

which generates the exact sequence

$$\begin{aligned} \cdots \rightarrow H^{l-1}(M; \mathcal{M}\mathcal{V}) \rightarrow H^{l-1}(M; \mathcal{M}\mathcal{V}/\mathcal{V}) \xrightarrow{\delta} H^l(M; \mathcal{V}) \\ \rightarrow H^l(M; \mathcal{M}\mathcal{V}) \rightarrow \cdots \end{aligned}$$

From this we obtain that the image of the Bockstein homomorphism δ is precisely the torsion subspace $\text{im}(\delta) = T^l$.

Finally, for $\alpha, \beta \in T^l(M)$ we can define

$$\{\alpha, \beta\} = (\delta^{-1}(\alpha) \cup \beta, [M]) \quad (7)$$

where the cup-product is taken with respect to the natural pairing

$$(\mathcal{M}\mathcal{V}/\mathcal{V}) \times \mathcal{V} \rightarrow \mathcal{M}/\mathcal{O}$$

determined by the Hermitian form $\langle, \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{O}$. One easily checks that formula (7) correctly defines a $(-1)^l$ -Hermitian form

$$\{, \} : T^l(M) \times T^l(M) \rightarrow \mathcal{M}/\mathcal{O}$$

which is \mathcal{O} -linear with respect to the first variable and \mathcal{O} -antilinear with respect to the second variable. Poincaré duality implies that it is non-degenerate.

We will explain in Sect. 2 that a linking form determines (in a purely algebraic way) a sequence of signature invariants

$$\sigma_1, \sigma_2, \sigma_3, \dots, \in \mathbb{Z}.$$

These appear in the following statement which is the main result of the present paper.

1.5 Theorem *Suppose that an analytic deformation (3) of a flat Hermitian vector bundle \mathcal{E} is given. Let B_t be the corresponding analytic family of Atiyah–Patodi–Singer operators (2) and let η_+, η_- and η_0 be the corresponding eta-invariants, cf. (4). Consider also the deformation of the monodromy representation \mathcal{V} (cf. 1.3), corresponding to ∇_t , where $t \in (-\varepsilon, \varepsilon)$ and the signatures $\sigma_1, \sigma_2, \dots$ of the linking form (6). Then the eta-invariant jumps are given by the formulae:*

$$\eta_+ = \eta_0 + \sum_{i \geq 1} \sigma_i, \quad \eta_- = \eta_0 + \sum_{i \geq 1} (-1)^i \sigma_i \quad (8)$$

In particular, we obtain that the jumps of the eta-invariant can be expressed through homotopy-theoretic (even homological) invariants.

From formulae (8) it follows that the *infinitesimal spectral flow* $\eta_+ - \eta_-$ is given by

$$\eta_+ - \eta_- = 2 \sum_{i \geq 1} \sigma_{2i-1} \quad (9)$$

Note that formula (9) involves only *odd* signatures. Another nice formula which follows immediately from Theorem 1.5 is

$$\frac{1}{2}(\eta_+ + \eta_-) - \eta_0 = \sum_{i \geq 1} \sigma_{2i} \quad (10)$$

It describes the deviation of η_0 from the mean value of η_+ and η_- and involves only *even* signatures.

Similar formulae were obtained by J. Levine (cf. [21], Theorem 2.3) for jumps of the signatures of knots; the present work actually emerged as the result of an attempt to understand the nature of those jump formulae.

The proof of Theorem 1.5 is given in Sect. 6. The Sects. 2–5 are devoted to auxiliary material needed for the proof.

2 Linking forms and their invariants

2.1 In this section we will consider algebraic invariants of Hermitian pairings of the form

$$\{, \} : T \otimes T \rightarrow \mathcal{M}/\mathcal{O} \quad (11)$$

where T is a finitely generated *torsion* \mathcal{O} -module. As explained in the previous section, such forms appear as linking forms describing deformations of monodromy representations. Very similar algebraic objects appear in knot theory as *Blanchfield pairings* of knots, cf. [21].

With any linking form (11) we will associate a *spectral sequence of quadratic forms* which will produce a set of numerical invariants.

Recall that \mathcal{O} denotes the ring of germs of holomorphic functions and \mathcal{M} denotes the field of germs of meromorphic functions. Thus, an element of \mathcal{O} has a representation in the form of a power series

$$f(t) = \sum_{n \geq 0} a_n t^n, \quad a_n \in \mathbb{C}$$

with non-zero radius of convergence; and element of \mathcal{M} can be represented by a similar power series having finitely many negative powers. \mathcal{O} and \mathcal{M} are considered together with the involutions induced by complex conjugation; t is assumed to be *real*, i.e., $\bar{t} = t$.

The \mathcal{O} -module T in (11) viewed as a vector space over \mathbb{C} is finite dimensional; its \mathcal{O} -module structure is given by a nilpotent \mathbb{C} -linear endomorphism

$t : T \rightarrow T$ of multiplication by $t \in \mathcal{O}$, with $t^n = 0$ for some n . The form (11) is assumed to be

- (a) *Hermitian*, i.e., $\{x, y\} = \overline{\{y, x\}}$ for all $x, y \in T$, where the bar denotes the involution of \mathcal{M}/\mathcal{O} induced from \mathcal{M} ;
- (b) *\mathcal{O} -linear*, i.e., $\{tx, y\} = t\{x, y\}$ for $x, y \in T$;
- (c) *non-degenerate*, i.e., the map $T \rightarrow \text{Hom}_{\mathcal{O}}(T, \mathcal{M}/\mathcal{O})$ is an isomorphism.

2.2 As an example consider the following pairing. Fix an integer $i \geq 1$ and a nonzero real c . Let T be $\mathcal{O}/t^i\mathcal{O}$; as a vector space over \mathbb{C} it has a basis of the form $x, tx, t^2x, \dots, t^{i-1}x$ where x represents the coset of $1 \in \mathcal{O}$. The pairing

$$\{, \}_{i,c} : T \otimes T \rightarrow \mathcal{M}/\mathcal{O} \quad (12)$$

is given by

$$\{x, x\}_{i,c} = ct^{-i} \bmod \mathcal{O}.$$

Note that in this example only the sign of c is important – changing c to λc with $\lambda > 0$ gives a congruent form.

2.3 Given a linking form (11), it defines the *scalar form*

$$[,] : T \otimes T \rightarrow \mathbb{C} \quad (13)$$

where

$$[x, y] = \text{Res}\{x, y\},$$

the residue of the meromorphic germ $\{x, y\}$. In terms of the scalar form $[,]$ one may write

$$\{x, y\} = [x, y]t^{-1} + [tx, y]t^{-2} + [t^2x, y]t^{-3} + \dots \quad (14)$$

for $x, y \in T$. Thus, the scalar form contains all the information. It has the following properties:

- (1) $[x, y] = \overline{[y, x]}$ (i.e., it is Hermitian);
- (2) the scalar form $[,]$ is non-degenerate;
- (3) $t : T \rightarrow T$ (the multiplication by $t \in \mathcal{O}$) is self-adjoint with respect to the scalar form i.e.,

$$[tx, y] = [x, ty]$$

for $x, y \in T$.

2.4 Denote

$$T_i = \{x \in T; t^i x = 0\}$$

for $i = 0, 1, 2, \dots$. We have $0 = T_0 \subset T_1 \subset T_2, \dots$ and $T = T_N = T_\infty$ for large N .

Note that $T_i \supset tT_{i+1}$, and the natural inclusion map $T_i \rightarrow T_{i+1}$ induces an inclusion $T_i/tT_{i+1} \rightarrow T_{i+1}/tT_{i+2}$. Denote

$$V_i = T_i/tT_{i+1};$$

for large i the space V_i is equal to $V_\infty = T/tT$. Thus we have the sequence of vector spaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\infty$$

On every V_i there is defined a Hermitian form

$$l_i : V_i \times V_i \rightarrow \mathbb{C}$$

where

$$l_i(x, y) = [t^{i-1}x, y]$$

for $x, y \in T_i$; one easily checks that this formula correctly defines a form on $V_i = T_i/tT_{i+1}$.

Lemma 2.5 *The annihilator of the form $l_i : V_i \times V_i \rightarrow \mathbb{C}$ is equal to V_{i-1} , i.e.,*

$$\ker(l_i) = V_{i-1}.$$

Proof. Using non-degeneracy of the scalar form $[,]$ one first checks that $T_i^\perp = t^i T$. If $x \in T_i$ and $[t^{i-1}x, y] = 0$ for any $y \in T_i$ then $t^{i-1}x \in T_i^\perp = t^i T$ and thus $t^{i-1}x = t^i z$ where $z \in T_{i+1}$. We obtain that $x = tz + u$ with $u \in T_{i-1}$. These arguments show that $V_{i-1} \supset \ker(l_i)$. The other inclusion is obvious. \square

2.6 Thus we obtain that any linking form

$$\{, \} : T \otimes T \rightarrow \mathcal{M}/\mathcal{O}$$

defines an algebraic object which we will call a *spectral sequence of quadratic forms* (because of its similarity to spectral sequences). It consists of a flag of finite dimensional vector spaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\infty = T/tT$$

(which actually stabilize, $V_i = V_\infty$ for large i) supplied with a sequence of Hermitian forms

$$l_i : V_i \times V_i \rightarrow \mathbb{C}, \quad i = 0, 1, 2, \dots$$

such that

- (1) l_i vanishes identically for large i ;
- (2) $V_{i-1} = \ker(l_i)$.

2.7 Using the spectral sequence of quadratic forms associated to a linking form one may construct a set of numerical invariants of linking forms. For any integer $i \geq 1$ let n_i^+ and n_i^- denote the number of positive and negative squares appearing in the diagonalization of the Hermitian form $l_i : V_i \times V_i \rightarrow \mathbb{C}$. Let σ_i denote the difference

$$\sigma_i = n_i^+ - n_i^-;$$

it is the signature of l_i . The numbers

$$n_i^+, n_i^-, \sigma_i$$

are obviously *invariants of the linking form*. The invariants σ_i will be the most important for the sequel; we will call them *signatures* of the form (11).

2.8 There is yet another spectral sequence of quadratic forms associated with any linking form (11), which is in fact more useful.

Denote $W_i = t^{i-1}T_i$ for $i = 1, 2, \dots$. Thus, $W_1 = T_1$ and we have a decreasing finite filtration

$$W_1 \supset W_2 \supset W_3 \supset \dots \supset W_\infty = 0.$$

For any integer $i \geq 1$ define a Hermitian form

$$\lambda_i : W_i \times W_i \rightarrow \mathbb{C}$$

by $\lambda_i(x, y) = \text{Res}\{a, y\}$ where $a \in T_i$ is an element with $t^{i-1}a = x$. Clearly, the form λ_i is correctly defined and is Hermitian.

The following statement is similar to Lemma 2.5; it states that the set of forms λ_i form a *spectral sequence of Hermitian forms*. Note that, the spectral sequence formed by W_i 's, grows in the opposite direction compared with the spectral sequence of 2.6.

Lemma 2.9 *The annihilator of the form $\lambda_i : W_i \times W_i \rightarrow \mathbb{C}$ is equal to W_{i+1} and the induced nondegenerate form on W_i/W_{i+1} has exactly n_i^+ positive squares and n_i^- negative squares, where the numbers n_i^+ and n_i^- are defined in 2.7. Thus, the signature σ_i can be also computed as the signature of the form λ_i .*

Proof. By Lemma 2.5, l_i induces a nondegenerate form on V_i/V_{i-1} ; let us denote this induced form \bar{l}_i . On the other hand, observe that $\lambda_i(x, y) = \text{Res}\{a, y\} = 0$ if $x \in W_i$ and $y \in W_{i+1}$. Thus, λ_i induces a form $\bar{\lambda}_i$ on the factor W_i/W_{i+1} . We claim that there is an isomorphism

$$\alpha_i : W_i/W_{i+1} \rightarrow V_i/V_{i-1}$$

which intertwines between \bar{l}_i and $\bar{\lambda}_i$. This would obviously imply the statement of the Lemma.

If $x \in W_i$, represent x as $t^{i-1}a$ for some $a \in T_i$ and define $\alpha_i(x)$ to be equal to the coset of a in $V_i/V_{i-1} = T_i/(T_{i-1} + tT_{i+1})$. One easily checks that this map is correctly defined and has the properties mentioned above. \square

2.10 As an example consider the linking form (12)

$$\{, \}_{i,c} : T \otimes T \rightarrow \mathcal{M}/\mathcal{O},$$

where $T = \mathcal{O}/t^i\mathcal{O}$. In this case we obtain that $V_j = 0$ for $j < i$ and $V_j = \mathbb{C}$ for $j \geq i$. All forms l_j with $j \neq i$ vanish; the i -th form l_i has signature equal to the sign of the number c .

Since the invariants n_i^+, n_i^-, σ_i are *additive* we obtain:

2.11 Corollary *Given a linking form (11) which is represented as an orthogonal sum of finitely many forms of type $\{, \}_{i,c}$ (with different i and c), the*

number n_i^+ is equal to the number of summands of type $\{, \}_{i,c}$ with c positive and the number n_i^- is equal to the number of summands $\{, \}_{i,c}$ with c negative in the above decomposition.

2.12 It is easy to show that any linking form (11) is diagonalizable, i.e. it is congruent to a direct sum of forms of the type $\{, \}_{i,c}$; we will not use this fact in the present paper and thus will leave it without proof. The uniqueness of this orthogonal decomposition follows from the previous arguments. Thus the numbers n_i^+ and n_i^- determine the type of the form (11).

A linking form

$$\{, \} : T \otimes T \rightarrow \mathcal{M}/\mathcal{O}$$

will be called *hyperbolic* if the \mathcal{O} -module T can be represented as a direct sum $T = A \oplus B$ such that the restrictions of the form $\{, \}$ on A and on B vanish: $\{x, y\} = 0$ if either $x, y \in A$ or $x, y \in B$ (this can be expressed by saying that A and B are Lagrangian direct summands).

Lemma 2.13 *All signatures σ_i , $i \geq 1$ of a hyperbolic linking form vanish.*

Proof. Suppose $T = A \oplus B$ where A and B are Lagrangian direct summands (over \mathcal{O}). Then for any integer $i \geq 1$ the vector space T_i (defined as in 2.4) is also a direct sum $T_i = A_i \oplus B_i$ of vector spaces defined in a similar way by A and B respectively; thus the vector space $V_i = V_i(T)$ is also given as a direct sum $V_i(T) = V_i(A) \oplus V_i(B)$. By Lemma 2.5 the pairing l_i induces a non-degenerate pairing \tilde{l}_i on $V_i(T)/V_{i-1}(T)$ and the signature of \tilde{l}_i is equal to σ_i . Thus we obtain that $V_i(T)/V_{i-1}(T)$ is a direct sum

$$V_i(A)/V_{i-1}(A) \oplus V_i(B)/V_{i-1}(B)$$

and the form \tilde{l}_i vanishes on $V_i(A)/V_{i-1}(A)$ and on $V_i(B)/V_{i-1}(B)$. This implies that $\sigma_i = 0$ for all $i \geq 1$. \square

2.14 We also mention the closely related notion of *metabolic* form. By definition, a form $\{, \}$ is metabolic if there is a submodule $A \subset T$, of half the dimension (as vector space over \mathbb{C}) of T , such that $\{, \}$, restricted to A , vanishes. (We say A is a *Lagrangian*). When $\{, \}$ is metabolic the individual signatures are not necessarily zero but we do have the property $\sum_{i \geq 1} \sigma_{2i-1} = 0$. In fact, this equation is a necessary and sufficient condition for $\{, \}$ to be metabolic. We will not prove these statements here since they will not be used in the present work. It is interesting to compare this to formula (9) above. The relations between hyperbolic and metabolic forms and related signature invariants are discussed more fully, in a special case, in [21].

2.15 As a concluding remark let us note that the study of skew-Hermitian forms (11) (i.e., forms satisfying $\{x, y\} = -\overline{\{y, x\}}$ for all $x, y \in T$ together with (b) and (c) of Subsect. 2.1) can be automatically reduced to the case of Hermitian forms discussed above by multiplying the form by $i = \sqrt{-1}$. Thus

we define signatures $\sigma_j, j \geq 1$ of a skew-Hermitian linking form (11) as the corresponding signatures of the form $i\{\cdot, \cdot\}$.

3 Jumps of the eta-invariant and signatures of the linking form determined by deformation of a self-adjoint operator

We are going to establish that an analytic deformation of a self-adjoint elliptic operator defines a linking form of the type studied in the previous section. We will call this linking form analytic in order to distinguish it from the algebraic linking form defined in 1.4. We will prove (it will be the main result of this section) that the jumps of the eta-invariant of the family of operators can be expressed through a combination of signatures associated to the analytic linking form. We will also compute explicitly the corresponding spectral sequence of quadratic forms in terms of the Taylor expansion of the family.

The above mentioned linking form is constructed by studying the action of the family of the operators on germs of analytic curves of sections of a vector bundle. The idea of considering the family of operators as a single operator acting on the space of curves is actually the principal technical novelty of the present paper.

We start this section by defining precisely the analytic curves we are going to use.

3.1 First we recall some standard definitions.

Let Ω be an open subset of \mathbb{C} and let V be a complex topological vector space. A function $f : \Omega \rightarrow V$ is said to be *weakly holomorphic* in Ω if vf is holomorphic in the ordinary sense for every continuous linear functional v on V . The function $f : \Omega \rightarrow V$ is called *strongly holomorphic* in Ω if the limit

$$\lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}$$

exists (in the topology of V) for every $z \in \Omega$. It is known that the above two notions of analyticity actually coincide if V is a Frechet space, cf. [29], Chapter 3.

A function $f : (a, b) \rightarrow V$ defined on a real interval (a, b) with values in a Frechet space V is called *analytic* (or *real analytic* or *holomorphic*) if it is a restriction of an analytic function $\Omega \rightarrow V$ defined in a neighbourhood $\Omega \subset \mathbb{C}$ of the interval (a, b) .

3.2 We will mainly consider analytic curves in spaces of smooth sections of vector bundles. Let M be a compact C^∞ Riemannian manifold (possibly with boundary) and let \mathcal{E} be a Hermitian vector bundle over M . For any integer k the symbol $\mathcal{H}_k(\mathcal{E})$ will denote the corresponding Sobolev space (defined as in Chapter 9 of [28]). Recall that the Sobolev space $\mathcal{H}_k(\mathcal{E})$ with $k \in \mathbb{Z}$ form a chain of Hilbertian spaces (in the terminology of [28]), which, in particular, means that $\mathcal{H}_k(\mathcal{E})$ is embedded into $\mathcal{H}_l(\mathcal{E})$ for $k > l$ (as a

topological vector space) and the intersection of all the spaces $\mathcal{H}_k(\mathcal{E})$ coincides with $\mathcal{H}_\infty(\mathcal{E}) = C^\infty(M)$.

3.3 Definition Let $f : (a, b) \rightarrow C^\infty(M)$ be a curve of smooth sections; we will say that f is *analytic* if for any integer k the curve f represents a (real) analytic curve considered as a curve in the Sobolev space $\mathcal{H}_k(\mathcal{E})$.

Note that any curve $f : (a, b) \rightarrow C^\infty(M)$ which is analytic in the sense of Sect. 3.1 (i.e., by viewing $C^\infty(M)$ as a Frechet space) will be obviously analytic in the sense of Definition 3.3. The converse is also true, although the proof of this fact is not elementary; we are grateful to V. Matsaev for explaining this to us. The proof suggested by V. Matsaev uses interpolation theory of Hilbert spaces. Since we wish to avoid these analytic subtleties, and since the Definition 3.3 is the most convenient and entirely sufficient for our purposes, we will accept it and will never use the equivalence of the above two definitions in the present paper.

Suppose that \mathcal{E} and \mathcal{F} are two Hermitian vector bundles over M . Then any differential operator $D : C^\infty(\mathcal{E}) \rightarrow C^\infty(\mathcal{F})$ of order ℓ defines a bounded linear map of Sobolev spaces $\mathcal{H}_k(\mathcal{E}) \rightarrow \mathcal{H}_{k-\ell}(\mathcal{F})$ (where $k \geq \ell$) and thus D maps analytic curves in $C^\infty(\mathcal{E})$ into analytic curves in $C^\infty(\mathcal{F})$.

3.4 We will now give the definition of analyticity for families of linear differential operators.

Suppose that \mathcal{E} and \mathcal{F} are two vector bundles over the manifold M and $D_t \in \text{Diff}_\ell(\mathcal{E}, \mathcal{F})$ is a family of linear differential operators of order ℓ , depending on a real parameter $t \in (a, b)$. Let $J^\ell(\mathcal{E})$ denote the jet bundle of order ℓ , cf. [28], Chapter IV, Sect. 2. Then by Theorem 1 on page 61 of [28], the set $\text{Diff}_\ell(\mathcal{E}, \mathcal{F})$ can be identified with $C^\infty(\text{Hom}(J^\ell(\mathcal{E}), \mathcal{F}))$. The latter is the set of smooth sections of a vector bundle; therefore we can consider analytic curves in this space of sections using the definition of analyticity given in 3.3.

We accept the following definition: a curve of linear differential operators $(a, b) \rightarrow \text{Diff}_\ell(\mathcal{E}, \mathcal{F})$ is called (*real*) *analytic* iff the corresponding curve of sections of the bundle $\text{Hom}(J^\ell(\mathcal{E}), \mathcal{F})$ is analytic.

The main property of analytic families of operators D_t , which we will constantly use, consists of the following: for any integer $k \geq \ell$ the family of bounded linear operators $D_t : \mathcal{H}_k(\mathcal{E}) \rightarrow \mathcal{H}_{k-\ell}(\mathcal{F})$ depends analytically on the parameter t (i.e., defines an analytic curve in the Banach space of bounded linear operators $\mathcal{H}_k(\mathcal{E}) \rightarrow \mathcal{H}_{k-\ell}(\mathcal{F})$ with the operator norm).

From the above remark it follows that if $f : (a, b) \rightarrow C^\infty(\mathcal{E})$ is an analytic curve of smooth sections and if $D : (a, b) \rightarrow \text{Diff}_\ell(\mathcal{E}, \mathcal{F})$ is an analytic curve of linear differential operators then the "evaluation curve" $t \mapsto D_t(f_t)$ is also analytic.

We will formulate now a few simple lemmas which will be used later. Roughly speaking, they represent different converses of the statement of the previous paragraph.

3.5 Lemma Let \mathcal{E} be a Hermitian vector bundle over a compact Riemannian manifold M without boundary and let $D_t \in \text{Diff}_\ell(\mathcal{E}, \mathcal{E})$ be an analytic (in the

sense of 3.4) family of elliptic self-adjoint operators of order $\ell > 0$ defined for $t \in (a, b)$. Suppose that $\ker D_t = 0$ for all $t \in (a, b)$. If $\phi, \psi : (a, b) \rightarrow C^\infty(\mathcal{E})$ are two curves such that $D_t(\phi(t)) = \psi(t)$ for any $t \in (a, b)$ and the curve ψ is analytic (in the sense of Definition 3.3), then the curve ϕ is also analytic.

Proof. Fix an integer k . Since $\ker D_t = 0$, the operator D_t defines a linear homeomorphism $D_t : \mathcal{H}_{k+\ell}(\mathcal{E}) \rightarrow \mathcal{H}_k(\mathcal{E})$ (by the open mapping theorem, cf. [29], p. 47) which depends analytically on t . Thus it follows that $\phi(t) = D_t^{-1}(\psi(t))$ is an analytic curve in the Sobolev space $\mathcal{H}_{k+\ell}(\mathcal{E})$. Since this is true for any k , the statement follows. \square

3.6 Lemma *Let \mathcal{E} be a Hermitian vector bundle over a compact Riemannian manifold M without boundary and let $D_t \in \text{Diff}_\ell(\mathcal{E}, \mathcal{E})$ be an analytic (in the sense of 3.4) family of elliptic self-adjoint operators of order $\ell > 0$ defined for $t \in (a, b)$. Suppose that $\phi, \psi : (a, b) \rightarrow C^\infty(\mathcal{E})$ are two curves such that $D_t(\phi(t)) = \psi(t)$ for any $t \in (a, b)$ and it is known that the curve ψ is analytic in the sense of Definition 3.3, while the curve ϕ is analytic in a weaker sense – as a curve in the Hilbert space $\mathcal{H}_0(\mathcal{E}) = L^2(\mathcal{E})$. Then the curve ϕ is analytic in the sense of Definition 3.3 as well.*

Proof. Choose a point $t_0 \in (a, b)$ and an integer $k \geq 0$. It is enough to prove analyticity of the curve $\phi : (t_0 - \delta, t_0 + \delta) \rightarrow \mathcal{H}_k(\mathcal{E})$ for some small $\delta > 0$ (the restriction of the original curve ϕ onto a neighbourhood of t_0 , considered as a curve in the Sobolev space $\mathcal{H}_k(\mathcal{E})$).

Let π denote the orthogonal projection of $\mathcal{H}_0(\mathcal{E})$ onto $\ker(D_{t_0}) \subset \mathcal{H}_\infty(\mathcal{E})$. The operator

$$D_t + \pi : \mathcal{H}_k(\mathcal{E}) \rightarrow \mathcal{H}_{k-\ell}(\mathcal{E})$$

is continuous, analytically depends on the parameter t , and is invertible for $t = t_0$. Thus it is invertible for $t \in (t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$. We have

$$(D_t + \pi)(\phi(t)) = \psi(t) + \pi(\phi(t)).$$

We claim that the right hand side of this equation is an analytic curve in the Sobolev space $\mathcal{H}_{k-\ell}(\mathcal{E})$. In fact, the first summand $\psi(t)$ is analytic in any Sobolev space, by assumption, while the second summand $\pi(\phi(t))$ belongs to a finite dimensional subspace $\ker D_{t_0}$, and it is given that it is analytic as a curve in Hilbert space $L^2(\mathcal{E}) = \mathcal{H}_0(\mathcal{E})$. Since all linear topologies on a finite dimensional vector space are equivalent, we conclude that the curve $\pi(\phi(t))$ is analytic as a curve in $\mathcal{H}_{k-\ell}(\mathcal{E})$.

Combining the remarks of the two previous paragraphs, we obtain that the curve $\phi : (t_0 - \delta, t_0 + \delta) \rightarrow \mathcal{H}_k(\mathcal{E})$ is analytic. \square

3.7 Suppose again that \mathcal{E} is a Hermitian vector bundle over a compact Riemannian manifold M without boundary and $D_t \in \text{Diff}_\ell(\mathcal{E}, \mathcal{E})$ is an analytic (in the sense of 3.4) family of elliptic self-adjoint operators of order $\ell > 0$ defined for $t \in I = (a, b)$. In this situation there exists a parametrized spectral decomposition (cf. [16], Theorem 3.9, Chapter VII, Sect. 3) which consists

of a sequence of analytic (in $L^2(\mathcal{E}) = \mathcal{H}_0(\mathcal{E})$) curves $\phi_n(t)$ and a sequence of analytic real valued functions $\mu_n(t)$ (defined for all $t \in I$) such that $\mu_n(t)$ represent all the repeated eigenvalues of D_t and $\phi_n(t)$ form the corresponding complete orthonormal family of eigenvectors of D_t acting on the Hilbert space $L^2(\mathcal{E}) = \mathcal{H}_0(\mathcal{E})$. By the regularity theorem for elliptic operators, the curves $\phi_n(t)$ actually belong to $C^\infty(\mathcal{E})$.

We claim now that *the curves of eigenfunctions $\phi_n(t)$, which appear in the parametrized spectral decomposition, are analytic in the sense of Definition 3.3, i.e., as curves in any Sobolev space $\mathcal{H}_k(\mathcal{E})$* . In fact, it is enough to apply Lemma 3.6 to the equation

$$(D_t - \mu_n(t))(\phi_n(t)) = 0$$

and observe that the operator $D_t - \mu_n(t)$ depends analytically on t while the curve on the right (the zero curve) is analytic.

3.8 Construction of the linking form

Let \mathcal{E} be a Hermitian vector bundle over a closed Riemannian manifold M . The symbol $\mathcal{O}C^\infty(\mathcal{E})$ will denote the set of germs, at $t = 0$, of all analytic curves $(-\varepsilon, \varepsilon) \rightarrow C^\infty(\mathcal{E})$ in the sense of Definition 3.3. It is a left module over the ring \mathcal{O} of germs of analytic curves on the complex plane (via the pointwise multiplication), which has already appeared in Sect. 2. There is also a “scalar product”

$$(\cdot, \cdot) : \mathcal{O}C^\infty(\mathcal{E}) \times \mathcal{O}C^\infty(\mathcal{E}) \rightarrow \mathcal{O} \quad (15)$$

(the pointwise scalar product of curves of sections), which is \mathcal{O} -linear with respect to the first variable and skew-linear with respect to the second variable.

Suppose that $D_t \in \text{Diff}_\ell(\mathcal{E}, \mathcal{E})$ is an analytic curve of elliptic self-adjoint differential operators of order $\ell > 0$ defined for $t \in (-\varepsilon, \varepsilon)$, cf. 3.4. Then it defines the following single map

$$\tilde{D} : \mathcal{O}C^\infty(\mathcal{E}) \rightarrow \mathcal{O}C^\infty(\mathcal{E}) \quad (16)$$

where for $\alpha \in \mathcal{O}C^\infty(\mathcal{E})$ the germ $\tilde{D}(\alpha)$ represents the curve $t \mapsto D_t(\alpha(t))$. It is clear that \tilde{D} is an \mathcal{O} -homomorphism. Consider the image of \tilde{D} and a larger \mathcal{O} -submodule $\mathfrak{C}l(\text{im}(\tilde{D})) \subset \mathcal{O}C^\infty(\mathcal{E})$ consisting of germs α with the property that $t^k \alpha$ belongs to $\text{im}(\tilde{D})$ for some $k > 0$; here t^k denotes the element of the ring \mathcal{O} represented by the curve $t \mapsto t^k$. Now define

$$T = \mathfrak{C}l(\text{im}(\tilde{D}))/\text{im}(\tilde{D}) ; \quad (17)$$

it is a module over \mathcal{O} .

Let \mathcal{M} denote the field of fractions of \mathcal{O} ; in other words, \mathcal{M} is the field of germs at 0 of meromorphic curves on \mathbb{C} . Let us define the *linking pairing*

$$\{, \} : T \times T \rightarrow \mathcal{M}/\mathcal{O} . \quad (18)$$

If α, β are two given elements of T , represent α and β by germs of curves f and g in $\mathcal{O}C^\infty(\mathcal{E})$ correspondingly; then $t^k f = \tilde{D}(h)$ for some $k > 0$ and $h \in \mathcal{O}C^\infty(\mathcal{E})$. Now we define

$$\{\alpha, \beta\} = t^{-k}(h, g) \in \mathcal{M}/\mathcal{O}. \quad (19)$$

One easily checks that the definition is correct. We will also refer to (18) as the *analytic linking form* associated with the deformation D_t .

3.9 Theorem (1) *The analytic linking pairing (18) (constructed out of an analytic family of elliptic self-adjoint operators $D_t : C^\infty(\mathcal{E}) \rightarrow C^\infty(\mathcal{E})$, where $-\varepsilon < t < \varepsilon$, acting on sections of a Hermitian vector bundle \mathcal{E} over a closed Riemannian manifold M), is Hermitian, \mathcal{O} -linear with respect to the first variable and non-degenerate and the \mathcal{O} -module T is finite-dimensional over \mathbb{C} ;* (2) *Let $\eta(D_t)$ denote the eta-invariant of the operator D_t (cf. [14]) and let η_\pm denote the limits*

$$\eta_\pm = \lim_{t \rightarrow \pm 0} \eta(D_t). \quad (20)$$

Then the following jump formulae hold:

$$\eta_+ = \eta_0 + \sum_{i \geq 1} \sigma_i, \quad \eta_- = \eta_0 + \sum_{i \geq 1} (-1)^i \sigma_i; \quad (21)$$

here η_0 denotes $\eta(D_0)$ and $\{\sigma_i\}$ denote the signatures of the linking pairing $\{\cdot, \cdot\}$, introduced in Sect. 2.

Proof. Consider the parametrized spectral decomposition $\phi_n(t)$, $\mu_n(t)$ where $n = 1, 2, \dots$, and $t \in I = (-\varepsilon, \varepsilon)$, of the analytic self-adjoint elliptic family D_t , mentioned in Subsect. 3.7. For every value of $t \in I$ the vectors $\phi_n(t)$ form a complete orthonormal system of eigenvectors of D_t with eigenvalues $\mu_n(t)$ and we know that $\mu_n(t)$ are analytic functions of $t \in I$ and the curves of sections $\phi_n(t)$ are analytic in the sense of Definition 3.3, as shown in 3.7.

From the ellipticity of D_t it follows that there exist only finitely many numbers n such that $\mu_n(0) = 0$. We can suppose that the numeration of the eigenfunctions and the corresponding eigenvalues has been arranged so that

- (1) $\mu_n(0) = 0$ but $\mu_n(t)$ is not identically zero for $n = 1, 2, \dots, N$;
- (2) $\mu_n(t) \equiv 0$ for $n = N + 1, N + 2, \dots, N_1$;
- (3) $\mu_n(0) \neq 0$ for all $n > N_1$.

The proof of the Theorem will be based on the following statement:

Claim. *For any curve $\psi : (-\varepsilon, \varepsilon) \rightarrow C^\infty(\mathcal{E})$, which is analytic in the sense of Definition 3.3 and satisfies*

$$(\phi_n(t), \psi(t)) = 0 \quad \text{for } n = 1, 2, \dots, N_1$$

(where we use the $L^2(\mathcal{E}) = \mathcal{H}_0(\mathcal{E})$ -scalar product), there exists a curve $\phi : (-\varepsilon, \varepsilon) \rightarrow C^\infty(\mathcal{E})$ satisfying

- (1) $D_t(\phi(t)) = \psi(t)$ for all $t \in (-\varepsilon, \varepsilon)$;
- (2) $(\phi_n(t), \phi(t)) = 0$ for $n = 1, 2, \dots, N_1$;

and the curve $\phi : (-\varepsilon, \varepsilon) \rightarrow C^\infty(\mathcal{E})$ is analytic in the sense of 3.3.

Such curve ϕ is actually unique but we will not need this fact.

Proof of the Claim. For any $t \in (-\varepsilon, \varepsilon)$ let $\pi_t : \mathcal{H}_0(\mathcal{E}) \rightarrow \mathcal{H}_\infty(\mathcal{E})$ denote the orthogonal projection onto the finite dimensional subspace generated by $\phi_1(t), \dots, \phi_{N_1}(t)$. Represent $\pi_t = \pi'_t + \pi''_t$ where π'_t is the projection onto the subspace generated by the curves $\phi_n(t)$, with $1 \leq n \leq N$, and π''_t is the projection onto the subspace generated by the curves $\phi_n(t)$ with $N+1 \leq n \leq N_1$. These operators naturally define operators acting on all Sobolev spaces. From the fact that the curves $\phi_n(t)$ are analytic in the sense of 3.3 (cf. 3.7) it follows that the operators

$$\pi'_t, \pi''_t : \mathcal{H}_k(\mathcal{E}) \rightarrow \mathcal{H}_\infty(\mathcal{E})$$

analytically depend on t for any integer k .

Consider the operators

$$(D_t + \pi_t) : \mathcal{H}_k(\mathcal{E}) \rightarrow \mathcal{H}_{k-\ell}(\mathcal{E}) ;$$

they form an analytic family of linear homeomorphisms (i.e., an analytic curve in the space of bounded linear operators $\mathcal{H}_k(\mathcal{E}) \rightarrow \mathcal{H}_{k-\ell}(\mathcal{E})$ with the operator norm topology). It follows that the equation

$$(D_t + \pi_t)(\xi(t)) = \psi(t)$$

admits a unique solution $\xi(t)$ which lies in $C^\infty(\mathcal{E})$ and is analytic in the sense of definition 3.3. Since $\pi_t(\psi(t)) = 0$ and $\pi'_t(D_t(\xi(t))) = 0$ we obtain

$$\pi'_t(D_t(\xi(t))) + \pi'_t(\xi(t)) = 0 \quad \text{and} \quad \pi''_t(\xi(t)) = 0 .$$

Moreover, since $\pi'_t(\xi(t))$ is an eigenvector of D_t with eigenvalue -1 and on the other hand it is a sum of eigenvectors each of whose eigenvalues has constant term 0, we obtain $\pi'_t(\xi(t)) = 0$. Thus, if we set $\phi(t) = \xi(t)$, then $D_t\phi = \psi$ and $\pi_t\phi = 0$. This completes the proof of the Claim.

Now we may easily finish the proof of Theorem 3.9. Given a germ of an analytic curve $f \in \mathcal{O}C^\infty(\mathcal{E})$ we can find (using the Claim) a germ $g \in \mathcal{O}C^\infty(\mathcal{E})$ with

$$f - \pi_t f = \tilde{D}(g), \quad \pi_t(g) = 0 .$$

Here we use the notation introduced in the proof of the Claim. Since the projection π''_t vanishes on $\mathbb{C}l(\text{im}(\tilde{D}))$ we obtain that:

(i) a germ $f \in \mathcal{O}C^\infty(\mathcal{E})$ belongs to $\text{im}(\tilde{D})$ if and only if $\pi'_t(f) \in \text{im}(\tilde{D})$ and $\pi''_t(f) = 0$;

(ii) a germ $f \in \mathcal{O}C^\infty(\mathcal{E})$ belongs to $\mathfrak{L}l(\text{im}(\tilde{D}))$ if and only if $\pi'_t(f) \in \mathfrak{L}l(\text{im}(\tilde{D}))$ and $\pi''_t(f) = 0$.

Now we see that the germs of eigenfunctions $\phi_1(t), \phi_2(t), \dots, \phi_N(t)$ generate T as a module over \mathcal{O} . The full set of relations of T is given by $\mu_n(t)\phi_n(t) = 0$, where $n = 1, 2, \dots, N$.

Computing the linking form on the elements $\phi_n \in T$, where $1 \leq n \leq N$, representing the eigenfunctions, we obtain

$$\{\phi_i, \phi_j\} = \begin{cases} 0, & \text{for } i \neq j \\ (\mu_i(t))^{-1} \bmod \mathcal{O}, & \text{for } i = j \end{cases} \quad (22)$$

We also obtain that the invariant n_i^+ of the linking form (18) (which was defined in Sect. 2) is equal to the number of eigenvalues $\mu_n(t)$ having the form $\mu_n(t) = t^i \bar{\mu}_n(t)$ where $\bar{\mu}_n(0)$ is positive. Similarly, the invariant n_i^- of the linking form is equal to the number of eigenvalues $\mu_n(t)$ having the form $\mu_n(t) = t^i \bar{\mu}_n(t)$ where $\bar{\mu}_n(0)$ is negative.

Since $\sigma_i = n_i^+ - n_i^-$, we obtain that the jump of the eta-invariant $\eta_+ - \eta_0$ is given by $\sum_{i \geq 1} \sigma_i$. The second formula follows similarly. \square

3.10 We are now going to explicitly compute the spectral sequence of quadratic forms (described in 2.8) of the linking form of the self-adjoint analytic family D_t , constructed in 3.8. We will show that this spectral sequence of quadratic forms can be expressed in terms of the kernel of the undeformed operator $\ker(D_0)$ (the space of “harmonic forms”) and the pairings on it given by the terms of the Taylor expansion of the family

$$D_t = D_0 + tD_1 + t^2D_2 + \dots$$

We want to emphasize that each form of the spectral sequence uses only finitely many derivatives D_i .

First, we are going to identify the initial term $W_1 = T_1 = \{\alpha \in T; t\alpha = 0\}$ of this spectral sequence, cf. 2.8. According to the definition of Subsect. 3.8, any element $\alpha \in T_1$ can be represented by an analytic germ $\alpha_t \in C^\infty(\mathcal{E})$ such that

$$t\alpha_t = D_t(\beta_t)$$

for some analytic germ $\beta_t \in C^\infty(\mathcal{E})$. The curve β_t determines α while adding to β_t a curve of the form $t\gamma_t$ does not change the class of α in T . Thus, we obtain that the initial term β_0 of the Taylor expansion

$$\beta_t = \beta_0 + t\beta_1 + t^2\beta_2 + \dots$$

determines the class $\alpha \in W_1$. The section β_0 must satisfy $D_0(\beta_0) = 0$. Let $\Sigma \subset \ker(D_0)$ denote the set of all $s \in C^\infty(\mathcal{E})$ satisfying $D_i(s) = 0$ for every $i \geq 0$. Thus we obtain that:

the initial term of the spectral sequence of quadratic forms associated with the linking form of an analytic self-adjoint family, is given by

$$W_1 = \ker(D_0)/\Sigma.$$

Let us now compute the first form $\lambda_1 : W_1 \times W_1 \rightarrow \mathbb{C}$, defined in 2.8. Suppose that $\beta_0, \beta'_0 \in \ker(D_0)/\Sigma$ are two elements representing $\alpha, \alpha' \in T_1 = W_1$ correspondingly. Then, we have $t\alpha_t = D_t(\beta_0)$ and $t\alpha'_t = D_t(\beta'_0)$ and, combining our definitions, we get

$$\lambda_1(\beta_0, \beta'_0) = \text{Res}\{\alpha, \alpha'\} = \text{Res } t^{-1}(\beta_t, \alpha'_t) = (\beta_0, D_1(\beta'_0)).$$

Thus, we obtain the following statement, which in the case of three dimensional manifolds was established by P. Kirk and E. Klassen [17]:

3.11 Corollary *The first Hermitian form $\lambda_1 : W_1 \times W_1 \rightarrow \mathbb{C}$ can be identified with the form on $\ker(D_0)/\Sigma$ induced by the first derivative D_1 . In particular, the first signature σ_1 , as well as the invariants n_1^+ and n_1^- , can be computed as the corresponding invariants of the Hermitian form on $\ker(D_0)$ given by $(x, y) \mapsto (D_1 x, y)$.*

By the construction of 2.8, the annihilator of λ_1 is W_2 , and there is a form λ_2 defined on W_2 with annihilator W_3 , and so on. We will describe all these forms λ_i as follows.

3.12 By 2.8, $W_i = t^{i-1}T_i \subset W_1$. An element α of T_i can be represented by a curve $\alpha_t \in C^\infty(\mathcal{E})$ such that $t^i\alpha_t = D_t(\beta_t)$ for some analytic curve $\beta_t \in C^\infty(\mathcal{E})$. We observe that the first i coefficients of the Taylor expansion $\beta_t = \beta_0 + t\beta_1 + t^2\beta_2 + \dots$ determine the class of $\alpha \in T_i$. In other words, an element of T_i can be described by a polynomial curve

$$\beta_t = \beta_0 + t\beta_1 + \dots + t^{i-1}\beta_{i-1}$$

of degree $i - 1$ satisfying the following system of equations

$$\begin{cases} D_0\beta_0 = 0 \\ D_1\beta_0 + D_0\beta_1 = 0 \\ \dots \dots \\ D_{i-1}\beta_0 + D_{i-2}\beta_1 + \dots + D_0\beta_{i-1} = 0 \end{cases} \quad (23i)$$

Thus, we obtain that *an element of $W_i = t^{i-1}T_i$ can be identified with the set of all $\beta_0 \in \ker(D_0)/\Sigma$, such that the system of equations (23i) admits a solution with the given β_0 .*

Suppose now that $\beta_0, \beta'_0 \in W_i$ are two such elements. We want to compute their product $\lambda_i(\beta_0, \beta'_0)$. Denote $\beta_t = \beta_0 + t\beta_1 + \dots + t^{i-1}\beta_{i-1}$ where $\beta_1, \beta_2, \dots, \beta_{i-1}$ form a solution of (23i) with given β_0 . Then $D_t\beta_t = t^i\alpha_t$ for some analytic curve α_t . We can similarly define β'_t and α'_t . According to our definitions given in 2.8, 3.8, we obtain

$$\begin{aligned} \lambda_i(\beta_0, \beta'_0) &= \text{Res}\{\alpha_t, t^{i-1}\alpha'_t\} = \text{Res}\{t^{i-1}\alpha_t, \alpha'_t\} \\ &= \text{Res } t^{-1}(\beta_t, \alpha'_t) = (\beta_0, \alpha'_0) \\ &= (\beta_0, D_i\beta'_0 + D_{i-1}\beta'_1 + \dots + D_1\beta'_{i-1}) \end{aligned}$$

Thus we have established

3.13 Corollary *The i -th term $W_i \subset \ker(D_0)/\Sigma$ of the spectral sequence of quadratic forms determined by the linking form (18) consists of all β_0 such that the system (23i) has a solution with the given β_0 . If $\beta_0, \beta'_0 \in W_i$ are two elements then their product $\lambda_i(\beta_0, \beta'_0) \in \mathbb{C}$ is given by*

$$\lambda_i(\beta_0, \beta'_0) = (\beta_0, D_i \beta'_0 + D_{i-1} \beta'_1 + \cdots + D_1 \beta'_{i-1})$$

where $\beta'_0, \beta'_1, \dots, \beta'_{i-1}$ form a solution of (23i).

3.14 *As an application of the above general results, consider deformations of flat bundles.*

Suppose that ∇_t , for $-\varepsilon < t < \varepsilon$, is an analytic family of flat connections on a vector bundle \mathcal{E} over a compact Riemannian manifold M of odd dimension $2l - 1$. We assume that all ∇_t preserve a fixed Hermitian metric on \mathcal{E} . Then

$$\nabla_t = \nabla + \sum_{i \geq 1} t^i \Omega_i,$$

where $\nabla = \nabla_0$ and $\Omega_i \in A^1(M; \text{End}(\mathcal{E}))$ are 1-forms with values in the bundle of endomorphisms. The latter bundle has a natural flat structure induced by ∇ , and from $\nabla_t^2 = 0$ it follows that the first form Ω_1 is flat $\nabla(\Omega_1) = 0$. Thus, it determines a cohomology class

$$\omega_1 = [\Omega_1] \in H^1(M; \text{End}(\mathcal{E})).$$

Consider now the following bilinear form

$$q : H^{l-1}(M; \mathcal{E}) \times H^{l-1}(M; \mathcal{E}) \rightarrow \mathbb{C}$$

given by

$$q(\alpha, \beta) = \langle \omega_1 \cdot \alpha \cup \beta, [M] \rangle,$$

where $\alpha, \beta \in H^{l-1}(M; \mathcal{E})$, the product $\omega_1 \cdot \alpha$ belongs to $H^l(M; \mathcal{E})$ and the cup-product \cup uses the Hermitian metric on \mathcal{E} .

The form q is $(-1)^l$ -Hermitian; this fact can be easily verified using the assumption that all ∇_t preserve the Hermitian metric. Let $\text{sign}(q)$ denote the signature of the form q , if l is even, and the signature of $iq = \sqrt{-1} \cdot q$, if l is odd.

The following result essentially coincides with Theorem 5.1 in [17].

3.15 Corollary *In the situation described above, consider the family of Atiyah–Patodi–Singer operators*

$$B_t : A^{ev}(M; \mathcal{E}) \rightarrow A^{ev}(M; \mathcal{E}), \quad -\varepsilon < t < \varepsilon,$$

where $B_t \phi = i^l (-1)^{p+1} (*\nabla_t - \nabla_t*)\phi$ acting on a form $\phi \in A^{2p}(M; \mathcal{E})$. Then the first signature σ_1 of the linking form 3.8 associated with this analytic self-adjoint family of elliptic operators B_t is equal to $\text{sign}(q)$.

Proof. Applying Corollary 3.11, we obtain that the first signature σ_1 can be computed as the signature of the pairing $(x, y) \mapsto (B_1 x, y)$, where $B_1 = \pm(*\Omega_1 - \Omega_1*)$, acting on the space of harmonic forms of even degrees with respect to the flat connection ∇ . We can identify the above space of harmonic forms with $H^{ev}(M; \mathcal{E})$. Our aim now is to find a form on the middle dimensional cohomology with the same signature.

We will use the following well-known fact: *let $l : X \times X \rightarrow \mathbb{C}$ be a Hermitian form on a finite dimensional vector space X and let $A \subset B \subset X$ be subspaces such that $A^\perp = B + K$, where K is the annihilator of l , $K = X^\perp$. Then the signature of the induced form on B/A is equal to $\text{sign}(l)$.*

Consider first the case when the number l is odd, i.e., $l = 2r - 1$, and so the dimension of M is $4r - 3$. Let A be the direct sum of all $H^{2k}(M; \mathcal{E})$ with $k < r - 1$; let B denote the direct sum of all $H^{2k}(M; \mathcal{E})$ with $k \leq r - 1$. Applying the above mentioned fact to this situation we get that the signature σ_1 is equal to the form on $H^{l-1}(M; \mathcal{E})$ given by

$$(\alpha, \beta) \mapsto i \cdot (*\Omega\alpha, \beta) = i \int_M \Omega\alpha \wedge \beta$$

as applied to the harmonic representatives. It is well-known that the last pairing can be expressed in terms of the cup-product on homology. This proves our statement in the case of odd l .

In the case of l even the arguments are similar. We apply the above lemma to the subspace A , being the sum of all $H^{2k}(M; \mathcal{E})$ with $k > r$, and to the subspace B , equal to the sum of all $H^{2k}(M; \mathcal{E})$ with $k \geq r$. By the lemma, we obtain that σ_1 is the signature of the form on $H^{2r}(M; \mathcal{E})$ given by

$$(\alpha, \beta) \mapsto (\Omega * \alpha, \beta) = \int_M \Omega * \alpha \wedge * \beta.$$

Identifying $H^{2r}(M; \mathcal{E})$ with $H^{l-1}(M; \mathcal{E})$ via $*$, we arrive at the statement of the theorem. \square

Higher terms of the spectral sequence of quadratic forms determined by the family B_i can also be computed using the arguments of 3.12 and Corollary 3.13. P. Kirk and E. Klassen showed in [19] that this spectral sequence can be expressed in terms of higher Massey products.

We will not continue along this line herè, because our Theorem 1.5 gives a simple general answer in homological terms.

4 Parametrized Hodge decomposition

4.1 Suppose that we have a closed oriented Riemannian manifold M of dimension n and a flat Hermitian vector bundle \mathcal{E} of rank m over M . Assume that an analytic deformation of the flat structure of \mathcal{E} is given; this means that we have an analytic one-parameter family of flat connections

$$\nabla_t : A^k(M; \mathcal{E}) \rightarrow A^{k+1}(M; \mathcal{E}), \quad k = 0, 1, \dots \quad (24)$$

where $t \in (-\varepsilon, \varepsilon)$, $\nabla_t^2 = 0$ and the Hermitian metric on \mathcal{E} is flat with respect to every ∇_t . Note that analyticity of the family ∇_t we understand as in Sect. 3.4; in the present situation this means that $\nabla_t = \nabla_0 + \sum_{i=1}^{\infty} t^i \Omega_i$ where $\Omega_i \in A^1(M; \text{End}(\mathcal{E}))$ and the power series converges in any Sobolev space.

These data determine a deformation germ of the twisted De Rham complex

$$\dots \rightarrow A^k(M; \mathcal{E}) \xrightarrow{\nabla_t} A^{k+1}(M; \mathcal{E}) \xrightarrow{\nabla_t} A^{k+2}(M; \mathcal{E}) \rightarrow \dots$$

Instead we will study the following single cochain complex of \mathcal{O} -modules

$$\dots \rightarrow \mathcal{O}A^k(M; \mathcal{E}) \xrightarrow{\nabla} \mathcal{O}A^{k+1}(M; \mathcal{E}) \xrightarrow{\nabla} \mathcal{O}A^{k+2}(M; \mathcal{E}) \rightarrow \dots \quad (25)$$

Here the symbol $\mathcal{O}A^k(M; \mathcal{E})$ denotes the set of germs of analytic curves in $A^k(M; \mathcal{E})$ (defined as in 3.8) and the map ∇ acts by the formula

$$(\nabla \alpha)(t) = \nabla_t(\alpha(t)) \quad (26)$$

for $\alpha \in \mathcal{O}A^k(M; \mathcal{E})$, $\alpha : (-\varepsilon, \varepsilon) \rightarrow A^k(M; \mathcal{E})$, $t \in (-\varepsilon, \varepsilon)$.

The chain complex (25) will be called *the germ-complex of the deformation*; it is the central object in our study of the deformation ∇_t . The purpose of this section is to prove a version of the Hodge decomposition theorem for this complex.

4.2 Recall first the operators which are defined on the twisted DeRham complex. Every space $A^k(M; \mathcal{E})$ of smooth k -forms carries a Hermitian scalar product determined by the metrics on M and on \mathcal{E} . The Hodge duality operator

$$* : A^k(M; \mathcal{E}) \rightarrow A^{n-k}(M; \mathcal{E}) \quad (27)$$

satisfies

$$(\phi, \psi) = \int_M \phi \wedge * \psi, \quad \phi, \psi \in A^k(M; \mathcal{E}), \quad (28)$$

where (ϕ, ψ) denotes the scalar product and \wedge denotes the exterior product

$$\wedge : A^k(M; \mathcal{E}) \times A^l(M; \mathcal{E}) \rightarrow A^{k+l}(M) \quad (29)$$

determined by the metric on \mathcal{E} (here $A^{k+l}(M)$ denotes the space of complex valued $(k+l)$ -forms on M). Note that this exterior product satisfies

$$\alpha \wedge \beta = (-1)^{kl} \overline{\beta} \wedge \alpha, \quad \alpha \in A^k(M; \mathcal{E}), \quad \beta \in A^l(M; \mathcal{E}).$$

These structures on the twisted De Rham complex determine (by pointwise extension) the following objects on the germ-complex (25). The scalar product (28) defines the following bilinear form

$$(\cdot, \cdot) : \mathcal{O}A^k(M; \mathcal{E}) \times \mathcal{O}A^k(M; \mathcal{E}) \rightarrow \mathcal{O} \quad (30)$$

where for germs of maps $\alpha, \beta : (-\varepsilon, \varepsilon) \rightarrow A^k(M; \mathcal{E})$ their product is given by

$$(\alpha, \beta)(t) = (\alpha(t), \beta(t)), \quad t \in (-\varepsilon, \varepsilon).$$

This form is \mathcal{O} -linear in the first variable, Hermitian

$$(\alpha, \beta) = \overline{(\beta, \alpha)}$$

and positive definite; the latter means that the scalar square $(\alpha, \alpha) \in \mathcal{O}$ is a germ of a *real* curve assuming *non-negative* values and $(\alpha, \alpha) = 0$ if and only if $\alpha = 0$.

In a similar way the Hodge duality operator (27) and the exterior product (29) determine the following maps

$$*: \mathcal{O}A^k(M; \mathcal{E}) \rightarrow \mathcal{O}A^{n-k}(M; \mathcal{E}) \quad (31)$$

$$\wedge : \mathcal{O}A^k(M; \mathcal{E}) \times \mathcal{O}A^l(M; \mathcal{E}) \rightarrow \mathcal{O}A^{k+l}(M) \quad (32)$$

where $(*\alpha)(t) = *(\alpha(t))$ for $\alpha \in \mathcal{O}A^k(M; \mathcal{E})$, $t \in (-\varepsilon, \varepsilon)$ and $(\alpha \wedge \beta)(t) = \alpha(t) \wedge \beta(t)$, $\beta \in \mathcal{O}A^l(M; \mathcal{E})$. The star-operator (31) is $\mathcal{O}A^0(M)$ -linear and satisfies

$$**\alpha = (-1)^{k(n-k)}\alpha$$

A relation between the star (31), the exterior product (32) and the “scalar product” (30) is given by the formula

$$(\alpha, \beta)(t) = \int_M (\alpha \wedge * \beta)(t) \quad (33)$$

Let

$$\nabla' : \mathcal{O}A^k(M; \mathcal{E}) \rightarrow \mathcal{O}A^{k-1}(M; \mathcal{E}) \quad (34)$$

be given by

$$\nabla'(\alpha) = (-1)^{n(k+1)+1} * \nabla * (\alpha)$$

for $\alpha \in \mathcal{O}A^k(M; \mathcal{E})$. If $\beta \in \mathcal{O}A^{k-1}(M; \mathcal{E})$ then

$$(\alpha, \nabla \beta) = (\nabla' \alpha, \beta) \in \mathcal{O}$$

which means that ∇' is dual to ∇ with respect to the product (30).

Let

$$\Delta : \mathcal{O}A^k(M; \mathcal{E}) \rightarrow \mathcal{O}A^k(M; \mathcal{E}) \quad (35)$$

be the Laplacian $\Delta = \nabla \nabla' + \nabla' \nabla$.

An element $\alpha \in \mathcal{O}A^k(M; \mathcal{E})$ is called *harmonic* if $\Delta \alpha = 0$. Since

$$(\Delta \alpha, \alpha) = (\nabla \alpha, \nabla \alpha) + (\nabla' \alpha, \nabla' \alpha)$$

and the scalar product is positive definite we obtain that a form $\alpha \in \mathcal{O}A^k(M; \mathcal{E})$ is harmonic if and only if $\nabla \alpha = 0$ and $\nabla' \alpha = 0$. The set of all harmonic forms in $\mathcal{O}A^k(M; \mathcal{E})$ will be denoted Har^k . It is an \mathcal{O} -module.

The following theorem is the main result of this section. In its statement we use the notation introduced in 3.8: if $X \subset Y$ is a \mathcal{O} -submodule of an \mathcal{O} -module Y then $\mathfrak{C}l(X)$ denotes the set of all $y \in Y$ such that fy belongs to X for some nonzero $f \in \mathcal{O}$.

4.3 Theorem *Suppose that an analytic deformation ∇_t of a flat Hermitian vector bundle \mathcal{E} over a compact manifold M is given, cf. Subsect. 4.1. Consider the germ-complex of \mathcal{O} -modules (25) determined by this deformation. Then:*

(1) *the following decomposition holds:*

$$\mathcal{O}A^k(M; \mathcal{E}) = \text{Har}^k \oplus \mathfrak{L}(\nabla(\mathcal{O}A^{k-1}(M; \mathcal{E}))) \oplus \mathfrak{L}(\nabla'(\mathcal{O}A^{k+1}(M; \mathcal{E}))) \quad (36)$$

and the summands of this decomposition are orthogonal to each other with respect to scalar product (30);

(2) *the \mathcal{O} -module Har^k of harmonic forms is free of finite rank;*

(3) *the factor-modules*

$$\mathfrak{L}(\nabla(\mathcal{O}A^{k-1}(M; \mathcal{E}))) / \nabla(\mathcal{O}A^{k-1}(M; \mathcal{E})) \quad (37)$$

and

$$\mathfrak{L}(\nabla'(\mathcal{O}A^{k+1}(M; \mathcal{E}))) / \nabla'(\mathcal{O}A^{k+1}(M; \mathcal{E})) \quad (38)$$

are finitely generated torsion \mathcal{O} -modules;

(4) *the homology of the complex (25) is finitely generated over \mathcal{O} and is isomorphic to*

$$\text{Har}^k \oplus \tau^k$$

where

$$\tau^k = \mathfrak{L}(\nabla(\mathcal{O}A^{k-1}(M; \mathcal{E}))) / \nabla(\mathcal{O}A^{k-1}(M; \mathcal{E})) \quad (39)$$

is the torsion part of the homology and Har^k is the free part of the homology.

(5) *the star operator (31) establishes \mathcal{O} -isomorphisms*

$$\text{Har}^k \xrightarrow{\sim} \text{Har}^{n-k}, \quad \tau^k \xrightarrow{\sim} \varrho^{n-k} \quad (40)$$

where we denote

$$\varrho^j = \mathfrak{L}(\nabla'(\mathcal{O}A^{j+1}(M; \mathcal{E}))) / \nabla'(\mathcal{O}A^{j+1}(M; \mathcal{E})) \quad (41)$$

Proof of a similar theorem in the more general situation of elliptic complexes is given in [10], Sect. 3. The last statement (5) is absent in [10] (since it is meaningless for general elliptic complexes), but this statement obviously follows from the definitions. \square

5 De Rham theorem for the germ-complex

In this section we are going to prove the following statement which is similar to the classical De Rham theorem.

5.1 Proposition *Let ∇_t be a deformation of flat a Hermitian bundle and let the $\mathcal{O}[\pi]$ -module \mathcal{V} be the corresponding deformation of monodromy representations as defined in Subsect. 1.3. Let $H^*(M; \mathcal{V})$ be the cohomology*

with coefficients in \mathcal{V} , cf. Subsect. 1.4. Then there is a canonical isomorphism between $H^*(M; \mathcal{V})$ and the cohomology of the germ-complex (25). Moreover, if the dimension of the manifold M is odd, $n = 2l - 1$, then the linking form (6) corresponds under this isomorphism to the $(-1)^l$ -Hermitian form

$$\tau^l \times \tau^l \rightarrow \mathcal{M}/\mathcal{O}$$

which is given on the classes $[f], [f'] \in \tau^l$ by the formula

$$\{[f], [f']\}(t) = t^{-k} \int_M g(t) \wedge f'(t) \bmod \mathcal{O} \quad (42)$$

where $g \in \mathcal{O}A^{k-1}(M; \mathcal{E})$ is any solution of the equation $t^k f = \nabla g$.

The proof (cf. Subsect. 5.3 below) is based on the following analytic fact:

5.2 Lemma *Let \mathcal{E} be a vector bundle over a closed n -dimensional ball M lying in Euclidean space \mathbb{R}^n and let ∇_t , with $t \in (-\varepsilon, \varepsilon)$, be a family of flat connections on \mathcal{E} which is analytic in the sense of 3.4. Fix a point $p \in M$ and a vector $e \in \mathcal{E}_p$ in the fiber above p . For every $t \in (-\varepsilon, \varepsilon)$ there exists a unique section $s_t \in C^\infty(\mathcal{E})$ such that $\nabla_t(s_t) = 0$ and $s_t(p) = e$, cf. [20], Chapter 1. Then the curve of sections $(-\varepsilon, \varepsilon) \rightarrow C^\infty(\mathcal{E})$, where $t \mapsto s_t$, is analytic in the sense of 3.3.*

The proof is given in 5.4.

5.3 Proof of Proposition 5.1 The arguments are standard; we will describe them briefly for completeness.

Let us define presheaves \mathfrak{F}^i , where $i = 0, 1, 2, \dots$, on M . For an open set $U \subset M$ let $\mathfrak{F}^i(U)$ denote $\mathcal{O}C^\infty((A^i T^*(M) \otimes \mathcal{E})|_U)$; in other words, the sections of \mathfrak{F}^i are germs of analytic curves of i -forms over U with values in \mathcal{E} . Analyticity of curves of forms over an open set U is understood in the following sense: for any compact submanifold $C \subset U$, the curves of restrictions to C are supposed to be analytic with respect to Definition 3.3. Note that $\mathfrak{F}^i(U)$ has the natural structure of an \mathcal{O} -module and the restriction maps of the presheaf are \mathcal{O} -homomorphisms.

The path of flat connections ∇_t on \mathcal{E} defines a homomorphism $\nabla : \mathfrak{F}^i \rightarrow \mathfrak{F}^{i+1}$ for every $i = 0, 1, 2, \dots$, acting as follows: let $\omega = (\omega_t)$ be an analytic curve of i -forms over U ; then $\nabla(\omega)$ is a curve of $(i+1)$ -forms represented by $t \mapsto \nabla_t(\omega_t)$. It is clear that $\nabla^2 = 0$.

Let \mathfrak{B} denote the following presheaf:

$$\mathfrak{B}(U) = \ker[\nabla : \mathfrak{F}^0(U) \rightarrow \mathfrak{F}^1(U)]. \quad (43)$$

According to Lemma 5.2, \mathfrak{B} is a locally trivial presheaf of flat analytic curves of sections of \mathcal{E} , i.e., analytic curves of sections s_t such that $\nabla_t(s_t) = 0$. In fact, if $U \subset M$ is a disk then $\mathfrak{B}(U)$ is isomorphic to $\mathcal{O}_{\mathcal{E}_p}$ —the set of analytic curves in the fiber over a point $p \in U$.

We claim now that for any disk $U \subset M$ the sequence

$$0 \rightarrow \mathfrak{B}(U) \xrightarrow{\nabla} \mathfrak{F}^0(U) \xrightarrow{\nabla} \mathfrak{F}^1(U) \xrightarrow{\nabla} \mathfrak{F}^2(U) \xrightarrow{\nabla} \dots \quad (44)$$

is exact. To prove this, fix a point $p \in U$ and a frame in the fiber $e_1, e_2, \dots, e_m \in \mathcal{E}_p$. By Lemma 5.2 we may find germs of curves $s_1, \dots, s_m \in \mathfrak{B}(U)$ such that $s_k(p) = e_k$ for $k = 1, 2, \dots, m$. Now, any $\omega \in \mathfrak{F}^i(U)$ can be uniquely represented as $\omega = \sum_{k=1}^m \omega_k s_k$ where $\omega_k \in \mathcal{O}A^i(U)$. We see that $\nabla(\omega) = 0$ if and only if $d\omega_k = 0$ for all $k = 1, 2, \dots, m$. The usual proof of the Poincaré Lemma (cf., for example, [31], 4.18) shows that if ω_k is an analytic curve of closed differential i -forms on U then there exists an analytic curve of $(i-1)$ -forms v_k with $dv_k = \omega_k$; the construction of such forms v_k described in [31], 4.18, uses contraction of the curve of forms ω_k with a vector field and then integration and so it produces curves of forms analytic with respect to the parameter. Denote $v = \sum_{k=1}^m v_k s_k$. Then $v \in \mathfrak{F}^{i-1}(U)$ and $\nabla v = \omega$.

Let $\mathfrak{U} = \{U_\alpha\}$ be a good finite cover of M , i.e., a cover by interiors of closed disks such that all intersections of the sets of \mathfrak{U} are also disks. Consider the Čech–De Rham complex $K = \bigoplus K^{p,q}$, where $K^{p,q} = C^p(\mathfrak{U}, \mathfrak{F}^q)$ is the module of Čech p -cochains of the cover \mathfrak{U} with coefficients in the presheaf \mathfrak{F}^q , cf. [7], Chapter 2. The bigraded complex K has differential $D = D' + D''$, where D' is the Čech differential (it has degree $(1,0)$) and D'' acting on $K^{p,q}$ is equal to $(-1)^p \nabla$. The differential D'' has degree $(0,1)$.

Let L_1 denote the Čech complex $C^p(\mathfrak{U}, \mathfrak{B})$ and let $i_1 : L_1 \rightarrow K$ denote the natural inclusion $C^p(\mathfrak{U}, \mathfrak{B}) \rightarrow C^p(\mathfrak{U}, \mathfrak{F}^0)$; let L_2 denote the germ-complex (25) and let $i_2 : L_2 \rightarrow K$ denote the natural inclusion $\mathcal{O}A^q(M; \mathcal{E}) \rightarrow C^0(\mathfrak{U}; \mathfrak{F}^q)$. Since the presheaves \mathfrak{F}^q are fine and the sequences (44) are exact over all intersections of the sets of \mathfrak{U} , the usual arguments, using the spectral sequence of the bicomplex K , cf., for example [7], Chapter 3, show that both maps i_1 and i_2 induce isomorphisms in cohomology. Thus, the cohomology of the germ-complex is isomorphic to the sheaf cohomology $H^*(\mathfrak{U}, \mathfrak{B})$ which, as it is well known, is isomorphic to cohomology with local coefficients $H^*(M, \mathcal{V})$, where \mathcal{V} is the local system determined by the monodromy of the locally constant sheaf \mathfrak{B} . Obviously, \mathcal{V} is the deformation of the monodromy representation as defined in 1.3.

We are left to prove the second statement of Proposition 5.1, concerning the linking pairings. To do this we have to compare multiplicative structures on the germ-complex and on the Čech complex $C^*(\mathfrak{U}, \mathfrak{B})$.

Let K' denote the complex constructed similarly to K with respect to the trivial line bundle instead of \mathcal{E} and the exterior derivative d instead of the path of flat connections ∇_t . In other words, $K'^{p,q} = C^p(\mathfrak{U}, \mathcal{O}\Omega^q)$, where Ω^q denotes the sheaf of differential forms. Similarly, let L'_1 denote $C^*(\mathfrak{U}, \mathcal{O})$ and L'_2 denote the complex $\mathcal{O}\Omega^*$ (the germs of curves in the De Rham complex). We have the obvious imbeddings $j_1 : L'_1 \rightarrow K'$ and $j_2 : L'_2 \rightarrow K'$ which induce isomorphisms in cohomology.

The multiplicative structure in the Čech–De Rham complex K , which uses the Hermitian scalar product on the bundle \mathcal{E} to define the wedge product

of form with values in \mathcal{E} , and is given by a formula similar to [7], p. 174, determines the chain map

$$K \otimes \bar{K} \rightarrow K'. \quad (45)$$

Here \bar{K} denotes the complex K with the \mathbb{C} -module structure twisted by complex conjugation.

Similarly, there are products $L_1 \otimes \bar{L}_1 \rightarrow L'_1$ and $L_2 \otimes \bar{L}_2 \rightarrow L'_2$, which are restrictions of the product (45). In other words, these products are related by the following commutative diagrams

$$\begin{array}{ccc} L_1 \otimes \bar{L}_1 & \longrightarrow & L'_1 \\ \downarrow i_1 \otimes i_1 & & \downarrow j_1 \\ K \otimes \bar{K} & \longrightarrow & K' \end{array} \quad \begin{array}{ccc} L_2 \otimes \bar{L}_2 & \longrightarrow & L'_2 \\ \downarrow i_2 \otimes i_2 & & \downarrow j_2 \\ K \otimes \bar{K} & \longrightarrow & K' \end{array}$$

Let $\theta_2 : L'_2 \rightarrow \mathcal{O}$ be the \mathcal{O} -homomorphism given by the fundamental class of M ; in other words

$$\theta_2(\omega_t) = \int_M \omega_t$$

(the above integral is defined to be zero if the degree of the forms ω_t is not equal to $n = \dim M$). θ_2 is a cocycle, i.e., it vanishes on coboundaries. Thus there exists a cocycle $\theta : K' \rightarrow \mathcal{O}$ of degree n such that $\theta|_{L'_2} = \theta_2$. Denote $\theta_1 = \theta|_{L'_1} : L'_1 \rightarrow \mathcal{O}$; it represents the fundamental class of M in Čech cohomology.

Each of these cocycles determines, in a standard way, a linking pairing on the corresponding complex. For example, the cocycle θ_1 together with the product $L_1 \otimes \bar{L}_1 \rightarrow L'_1$ determine the linking pairing on Čech cohomology, which is identical with the pairing (7) described in Subsect. 1.4. The linking pairing corresponding to the cocycle θ_2 and to the product $L_2 \otimes \bar{L}_2 \rightarrow L'_2$ is the one which was described in the statement of Proposition 5.1. They are isomorphic to the linking pairing on the torsion subgroups of the cohomology of K which is constructed by using the product $K \otimes \bar{K} \rightarrow K'$ and the cocycle θ . \square

5.4 Proof of Lemma 5.2 We are thankful to V. Matsaev who suggested the idea of the following arguments.

We may suppose that the vector bundle \mathcal{E} has been trivialized and that the connections ∇_t are of the form $d + \Omega$, where $\Omega = \Omega(x, t)$ is a matrix valued 1-form on M which depends on the parameter t analytically (in sense of Definition 3.3). We will also suppose that M is the ball of radius 1 in \mathbb{R}^n and that $p = 0$ is its center.

Let

$$Z = \sum_{i=1}^n r_i \frac{\partial}{\partial r_i}$$

be the radial vector field and let the matrix valued function $F(x, t)$ be defined as evaluation of $\Omega(x, t)$ on the vector field Z . Then solving the system of linear differential equations of parallel displacement along the radius joining a point $x \in M$ with the center $0 \in M$, we obtain the following formula for the flat section $s_t(x)$

$$s_t(x) = \sum_{k=0}^{\infty} (-|x|)^k \int_{0 \leq \tau_k \leq \dots \leq \tau_1 \leq 1} F(\tau_1 x, t) \dots F(\tau_k x, t) d\tau_1 \dots d\tau_k \cdot e. \quad (46)$$

This power series converges absolutely, cf. [13], Chapter 14, Sect. 5.

Let $W \subset \mathbb{C}$ be a neighbourhood of the interval $(-\varepsilon, \varepsilon)$ such that the curve of 1-forms $\Omega(x, t)$, where $t \in (-\varepsilon, \varepsilon)$, can be extended to an analytic curve $t \mapsto A^1(M; \text{End}(\mathcal{E}))$ defined for all $t \in W$, where $A^1(M; \text{End}(\mathcal{E}))$ is supplied with the $L^2 = \mathcal{H}_0$ -topology. Each term of the series (46) is an analytic function mapping W to $\mathcal{H}_0(\text{End}(\mathcal{E}))$; thus the series represents an analytic function and, therefore, the curve of sections s_t is analytic as a curve in $\mathcal{H}_0(\mathcal{E})$.

We will show now that s_t is analytic as a curve in any Sobolev space $\mathcal{H}_k(\mathcal{E})$. Let ∇ denotes the connection ∇_t for $t = 0$. Then the covariant derivative of s_t along any vector field X is equal to

$$\nabla_X(s_t) = -\Omega(X) \cdot s_t. \quad (47)$$

Since the right hand side is analytic as a curve in $\mathcal{H}_0(\mathcal{E})$ we obtain that s_t is analytic as a curve in $\mathcal{H}_1(\mathcal{E})$. For any other vector field Y , differentiating (47) we obtain

$$\nabla_Y \nabla_X(s_t) = -\nabla_Y(\Omega(X)) \cdot s_t + \Omega(X)\Omega(Y) \cdot s_t$$

By our assumptions the curve of 1-forms Ω and all its derivatives are analytic as curves in \mathcal{H}_0 . Thus we obtain from the last equation that s_t is analytic as a curve in $\mathcal{H}_2(\mathcal{E})$.

Continuing similarly, by induction, we establish analyticity of s_t in the sense of 3.3. \square

6 Proof of Theorem 1.5

6.1 Suppose that M is a closed oriented Riemannian manifold of odd dimension $2l - 1$ and \mathcal{E} is a flat Hermitian vector bundle over M . Suppose also that a deformation of the flat structure of \mathcal{E} is given. This means that we have a family of flat connections ∇_t , preserving the Hermitian metric on \mathcal{E} . Then the family of Atiyah-Patodi-Singer operators B_t is defined

$$B_t \phi = i^l (-1)^{p+1} (*\nabla_t - \nabla_t*)\phi, \quad B_t : A^{ev}(M; \mathcal{E}) \rightarrow A^{ev}(M; \mathcal{E}),$$

all B_t being elliptic and self-adjoint. We want to compute the infinitesimal spectral flow of this family. According to our general results (cf. Sect. 3, Theorem 3.9) we have to study the signatures of the *analytic* linking form

of this deformation which is given by the general construction of Sect. 3 applied to the self-adjoint family B_t . Our aim here is to compare this analytic linking form with the *algebraic* or *homological* linking form (6) constructed in 1.4.

In general, the analytic linking form can be understood if we know the action of the deformation on germs of holomorphic curves with values in $A^{ev}(M; \mathcal{E})$, cf. (16). The action of the Atiyah–Patodi–Singer operators B_t on holomorphic curves with values in $A^k(M; \mathcal{E})$ can be deduced from Theorem 4.3. We will see that the information given by this theorem is complete enough for our purposes.

In order to simplify the notations we will denote the modules appearing in Theorem 4.3 as follows

$$X^k = \nabla(\mathcal{O}A^{k-1}(M; \mathcal{E})),$$

$$Y^k = \nabla'(\mathcal{O}A^{k+1}(M; \mathcal{E}))$$

Then the decomposition of Theorem 4.3 looks

$$\mathcal{O}A^k(M; \mathcal{E}) = \text{Har}^k \oplus \mathfrak{L}I(X^k) \oplus \mathfrak{L}I(Y^k) \quad (48)$$

Recall the notation introduced in Theorem 4.3:

$$\mathfrak{L}I(X^k)/X^k = \tau^k \quad \text{and} \quad \mathfrak{L}I(Y^k)/Y^k = \varrho^k.$$

Note also that τ^k coincides with the \mathcal{O} -torsion part of the homology of the germ-complex (25) associated to the deformation.

We find it convenient to consider separately the cases of even and odd l .

6.2 Suppose first that l is even, $l = 2r$, i.e., the dimension of M is $4r - 1$.

Consider the result of applying the family B_t to each term of the decomposition (48). We obtain that the action of the deformation B_t on $\mathcal{O}A^k(M; \mathcal{E})$ splits into two sequences of *epimorphisms*:

$$(-1)^{r+p} \nabla_* : \mathfrak{L}I(X^{2p}) \rightarrow X^{4r-2p} \quad (49)$$

$$(-1)^{r+p+1} * \nabla : \mathfrak{L}I(Y^{2p}) \rightarrow Y^{4r-2p-2} \quad (50)$$

Applying the construction of Subsect. 3.8 to B_t we obtain easily that the linking form of this deformation splits into the following orthogonal sum:

$$\bigoplus_{p=0}^{r-1} (\tau^{2p} \oplus \tau^{4r-2p}) \perp \bigoplus_{p=0}^{r-1} (\varrho^{2p} \oplus \varrho^{4r-2p-2}) \perp \tau^{2r} \quad (51)$$

where all forms $\tau^{2p} \oplus \tau^{4r-2p}$ and $\varrho^{2p} \oplus \varrho^{4r-2p-2}$ for $0 \leq p \leq r-1$ (except the last one on τ^{2r}) are *hyperbolic*, cf. Subsect. 2.12. Actually, their representation as sums of two Lagrangian direct summands is given by (51). Lemma 2.13 and the additivity of the signatures σ_i imply that *the signatures of the analytic linking pairing corresponding to a deformation of the Atiyah–Patodi–Singer operators B_t are equal to the signatures of the linking form on the middle-dimensional torsion submodule τ^1 given by the operator ∇_* , cf. (49).*

Let us compute the last form explicitly. By the construction of Sect. 3.8 the value of the form on holomorphic germs $f, f' \in \tau^{2r}$ is given by

$$\{f, f'\} = t^{-k}(h, f') \bmod \mathcal{O}$$

where the germ h is a solution of the equation $t^k f = \nabla * h$. Using formulae (31) and (33) we obtain that

$$\{f, f'\} = t^{-k} \int_M (*h \wedge f') \bmod \mathcal{O}$$

Using Theorem 5.1 we obtain from the last formula that *the analytic linking form on τ^{2r} determined by the operator $\nabla*$ coincides with the homological linking pairing on the torsion part of the homology as defined in 1.4.*

Now an application of Theorem 3.9 finishes the proof of Theorem 1.5 in the case when l is even.

6.3 In the case when l is odd the arguments are similar. Assume that $l = 2r + 1$; so the dimension of M is $4r + 1$. As in the previous case we obtain that the family of Atiyah–Patodi–Singer operators B_t acting on the germs of holomorphic curves, splits into the following sequence of \mathcal{O} -epimorphisms:

$$i(-1)^{r+p} \nabla * : \mathfrak{C} l(X^{2p}) \rightarrow X^{4r-2p+2}, \quad (52)$$

$$i(-1)^{r+p+1} * \nabla : \mathfrak{C} l(Y^{2p}) \rightarrow Y^{4r-2p}, \quad (53)$$

where $i = \sqrt{-1}$. This implies that the linking form of the self-adjoint family B_t is the following orthogonal sum:

$$\bigoplus_{p=0}^{r-1} (\tau^{2p} \oplus \tau^{4r-2p+2}) \perp \bigoplus_{p=0}^{r-1} (q^{2p} \oplus q^{4r-2p}) \perp q^{2r}. \quad (54)$$

Again, all forms in this decomposition except the middle-dimensional form on q^{2r} (corresponding to the operator $(-i) * \nabla$) are *hyperbolic* and so have zero signatures by Lemma 2.13. Thus, we obtain that the signatures of the linking form corresponding to a deformation of the Atiyah–Patodi–Singer operator are equal to the signatures of the linking pairing on q^{2r} which acts as follows: for $f, f' \in q^{2r}$ their product $\{f, f'\} \in \mathcal{M}/\mathcal{O}$ is given by $\{f, f'\} = t^{-k}(g, f')$ where $g \in \mathcal{O}A^{2r}(M; \mathcal{E})$ solves the equation $t^k f = -i * \nabla g$. Applying $*$ we get

$$t^k * f = -i \nabla_g = \nabla(-ig) \quad (55)$$

and

$$\{f, f'\} = i \times t^{-k} \int_M (-ig) \wedge * f' = i \times \{ * f, * f' \}' \quad (56)$$

In (56) the brackets $\{, \}'$ denote the *homological* linking form

$$\{, \}' : \tau^{2r+1} \times \tau^{2r+1} \rightarrow \mathcal{M}/\mathcal{O}$$

constructed as in Subsect. 1.4. The formula (56) shows that the star-operator $* : q^{2r} \rightarrow \tau^{2r+1}$ establishes an isomorphism between the “analytic” form $\{, \}$

on q^{2r} and the algebraic form $\{, \}'$ on τ^{2r+1} multiplied by i . The analytic form is Hermitian and the algebraic form is skew-Hermitian; taking into account our convention 2.15 on signatures of skew-Hermitian linking forms, we obtain finally that *the signatures σ_i of the analytic linking form corresponding to the deformation of the Atiyah–Patodi–Singer operator B_t , cf. Sect. 3, coincide with the corresponding signatures of the algebraic linking form (7) constructed in Subsect. 1.4.*

An application of Theorem 3.9 completes the proof of Theorem 1.5. \square

7 Variation of the eta-invariant modulo \mathbb{Z}

We now examine the behavior of the eta-invariant reduced modulo \mathbb{Z} . This problem is rather well-understood, even (implicitly) in the original work of Atiyah–Patodi–Singer [2]; also see, for example, [8] and [24]. Much more general results of this kind were proven by P. Gilkey [14] (cf. Theorem 4.4.6 of [14] for example). For the convenience of the reader, and in order to emphasize the explicit dependence on the homotopy type of M , we give a complete and independent treatment here. These results, together with Theorem 1.5, will be important for our study in Sect. 10 of the problem of homotopy invariance of the ρ -invariant.

7.1 Theorem *Let \mathcal{E} be a Hermitian line bundle over a closed oriented Riemannian manifold M of odd dimension $2l - 1$. Suppose that two flat connections ∇_0 and ∇_1 on \mathcal{E} preserving the Hermitian metric are given. Let $\bar{\eta}_0$ and $\bar{\eta}_1$ denote the reductions modulo 1 of the eta-invariants of the corresponding Atiyah–Patodi–Singer operators (2). Consider the difference*

$$\frac{1}{2\pi i}(\nabla_1 - \nabla_0);$$

this is a closed 1-form on M with real values. Let $\xi \in H^1(M; \mathbb{R})$ denote the corresponding cohomology class. Then the following formulae hold:

$$\bar{\eta}_1 - \bar{\eta}_0 = \begin{cases} 0, & \text{if } l \text{ is even} \\ 2\langle \xi \cup L(M), [M] \rangle \bmod 1, & \text{if } l \text{ is odd,} \end{cases}$$

where $L(M)$ denotes the Hirzebruch polynomial in the Pontrjagin classes of M .

Note that the connections ∇_0 and ∇_1 are gauge-equivalent iff the class ξ is integral, $\xi \in H^1(M; \mathbb{Z})$. In this case we obviously obtain $\bar{\eta}_0 = \bar{\eta}_1 \bmod 1$.

7.2 Theorem 7.1 can also be interpreted in the following way. Assume that l is odd, $l = 2r + 1$, and so the dimension of M is $4r + 1$. Let N_1, N_2, \dots, N_k be a set of oriented submanifolds of M realizing a basis in the homology group $H_{4r}(M; \mathbb{Z})$; here k is the first Betti number of M . Denote by τ_i the signature of N_i , $1 \leq i \leq k$. Then Theorem 7.1 gives

$$\bar{\eta}_1 - \bar{\eta}_0 = 2 \sum_{i=1}^k \tau_i x_i \bmod 1$$

where the numbers x_1, x_2, \dots, x_k are obtained as the coefficients of the decomposition of $\xi \cap [M]$ (the Poincare dual of ξ) in terms of the basis formed by the classes $[N_i] \in H_{4r}(M; \mathbb{Z})$:

$$\xi \cap [M] = \sum_{i=1}^k x_i [N_i] \quad \text{in } H_{4r}(M; \mathbb{R})$$

7.3 Note that the space of flat structures on a given line bundle \mathcal{E} up to gauge equivalence is a torus $H^1(M; \mathbb{R})/H^1(M; \mathbb{Z})$; its dimension is equal to the first Betti number of M . The eta-invariant $\bar{\eta}$ comprises a function on this torus with values in \mathbb{R}/\mathbb{Z} . Theorem 7.1 states that this function is *linear* if l is odd and is *constant* if l is even. Theorem 7.1 implies also the following statement:

7.4 Corollary *If l is odd, $l = 2r + 1$, the reduced eta-invariant $\bar{\eta}_{\nabla} \in \mathbb{R}/\mathbb{Z}$ is constant (i.e., does not depend on choice of the flat structure ∇ on \mathcal{E}) if and only if all $4r$ -dimensional compact submanifolds $N^{4r} \subset M^{4k+1}$ have vanishing signatures.*

7.5 Consider now deformations of flat Hermitian bundles of arbitrary rank.

Again, let M denote a compact oriented Riemannian manifold of odd dimension $2l - 1$ and \mathcal{E} a vector bundle of rank m over M . Let $\det(\mathcal{E})$ denote the line bundle $\wedge^m(\mathcal{E})$, the m -th exterior power of \mathcal{E} . Any connection ∇ on \mathcal{E} determines canonically a connection on the line bundle $\det(\mathcal{E})$ which will be denoted $\det(\nabla)$.

Given a flat connection ∇ , the symbol $\bar{\eta}_{\nabla}$ will denote the reduced modulo 1 eta-invariant of the corresponding Atiyah–Patodi–Singer operator (2).

7.6 Theorem *In the situation described above, suppose that we have two flat connections ∇_0 and ∇_1 which can be joined by a smooth path of flat connections ∇_t , $0 \leq t \leq 1$ on \mathcal{E} . Then*

$$\bar{\eta}_{\nabla_1} - \bar{\eta}_{\nabla_0} = \bar{\eta}_{\det(\nabla_1)} - \bar{\eta}_{\det(\nabla_0)} \in \mathbb{R}/\mathbb{Z}$$

where $\det(\nabla_1)$ and $\det(\nabla_0)$ are the corresponding flat connections on $\det(\mathcal{E})$.

In particular, we obtain that if the dimension of M is of the form $4r - 1$, the eta-invariant $\bar{\eta}_{\nabla}$ assumes a constant value (in \mathbb{R}/\mathbb{Z}) on connected components of the space flat connections.

If the dimension of M is of the form $4r + 1$ then

$$\bar{\eta}_{\nabla_1} - \bar{\eta}_{\nabla_0} = 2\langle \xi \cup L(M), [M] \rangle$$

where $\xi \in H^1(M; \mathbb{R})$ is the cohomology class represented by the following closed 1-form

$$\frac{1}{2\pi i} (\det(\nabla_1) - \det(\nabla_0)) = \frac{1}{2\pi i} \text{tr}(\nabla_1 - \nabla_0);$$

thus, the reduced eta-invariant $\bar{\eta}_{\nabla}$ is constant on connected components of the space of flat connections if and only if all $4r$ -dimensional submanifolds $N^{4r} \subset M^{4r+1}$ have vanishing signatures.

We refer to [20], p. 18 for general information on determinant line bundles.

7.7 It can be useful in applications to express the class ξ which appears in Theorems B and C in terms of the monodromy representations.

Fix a base point $x \in M$. Let

$$\rho_v : \pi = \pi_1(M, x) \rightarrow U(\mathcal{E}_x), \quad v = 0, 1$$

denote the monodromy representation corresponding to the connections ∇_v where $v = 0, 1$. The monodromy representations of the line bundle $\wedge^m \mathcal{E}_x$ corresponding to the connection $\det(\nabla_v)$, $v = 0, 1$, is equal to the composition

$$\det \circ \rho_v : \pi \rightarrow U(\mathcal{E}_x) \xrightarrow{\det} U(\wedge^m \mathcal{E}_x)$$

Let

$$\arg : U(\wedge^m \mathcal{E}_x) \rightarrow \mathbb{R}/\mathbb{Z}$$

denote the argument function. Then for any element $g \in \pi$ we have

$$\frac{1}{2\pi} (\arg \det(\rho_1(g)) - \arg \det(\rho_0(g))) = -\langle \xi, g \rangle \bmod \mathbb{Z}.$$

The last equality determines the coset of the class ξ in $H^1(M; \mathbb{R})/H^1(M; \mathbb{Z})$.

The results of Theorems 7.1 and 7.6 were obtained in [22], [23] in some special cases.

7.8 *Proofs of Theorems 7.1 and 7.6.* Suppose that we are in the situation of Theorem 7.6. Namely, we suppose that \mathcal{E} is a vector bundle of rank m over a closed oriented Riemannian manifold M of odd dimension $2l - 1$ and ∇_t with $0 \leq t \leq 1$ is a path of flat connections on \mathcal{E} . Consider the vector bundle $\tilde{\mathcal{E}}$ over the product $I \times M$ (where I denotes the interval $[0, 1]$) induced from \mathcal{E} by the projection $I \times M \rightarrow M$. The path of connections ∇_t determines a unique connection

$$\tilde{\nabla} : A^0(I \times M; \tilde{\mathcal{E}}) \rightarrow A^1(I \times M; \tilde{\mathcal{E}}) \quad (57)$$

on the bundle $\tilde{\mathcal{E}}$ where

$$(\tilde{\nabla}s)(t, x) = (\nabla_t s(t, \cdot))(x) + dt \wedge \frac{\partial s(t, x)}{\partial t}$$

for $s \in A^0(I \times M; \tilde{\mathcal{E}})$, cf. [4], p. 48.

Consider the *generalized signature operator* on the manifold $I \times M$ (supplied with the Riemannian metric which is the product of the metric of M and the Euclidean metric on I)

$$A : \Omega^+ \rightarrow \Omega^-$$

as defined by Atiyah, Bott and Patodi in [1], p. 309. Recall that here $\Omega^+ \oplus \Omega^- = A^*(I \times M; \tilde{\mathcal{E}})$ and Ω^\pm are the \pm -eigenspaces of the involution

$$\tau(\alpha) = i^{p(p-1)+l} * (\alpha), \quad \alpha \in A^p(I \times M; \tilde{\mathcal{E}})$$

and the operator A is $\tilde{\nabla} + \tilde{\nabla}^*$. Applying to this operator A the index theorem of Atiyah–Patodi–Singer [2], part I, and taking into account computation of the index density in [1], Sect. 6, (cf. also [4]) we obtain the following equality:

$$\text{index } A = 2^l \int_{I \times M} \text{ch}(\tilde{\mathcal{E}}) \cdot \mathcal{L}(I \times M)[M] - 1/2(h_1 - h_0) - 1/2(\eta_1 - \eta_0). \quad (58)$$

Here $\text{ch}(\tilde{\mathcal{E}})$ denotes the Chern character form of the connection $\tilde{\nabla}$ on $\tilde{\mathcal{E}}$; $\mathcal{L}(M)$ denotes the Hirzebruch polynomial in Pontrjagin forms which corresponds to $\prod \frac{x_j/2}{\tanh(x_j/2)}$; for $i = 0, 1$ the numbers h_i and η_i denote respectively the dimension of the kernel and the eta-invariant of the Atiyah–Patodi–Singer operator (2)

$$\pm(*\nabla_i - \nabla_i^*) : A^*(M; \mathcal{E}) \rightarrow A^*(M; \mathcal{E}),$$

acting on the full twisted De Rham complex.

Consider (58) modulo 1. Note that the numbers h_i are equal to the sum of all Betti numbers of M with coefficients in the flat vector bundle \mathcal{E} determined by the connection ∇_i ; the alternating sum of those Betti numbers does not depend on the flat structure and is zero by Poincaré duality. This proves that *the numbers h_i are even, $i = 0, 1$* . Thus h_i will disappear from (58) if we consider it modulo 1.

Let η_{∇_i} denote the eta-invariant of the Atiyah–Patodi–Singer operator (2)

$$\pm(*\nabla_i - \nabla_i^*) : A^{ev}(M; \mathcal{E}) \rightarrow A^{ev}(M; \mathcal{E}). \quad (59)$$

Then $2\eta_{\nabla_i} = \eta_i$ and we obtain from (58)

$$\eta_{\nabla_1} - \eta_{\nabla_0} = 2^l \int_{I \times M} \text{ch}(\tilde{\mathcal{E}}) \cdot \mathcal{L}(I \times M) \bmod 1 \quad (60)$$

Let us compute the Chern character $\text{ch}(\tilde{\mathcal{E}})$. The curvature of the connection $\tilde{\nabla}$ is equal to

$$K = dt \wedge \frac{d\nabla_t}{dt}.$$

Here $\frac{d\nabla_t}{dt} \in A^1(I \times M; \text{End}(\mathcal{E}))$. Thus we obtain

$$\text{ch}(\tilde{\mathcal{E}}) = m - (2\pi i)^{-1} \text{tr}(K) = m - (2\pi i)^{-1} dt \wedge \text{tr} \left(\frac{d\nabla_t}{dt} \right)$$

Substituting this into (60) and using the fact that

$$\int_{I \times M} \mathcal{L}(I \times M) = 0$$

(since the Pontrjagin forms of $I \times M$ do not contain dt) we obtain

$$\eta_{\nabla_1} - \eta_{\nabla_0} = \frac{2^{l-1}}{\pi i} \int_{I \times M} dt \wedge \text{tr} \left(\frac{d\nabla_t}{dt} \right) \wedge \mathcal{L}(I \times M) \bmod 1 \quad (61)$$

Integrating the last formula with respect to t we get

$$\eta_{\nabla_1} - \eta_{\nabla_0} = \frac{2^{l-1}}{\pi i} \int_M \text{tr}(\nabla_1 - \nabla_0) \wedge \mathcal{L}(M) \bmod 1 \quad (62)$$

Note that here $\text{tr}(\nabla_1 - \nabla_0) \in A^1(M)$. This shows that only the component of $\mathcal{L}(M)$ having dimension $2l-2$ appears in (62). Thus the LHS of (62) vanishes for l even.

If l is odd, $l = 2r + 1$, then we have

$$2^{2r} \mathcal{L}(M)_{4r} = L(M)_{4r}, \quad (63)$$

where $L(M)$ denotes the Hirzebruch polynomial in the Pontrjagin classes defined by the generating series $x/\tanh(x)$ and the subscript refers to the corresponding homogeneous components. On the other hand,

$$\det(\nabla_1) - \det(\nabla_0) = \text{tr}(\nabla_1 - \nabla_0) \quad (64)$$

(cf. [20], p. 18); thus $\text{tr}(\nabla_1 - \nabla_0)$ is a closed 1-form realizing the class $2\pi i \xi$, cf. Theorem 7.1. This gives

$$\eta_{\nabla_1} - \eta_{\nabla_0} = 2 \langle \xi \cup L(M), [M] \rangle \bmod 1$$

and finishes the proof of Theorems 7.1 and 7.6. \square

8 An example

In this section we consider the simplest possible example of the circle. This example was calculated analytically in [2], II, pages 410–411. We wish to apply our Theorems 1.5 and 7.1 to replicate this calculation.

Let M be S^1 , the circle, and let \mathcal{E} be the trivial line bundle over M . Suppose that ∇_a is an analytic family of flat connections on \mathcal{E} defined for $0 \leq a < 1$ such that the induced family of monodromy representations is given by

$$\rho_a : \pi \rightarrow U(1) = S^1, \quad \rho(\tau) = \exp(2\pi i a)$$

where $\pi = \pi_1(M)$ and $\tau \in \pi$ is a generator. If η_a denotes the eta-invariant of the corresponding Atiyah–Patodi–Singer operator, which in the present case is

$$-i * \nabla_a : C^\infty \rightarrow C^\infty,$$

then the computation in [2] gives:

$$\eta_a = \begin{cases} 0, & \text{if } a = 0 \\ 1 - 2a, & \text{if } 0 < a < 1. \end{cases}$$

Thus, for this family of flat connections, the corresponding monodromy representations are parametrized by the circle and the eta-invariant has a jump at

the trivial representation (which corresponds to the value $a = 0$) but is smooth near all other representations. Note that, at the trivial representation we have

$$\eta_0 = 0, \quad \eta_+ = 1, \quad \eta_- = -1$$

where the notation introduced in Sect. 1 is used.

Let us compute the value of the jump using Theorem 1.5. In order to do this we have to:

- (i) find the corresponding deformation of the monodromy representation;
- (ii) find the cohomology with local coefficients $H^*(\text{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{M}), \mathcal{V}))$;
- (iii) calculate the linking form (6) on the middle-dimensional torsion;
- (iv) find the signatures σ_i , $i \geq 1$, cf. Sect. 2.

The deformation of the monodromy representation near a representation ρ_a is given by the $\mathcal{O}[\pi]$ -module \mathcal{O} with the action of π determined by

$$\tau \cdot f = \exp(2\pi i(t + a)) \cdot f, \quad t \in (-\varepsilon, \varepsilon), \quad f \in \mathcal{O}$$

where $\tau \in \pi$ is a generator.

To find the cohomology with local coefficients (ii) consider the cell decomposition of $M = S^1$ consisting of one zero-dimensional cell e^0 and one one-dimensional cell e^1 . Then the chain complex $C_*(\tilde{M})$ of the universal cover \tilde{M} is

$$0 \rightarrow \mathbb{Z}[\tau, \tau^{-1}]e^1 \xrightarrow{d} \mathbb{Z}[\tau, \tau^{-1}]e^0 \rightarrow 0$$

where $d(pe^1) = (\tau - 1)pe^0$ for $p \in \mathbb{Z}[\tau, \tau^{-1}]$. Thus, the cochain complex

$$\text{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{M}), \mathcal{V})$$

in this case is

$$0 \leftarrow \mathcal{O} \xleftarrow{\delta} \mathcal{O} \leftarrow 0,$$

where $\delta(f) = \exp(2\pi i(t + a)) - 1 \cdot f$ for $f \in \mathcal{O}$. We see that for $a \neq 0$ this complex is acyclic, confirming that *there are no jumps at any nontrivial representation*.

In the case $a = 0$ we obtain

$$H^1 = \mathbb{C} = \mathcal{O}/t\mathcal{O};$$

if α denotes the generator of this group then the linking form is given by

$$\{\alpha, \alpha\} = (\exp(2\pi i t) - 1)^{-1} \in \mathcal{M}/\mathcal{O}.$$

The linking form is skew-Hermitian (in this dimension) and in order to compute its signatures we have first to multiply it by $i = \sqrt{-1}$, cf. 2.15. Then (using the notation of Sect. 2) we obtain

$$[\alpha, \alpha] = \text{Res } i\{\alpha, \alpha\} = (2\pi)^{-1}$$

and thus

$$\sigma_1 = 1, \quad \text{and} \quad \sigma_j = 0 \quad \text{for } j > 1$$

Theorem 1.5 gives now the correct jump formulae near the trivial representation.

Theorem 7.1 together with Remark 7.7 give the correct reduction of the eta-invariant modulo 1.

9 Deformations of flat line bundles and Blanchfield pairings

In this section we present a more general example as an application of Theorem 1.5. We will show that the signatures derived from studying the Blanchfield pairings, which are among the most standard tools of the knot theory, are special cases of the signatures studied in Sect. 3, corresponding to some particular curves of deformations of line bundles. The results of this section relate Theorem 1.5 to the work of W. Neumann [26] who studied the eta-invariants of flat line bundles and the Blanchfield pairings.

We will first recall the construction of the Blanchfield pairing (introduced by Blanchfield [B]) and its local version in the form convenient for the sequel.

9.1 Let M denote a compact oriented manifold of odd dimension $2l - 1$ and let

$$\phi : \pi_1(M) \rightarrow \mathbb{Z} \tag{65}$$

be a fixed epimorphism. Here \mathbb{Z} will be understood as the multiplicatively written infinite cyclic group, whose generator will be denoted by τ . We will also fix a subfield K in the field of complex numbers.

Consider the infinite cyclic covering \tilde{M} of M corresponding to the kernel of the homomorphism ϕ . Let C denote the simplicial chain complex (with coefficients in K) of \tilde{M} . C is a complex of free finitely generated left Λ -modules, where $\Lambda = K[\mathbb{Z}] = K[\tau, \tau^{-1}]$ is the ring of Laurent polynomials of τ . For $0 \leq k \leq 2l - 1$, the cohomology

$$H^k(M; \Lambda) = H^k(\text{Hom}_\Lambda(C, \Lambda)) \tag{66}$$

is a finitely generated Λ -module and (since Λ is a principal ideal domain) we have the following decomposition

$$H^k(M; \Lambda) = \mathcal{T}^k \oplus F^k \tag{67}$$

where \mathcal{T}^k denotes the torsion part of the Λ -module $H^k(M; \Lambda)$ and F^k denotes its free part.

Denote by \mathcal{R} the field of rational functions of τ with coefficients in K . We will consider \mathcal{R} with the involution $\mathcal{R} \rightarrow \mathcal{R}$ which is the composition of the complex conjugation and substitution $\tau \mapsto \tau^{-1}$. We will denote this involution by an overline. The ring Λ is embedded into \mathcal{R} and the involution preserves Λ .

The *Blanchfield form* is the map

$$\{, \} : \mathcal{T}^l \otimes \mathcal{T}^l \rightarrow \mathcal{R}/A \quad (68)$$

which is constructed as follows. The extension

$$0 \rightarrow A \rightarrow \mathcal{R} \rightarrow \mathcal{R}/A \rightarrow 0$$

generates the exact sequence

$$\dots \rightarrow H^{l-1}(M; \mathcal{R}) \rightarrow H^{l-1}(M; \mathcal{R}/A) \xrightarrow{\delta} H^l(M; A) \rightarrow H^l(M; \mathcal{R}) \rightarrow \dots$$

where δ is the Bockstein homomorphism. The image of δ is precisely the torsion submodule \mathcal{T}^l . For $\alpha, \beta \in \mathcal{T}^l$ one defines

$$\{\alpha, \beta\} = \langle \delta^{-1}(\alpha) \cup \beta, [M] \rangle \in \mathcal{R}/A \quad (69)$$

where the cup-product is taken with respect to the coefficient pairing

$$\mathcal{R}/A \times A \rightarrow \mathcal{R}/A, (f, g) \mapsto f \cdot \bar{g}, \quad (70)$$

the dot stands for multiplication of functions.

It is well known that the resulting pairing is *well-defined, non-degenerate* and $(-1)^l$ -*Hermitian* with respect to the induced involution on \mathcal{R}/A .

9.2 The *local version* of the Blanchfield pairing (68) corresponding to a prime ideal \mathfrak{p} in A , is given by the map

$$\{, \}_{\mathfrak{p}} : \mathcal{T}_{\mathfrak{p}}^l \otimes \mathcal{T}_{\mathfrak{p}}^l \rightarrow \mathcal{R}/A_{\mathfrak{p}}. \quad (71)$$

Here $A_{\mathfrak{p}}$ denotes the localization of the ring A with respect to the complement of the ideal \mathfrak{p} and $\mathcal{T}_{\mathfrak{p}}^l$ denotes the \mathfrak{p} -torsion part of the cohomology module $H^l(M, A)$ i.e., the set of elements $z \in H^l(M, A)$ such that for any $q \in \mathfrak{p}$ there exists n such that $q^n z = 0$. Clearly, $\mathcal{T}_{\mathfrak{p}}^l$ is a module over $A_{\mathfrak{p}}$.

The prime ideal \mathfrak{p} is supposed to be invariant under the involution. The local Blanchfield pairing (71) is defined as the restriction of the pairing (68) to the \mathfrak{p} -torsion subgroup $\mathcal{T}_{\mathfrak{p}}^l$ and then reducing the values modulo $A_{\mathfrak{p}}$, i.e.,

$$\{\alpha, \beta\}_{\mathfrak{p}} = \{\alpha, \beta\} \bmod A_{\mathfrak{p}}.$$

The global Blanchfield pairing (68) is a direct orthogonal sum of the local pairings (71).

9.3 In the case when the field K is \mathbb{C} , the field of complex numbers, the prime ideals \mathfrak{p} in A are in one-to-one correspondence with the complex numbers $\xi \in \mathbb{C}$, $\xi \neq 0$; the point $\xi \in \mathbb{C}$ represents the principal ideal $\mathfrak{p} \subset A$ generated by $\tau - \xi$.

In this case we will write \mathcal{T}_{ξ}^l instead of $\mathcal{T}_{\mathfrak{p}}^l$ and A_{ξ} instead of $A_{\mathfrak{p}}$. Thus the local Blanchfield form in case $K = \mathbb{C}$ is denoted by

$$\{, \}_{\xi} : \mathcal{T}_{\xi}^l \otimes \mathcal{T}_{\xi}^l \rightarrow \mathcal{R}/A_{\xi}. \quad (72)$$

9.4 Now we are going to apply Theorem 1.5 in the following situation. Suppose that M is a compact manifold of odd dimension $2l-1$ and $\phi : \pi_1(M) \rightarrow \mathbb{Z}$ is a fixed epimorphism. Consider the following loop of one-dimensional representations:

$$\rho_t : \pi = \pi_1(M) \rightarrow S^1 = U(1), \quad \rho_t = \mu_t \circ \phi \quad (73)$$

where

$$\mu_t : \mathbb{Z} \rightarrow S^1, \quad \mu_t(\tau) = \exp(2\pi it), \quad 0 \leq t \leq 1. \quad (74)$$

Here, as above, \mathbb{Z} denotes the infinite cyclic group written multiplicatively and τ denotes its fixed generator.

Let ω be a closed 1-form on M with real values representing the De Rham cohomology class determined by ϕ . In other word, ω has the following property: for any closed loop α in M

$$\phi([\alpha]) = \tau^l \text{ where } l = \int_{\alpha} \omega.$$

Then for every $0 \leq t \leq 1$ the operator

$$\nabla_t = d - 2\pi it \omega \wedge$$

is a flat connection on the trivial complex line bundle \mathcal{E} over M . Note that the monodromy of the connection ∇_t is ρ_t .

Thus, we have an analytic curve of flat connections. We intend to compute the jumps of the eta-invariant η_t by using Theorem 1.5.

9.5 Fix a point $\xi \in S^1 \subset \mathbb{C}$ on the unit circle. It determines the following representation

$$v_{\xi} : \pi \rightarrow U(1), \quad v_{\xi}(g) = \xi^n \text{ where } g \in \pi \text{ and } \phi(g) = \tau^n \quad (75)$$

Then the 1-parameter family of representations

$$\mu_t = v_{\xi} \exp(2\pi it) : \pi_1(M) \rightarrow U(1), \quad -\varepsilon < t < \varepsilon \quad (76)$$

is a deformation of the representation v_{ξ} .

As explained in (1.3), a deformation of representations determines a module over the group ring $\mathcal{O}[\pi]$ with coefficients in \mathcal{O} . Since all representations under consideration factor through $\phi : \pi \rightarrow \mathbb{Z}$, it is enough for our purposes to consider the following $\mathcal{O}[\mathbb{Z}] = \mathcal{O}[\tau, \tau^{-1}]$ -module V_{ξ} . Here V_{ξ} is equal to \mathcal{O} as an \mathcal{O} -module and the action of τ is given by the formula

$$(\tau \cdot f)(t) = \xi \exp(2\pi it) f(t) \text{ for } f \in \mathcal{O}.$$

Then the cohomology of M with coefficients in V_{ξ} can be computed as the cohomology of the following complex

$$\text{Hom}_{\mathcal{O}[\tau, \tau^{-1}]}(C(\tilde{M}); V_{\xi})$$

where \tilde{M} denotes the space of the infinite cyclic covering corresponding to the kernel of the homomorphism ϕ and $C(\tilde{M})$ denotes the simplicial chain complex of \tilde{M} .

We want to show that *the linking form of this particular deformation μ_t essentially coincides with the local Blanchfield form (72)*. This result is formulated more precisely in 9.8. Note that W. Neumann [26] has already shown that the Blanchfield pairing determines the η -invariant in this case.

Note, that we have the following ring homomorphism

$$\alpha_\xi : \Lambda \rightarrow \mathcal{O}, \quad \text{where } \tau \mapsto \xi \exp(2\pi i t) \in \mathcal{O}, \quad (77)$$

and V_ξ is just \mathcal{O} considered as a Λ -module via α_ξ . We observe next that α_ξ has a unique extension to $\alpha_\xi : \Lambda_\xi \rightarrow \mathcal{O}$ and also $\alpha_\xi : \mathcal{R} \rightarrow \mathcal{M}$. The last map is a field extension.

We have the following commutative diagram

$$\begin{array}{ccccccc} \Lambda & \longrightarrow & \Lambda_\xi & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{R}/\Lambda_\xi \\ & & \downarrow \alpha_\xi & & \downarrow \alpha_\xi & & \downarrow \alpha_\xi \\ & & \mathcal{O} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}/\mathcal{O} \end{array} \quad (78)$$

9.6 Lemma V_ξ is flat as a Λ -module.

Proof. Since Λ is a principal ideal domain it is enough to show that V_ξ has no Λ -torsion. But this can be easily checked. \square

9.7 Lemma Let X be a finitely generated Λ -module and let X_ξ denote its ξ -torsion, i.e., $X_\xi = \ker((\tau - \xi)^n : X \rightarrow X)$, where n is large. Then the map

$$X_\xi \rightarrow X \otimes_\Lambda V_\xi, \quad x \mapsto x \otimes 1 \quad (79)$$

establishes an isomorphism between X_ξ and the \mathcal{O} -torsion submodule of $X \otimes_\Lambda V_\xi$.

Proof. Since the statement is clear when X is free, it is enough to prove it in the case when $X = \Lambda/(\tau - \xi)^n \Lambda$. Then $X \otimes V_\xi$ is isomorphic to

$$\mathcal{O}/(\exp(2\pi i t) - 1)^n \mathcal{O} \simeq \mathcal{O}/t^n \mathcal{O}$$

and the map

$$\Lambda/(\tau - \xi)^n \Lambda \rightarrow \mathcal{O}/t^n \mathcal{O}, \quad \tau \mapsto \xi \exp(2\pi i t)$$

is an isomorphism. \square

9.8 Proposition The linking form of the deformation v_t coincides with the composition

$$\mathcal{T}_\xi^I \times \mathcal{T}_\xi^I \rightarrow \mathcal{R}/\Lambda_\xi \xrightarrow{\alpha_\xi} \mathcal{M}/\mathcal{O}, \quad (80)$$

where \mathcal{T}_ξ^l is the ξ -torsion submodule of the Λ -module $H^l(M, \Lambda)$, the first map is the Blanchfield pairing localized at ξ (cf. (72)), and the map α_ξ is given by $f \mapsto f(\xi \exp(2\pi it))$, $f \in \mathcal{R}$.

Proof. Using Lemma 9.6 we obtain the following isomorphisms

$$H^l(M; V_\xi) \simeq H^l(\text{Hom}(C, V_\xi)) \simeq H^l(\text{Hom}(C, \Lambda) \otimes V_\xi) \simeq H^l(M; \Lambda) \otimes V_\xi$$

where C denotes the simplicial chain complex of \tilde{M} and the tensor product is taken over Λ . Then using Lemma 9.7 we may identify the \mathcal{O} -torsion in $H^l(M; V_\xi)$ with the ξ -torsion in $H^l(M; \Lambda)$, i.e., with \mathcal{T}_ξ^l .

Two Bockstein homomorphisms (one which is used in the definition of the Blanchfield form and the other which is used in the construction of the linking form of the deformation) appear in the following commutative diagram

$$\begin{array}{ccc} H^{l-1}(M; \mathcal{R}/\Lambda_\xi) & \xrightarrow{\delta} & H^l(M; \Lambda_\xi) \\ \alpha_\xi \downarrow & & \downarrow \alpha_\xi \\ H^{l-1}(M; \mathcal{M}/\mathcal{O}) & \xrightarrow{\delta} & H^l(M; V_\xi) \end{array}$$

There is also a commutative diagram

$$\begin{array}{ccc} \mathcal{R}/\Lambda_\xi \times \Lambda_\xi & \longrightarrow & \mathcal{R}/\Lambda_\xi \\ \alpha_\xi \times \alpha_\xi \downarrow & & \downarrow \alpha_\xi \\ \mathcal{M}/\mathcal{O} \times \mathcal{O} & \longrightarrow & \mathcal{M}/\mathcal{O} \end{array}$$

where the horizontal maps are the pairings which are used in the constructions of the cup-products used in the definition of the local Blanchfield form and the linking form of the deformation. All these facts taken together complete the proof. \square

10 On homotopy-invariance of the ρ -invariant

We now apply our results to the ρ -invariant.

10.1 Recall the definition from [2]. If M is a closed smooth oriented odd-dimensional manifold and α a k -dimensional unitary representation of $\pi_1(M)$, then

$$\rho_\alpha(M) = \eta_\alpha(M) - \eta_0(M)$$

where $\eta_\alpha(M)$ denotes the eta-invariant of a flat connection with monodromy α , and 0 denotes the trivial k -dimensional representation. The definition requires a choice of Riemannian metric on M but it is shown in [2], as an easy consequence of the Index Theorem, that $\rho_\alpha(M)$ is independent of this choice. [2] poses the problem of finding a direct “topological” definition of $\rho_\alpha(M)$. In the special case that (M, α) bounds, i.e., $M = \partial V$, where V compact and oriented and α extends to a unitary representation β of $\pi_1(V)$, then

$$\rho_\alpha(M) = k \operatorname{sign}(V) - \operatorname{sign}_\beta(V)$$

where $\operatorname{sign}_\beta(V)$ is the signature of the intersection pairing on the homology of V with twisted coefficients defined by β , and $\operatorname{sign}(V)$ is the usual signature of V – see [2] for more details.

The related question of when $\rho_\alpha(M)$ is an oriented homotopy invariant of M has been of some interest. W. Neumann [26] showed this to be true when α factors through a free abelian group and S. Weinberger [32] extended this to a much larger class of torsion-free groups. However it was already shown in [30] that the ρ -invariant could distinguish (even mod \mathbb{Z}) homotopy equivalent fake lens spaces. S. Weinberger [32] finds such examples for a larger class of fundamental groups with torsion.

Our goal is to apply Theorems 1.5 and 7.6 to give an explicit homotopy invariant definition of $\rho_\alpha(M)$, but with an indeterminacy expressed by a function on the space of unitary representations of $\pi_1(M)$, which is constant on connected components – zero at the component of the trivial representation. If the representation space is connected (e.g. $\pi_1(M)$ is free or free abelian) then the definition is complete and it follows that ρ_α is a homotopy invariant in this case. The homotopy invariance of ρ_α up to a similar, but rational-valued, indeterminacy was proved in [32] when $\pi_1(M)$ satisfies the Novikov conjecture – we will also recapture this result.

10.2 We will treat the ρ -invariant for a fixed closed oriented smooth manifold M of odd dimension as a function

$$\rho(M) : \mathcal{R}_k(\pi) \rightarrow \mathbb{R}, \quad \rho(M) \cdot \alpha = \rho_\alpha(M),$$

where $\mathcal{R}_k(\pi)$ is the real algebraic variety of k -dimensional unitary representations of $\pi = \pi_1(M)$. This function is piecewise continuous in the sense that there is a stratification of $\mathcal{R}_k(\pi)$ by subvarieties

$$\mathcal{R}_k(\pi) = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$$

such that $\rho(M)|_{V_i - V_{i+1}}$ is continuous (see [23]). In fact, the discontinuities are integral jumps so that the reduced function

$$\bar{\rho}(M) : \mathcal{R}_k(\pi) \rightarrow \mathbb{R}/\mathbb{Z}$$

is continuous. We can rephrase Theorem 7.6 to give an explicit description of $\bar{\rho}(M)$ up to an indeterminacy expressed as a function on $\mathcal{R}_k(\pi)$ which is

constant on connected components and 0 at the trivial representation. We will henceforth refer to such functions (with values in \mathbb{R} or \mathbb{R}/\mathbb{Z}) as *quasi-null*.

10.3 Define

$$\tilde{\rho}(M) : \mathcal{R}_k(\pi) \rightarrow \mathbb{R}/\mathbb{Z}$$

by the formula:

$$\tilde{\rho}(M) \cdot \alpha = \begin{cases} -2 \langle (\arg \det \alpha) \cup \tilde{L}^{4r}(M), [M] \rangle & \text{if } \dim M = 4r + 1 \\ 0 & \text{if } \dim M \equiv 3 \pmod{4} . \end{cases}$$

We explain the terms in this formula: $\arg \det \alpha$ is the element of $H^1(M; \mathbb{R}/\mathbb{Z})$ defined by the homomorphism $\pi \rightarrow S^1$ given by $g \mapsto \det \alpha(g)$, $g \in \pi$, and $S^1 \approx \mathbb{R}/\mathbb{Z}$ given by $\exp(2\pi i t) \leftrightarrow t$. $\tilde{L}^{4r}(M)$ is a lift to $H^{4r}(M; \mathbb{Z})$ of $L^{4r}(M)$, the Hirzebruch polynomial in the Pontrjagin classes of M . The existence of the integral lift was proven by Novikov [27]. More explicitly, he observed that for any $\xi \in H^1(M; \mathbb{Z})$

$$\langle L^{4r}(M) \cup \xi, [M] \rangle = \text{sign}(N)$$

where N is any closed oriented submanifold of M dual to ξ (and so $\dim N = 4r$). Note that $\tilde{L}^{4r}(M)$ is not unique, but can be varied by a torsion class in $H^{4r}(M; \mathbb{Z})$. This may change $\tilde{\rho}(M) \cdot \alpha$ by an element of finite order which depends continuously on α . Since $\tilde{\rho}(M) \cdot 0 = 0$ unambiguously, we conclude that $\tilde{\rho}(M)$ is well defined up to a quasi-null function with values in \mathbb{Q}/\mathbb{Z} .

We also point out that Novikov's formula gives an alternative definition of $\tilde{\rho}(M)$. Choose classes $z_1, \dots, z_n \in H_1(M; \mathbb{Z})$ which define a basis of $H_1(M; \mathbb{Z})/\text{torsion}$ and let z'_1, \dots, z'_n be the dual basis of $H^1(M; \mathbb{Z})$. Choose N_1, \dots, N_n closed oriented submanifolds of M Poincare dual to z'_1, \dots, z'_n . Then we have

$$\tilde{\rho}(M) \cdot \alpha = -2 \sum_{i=1}^n \text{sign}(N_i) \arg \det \alpha(z_i) .$$

The choice of z_1, \dots, z_n produces the indeterminacy of $\tilde{\rho}(M)$, from this point of view.

Finally we point out that $\tilde{\rho}(M)$ is an oriented homotopy invariant of M . This follows from the result of Novikov [27] that for any $\xi \in H^1(\pi; \mathbb{R})$, the invariant

$$(M, \phi) \mapsto \langle \phi^*(\xi) \cup L^{4r}(M), [M] \rangle ,$$

where $\phi : \pi_1(M) \rightarrow \pi$, is a homotopy invariant of (M, ϕ) . If we choose an integer m so that $mH^1(M; \mathbb{R}/\mathbb{Z}) \subseteq H^1(M; \mathbb{R})/H^1(M; \mathbb{Z})$, then $m \arg \det \alpha$ can be lifted to $H^1(M; \mathbb{R})$ for any α and Novikov's result implies that $m\tilde{\rho}(M)$ is a homotopy invariant of M . Thus, if M' is homotopy equivalent to M , then $m(\tilde{\rho}(M) - \tilde{\rho}(M')) = 0$ and so $\tilde{\rho}(M) - \tilde{\rho}(M')$ is a continuous function into a discrete subset of \mathbb{R}/\mathbb{Z} . Thus $\tilde{\rho}(M) - \tilde{\rho}(M')$ is a quasi-null function into \mathbb{Q}/\mathbb{Z} .

We now rephrase Theorem 7.6 as:

10.4 Theorem *The difference*

$$\tilde{\rho}(M) - \bar{\rho}(M) : \mathcal{R}_k(\pi) \rightarrow \mathbb{R}/\mathbb{Z}$$

is a quasi-null function.

This is clear since Theorem 7.6 says

$$\bar{\rho}(M) \cdot \alpha - \bar{\rho}(M) \cdot \beta = \tilde{\rho}(M) \cdot \alpha - \tilde{\rho}(M) \cdot \beta$$

if α and β can be connected by a path in $\mathcal{R}_k(\pi)$ and $\bar{\rho}(M) = \tilde{\rho}(M) = 0$ on the trivial representation.

10.5 Corollary *If M and M' are oriented homotopy equivalent, then $\bar{\rho}(M) = \bar{\rho}(M')$ up to a quasi-null function. Moreover, if α factors through a group satisfying the Novikov conjecture, then $\bar{\rho}(M) \cdot \alpha - \bar{\rho}(M') \cdot \alpha \in \mathbb{Q}/\mathbb{Z}$.*

Recall that the Novikov conjecture for a group π asks that for any homomorphism $\theta : \pi_1(M) \rightarrow \pi$, M a closed oriented manifold, the homology class $\theta_*(l(M)) \in H_*(\pi; \mathbb{Q})$ depends only on the oriented homotopy class of M ($l(M)$ is the total homology class dual to the Hirzebruch class $L^*(M)$). Now $\alpha \mapsto \bar{\rho}_{\alpha\theta}(M)$ defines a continuous function $\mathcal{R}_k(\pi) \rightarrow \mathbb{R}/\mathbb{Z}$ (if M is odd-dimensional). Consider the subspace $\mathcal{R}_k^0(\pi) \subseteq \mathcal{R}_k(\pi)$, a union of some connected components, consisting of all α whose associated flat bundle ξ_α over $B\pi$ is trivial on all finite subcomplexes of $B\pi$. Now, following [2], part III, each $\alpha \in \mathcal{R}_k^0(\pi)$ defines an element $v_\alpha \in (K^{-1}(B\pi) \otimes \mathbb{R})/K^{-1}(B\pi)$, using a path of connections from α to a trivial connection (i.e., trivial monodromy), and then we have the formula:

$$\bar{\rho}_{\alpha\theta}(M) \equiv \langle \theta^* \text{ch}(v_\alpha) \cup \mathcal{L}^*(M), [M] \rangle \bmod \mathbb{Z}$$

where $4^r \mathcal{L}^{4r}(M) = L^{4r}(M)$ for each nonnegative integer r . If π satisfies the Novikov conjecture, then the right side is a homotopy invariant of (M, θ) . For general α we have non-trivial ξ_α , but, since it is a flat bundle, all the Chern classes vanish and some multiple $m\xi_\alpha$ is (stably) trivial on any finite subcomplex. Thus $m\bar{\rho}_{\alpha\theta}(M) = \bar{\rho}_{(m\alpha)\theta}(M)$ is a homotopy invariant of (M, θ) and the second sentence of the corollary follows.

10.6. We now return to the unreduced ρ -invariant. We recall Theorem 1.5 with some new notation. Let γ be a curve in $\mathcal{R}_k(\pi)$, which is holomorphic at 0, where $\pi = \pi_1(M)$ as usual. Set

$$\lambda_M(\gamma) = \lim_{t \rightarrow 0^+} \rho(M) \cdot \gamma(t) - \rho(M) \cdot \gamma(0)$$

If γ is induced by a holomorphic curve of flat connections, then Theorem 1.5 asserts that $\lambda_M(\gamma)$ is a sum of signatures associated to a linking pairing defined on the torsion submodule of the homology of M with a twisted coefficient system of $\mathbb{C}[[t]]$ -modules defined by γ . Since this depends only on duality and cup-product in M , Theorem 1.5 gives an explicit homotopy theoretic formula

for $\lambda_M(\gamma)$. In fact a recent preprint of B. Fine, P. Kirk and E. Klassen [12] shows that any holomorphic γ can be lifted to a holomorphic curve of flat connections.

We now state the main result of this section.

10.7 Theorem

(1) *The function $\rho(M): \mathcal{R}_k(\pi) \rightarrow \mathbb{R}$ is uniquely determined, up to a quasi-null function with values in \mathbb{Z} , by λ_M and $\bar{\rho}(M)$. Therefore, $\rho(M)$ is uniquely determined up to a quasi-null function by λ_M and $\bar{\rho}(M)$.*

(2) *If M and M' are oriented and homotopy equivalent, then the difference*

$$\rho(M) - \rho(M'): \mathcal{R}_k(\pi) \rightarrow \mathbb{R}$$

is a quasi-null function. If α factors through a group satisfying the Novikov conjecture, then $\rho(M) - \rho(M')$ admits a rational value on the component of the representation space $\mathcal{R}_k(\pi)$ containing α .

Theorem 10.7 will follow from the following elementary consequence of the “curve selection lemma” (see [6], Proposition 8.1.17):

10.8 Lemma *Let V be a real algebraic set, and $\phi: V \rightarrow \mathbb{R}$ a function satisfying:*

(1) *ϕ is piecewise-continuous, i.e., there is a stratification of V by subvarieties $V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n = \emptyset$ such that $\phi|_{V_i - V_{i+1}}$ is continuous;*

(2) *the reduction $\phi: V \rightarrow \mathbb{R}/\mathbb{Z}$ is locally constant;*

(3) *for any curve γ in V which is holomorphic at 0,*

$$\lim_{t \rightarrow 0^+} \phi(\gamma(t)) = \phi(\gamma(0)).$$

Then ϕ is locally constant.

Proof. We may assume, by (2), that $\bar{\phi} = 0$ after adding a locally constant function. Thus $\phi(V) \subset \mathbb{Z}$ and we only need to show ϕ is continuous. Suppose that $\phi|_{V_{i+1}}$ is continuous and $x \in V_i$ is a discontinuity of $\phi|_{V_i}$. By (1) we conclude that $x \in V_{i+1}$. Let $\{x_n\}$ be a sequence of points in V_i converging to x such that $\phi(x_n) \neq \phi(x)$ for all n . Since $\phi|_{V_{i+1}}$ is continuous, we may assume $\{x_n\} \subseteq V_i - V_{i+1}$. Now $V_i - V_{i+1}$ has a finite number of components, each of which is a semi-algebraic set ([6], Theorem 2.4.5) and so we may assume that $\{x_n\}$ is contained in one of them C . Since $\phi|_C$ is continuous, it is constant, and so $\phi(C) \neq \phi(x)$. Now we apply the curve selection lemma to obtain a holomorphic curve γ in V_i such that $\gamma(0) = x$ and $\gamma(t) \in C$ for all $t \in (0, \varepsilon)$ for some $\varepsilon > 0$. But now (3) contradicts $\phi(C) \neq \phi(x)$.

One can conjecture that the quasi-null functions of Theorems 10.4 and 10.7(1) are \mathbb{Q} -valued.

In the Appendix, written by S. Weinberger, it is proved that the quasi-null function of Corollary 10.5 and of Theorem 10.7(2) are \mathbb{Q} -valued.

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Appendix

Rationality of ρ -invariants

Shmuel Weinberger[★]

Department of Mathematics, The University of Chicago, Chicago, Ill, USA

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Here we observe some rationality statements that follow from [3] and known facts regarding the Novikov conjecture.

Theorem

(i) If $h: M' \rightarrow M$ is a homotopy equivalence, then $\rho(M') - \rho(M)$ take values in \mathbb{Q} .

(ii) If a nontrivial finite group acts freely and homologically trivially on M , then $\rho(M)$ takes values in \mathbb{Q} .

In (ii) we assume that a nontrivial finite group G acts freely on M such that the sequence $1 \rightarrow \pi_1(M) \rightarrow \pi_1(M/G) \rightarrow G \rightarrow 1$ splits and the action of G on $H_*(M; \mathbb{Q}[\pi_1(M)])$ is trivial.

Case (ii) is an analogue of a conjecture of Cheeger and Gromov [2] in light of the existence of “F-structure” choppings of complete manifolds with bounded curvature and finite volume.

Proof. (i) follows from [3], Corollary 10.5 and the following propositions:

Proposition 1 *The Novikov conjecture is correct for $\Gamma \subset GL_n(\overline{\mathbb{Q}})$ (where $\overline{\mathbb{Q}}$ is the algebraic closure of \mathbb{Q}).*

This is proven in [5] and [4].

Proposition 2 *If π is a finitely presented group then every component of $\mathcal{R}_k(\pi)$ contains a point defined over $\overline{\mathbb{Q}}$. The corresponding representation is a homomorphism $\rho: \pi \rightarrow U_k(\overline{\mathbb{Q}})$.*

Proof. Indeed, $\mathcal{R}_k(\pi)$ is a real algebraic variety, defined over \mathbb{Q} . It is the case that the \mathbb{Q} -points of any real variety defined over \mathbb{Q} are dense in the

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\mathbb{R} -points! (Since the variety is triangulable, density implies that there are $\overline{\mathbb{Q}}$ points in each component). According to the Tarski-Seidenberg Theorem (see e.g. [1]) a system of \mathbb{Q} -equations and inequalities has an \mathbb{R} -solution iff it has a \mathbb{Q} -solution. If $p = (p_1, \dots, p_n)$ is an \mathbb{R} -point in coordinates, let r_i, q_i be any rational numbers with $r_i < p_i < q_i$. There is a \mathbb{R} -point satisfying the equations of the variety and the inequalities $r_i < p_i < q_i$ (namely p) so there is such a \mathbb{Q} -point, which is exactly density. \square

To prove (ii) one relies on [6] which shows that the classes in oriented bordism $\Omega(B\pi) \otimes \mathbb{Q}$ represented by manifolds with homologically trivial action are the same as those represented by differences of homotopy equivalent manifolds, and (the obvious fact) that $\rho \bmod \mathbb{Q}$ only depends on the class in $\Omega(B\pi) \otimes \mathbb{Q}$. \square

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