EXACTNESS OF THE NOVIKOV INEQUALITIES

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1\. 1.1. In his investigation of periodic solutions of certain classical problems of mechanics, Novikov [8] constructed and applied an analog of Morse theory, in which, instead of functionals appear multivalued functionals, given through closed 1-forms. The following problem serves as a finite-dimensional topological model of the considerations of Novikov [7, 8]: what is the minimal number of critical points that a Morse map of a finite-dimensional smooth manifold M in the circle S^1, which realizes a given cohomology class $\xi \in H^1(M; \mathbf{Z})$, may have? In the case $\xi = 0$ this question is equivalent to the classical problem of the critical points of smooth functions; as is known, the answer in this case is given, on one hand, by the Morse inequalities, and on the other, by Smale's theorem [9] asserting that on every simply connected manifold there exists an exact function, i.e., a function with the smallest possible number of critical points of each index. In the general case, where ξ is an arbitrary one-dimensional cohomology class, Novikov derived [7, 8] inequalities that estimate from below the number of critical points. The aim of this paper is to prove the analog of Smale's theorem for $\xi \neq 0$.

1.2. Novikov Numbers. Let $\Lambda = \mathbf{Z}[t, t^{-1}]$ be the ring of Laurent polynomials over the integers. Consider the multiplicative subset $S \subset \mathbf{Z}[t] \subset \Lambda$ consisting of polynomials in t with free term 1. The ring of fractions $\Gamma = S^{-1}\Lambda$ is a principal ideal ring (see Corollary 2.4 below), and hence every finitely generated Γ-module A is isomorphic to a direct sum of monogenic modules Γ/α_k, where $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_n \neq \Gamma$ are ideals of the ring Γ, uniquely determined by A. As is known, the number of zero ideals α_k is called the rank of the Γ-module A. The number of nonzero ideals α_k equals the minimal number of generators of the torsion submodule $\mathrm{Tors}_\Gamma A$.

For every connected cellular space X and the one-dimensional class $\xi \in H^1(X; \mathbf{Z})$ there is defined a covering $p_\xi : \tilde{X}_\xi \to X$ induced by any mapping $f: X \to S^1$ realizing ξ from the universal covering $\mathbf{R} \to S^1$, $a \to \exp(2\pi i a)$. Space $\tilde{X}_\xi$ is, by definition, $\{(x, a) \in X \times \mathbf{R};\ f(x) = \exp(2\pi i a)\}$. Let $t: \tilde{X}_\xi \to X$ be the generator of the infinite cyclic group of covering transformations of covering p_ξ, acting by the formula $t(x, a) = (x, a - 1)$, $(x, a) \in \tilde{X}_\xi$. Thanks to the action of t, the homology groups $H_*\tilde{X}_\xi$ are endowed with structures of Λ-modules. Since Λ is Noetherian, the Λ-modules $H_i\tilde{X}_\xi$ are finitely generated, provided that X has finitely many cells for each dimension. Assuming that the last condition is satisfied, we denote by $b_i(X, \xi)$ and $q_i(X, \xi)$ the rank of the Γ-module $\Gamma \otimes_\Lambda H_i\tilde{X}_\xi$, and, respectively, the minimal number of generators of its torsion Γ-module $\mathrm{Tors}_\Gamma(\Gamma \otimes_\Lambda H_i\tilde{X}_\xi)$.

Also, let $c_i(f)$ denote the number of critical points of index i of the Morse map f of a given manifold into S^1.

The next theorem is the main result of the paper.

1.3. THEOREM. Let M^n be a smooth connected closed manifold of dimension $n \geqslant 6$, with infinite cyclic fundamental group, and let $\xi \in H^1(M; \mathbf{Z})$ be a nonzero cohomology class. Then there exists a Morse map $f: M \to S^1$ realizing ξ, such that $c_i(f) = b_i(M, \xi) + q_i(M, \xi) + q_{i-1}(M, \xi)$ for all $i = 0, 1, \ldots, n$.

As Novikov showed [7, 8], the number of critical points of index i of any Morse map M S^1 realizing ξ cannot be smaller than $b_i(M, \xi) + q_i(M, \xi) + q_{i-1}(M, \xi)$. Thus, the map f whose existence is asserted by Theorem 1.3 has the smallest possible number of critical points of each index.

The proof of Theorem 1.3 is given in Sec. 4. The necessary auxiliary assertions are proved in Secs. 2 and 3.

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1.4. The definitions of numbers $b_i(X, \xi)$ and $q_i(X, \xi)$ given in Subsec. 1.2 differ somewhat from those given in Novikov's works [7, 8], and we presently show that actually the numbers defined here coincide with his. Another definition of the numbers $b_i(X, \xi)$, more suitable for their calculation, is given in the Appendix.

We remark that the ring $\hat{\Lambda} = \mathbf{Z}[[t]][t^{-1}]$ is a flat Λ-module. In fact, on the category of Λ-modules the functor $\hat{\Lambda} \otimes_\Lambda$ can be identified with $\hat{\Lambda} \otimes_P$, and the latter factorizes as the composition of the three functors $K \otimes_P$, $\hat{P} \otimes_K$, and $\hat{\Lambda} \otimes_{\hat{P}}$, where $\hat{P} = \mathbf{Z}[t]$, $K = S^{-1}P$, $\hat{P} = \mathbf{Z}[[t]]$. The first and third functors in this factorization are exact, by Theorem 1 of Sec. 2, Chap. II of [3], while the second is exact because K is a Zariski ring in the tK-adic topology and $\hat{P}$ is its tK-adic completion (see Proposition 9 in [3, Chap. III, Sec. 3]).

If X is a cellular space, $\xi \in H^1(X; \mathbf{Z})$ and $\tilde{X}_\xi \to X$ is the corresponding covering, then the $\hat{\Lambda}$-module $H_i(\hat{\Lambda} \otimes_\Lambda C_*(\tilde{X}_\xi))$ is isomorphic to $\hat{\Lambda} \otimes_\Lambda H_i \tilde{X}_\xi$ (by the assertion of the preceding paragraph), and its rank and minimal number of generators of its torsion submodule are identical to the rank and, respectively, the minimal number of generators of the torsion submodule of the Γ-module $\Gamma \otimes_\Lambda H_i \tilde{X}_\xi$. The last assertion follows from the fact that Γ and $\hat{\Lambda}$ are principal ideal rings, Γ is naturally embedded in $\hat{\Lambda}$, and an arbitrary element $\gamma \in \Gamma$ is invertible in $\hat{\Lambda}$ if and only if it is invertible in Γ. But the rank of the $\hat{\Lambda}$-module $H_i(\hat{\Lambda} \otimes_\Lambda C_*(\tilde{X}_\xi))$ and the minimal number of generators of its torsion submodule are exactly the numbers $b_i(X, \xi)$ and, respectively, $q_i(X, \xi)$, according to Novikov's definition [7, 8].

1.5. We remark that Theorem 1.3 yields immediately a theorem of Browder and Levine [1], asserting that under the assumptions of Theorem 1.3 the manifold M is fibered over S^1 if and only if the homologies $H_* \tilde{M}_\xi$ are finitely generated over $\mathbf{Z}$, where $\xi \in H^1(M; \mathbf{Z})$ is a generator. In fact, if the module $H_i \tilde{M}_\xi$ is finitely generated over Z, then $\Gamma \otimes_\Lambda H_i \tilde{M}_\xi = 0$ (this follows, for example, from Lemma 2.1 below), and by Theorem 1.3 class ξ can be realized by a smooth map with no critical points, i.e., by a bundle. The converse is obvious.

2. Algebraic Lemmas

Let $P = Z[t]$, $\Lambda = \mathbf{Z}[t, t^{-1}]$, $K = S^{-1}P$, and $\Gamma = S^{-1}\Lambda$, where $S \subset P$ is the multiplicative subset consisting of the polynomials $p(t) \in P$ with $p(0) = 1$. The present section is devoted to the following question: what is the relationship between the minimal number of generators of the Γ-module $\Gamma \otimes_P A$ and the minimal numbers of generators of the Abelian group A/tA, where A is a finitely generated P-module? To simplify formulations we denote by $\mu_R(X)$ the minimal number of generators of the module X over the ring R. If A is a P- or a K-module, then A_t and ${}^tA$ will designate, respectively, A/tA and the set of all elements $a \in A$, such that $ta = 0$. Finally, r(A) designates the rank of the Abelian group A.

2.1. LEMMA. If A is a finitely generated P-module, then

$$\mu_{\mathbf{Z}}(A_t) = \mu_K(K \otimes_P A).$$

Proof. Let $A' = K \otimes_P A$. Since the P-modules P/tP and K/tK are canonically isomorphic, the module $A_t \approx P/tP \otimes_P A$ can be identified with

$$K/tK \otimes_P A \approx K/tK \otimes_K (K \otimes_P A) \approx A'_t.$$

Therefore, it suffices to show that $\mu_Z(A'_t) = \mu_K(A')$. Obviously, the cosets of the elements $a_1, \ldots, a_m \in A'$, which generate A' over K, generate A'_t over $Z = K/tK$. Conversely, suppose that the cosets of elements $a_1, \ldots, a_m \in A'$ generate A'_t over $\mathbf{Z}$, and consider the K-submodule $N \subset A'$, generated by $a_1, \ldots, a_m$. Then $N + tA' = A'$, and since t is contained in the radical of the ring K, Nakayama's lemma (see [2, Chap. VIII, Sec. 6, Corollary 2 of Proposition 6]) implies that N coincides with A'. The lemma is proved.

2.2. LEMMA. Let A be a P-module of finite type. Then the number $\mu_\Gamma(\Gamma \otimes_P A)$ coincides with the infimum of the numbers $\mu_Z(B_t)$, where B runs through those submodules $B \subset A$ which contain t^rA for some $r \geqslant 0$ (which depends on B).

Proof. If $t^rA \subset B \subset A$, then clearly $\Gamma \otimes_P A \approx \Gamma \otimes_P B$ and hence, by the preceding lemma, $\mu_{\mathbf{Z}}(B_t) = \mu_K(K \otimes_P B) \geqslant \mu_\Gamma(\Gamma \otimes_P B) = \mu_\Gamma(\Gamma \otimes_P A)$. Therefore, it suffices to show that there is a submodule $B \subset A$, which contains t^rA for some integer $r \geqslant 0$ and has the property that $\mu_K(K \otimes_P B) \leqslant \mu_\Gamma(\Gamma \otimes_P A)$.

Recall that the P-submodule $B \subset A$ is called saturated with respect to the multiplicative subset $S \subset P$ if $sa \in B$, with $s \in S$, implies $a \in B$. We denote by $[B]_S$ the smallest saturated submodule containing a given submodule $B \subset A$. From the results of part 4, Sec. 2, Chap. II of [3] it follows that $\mu_K(K \otimes_P A)$ is the infimum of all numbers $\mu_P(B)$, where B runs through all submodules $B \subset A$ with $[B]_S = A$. In a similar manner, the ring Γ is equal to $S_1^{-1}P$, where $S_1 \subset P$ is the multiplicative subset of polynomials of the form $t^r p(t)$ with $p(0) = 1$, and $\mu_\Gamma(\Gamma \otimes_P A) = \min \mu_P(B)$, where B runs through all submodules $B \subset A$ with $[B]_{S_1} = A$.

Let $m = \mu_\Gamma(\Gamma \otimes_P A)$ and let $a_1, \ldots, a_m \in A$ be elements with the property that the P-submodule X that they generate is such that $[X]_{S_1} = A$. Set $B = [X]_S$. Then $\mu_K(K \otimes_P B) \leqslant m$, and since $[X]_{S_1} = A$, we conclude that B contains $t^r A$ for some $r \geqslant 0$. The lemma is proved.

2.3. LEMMA. If A is a P-module of finite type such that ${}^tA = 0$ and the group A_t is free, then

$$\mu_Z(A_t) = \mu_K(K \otimes_P A) = \mu_\Gamma(\Gamma \otimes_P A).$$

Proof. Suppose that the cosets of the elements $a_1, \ldots, a_m \in A$ generate freely A_t. Construct a homomorphism of the free P-module P^m of rank m into A by sending its generators into $a_1, \ldots, a_m$. Since A_t is free and ${}^tA = 0$, this homomorphism is monomorphic. Upon tensoring it by $K \otimes_P$, we obtain, in view of Nakayama's lemma and Theorem 1 [3, Chap. II, Sec. 2], an isomorphism. Thus, the K-module $K \otimes_P A$ is free and has rank m, which proves the lemma.

2.4. COROLLARY. Γ is a principal ideal ring.

Proof. Suppose J is an ideal in Γ. The set $I = J \cap K$ is an ideal in K; moreover, if $a \in I$ and $a \in tK$, then $a \in tI$. Consequently, the embedding $I \to K$ induces a monomorphism $I_t \to K_t \approx \mathbf{Z}$, and hence I_t either reduces to zero or is isomorphic to $\mathbf{Z}$. By Nakayama's lemma and arguments analogous to those used in the proof of Lemma 2.3, either $I = 0$ or $I \approx K$. Consequently, $I = aK$ for some $a \in K$, and hence $J = a\Gamma$ is a principal ideal ring.

2.5. LEMMA. If A is a finitely generated P-module, then

$$\mu_Z(\mathrm{Tors}_Z A_t) = \mu_\Gamma(\mathrm{Tors}_\Gamma(\Gamma \otimes_P A)) + [\mu_K(K \otimes_P A) - \mu_\Gamma(\Gamma \otimes_P A) - r({}^tA)].$$

Proof. Let

$$0 \to F_2 \to F_1 \to F_0 \to A \to 0$$

be a free resolvent of the P-module A. Consider the following complex C of free Abelian groups:

$$0 \to (F_2)_t \to (F_1)_t \to (F_0)_t \to 0.$$

It has the cohomologies: $H_0 = A_t$, $H_1 = {}^tA$, $H_2 = 0$. For a prime number q we consider the complex $C \otimes \mathbf{Z}_q$ of vector spaces over $\mathbf{Z}_q$. By the universal coefficient formulas

$$H_0(C \otimes \mathbf{Z}_q) = A_t \otimes \mathbf{Z}_q, \quad H_1(C \otimes \mathbf{Z}_q) = ({}^tA \otimes \mathbf{Z}_q) \oplus (A_t * \mathbf{Z}_q),$$
$$H_2(C \otimes \mathbf{Z}_q) = {}^tA * \mathbf{Z}_q,$$

and the Euler–Poincaré theorem gives the equality

$$\mu_P(F_0) - \mu_P(F_1) + \mu_P(F_2) = \mu_{\mathbf{Z}_q}(A_t \otimes \mathbf{Z}_q) - [\mu_{\mathbf{Z}_q}({}^tA \otimes \mathbf{Z}_q) + \mu_{\mathbf{Z}_q}(A_t * \mathbf{Z}_q)] + \mu_{\mathbf{Z}_q}({}^tA * \mathbf{Z}_q).$$

Suppose now that q is selected so that $\mu_{\mathbf{Z}_q}(A_t \otimes \mathbf{Z}_q) = \mu_Z(A_t)$. Then $\mu_{\mathbf{Z}_q}(A_t * \mathbf{Z}_q) = \mu_Z(\mathrm{Tors}_Z A_t)$ and $\mu_{\mathbf{Z}_q}({}^tA \otimes \mathbf{Z}_q) - \mu_{\mathbf{Z}_q}({}^tA * \mathbf{Z}_q) = r({}^tA)$, and we obtain the equality

$$\mu_Z(\mathrm{Tors}_Z A_t) = \mu_K(K \otimes_P A) - \mu_P(F_0) + \mu_P(F_1) - \mu_P(F_2) - r({}^tA).$$

Since Γ is a principal ideal ring,

$$\mu_\Gamma(\Gamma \otimes_P A) - \mu_P(F_0) + \mu_P(F_1) - \mu_P(F_2) = \mu_\Gamma(\mathrm{Tors}_\Gamma(\Gamma \otimes_P A)).$$

The last two relations prove the lemma.

3. Geometric Lemma

Suppose M^n is a closed manifold, $\xi \in H^1(M; \mathbf{Z})$ is a cohomology class, and $V^{n-1} \subset M$ is a closed submanifold with oriented normal bundle. We say that V realizes the class ξ if for every class $\alpha \in H_1 V$ the intersection index $V \cdot \alpha$ equals $\langle \xi, \alpha \rangle$. It is well known that every indivisible class ξ on a connected manifold M can be realized by a connected framed submanifold.

If the connected manifold V realizes the indivisible class $\xi \in H^1(M; \mathbf{Z})$, then V can be lifted to the covering $\tilde{M}_\xi \to M$ defined by the class ξ, and the complement $\tilde{M}_\xi - V$ has two components. Let X_V denote the closure of the component with the property that the positive normal to V points towards its exterior. Then $tX_V \subset \operatorname{int} X_V$, where $t: \tilde{M}_\xi \to \tilde{M}_\xi$ is a generator of the group of covering transformations, selected as in Subsec. 1.2. The homologies $H_* X_V$ are P-modules.

The next two lemmas describe variations of the manifold V which lead to simpler P-modules $H_* X_V$.

3.1. LEMMA. Suppose that $\pi_1 M^n = \mathbf{Z}$, $n \geqslant 6$, and let $V^{n-1} \subset M$ be a connected, simply connected submanifold which realizes the generator $\xi \in H^1(M; \mathbf{Z})$. If ${}^t H_j X_V = 0$ for $j < i$, where $2 \leqslant i \leqslant n-3$, there exists a connected, simply connected submanifold $W^{n-1} \subset M$ which realizes ξ, such that the P-module $H_j X_W$ is isomorphic to $H_j X_V$ for all $j < i$, and there is a P-epimorphism $H_i X_V \to H_i X_W$, whose kernel coincides with the set of all classes $v \in H_i X_V$, such that $t^r v = 0$ for some $r \geqslant 0$.

Proof. Consider the manifold $N_V = X_V - \operatorname{int}(tX_V)$; it is a cobordism with boundaries tV and V. According to Smale's theorem [9], on N_V there exists an "exact" Morse function $f: N_V \to [0, 1]$, equal to zero on tV and to one on V, and such that the number of its critical points of index j equals $b_j + q_j + q_{j-1}$, where b_j is the rank of the group $H_j(N_V, tV)$, and q_j is the minimal number of generators of its torsion subgroup. Function f yields a decomposition of manifold N_V into handles that are attached successively to some collar of tV in N_V. Let Y' denote the manifold obtained by attaching all handles of indices $\leqslant i-1$. In the group C_i of i-dimensional chains of the chain complex generated by f, one can choose the following basis

$$z_1^i, z_2^i, \ldots, z_{b_i+q_i}^i, \beta_1^i, \beta_2^i, \ldots, \beta_{q_{i-1}}^i,$$

in which the first $b_i + q_i$ elements form a basis in the group Z_i of i-dimensional cycles, and the boundaries of the last q_{i-1} elements form a basis in the group B_{i-1} of $(i-1)$-dimensional boundaries. By Theorem 7.6 of [5], we can assume that this is precisely the basis realized by the i-dimensional handles of the decomposition of N_V. The handles realizing the first $b_i + q_i$ (last q_{i-1}) elements will be denoted by $H_1^i, H_2^i, \ldots, H_{b_i+q_i}^i$ (respectively, $h_1^i, h_2^i, \ldots, h_{q_{i-1}}^i$).

An analogous choice of basis in the group C_{i+1} yields the handles $H_1^{i+1}, H_2^{i+1}, \ldots, H_{b_{i+1}+q_{i+1}}^{i+1}, h_1^{i+1}, h_2^{i+1}, \ldots, h_{q_i}^{i+1}$. Now notice that, since the handles H_λ^{i+1} realize cycles, the intersection indices of the i-dimensional spheres, lying at their bases, with the boundaries of all comiddle disks of all handles of index i vanish. Therefore, by Whitney's theorem, one can modify the handles H_λ^{i+1} by isotopies so that they will not intersect the handles of index i. (Notice that since V is simply connected, so are X_V and N_V. Consequently, $b_1 = q_1 = 0$ and, by duality, $b_{n-1} = q_{n-1} = q_{n-2} = 0$. Therefore, in the decomposition of N_V there appear only handles of indices in the range from 2 through $n-2$; hence, all noncritical level manifolds of function f are simply connected, and Whitney's lemma applies for $i \geqslant 3$, $n - i - 1 \geqslant 3$. If $i = 2$, Whitney's lemma applies (see [5], Theorem 6.6) if the complements of the boundaries of the comiddle disks of handles of index i in the level manifold are simply connected; but this follows from general position considerations and the assumption that $n \geqslant 6$. The case $i = n-3$ is dealt with by analogous arguments.)

Assuming that the bundles H_λ^{i+1} do not intersect the handles of index i, we consider the submanifold $Y \subset X_V$ described as the union

$$tX_V \cup Y' \cup h_1^i \cup h_2^i \cup \ldots \cup h_{q_{i-1}}^i \cup H_1^{i+1} \cup \ldots \cup H_{b_{i+1}+q_{i+1}}^{i+1}.$$

Then the homomorphism $H_j(Y, tX_V) \to H_j(X_V, tX_V)$ induced by the embedding is an isomorphism for $j \leqslant i-1$ and an epimorphism for $j = i+1$. Moreover, $H_i(Y, tX_V) = 0$. From the exact homology sequence

$$\ldots \to H_j(Y, tX_V) \to H_j(X_V, tX_V) \to H_j(X_V, Y) \to \ldots$$

we get $H_j(X_V, Y) = 0$ for $j \leqslant i-1$ and for $j = i+1$, and $H_i(X_V, tX_V) \approx H_i(X_V, Y)$. This implies immediately that the imbedding $Y \to X_V$ induces an isomorphism $H_j Y \to H_j X_V$ for $j \leqslant i-2$. Let us show that this is true for $j = i-1$ too. To this end, we consider the commutative diagram

$$\begin{array}{ccc} H_i(X_V, Y) & \xrightarrow{\partial} & H_{i-1}Y \\ \uparrow\approx & & \uparrow \\ H_i(X_V, tX_V) & \xrightarrow{\partial_1} & H_{i-1}(tX_V). \end{array}$$

As we have already remarked, the left vertical homomorphism is an isomorphism, whereas ∂_1 is the null homomorphism, because ${}^tH_{i-1}X_V = 0$. Therefore, $\partial = 0$, and since, in addition, $H_{i-1}(X_V, Y) = 0$, the embedding $Y \to X_V$ induces an isomorphism of $(i-1)$-dimensional homologies. Thus, $H_jX_V \approx H_jY$ for all $j \leqslant i-1$.

The homology sequence of the pairs (Y, tX_V) and (X_V, tX_V) yield the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} H_{i+1}(Y, tX_V) & \xrightarrow{\varkappa} & H_i(tX_V) & \xrightarrow{\gamma} & H_iY & \to & 0 \\ \downarrow\alpha & & \downarrow= & & \downarrow & & \\ H_{i+1}(X_V, tX_V) & \xrightarrow{\beta} & H_i(tX_V) & \xrightarrow{\sigma} & H_iX_V. & & \end{array}$$

Since α is an epimorphism,

$$\ker\sigma = \operatorname{im}\beta = \operatorname{im}\beta\circ\alpha = \operatorname{im}\varkappa = \ker\gamma.$$

On the other hand, if we identify $H_i(tX_V)$ with H_iX_V, then $\ker\sigma = {}^tH_iX_V$. Obviously, $Y = X_{V_1}$ for some connected, simply connected submanifold $V_1^{n-1} \subset M$, which realizes ξ. As have already showed, $H_j(X_{V_1}) \approx H_jX_V$ for $j < i$, and there is an epimorphism $\gamma: H_iX_V \to H_iX_{V_1}$, whose kernel equals ${}^tH_iX_V$.

Applying the same procedure to V_1, we obtain a submanifold V_2, then applying it to V_2 we obtain V_3, and so on. It is clear that, for m large enough, submanifold V_m will possess the property that ${}^tH_i(X_{V_m}) = 0$, and we can set $W = V_m$. The lemma is proved.

3.2. LEMMA. Suppose that $\pi_1M^n = \mathbf{Z}$, $n \geqslant 6$, and $V^{n-1} \subset M$ is a connected, simply connected, closed submanifold which realizes the generator $\xi \in H^1(M; \mathbf{Z})$. If $i \leqslant n-3$ and ${}^tH_{i-1}(X_V) = {}^tH_i(X_V) = 0$, then for every P-submodule $B \subset A = H_iX_V$ containing t^rA for some $r \geqslant 0$ one can find a connected, simply connected, closed submanifold $W^{n-1} \subset M$ which realizes ξ and is such that the P-module H_jX_W is isomorphic to H_jX_V for all $j < i$, and the P-module H_iX_W is isomorphic to B.

Proof. It suffices to examine the case where $tA \subset B \subset A$ and the quotient group B/tA is cyclic; the general case follows from this one by induction.

Thus, suppose that $tA \subset B \subset A$ and the quotient group B/tA is cyclic, generated by some element $b \in A/tA$. As in the proof of the preceding lemma, we consider the cobordism $(N_V; tV, V)$ and its minimal handle decomposition. Let Y' denote the union of tX_V, all handles of indices $\leqslant i-1$, and the handles $h_1^i, h_2^i, \ldots, h_{q_{i-1}}^i$ (we follow the notations used in the proof of the preceding lemma). Then $H_j(Y', tX_V) = 0$ for $j \geqslant i$, whereas for $j \leqslant i-1$ we have the isomorphism $H_j(Y', tX_V) \to H_j(X_V, tX_V)$ induced by embedding. From the exact homology sequence

$$\ldots \to H_j(Y', tX_V) \to H_j(X_V, tX_V) \to H_j(X_V, Y') \to \ldots$$

we find that $H_j(X_V, Y') = 0$ for $j \leqslant i-1$, and $A/tA \approx H_i(X_V, tX_V) \approx H_i(X_V, Y')$. Let $Q = X_V - \operatorname{int} Y'$. Then Q and ∂Q are simply connected, $\partial Y' \subset \partial Q$, and the pair $(Q, \partial Y')$ is $(i-1)$-connected. The element $b \in A/tA$ determines a class $b' \in \pi_i(Q, \partial Y') \approx H_i(Q, \partial Y') \approx A/tA$. Since $i \leqslant n-3$, Corollary 1.1 of Hudson's work [4] shows that class b' can be realized by a smoothly embedded disk $D^i \subset Q$ with $D^i \cap \partial Q = S^{i-1} \subset \partial Y'$. Let Y denote the union of Y' and a small tubular neighborhood of D^i in Q.

Calculations, analogous to those carried out above, show that $H_j(X_V, Y) = 0$ for $j \leqslant i-1$, $H_j(Y, tX_V) = 0$ for $j \geqslant i+1$, and there is the exact sequence

$$H_i(Y, tX_V) \to H_i(X_V, tX_V) \to H_i(X_V, Y) \to 0.$$

This shows that the embedding $Y \to X_V$ induces an isomorphism $H_jY \to H_jX_V$ for $j \leqslant i-2$. In the commutative diagram

$$\begin{array}{ccccccc} H_i(X_V, Y) & \xrightarrow{\partial} & H_{i-1}Y & \to & H_{i-1}X_V & \to & 0 \\ \uparrow\alpha & & \uparrow & & & & \\ H_i(X_V, tX_V) & \xrightarrow{\partial_1} & H_{i-1}(tX_V) & & & & \end{array}$$

the row is exact, while $\partial_1 = 0$ because ${}^tH_{i-1}X_V = 0$. Moreover, α is an epimorphism. Consequently, $\partial = 0$ and hence $H_{i-1}Y \approx H_{i-1}X_V$.

Now consider the homomorphism $\varphi: H_iY \to H_iX_V = A$ induced by embedding. An analysis of the commutative diagram

$$\begin{array}{ccccccc} 0 \to H_i(tX_V) & \xrightarrow{\psi} & H_iY & \to & H_i(Y, tX_V) & \to 0 \\ \downarrow = & & \downarrow \varphi & & \downarrow & \\ 0 \to H_i(tX_V) & \to & H_iX_V & \to & H_i(X_V, tX_V) & \to 0 \\ & & \downarrow & & \downarrow & \\ & & H_i(X_V, Y) & \xrightarrow{=} & H_i(X_V, Y) & \end{array}$$

with exact rows and columns shows that $\operatorname{im} \varphi = B \subset A$, $\ker \varphi = {}^tH_iY$.

We notice that $Y = X_{V_1}$ for a connected simply connected submanifold $V_1^{n-1} \subset M^n$ which realizes the class ξ. The manifold V_1 has the property that there is a homomorphism φ: $H_iX_{V_1} \to A$ with $\operatorname{im} \varphi = B$ and $\ker \varphi = {}^tH_iX_{V_1}$. Applying to V_1 the reduction procedure described in Lemma 3.1 we obtain the desired manifold W.

The lemma is proved.

4. Proof of Theorem 1.3

4.1. If $\xi = \lambda\eta$, where $\lambda \in \mathbf{Z}, \lambda > 0$, and $\eta \in H^1(M; \mathbf{Z})$ is an indivisible cohomology class, then the covering space $\tilde{M}_\xi$ has λ components $\tilde{M}_1, \ldots, \tilde{M}_\lambda$, all homeomorphic to $\tilde{M}_\eta$. Moreover, the generator of the group of covering transformations acts as follows: it maps the first component onto the second, the second onto the third, and so on, and the last components $\tilde{M}_\lambda$ onto $\tilde{M}_1$. The composite mapping $\tilde{M}_1 \to \tilde{M}_2 \to \ldots \to \tilde{M}_\lambda \to \tilde{M}_1$ is identical with the generator of the group of covering transformations, $t: \tilde{M}_\eta \to \tilde{M}_\eta$. This readily implies that for all i

$$b_i(M, \xi) = b_i(M, \eta), \quad q_i(M, \xi) = q_i(M, \eta).$$

If there is a Morse map $g: M \to S^1$ that realizes η and has exactly $b_i(M, \eta) + q_i(M, \eta) + q_{i-1} \times (M, \eta)$ critical points of index i for all i, then one can take as $f: M \to S^1$ the composition $M \xrightarrow{g} S^1 \xrightarrow{\lambda} S^1$, where λ is the canonical λ-sheeted covering. Then f is a Morse map which realizes ξ and satisfies the condition of Theorem 1.3.

4.2. Thus, it suffices to prove Theorem 1.3 under the supplementary assumption that ξ is a generator. As it is shown in [1], class ξ can be realized by a connected, simply connected, framed submanifold $V^{n-1} \subset M^n$. Successive application of Lemmas 2.2, 3.1, and 3.2 yields a connected, simply connected, framed submanifold $W^{n-1} \subset M$ which realizes ξ and, in addition, enjoys the following properties for $i = 2, 3, \ldots, n-3$:

$$ {}^tH_i(X_W) = 0, \quad \mu_\Gamma(\Gamma \otimes_P H_i(X_W)) = \mu_\mathbf{Z}(H_i(X_W)_t). \tag{1}$$

For $i = 0, 1, n-1, n$, these relations are clearly satisfied. Let us show that they are satisfied for $i = n-2$ too. In fact, in the exact homology sequence

$$H_{n-1}(X_W, tX_W) \to H_{n-2}(X_W) \xrightarrow{t} H_{n-2}(X_W) \to H_{n-2}(X_W, tX_W)$$

the group $H_{n-1}(X_W, tX_W)$ vanishes, because it is isomorphic to $H_{n-1}(N_W, tW) \approx H^1(N_W, W) = 0$, where $N_W = X_W - \operatorname{int}(tX_W)$. Similarly, the group $H_{n-2}(X_W, tX_W)$ is isomorphic to $H^2(N_W, W)$ and hence torsion-free. Now it follows from Lemma 2.3 that relations (1) are true also for $i = n-2$.

Notice that for every closed submanifold $V^{n-1} \subset M^n$ which realizes ξ, the Λ-module $H_i\tilde{M}_\xi$ is isomorphic to $\Lambda \otimes_P H_iX_V$. Consequently, the Γ-modules $\Gamma \otimes_\Lambda H_i\tilde{M}_\xi$ and $\Gamma \otimes_P H_iX_V$ are isomorphic.

For the manifold W constructed above we have

$$\mu_\mathbf{Z}(H_i(N_W, tW)) = \mu_\Gamma(\Gamma \otimes_\Lambda H_i\tilde{M}_\xi), \tag{2}$$

because

$$H_i(N_W, tW) \approx H_i(X_W, tX_W) \approx (H_iX_W)_t.$$

Similarly, using Lemma 2.5, we get

$$\mu_\mathbf{Z}(\operatorname{Tors}_\mathbf{Z} H_i(N_W, tW)) = \mu_\Gamma(\operatorname{Tors}_\Gamma(\Gamma \otimes_\Lambda H_i\tilde{M}_\xi)). \tag{3}$$

If $\bar{b}_i$ and $\bar{q}_i$ designate, respectively, the rank of the group $H_i(N_W, tW)$ and the minimal number of its torsion submodule, then it follows from (2) and (3) that for i = 0, 1,...,n

$$\bar{b}_i = b_i(M, \xi), \quad \bar{q}_i = q_i(M, \xi). \tag{4}$$

According to Smale's theorem [9] there is a Morse function $\varphi\colon N_W \to [0, 1]$, which equals zero on tW, one on W, and has exactly $\bar{b}_i + \bar{q}_i + \bar{q}_{i-1}$ critical points of index i for all i = 0, 1,...,n. Since $\bigcup_{j=-\infty}^{\infty} t^j N_W = \tilde{M}_\xi$, we can define a function $\tilde{\varphi}\colon \tilde{M}_\xi \to \mathbf{R}$, by the rule $\tilde{\varphi}(x) = \varphi(t^j x) - j$, where $x \in t^{-j}N_W$. Function $\tilde{\varphi}$ defined in this manner is continuous but not necessarily smooth. However, it is clear that upon modifying φ, if necessary, in a neighborhood of $W \cup tW$, we obtain a function ψ with exactly the same number of critical points for all indexes, and such that the analogous function $\tilde{\psi}\colon \tilde{M}_\xi \to R$ is smooth. We can now build the desired map $f\colon M \to S^1$, setting

$$f(x) = \exp(2\pi i \tilde{\psi}(p_\xi^{-1}(x))) \in S^1,$$

where $p_\xi\colon \tilde{M}_\xi \to M$ is the covering projection. By construction,

$$c_i(f) = c_i(\varphi) = \bar{b}_i + \bar{q}_i + \bar{q}_{i-1} = b_i(M, \xi) + q_i(M, \xi) + q_{i-1}(M, \xi).$$

The theorem is proved.

Appendix

A. Here we make several remarks on the Novikov numbers.

1. If X is a finite cellular space and $\xi \in H^1(X; \mathbf{Z})$ is a nonzero cohomology class, then $b_i(X, \xi)$ equals the i-dimensional Betti number of X and $q_i(X, \xi)$ equals the minimal number of generators of the torsion subgroup of H_iX.

In fact, the covering space $\tilde{X}_\xi$ is homeomorphic to $\mathbf{Z} \times X$, and hence $H_i\tilde{X}_\xi \approx \Lambda \otimes_{\mathbf{Z}} H_iX$. Consequently, $\Gamma \otimes_\Lambda H_i\tilde{X}_\xi \approx \Gamma \otimes_{\mathbf{Z}} H_iX$. To every decomposition of the group H_iX into a direct sum of cyclic components there corresponds a decomposition of the module $\Gamma \otimes_{\mathbf{Z}} H_iX$ into a direct sum of monogenic modules. This proves our assertion.

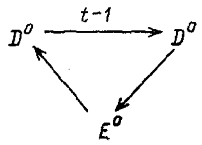
2. Suppose now that $\xi \in H^1(X; \mathbf{Z})$ is an arbitrary cohomology class. Milnor showed [6] that there is an exact sequence

$$\ldots \to H_{i+1}X \to H_i\tilde{X}_\xi \xrightarrow{t-1} H_i\tilde{X}_\xi \to H_iX \to \ldots$$

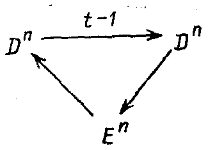
Letting

$$D^0 = \bigoplus_{i=0}^{\infty} H_i\tilde{X}_\xi, \quad E^0 = \bigotimes_{i=0}^{\infty} H_iX,$$

we obtain the following exact Massey pair



which, as is known, generates a sequence of derived pairs. For sufficiently large n the differential of the n-th pair



vanishes, and hence $E^n = E^\infty$. We presently show that the Novikov number $b_i(X, \xi)$ equals the rank of the group E_i^∞.

In fact, since the submodule $\mathfrak{B} \subset D^0$, consisting of the elements which are annihilated by some power $(t - 1)^m$, is finitely generated, one can find n such that $a \in \mathfrak{B}$ implies $(t - 1)^n a = 0$. Next, since $D^n = (t - 1)^n D^0$, multiplication by $t - 1$ is a monomorphism $D^n \to D^n$, and hence $E^n = E^\infty$. Consider the exact sequence

$$0 \to D_i^n \xrightarrow{t-1} D_i^n \to E_i^n \to 0.$$

Let $T \subset \Lambda$ be the multiplicative subset consisting of the Laurent polynomials $p(t) \in \Lambda$ with $p(1) \neq 0$. From Nakayama's lemma it follows that the rank of the group E_i^n equals the minimal number of generators of the $T^{-1}\Lambda$-module $T^{-1}D_i^n$ (see the analogous reasoning in Sec. 2). Now $T^{-1}\Lambda$ is a discrete valuation ring, and the $T^{-1}\Lambda$-module $T^{-1}D_i^n$ is torsion-free. Consequently, $T^{-1}D_i^n$ is free and its rank equals the dimension of the vector space $F \otimes_\Lambda D_i^n$ over the field of fractions F of the ring Λ. But $F \otimes_\Lambda D_i^n \approx F \otimes_\Lambda (t-1)^n D_i^0 \approx F \otimes_\Lambda H_i \tilde{X}_\xi$, and the dimension of the last vector space is $b_i(X, \xi)$ according to the definition in Subsec. 1.2.

The assertion we have just proved has as a corollary the relation

$$\sum_{i=0}^{\infty} (-1)^i b_i(X, \xi) = \chi(X).$$

3. The Novikov numbers $q_i(X, \xi)$ with $\xi \neq 0$ do not admit such a simple interpretation in terms of the homology structure of X.

To see this, we construct for arbitrary n, i, and N, with $1 < i < n/2$, an n-dimensional smooth closed manifold M^n with $\pi_1 M = \mathbf{Z}$ and $M_* M \approx H_*(S^1 \times S^{n-1})$, such that $q_j(M, \xi) = 0$ for $j < i$, and $q_i(M, \xi) = N$, where $\xi \in H^1(M; \mathbf{Z})$ is a generator. Let the Λ-module A be the direct sum of N copies of $\Lambda/(t-2)\Lambda$. By a theorem of Kervaire [10] there is an embedding $S^{n-2} \subset S^n$, with the following properties: $\pi_j E \approx \pi_j S^1$ for $j < i$, and the $\Lambda = \mathbf{Z}[\pi_1 E]$-module $\pi_i E$ is isomorphic to A, where $E = S^n - S^{n-2}$. Let the closed manifold M^n be obtained from S^n by a spherical modification along S^{n-2}. Then $\pi_1 M = \mathbf{Z}$, and for the generator $\xi \in H^1(M; \mathbf{Z})$ we have $H_j \tilde{M}_\xi = 0$ for $j < i$, and $H_i \tilde{M}_\xi = A$. This implies that $q_j(M, \xi) = 0$ for $j < i$ and $q_i(M, \xi) = N$.

B. Here we give a new proof of Novikov's inequalities [7, 8], which rests on the arguments used in the proof of Theorem 1.3.

THEOREM [7, 8]. Let $f: M^n \to S^1$ be a Morse map of the smooth connected closed manifold M into the circle $S^1 \subset \mathbf{C}$ (oriented counterclockwise). Then

$$c_i(f) \geqslant b_i(M, \xi) + q_i(M, \xi) + q_{i-1}(M, \xi),$$

where $\xi \in H^1(M; \mathbf{Z})$ is the cohomology class realized by f.

Proof. Consider the space $\tilde{M}_\xi = \{(x, a) \in M \times \mathbf{R}; f(x) = \exp(2\pi i a)\}$ and the commutative diagram

$$\begin{array}{ccc} \tilde{M}_\xi & \xrightarrow{\tilde{f}} & \mathbf{R} \\ {\scriptstyle p_\xi}\downarrow & & \downarrow{\scriptstyle \tau} \\ M & \xrightarrow{f} & S^1 \end{array}$$

where $p_\xi(x, a) = x$, $\tilde{f}(x, a) = a$ for $(x, a) \in \tilde{M}_\xi$, and $(a) = \exp(2\pi i a)$ for $a \in \mathbf{R}$. Then p_ξ is a covering projection, and hence $\tilde{M}_\xi$ is a smooth manifold, and $\tilde{f}$ is a Morse map. Let $\alpha \in \mathbf{R}$ be a regular value of $\tilde{f}$. Set $X = \tilde{f}^{-1}((-\infty, \alpha])$, $V = \tilde{f}^{-1}(\alpha)$, $N = \tilde{f}^{-1}([\alpha - 1, \alpha])$. The generator $t: \tilde{M}_\xi \to \tilde{M}_\xi$ of the group of covering transformations (see Subsec. 1.2) maps X into itself. Moreover, $\partial X = V$, $\partial N = V \cup tV$, and $N = X - \operatorname{int}(tX)$. The restriction of $\tilde{f}$ to N is a Morse map $N \to [\alpha - 1, \alpha]$, and it is clear that for all i:

$$c_i(f) = c_i(\tilde{f}|_N). \tag{5}$$

On the other hand, Morse's theory yields the inequality

$$c_i(\tilde{f}|_N) \geqslant \mu(H_i(N, tV)) + \mu(\operatorname{Tors} H_{i-1}(N, tV)), \tag{6}$$

where, for brevity, we write μ instead of $\mu_{\mathbf{Z}}$ (see the beginning of Sec. 2). By the excision axiom, $H_j(N, tV) \approx H_j(X, tX)$ and the last group appears in the short exact sequence

$$0 \to (H_j(X))_t \to H_j(X, tX) \to {}^t(H_{j-1}(X)) \to 0$$

[which is obtained from the homology sequence of the pair (X, tX)]. Consequently,

$$\mu(H_j(X, tX)) \geqslant \mu((H_j(X))_t) + r({}^t(H_{j-1}(X))), \tag{7}$$

$$\mu(\operatorname{Tors} H_j(X, tX)) \geqslant \mu(\operatorname{Tors}((H_j(X))_t)). \tag{8}$$

Using Lemmas 2.1 and 2.5 and the fact that the Γ-modules $\Gamma \otimes_P H_j(X)$ and $\Gamma \otimes_\Lambda H_j(\tilde{M}_\xi)$ are isomorphic (see Subsec. 4.2), we get from (7) and (8) that

$$\mu(H_i(X, tX)) + \mu(\mathrm{Tors}\, H_{i-1}(X, tX)) \geqslant \mu((H_i(X))_t) + r({}^t(H_{i-1}(X))) + \mu(\mathrm{Tors}\,((H_{i-1}(X))_t)) =$$
$$= \mu_K(K \otimes_P H_i(X)) + \mu_\Gamma(\mathrm{Tors}_\Gamma(\Gamma \otimes_P H_{i-1}(X))) + \mu_K(K \otimes_P H_{i-1}(X)) - \mu_\Gamma(\Gamma \otimes_P H_{i-1}(X)) =$$
$$= \mu_\Gamma(\Gamma \otimes_\Lambda H_i(\widetilde{M}_\xi)) + \mu_\Gamma(\mathrm{Tors}_\Gamma(\Gamma \otimes_\Lambda H_{i-1}(\widetilde{M}_\xi))) + [\mu_K(K \otimes_P H_i(X)) - \mu_\Gamma(\Gamma \otimes_P H_i(X))] +$$
$$+ [\mu_K(K \otimes_P H_{i-1}(X)) - \mu_\Gamma(\Gamma \otimes_P H_{i-1}(X))].$$

By the definitions of Subsec. 1.2, the first and second terms of the last sum equal $b_i(M, \xi) + q_i(M, \xi)$ and $q_{i-1}(M, \xi)$, respectively, whereas the terms in brackets are both nonnegative (for obvious reasons). In conjunction with relations (5) and (6) obtained above, this yields the asserted inequality

$$c_i(f) \geqslant b_i(M, \xi) + q_i(M, \xi) + q_{i-1}(M, \xi).$$

Remark Added in Proofs. One can show that under the assumptions of the theorem given in addendum B one has the inequalities

$$\sum_{i=0}^{m} (-1)^{m-i} c_i(f) \geqslant q_m(M, \xi) + \sum_{i=0}^{m} (-1)^{m-i} b_i(M, \xi), \text{ where } m = 0, 1, \ldots, n,$$

which have as consequences the inequalities of Novikov.

The results of this paper and some of their applications were announced in [11].

LITERATURE CITED

1. W. Browder and J. Levine, "Fibering manifolds over S^1," Commun. Math. Helv., 40, 153-160 (1966).
2. N. Bourbaki, Algebra: Modules, Rings, Forms [Russian translation], Nauka, Moscow (1966).
3. N. Bourbaki, Commutative Algebra, Addison-Wesley (1972).
4. J. F. P. Hudson, "Embeddings of bounded manifolds," Proc. Cambr. Phil. Soc., 72, 11-20 (1972).
5. J. Milnor, Lectures on the h-Cobordism Theorem, Princeton Math. Notes, Princeton Univ. Press, Princeton, N.J. (1965).
6. J. W. Milnor, "Infinite cyclic coverings," in: Conf. on the Topology of Manifolds, J. G. Hocking, Ed., Prindle, Weber and Smidt (1968), pp. 115-133.
7. S. P. Novikov, "Multivalued functions and functionals. An analog of Morse theory," Dokl. Akad. Nauk SSSR, 260, No. 1, 31-35 (1981).
8. S. P. Novikov, "The Hamiltonian formalism and a multivalued analog of Morse theory," Usp. Mat. Nauk, 37, No. 5, 3-49 (1982).
9. S. Smale, "On the structure of manifolds," Am. J. Math., 84, 387-399 (1962).
10. M. A. Kervaire, "Les noeuds des dimensions superieures," Bull. Soc. Math. France, 93, 225-271 (1965).
11. M. Sh. Farber, "Mappings into the circle with the minimal number of critical points, and higher-dimensional knots," Dokl. Akad. Nauk SSSR, 279, No. 1, 43-46 (1984).