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DOUBLY NULL CONCORDANT KNOTS HAVE HYPERBOLIC STABLE ISOMETRY STRUCTURES

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ABSTRACT

We shall show that the stable isometry structure determined by a suitable Seifert hypersurface of a doubly null concordant knot is hyperbolic and we prove a converse for stable knots. This suggests a "universal" source for the known homological invariants of DNC-equivalence. As an application of our main result we shall show that if the homology of the universal cover of the complement of a stable n-knot is torsion, involving only primes $> (n+10)/6$, and is 0 in the middle dimensions then the knot is doubly null concordant.

Keywords: antisimple, doubly null concordant, fibred knot, hyperbolic, Moore space, stable.

1. Introduction

An n-knot is a locally flat PL embedding K of S^n in S^{n+2}. (We shall assume that all spheres have fixed orientations). The set of ambient isotopy classes of n-knots is a commutative semigroup with respect to connected sum; we obtain a group on factoring out concordance. The construction which shows that the reflected inverse $-K$ of an n-knot K represents the inverse of the class of K in the knot concordance group actually shows more: $K\sharp -K$ is not merely a slice knot, but is a slice of the 1-twist spin of K, which is a trivial $(n+1)$-knot, and so is doubly null concordant [26]. Thus we may factor out the semigroup of n-knots by a finer equivalence relation (DNC-equivalence) and obtain a group which measures the obstructions to an n-knot being doubly null concordant. Most of the hitherto known homological conditions for a knot to be doubly null concordant assert that some pairing is hyperbolic (cf. [16, 18, 20, 25, 26]). We shall show that there is a universal homomorphism from the group of DNC-equivalence classes of n-knots into a "Witt group" of n-isometries from which the homological conditions should derive.

We begin in Sec. 2 by establishing our notation and reviewing the key definitions. In Theorem 1 we show that the n-isometry determined by a Seifert hypersurface of a null concordant n-knot is metabolic, and in Theorem 2 this result is used to show that if the knot is doubly null concordant and the Seifert hypersurface is the transverse intersection of an $(n+2)$-disc with the equator in S^{n+3} then the corresponding n-isometry is hyperbolic. When the knot is fibred the fibre is such a Seifert hypersurface (Theorem 3). Theorems 4 and 5 lead up to Theorem 6, in which we show that if the n-isometry associated to some highly connected Seifert hypersurface of a stable knot is hyperbolic then the knot is doubly null concordant. In Sec. 5 we consider the algebraic invariants, and in Sec. 6 we consider simple knots. As an application of our main results we show in Theorem 8 that any fibred stable n-knot whose fibre is $\mathbb{Z}/p\mathbb{Z}$-acyclic for all primes $p \leq (n+10)/6$ and has trivial integral homology in the middle dimensions must be doubly null concordant.

2. Knots and Stable Isometry Structures

The exterior $X = X(K)$ of the knot is the complement of an open regular neighbourhood, and is a compact oriented $(n+2)$-manifold with boundary homeomorphic to $S^n \times S^1$. By Alexander duality X has the homology of S^1, and the inclusion of ∂X into X induces an isomorphism on homology below degree n. The knot is *r-simple* if this inclusion is r-connected, and is *stable* if it is r-simple for some r with $3r > n \geq 5$. The infinite cyclic cover $\tilde{X}$ is the pullback of the universal cover of S^1 by the map generating $[X; S^1] = H^1(X;\mathbb{Z}) \cong \mathbb{Z}$. The knot is *fibred* if there is a fibre bundle projection $p : X \to S^1$; in this case $\tilde{X}$ is homeomorphic to $V \times R$ where $V = p^{-1}(1)$ is the fibre. A *Seifert hypersurface* for K is an oriented locally flat codimension 1 submanifold of S^{n+2} with (oriented) boundary $\partial V = K$. By a standard transversality argument, these always exist. An r-simple n-knot K has an r-connected Seifert hypersurface (cf. [10]). If $3r > n$ we shall say that such a Seifert hypersurface is *stably connected.*

An n-knot K in S^{n+2} is *doubly null concordant* (or DNC) if it is the transverse intersection of a trivial $(n+1)$-knot in S^{n+3}, bounding an $(n+2)$-disc Δ say, with the equator S^{n+2} in S^{n+3}. We shall say that the Seifert hypersurface $\Delta \cap S^{n+2}$ is a *disc section.* Two n-knots K and K' are DNC-*equivalent* if there are doubly null concordant knots L and L' such that $K \sharp L$ and $K' \sharp L'$ are ambient isotopic. This is an equivalence relation. Since $K \sharp - K$ is always doubly null concordant [26], the set of DNC-equivalence classes of n-knots is an abelian group (with respect to knot sum); we shall let DNC_n denote this group. The knot concordance group C_n is a natural factor group of DNC_n. (It is not known to the authors whether a knot which is DNC-equivalent to the unknot must be doubly null concordant. This has been proven for simple odd-dimensional knots by Bayer-Flückiger and Stoltzfus [2]).

Let $\mathbf{S} = Stab_o$ be the stable category of finite complexes and their formal desuspensions considered in [4, 7] and let S denote suspension. If P and P^* are finite subcomplexes of S^N such that P^* is a deformation retract of $S^N \backslash P$ then v_P shall denote the Spanier-Whitehead duality pairing from $P \wedge P^*$ to S^{N-1}. Let $D(P) = S^{1-N}(P^*)$ be the normalised dual of P. These pairings determine natural isomorphisms $\psi_P : \{X, D(P)\} \to \{P \wedge X, S^0\}$ by $\psi_P(k) = S^{1-N}(v_P(1_P \wedge S^{N-1}k))$ for any stable map $k : X \to D(P)$. If $f : Q \to P$ is a stable map between finite stable complexes then $\psi_P(k)(f \wedge 1_X) = \psi_Q(D(f)k)$, and so these pairings are also natural with respect to the variable P.

A *stable isometry structure* of dimension n (or *n-isometry*, for short) is by definition a triple (V, u, z), where V is an object of $\mathbf{S}$ and $u : V \wedge V \to S^{n+1}$ and $z : V \to V$ are $\mathbf{S}$-maps such that:
(a) u is a duality;
(b) $u' = (-1)^{n+1}u$ (where $u' = u\gamma$ with γ interchanging the factors of $V \wedge V$); and
(c) $u(z \wedge 1) + u(1 \wedge z) = u$.
(There is an equivalent formulation based on the triple $(V, z, \phi = S^{-1}h)$ [15], where h is defined in the next paragraph, and which we shall use in Secs. 5–7 below as it has some advantages on the algebraic side).

A *Seifert hypersurface* V for an n-knot K determines an n-isometry (V, u, z) as follows. If i_+ and i_- are small translations of V into its complement $Y = S^{n+2} \backslash V$ along positive and negative normal vector fields then the map $h = Si_+ - Si_-$ is a homotopy equivalence from SV to SY, and the *carving map* z defined by $hSz = Si_+$ is a stable homotopy class of self maps of V. The pairing u is then determined by the equation $u = v_V(1 \wedge S^{-1}h)$. The relation of *R-equivalence* between n-isometries associated to Seifert hypersurfaces of isotopic knots is described in [5]. The R-equivalence class of the n-isometry is a complete invariant for stable knots, and any such n-isometry with the connectivity of V greater than its homotopy length (= dimension − connectivity) may be realized by such a knot [5-9]. A stable *fibred* knot K is determined up to isotopy by the *isomorphism* class of the triple (V, u, z) where V is the fibre. In this case z is also a stable homotopy equivalence.

A *kernel* for an n-isometry (V, u, z) is a triple (U, z_U, f) where U is a based finite complex and $z_U : U \to U$ and $f : U \to V$ are $\mathbf{S}$-maps such that (i) $fz_U = zf$ and (ii) for every finite complex X the sequence

$$\{X, U\} \to \{X, V\} \to \{U \wedge X, S^{n+1}\}$$

is exact, where the map on the left is given by composition with f and the map on the right sends $\beta : X \to V$ to $u(f \wedge \beta)$. The n-isometry is *metabolic* if it has a kernel. We will say that a kernel (U, z_U, f) is *strong* if in the above exact sequence one may add zeroes preserving the exactness:

$$0 \to \{X, U\} \to \{X, V\} \to \{U \wedge X, S^{n+1}\} \to 0.$$

The n-isometry (V, u, z) is *hyperbolic* if there are two kernels (U_1, z_1, f_1) and (U_2, z_2, f_2) such that $f_1 \vee f_2 : U_1 \vee U_2 \to V$ is a stable homotopy equivalence. In this case we shall say that the n-isometry has the *hyperbolic decomposition*

$$V = U_1 \vee U_2.$$

In this case both kernels (U_1, z_1, f_1) and (U_2, z_2, f_2) are necessarily strong. With respect to the hyperbolic decomposition $V = U_1 \vee U_2$ the **S**-maps u and z have the following form

$$u = \begin{pmatrix} 0 & v \\ (-1)^{(n+1)}v' & 0 \end{pmatrix}, \; z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix},$$

where $v : U_1 \wedge U_2 \to S^{n+1}$ is a duality. (Here $v' = v\gamma$ where γ interchanges the factors of $U_1 \wedge U_2$).

The terminology has been chosen to underline the analogies between n-isometries and hermitean pairings.

3. Double Null Concordance Implies Hyperbolicity

The following lemma extends an argument used in the proof of Theorem 1.4 of [5].

Lemma 1. *Let M be a compact $(n+2)$-dimensional submanifold of D^{n+3} such that $\partial M = V \cup W$, where $V = M \cap \partial D^{n+3}$, $V \cap W = \partial V = \partial W$ and W is homeomorphic to D^{n+1}. Let Z denote the complement of M in D^{n+3}. Let $i_+, i_- : M \to Z$ be the maps given by small normal shifts to either side. Then $i_+ - i_- : M \to Z$ is an* **S**-*equivalence.*

Proof. Consider the isomorphism $\psi : H_k(Z) \to H_k(M, W)$, for $k > 0$, given by the following composition of isomorphisms:

$$H_k(Z) \xrightarrow{\partial^{-1}} H_{k+1}(D^{n+3}, Z) \to H_{k+1}(N, \partial_+ N) \to H_k(M, W)$$

where N is a tubular neighbourhood of M in D^{n+3} and $\partial_+ N = \partial N \cap int D^{n+3}$. Since $(N, \partial_+ N) \cong (M, W) \times (I, \partial I)$ we obtain the last isomorphism.

Observe that the composite $\psi(i_+ - i_-) : H_k(M) \to H_k(Z) \to H_k(M, W)$ coincides with the homomorphism induced by the inclusion of M into (M, W). Since W is a disc, this induced homomorphism is an isomorphism and hence so is $i_+ - i_- : H_k(M) \to H_k(Z)$, for all $k > 0$. □

Theorem 1. *Let K be a null concordant n-knot. Then the n-isometry corresponding to any connected Seifert hypersurface V for K is metabolic.*

Proof. Let $W \subset D^{n+3}$ be a slicing $(n+1)$-disk for K. (Thus $W \cap \partial D^{n+3} = \partial W = K$). By a relative transversality argument we may assume that there is a "Seifert hypersurface" for W which extends V, i.e. that there is a compact oriented $(n+2)$-dimensional submanifold M of D^{n+3} such that $\partial M = V \cup W$, $V = M \cap S^{n+2} = \partial M \cap S^{n+2}$ and $K = \partial V = \partial W$. Let $U = S^{-1}(M/V)$ and

$f : U \to V$ be the desuspension of the canonical map from M/V to SV. We shall show that there is a self-map z_U of U such that (U, z_U, f) is a kernel for the n-isometry corresponding to V.

Let N be a regular neighbourhood of M in D^{n+3} such that $N_o = N \cap S^{n+2}$ is a regular neighbourhood of V in S^{n+2}. Let $Z = D^{n+3} \backslash intN$ and $Y = S^{n+2} \backslash intN_o$. Let $J : V \to M$ and $j : Y \to Z$ be the inclusions and let I_+, $I_- : M \to Z$ and i_+, $i_- : V \to Y$ denote small translations along positive and negative normal vectorfields. We may assume that $I_+J = ji_+$ and $I_-J = ji_-$. By Lemma 1 and the argument of Sec. 1.1 of [6] the homomorphisms $I_{+*} - I_{-*}$ and $i_{+*} - i_{-*}$ induce isomorphisms on homology. In particular the stable map $I_+ - I_-$ is a stable homotopy equivalence and so we may define a self-map $z_M : M \to M$ by the equation $(I_+ - I_-)z_M = I_+$. Let z be the carving map for V. Then we also have $(i_+ - i_-)z = i_+$ and so $(I_+ - I_-)z_M J = I_+ J = ji_+ = j(i_+ - i_-)z = (I_+ - I_-)Jz$. Hence $z_M J = Jz$ and so z_M induces a self-map of $SU = M/V$. We define z_U as the desuspension of the latter map. The maps z, z_M and Sz_U determine a map from the cofibration sequence of the map J to itself. In particular, $(Sf)(Sz_U) = (Sz)(Sf)$ and so $fz_U = zf$. It remains for us to prove that for any finite complex X the sequence $\{X, U\} \to \{X, V\} \to \{U \wedge X, S^{n+1}\}$ is exact. We shall deduce this from the following lemma.

Lemma 2. *There is a duality pairing* $w : U \wedge M \to S^{n+1}$ *such that* $w(1_U \wedge J) = v_V(f \wedge (i_+ - i_-)) = u(f \wedge 1_V)$.

Let $S^{n+3} = D_\alpha \cup D_\beta$ be the decomposition of S^{n+3} into two hemispheres and let p and p' be the centres of D_α and D_β respectively. Identify D^{n+3} with D_α. Thus we assume that W and M lie in D_α. The set $M \cup C'(V)$ is a subcomplex of S^{n+3}, where $C'(V)$ is the cone over V with vertex p'. This subcomplex is homotopy equivalent to M/V and its complement in S^{n+3} has Z as a strong deformation retract. Therefore there is a Spanier-Whitehead duality pairing $v_{M/V} : (M \cup C'(V)) \wedge Z \to S^{n+2}$. Similarly, Y is a strong deformation retract of the complement of $SV = C(V) \cup C'(V)$ (here $C(V)$ denotes the cone over V with vertex p) in S^{n+3} and the Spanier-Whitehead duality pairing $v_{SV} : SV \wedge Y \to S^{n+2}$ is the suspension of v_V. It follows easily from the definitions that $v_{M/V}(1_{M \cup C'(V)} \wedge j) = v_{SV}(Sf \wedge 1_Y)$. Identify $M \cup C'(V)$ with SU and set $w_1 = S^{-1}v_{M/V}$. Then w_1 is also a duality pairing, and $v_V(f \wedge 1_Y) = w_1(1_U \wedge j)$. Hence $w = w_1(1_U \wedge (I_+ - I_-)) : U \wedge M \to S^{n+1}$ is also a duality pairing. Since $w(1_U \wedge J) = w_1(1_U \wedge (I_+ - I_-)J) = w_1(1_U \wedge j(i_+ - i_-)) = v_V(f \wedge (i_+ - i_-)) = u(f \wedge 1_V)$ the lemma is proven.

We may now finish the proof of Theorem 1. The cofibration sequence $U \to V \to M$ gives an exact sequence $\{X, U\} \to \{X, V\} \to \{X, M\}$. The duality pairing w defines an isomorphism from $\{X, M\}$ to $\{U \wedge X, S^{n+1}\}$. Let $F : \{X, V\} \to \{U \wedge X, S^{n+1}\}$ be the composition of the latter two homomorphisms. If δ is a stable map in $\{X, V\}$ then by the lemma $F(\delta) = w(1_U \wedge J\delta) = u(f \wedge 1_V \delta)$, and hence the theorem is proven. □

Theorem 2. *Let K be a doubly null concordant n-knot. Then the n-isometry corresponding to any Seifert hypersurface for K which is a disc section is hyperbolic.*

Proof. Let Δ be an $(n+2)$-disc in S^{n+3} which meets the equatorial zone $S^{n+2}\times I$ transversely in $V\times I$, where V is a disc section for K. Let D_α and D_β be the two hemispheres of S^{n+3} and let $M_\alpha=\Delta\cap D_\alpha$ and $M_\beta=\Delta\cap D_\beta$. Then $M_\alpha\cap M_\beta=V$, while $M_\alpha\cup M_\beta=\Delta$. By virtue of Theorem 1 we have two kernels for (V,u,z), namely $(U_\alpha,z_\alpha,f_\alpha)$ and $(U_\beta,z_\beta,f_\beta)$, where $U_\alpha=S^{-1}(M_\alpha/V)$ and $U_\beta=S^{-1}(M_\beta/V)$, and f_α and f_β are desuspensions of the canonical maps from M_α/V and M_β/V to SV. Now Δ/V is homeomorphic to $(M_\alpha/V)\vee(M_\beta/V)$ and there is a homotopy equivalence from Δ/V to SV. Therefore the map $(f_\alpha,f_\beta):U_\alpha\vee U_\beta\to V$ is a stable homotopy equivalence, which satisfies all necessary conditions. □

The above two theorems require the choice of a Seifert hypersurface, and in order to obtain a more intrinsic result we must work with R-equivalence classes. On the level of stable homotopy this may be achieved by localization. (See Sec. 5 below). It would be of interest to have formulations and proofs that worked directly with the $\mathbb{Z}$-equivariant SW-duality of the infinite cyclic cover of the knot exterior (cf. Sec. 3 of [22]). In the fibred case the situation is simpler as there is then a Seifert hypersurface, unique up to h-cobordism, which is homotopy equivalent to $\tilde{X}(K)$ (namely, the fibre) and the appropriate equivalence relation is just isomorphism.

Theorem 3. *Let K be a 1-simple fibred n-knot which is doubly null concordant. Then the n-isometry determined by the fibre and its associated maps u and z is hyperbolic.*

Proof. We may assume that $n\geq 3$, for otherwise the fibre is contractible. Let Σ be an unknotted $(n+1)$-sphere in S^{n+3} such that K is the transverse intersection of Σ with S^{n+2} in S^{n+3}, and let $\Xi_\alpha=D_\alpha\backslash\Sigma$ and $\Xi_\beta=D_\beta\backslash\Sigma$. Then the universal cover of $S^{n+3}\backslash\Sigma$ restricts to infinite cyclic covers $\tilde{X}$, $\tilde{\Xi}_\alpha$ and $\tilde{\Xi}_\beta$ of X, Ξ_α and Ξ_β respectively. By Van Kampen's theorem and the Mayer-Vietoris sequence, we see that $\tilde{\Xi}_\alpha$ and $\tilde{\Xi}_\beta$ are simply-connected and have finitely generated homology, and so Ξ_α and Ξ_β fibre over S^1 by maps extending the fibration of X [3]. Thus there is a fibration ϕ of $S^{n+3}\backslash\Sigma$ over S^1 extending the fibration of $S^{n+2}\backslash K$. The fibre Δ of ϕ is a homotopy disc with boundary S^{n+1}, and so is an $(n+2)$-disc. Clearly the disc section $\Delta\cap S^{n+2}$ is a fibre for K. The theorem now follows from Theorem 2. □

4. Hyperbolic Implies Doubly Null Concordant for Stable Knots

In this section we shall provide a converse for Theorem 2 above, for stable knots. Our argument uses the following relative version of Wall's embedding theorem [27].

Theorem 4. *Suppose that M is an m-manifold, P a codimension-0 submanifold of ∂M with $\pi_1(\partial P)=\pi_1(P)$, (K,L) is a finite CW-pair of relative dimension k and*

$$f:(K,L)\to(M,P)$$

is a map. Assume that
(1) $f|_L : L \to P$ *is a simple homotopy equivalence;*
(2) $f : K \to M$ *is* $(2k - m + 1)$*-connected; and*
(3) $k \leq m - 3$.
Then there is a compact codimension-0 submanifold N *of* M *and a simple homotopy equivalence* $g : K \to N$ *such that*
(a) $N \cap \partial M = P$;
(b) $\pi_1(\partial N \backslash P) \cong \pi_1(N)$;
(c) $g|_L = f|_L$; *and*
(d) f *and* g *are homotopic relative to* L *as maps from* K *to* M.

Proof. The proof is similar to that given by Wall on pages 84-86 of [27]. The only difference is that we start from a collar of P in M and then add handles inside M to the part of the boundary of N lying in the interior of M by embedding subsequently cells of $K \backslash L$. □

Theorem 5. *Let* $V^{n+1} \subset S^{n+2}$ *be a stably connected Seifert hypersurface of a knot* $K = \partial V \subset S^{n+2}$, *and suppose that the corresponding n-isometry* (V, u, z) *has a hyperbolic decomposition* $V = X_1 \vee X_2$. *Realize* S^{n+2} *as the equatorial sphere of* S^{n+3} *which separates* S^{n+3} *into two discs* D_1^{n+3}, $D_2^{n+3} \subset S^{n+3}$. *Then for* $i = 1, 2$ *there is a submanifold* $M_i^{n+2} \subset D_i^{n+3}$ *with*
(a) $M_i \cap \partial D_i^{n+3} = M_i \cap S^{n+2} = V$;
(b) ∂M_i *has a corner along* $\partial V \subset \partial M_i$; *and*
(c) M_i *and* ∂M_i *are simply connected and there is a stable homotopy equivalence* $s_i : X_i \to M_i$ *such that the composite*

$$(s_i) \circ (pr_i) : V = X_1 \vee X_2 \to X_i \to M_i$$

is homotopic to the inclusion.

Proof. Recall that to say $V = X_1 \vee X_2$ is a hyperbolic decomposition for the n-isometry (V, u, z) means that u and z are given by matrices of the form

$$u = \begin{pmatrix} 0 & v \\ (-1)^{(n+1)}v' & 0 \end{pmatrix}, \; z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix},$$

where $v : X_1 \wedge X_2 \to S^{n+1}$, $z_1 : X_1 \to X_1$ and $z_2 : X_2 \to X_2$.

It is sufficient to treat the case $i = 1$, as the case $i = 2$ is similar.

We may assume that $V = X_1 \vee X_2$ is r-connected, where $3r \geq n + 1 \geq 6$, and hence that X_1 and X_2 are r-connected true complexes, by Corollary 1.6 of [8]. Consider an embedding of $V \times I$ in S^{n+2} extending $V = V \times \{1\}$; denote its image by N_o. We are going to apply Theorem 4 with $L = V = X_1 \vee X_2$, $K = V \cup CX_2$ (homotopy equivalent to X_1), $P = N_o$, $M = D_1^{n+3}$, $f : (K, L) \to (D_1^{n+3}, N_o)$ a map with $f|_L : L \to P$ the inclusion of V as $V \times \{1\}$, and $f : K \to D_1^{n+3}$ being an extension of it.

By Theorem 4 we obtain a submanifold $N^{n+3} \subset D_1^{n+3}$ and a homotopy equivalence $g : X_1 \to N$ such that $g \circ pr_1 : V = X_1 \vee X_2 \to X_1 \to N$ is homotopic to the inclusion $V \subset N_o \subset N$. Let $\partial_+ N = \partial N \backslash int N_o$ be the side surface of N. Let also $Y = D_1^{n+3} \backslash N$ and let $i : \partial_+ N \to Y$, $j : \partial_+ N \to N$ be the inclusions.

Claim. There exists a map $f : X_1 \to \partial_+ N$ such that $f \circ pr_1 : V = X_1 \vee X_2 \to X_1 \to \partial_+ N$ represents the inclusion of V into $\partial_+ N$ in **S**.

As Lemma 1 implies that the map $i \vee (-j) : \partial_+ N \to N \vee Y$ is a stable homotopy equivalence, to prove the claim it is enough to show the existence of two maps $f_1 : X_1 \to N$ and $f_2 : X \to Y$ such that the composites $f_1 \circ pr_1 : V \to N$ and $f_2 \circ pr_1 : V \to Y$ are each homotopic to the corresponding inclusion.

It is clear that we may take $f_1 = g$, where g is as defined above. It remains for us to construct f_2. Since $V = X_1 \vee X_2$, the existence of f_2 is equivalent to the statement that the map $k : X_2 \to Y$ given by the composite of the inclusions $X_2 \to V \to Y$ is null-homotopic. To do this consider the canonical Spanier-Whitehead duality $w : (N/V) \wedge Y \to S^{n+2}$. Let $Y_o = Y \cap S^{n+2}$ and $w_o : V \wedge Y_o \to S^{n+1}$ be the canonical duality pairing, and let $\delta : N/V \to SV$ be the map from the Puppe sequence. Then $(Sw_o)(\delta \wedge 1_{Y_o}) = w(1_{N/V} \wedge incl)$ as **S**-maps (here $incl : Y_0 \to Y$).

By definition, the composite $w_o(1_V \wedge incl) : V \wedge V \to V \wedge Y_o \to S^{n+1}$ is the homotopy Seifert pairing θ of V. Thus we obtain $w(1_{N/V} \wedge k) = (S\theta)(\delta \wedge i_2)$, where $i_2 : X_2 \to V = X_1 \vee X_2$ is the inclusion. Since $\theta = u(1_V \wedge z)$ the last formula might be written as $w(1_{N/V} \wedge k) = (Su)(\delta \wedge z i_2)$. The next stage is to understand δ. From the equation $g \circ pr_1 = incl : V \to N$ we obtain a commutative diagram of maps between the Puppe sequences for pr_1 and for $incl$:

$$\begin{array}{ccccccc} V & \longrightarrow & X_1 & \longrightarrow & SX_2 & \longrightarrow & SX_1 \vee SX_2 \\ \big\downarrow & & {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow & & {\scriptstyle =}\big\downarrow \\ V & \longrightarrow & N & \longrightarrow & N/V & \xrightarrow{\delta} & SV \end{array}$$

As g is an equivalence it follows that h is an equivalence. Hence $\delta \circ h = Si_2$ and we obtain $w(h \wedge k) = (Su)(\delta h \wedge z i_2) = (Su)(Si_2 \wedge z i_2)$, which is 0 (since $z i_2 = i_2 z_2$ and $u(i_2 \wedge i_2) = 0$, by assumption). Since h is an equivalence and w is a duality we conclude that $k \sim 0$ and this proves the claim. We may now finish the proof of Theorem 4. We have constructed a map $f : X_1 \to \partial_+ N$ such that $f \circ pr_1 : V = X_1 \vee X_2 \to X_1 \to \partial_+ N$ is homotopic to the inclusion $V \to \partial_+ N$. Let us apply Theorem 4 again. We obtain a submanifold $M_1^{n+2} \subset \partial_+ N$ with $M_1 \cap S^{n+2} = V$ and a homotopy equivalence $\alpha : X_1 \to M_1$ such that $\alpha \circ pr_1$ is homotopic to the inclusion $V \to M_1$. This completes the construction of $M_1 \subset D_1^{n+3}$. The other manifold $M_2 \subset D_2^{n+3}$ may be constructed similarly. □

Theorem 6. *Let K be a stable n-knot with a stably connected Seifert hypersurface V such that the n-isometry (V, u, z) is hyperbolic. Then V is a disc section, and so K is doubly null concordant.*

Proof. Let $V = X_1 \vee X_2$ be a hyperbolic decomposition of (V, u, z). Let $M_i \subset D_i^{n+3}$ be corresponding submanifolds, given by Theorem 5, and let $\Delta = M_1 \cup M_2$. Using condition (c) of Theorem 5 and the Mayer-Vietoris sequence we find that Δ is contractible and so it is a disc of dimension $(n+2)$. Since $V = \Delta \cap S^{n+3}$, the theorem is proven. □

To what extent is this theorem independent of the choice of Seifert hypersurface? In particular, if the n-isometry determined by some Seifert hypersurface of a stable knot is hyperbolic, is there a stably connected Seifert hypersurface with hyperbolic n-isometry?

Combining Theorems 3 and 6 with the arguments used in the proof of Theorem 3 we obtain:

Corollary. *A stable fibred knot K is doubly null concordant if and only if the stable isometry structure determined by the fibre is hyperbolic.* □

5. Algebraic Invariants

In this section we shall use localization as an algebraic substitute for dealing directly with equivariant SW-duality. The category **S** is additive, i.e., the *Hom*-sets in **S** are $\mathbb{Z}$-modules and composition of morphisms is bilinear. Similarly, the category **A** of endomorphisms of objects of **S** is P-linear, where $P = \mathbb{Z}[z]$, for the *Hom*-sets in **A** are P-modules and composition of morphisms is P-bilinear. We may localize these modules to obtain a category of fractions **LA** which has the same objects as **A** and in which $Hom_{\mathbf{LA}}(M, N) = L \otimes_P Hom_{\mathbf{A}}(M, N)$, where $L = \mathbb{Z}[z, z^{-1}, (1-z)^{-1}]$.

Spanier-Whitehead duality in **S** gives rise to a family of duality functors D_k on **A**. The stable map $\phi = S^{-1}h \in \{V, Y\}$ associated to an n-knot with Seifert hypersurface V may be viewed as giving a $(-1)^{n+1}$-hermitian isomorphism of the object (V, z) of **A** with its dual $D_n(V, z)$ [15]. The duality functor D_n on **A** induces a duality on **LA**, giving another additive category with involution.

An additive functor F from the stable category **S** to (a subcategory of) the category $\mathbf{Mod}(\mathbb{Z})$ of $\mathbb{Z}$-modules gives rise to a P-linear functor PF from **A** to $\mathbf{Mod}(P)$ in an obvious way (i.e., by letting $PF(V, z) = F(V)$ with $z \in P$ acting via the endomorphism $F(z)$). This in turn extends to an L-linear functor $L \otimes F$ from **LA** to $\mathbf{Mod}(L)$ (by $L \otimes F(V, z) = L \otimes_P PF(V, z)$). As in [9], if F is defined on a larger category including all stable finite dimensional complexes, is covariant and commutes with direct limits then we have $L \otimes F(V, z) = F(\tilde{X}(K))$. (This is not true for all additive functors.)

For instance, we may take the q^{th} singular homology functor, given by $H_q(W) = H_q(W; \mathbb{Z})$, for the space W. Let $t_q(W)$ be the $\mathbb{Z}$-torsion subgroup of $H_q(W)$ and $B_q(W) = H_q(W)/t_q(W)$, the q^{th} Betti group of W. The subgroup $t_q(V)$ is preserved by any self-map z of V, and so the exact sequence $0 \to t_q(V) \to H_q(V) \to B_q(V) \to 0$ is then a sequence of P-modules. Since localization is exact we then have an exact sequence of L-modules $0 \to L \otimes t_q(V,z) \to L \otimes H_q(V,z) \to L \otimes B_q(V,z) \to 0$. Now $L \otimes t_q(V,z)$ is finite while $L \otimes B_q(V,z)$ is torsion free. Therefore $L \otimes t_q(V,z)$ is the maximal finite sub-L-module of $L \otimes H_q(V,z)$. (In the knot theoretic case this is just $t_q(\tilde{X}(K))$, and hence $L \otimes B_q(V,z) = B_q(\tilde{X}(K))$.)

Let $WQ_0^\epsilon(\mathbf{LA}; D_n)$ be the Witt group of the additive category $\mathbf{LA}$ with respect to the duality D_n and sign $\epsilon = (-1)^{n+1}$, as defined in [21].

Theorem 7. *The function which sends the ambient isotopy class of an n-knot K to the isomorphism class of the localization of the n-isometry determined by (any) one of its Seifert hypersurfaces is well-defined, and carries knot sums to orthogonal sums of n-isometries. The Witt class of the n-isometry determines a homomorphism from* DNC_n *to* $WQ_0^\epsilon(\mathbf{LA}; D_n)$, *where* $\epsilon = (-1)^{n+1}$.

Proof. It is clear from the definition of contiguity that the localizations of contiguous n-isometries are isomorphic. Since R-equivalence is the equivalence relation generated by contiguity it follows that the isomorphism class of the localization of an n-isometry of an n-knot is a well defined invariant of the knot. If V_1 and V_2 are Seifert hypersurfaces for K_1 and K_2, respectively, then the boundary connected sum $V_1 \natural V_2$ is a Seifert hypersurface for $K_1 \sharp K_2$, and hence the invariant is additive. Theorem 2 implies that the image of such an n-isometry in the Witt group $WQ_0^\epsilon(\mathbf{LA}; D_n)$ is an invariant of DNC-equivalence. □

Stoltzfus has defined Witt groups $CH^\epsilon(\mathbb{Z})$ and $CH^\epsilon(\mathbb{Q}/\mathbb{Z})$ of ϵ-hermitean forms, corresponding to the Seifert forms and Farber-Levine torsion linking pairings of knot theory [25]. Let $\mathbf{M}$ be the category of torsion L-modules of projective dimension 1 which are finitely generated as modules over $\Lambda = \mathbb{Z}[t, t^{-1}]$, where $t = 1 - z^{-1}$, and define a duality $*$ on $\mathbf{M}$ by $M^* = Hom_L(M, \mathbb{Q}(z)/L)$, with the L-module structure determined by $zf(m) = f((1-z)m)$ for all f in M^* and m in M. Then the correspondance between Seifert forms and Blanchfield pairings determines an isomorphism of $CH^\epsilon(\mathbb{Z})$ with the Witt group $WQ_0^\epsilon(\mathbf{M}, *)$, cf. [16]. Similarly, the group $CH^\epsilon(\mathbb{Q}/\mathbb{Z})$ may also be interpreted as the Witt group of the category of finite L-modules, with respect to the duality given by $N^* = Hom_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$, where $zg(n) = g((1-z)n)$ for all g in N^* and n in N.

Corollary. (i) [16, 25, 26] *The Blanchfield pairing determines a homomorphism from* DNC_{2q+1} *to* $CH^\epsilon(\mathbb{Z}) = WQ_0^\epsilon(\mathbf{M}, *)$, *where* $\epsilon = (-1)^{q+1}$, *and which is onto if* $q > 2$.

(ii) [25] *The Farber-Levine pairing determines a homomorphism from* DNC_{2q} *to* $CH^\epsilon(\mathbb{Q}/\mathbb{Z})$, *where* $\epsilon = (-1)^{q+1}$, *and which is onto if* $q > 2$.

Proof. The Betti functor B_q is an additive functor from $\mathbf{S}$ to $\mathbf{Mod}(\mathbb{Z})_{fgf}$, the category of finitely generated free abelian groups, which is compatible with the dualities D_{2q+1} of $\mathbf{S}$ and $Hom_{\mathbb{Z}}(-, \mathbb{Z})$ of $\mathbf{Mod}(\mathbb{Z})_{fgf}$, by Alexander duality in S^{2q+3}. On applying B_q to a $(2q+1)$-isometry (V, z, ϕ) of a $(2q+1)$-knot we obtain the corresponding Seifert form for the knot. This gives rise to the Blanchfield pairing of the knot as in Sec. 14 of [19]. Theorem 2 implies that the Witt class of the Blanchfield pairing of a doubly null concordant $(2q-1)$-knot must be trivial. If $q > 2$ every $(-1)^{q+1}$-hermitean symmetric Blanchfield pairing may be realised by some simple $(2q+1)$-knot and so the homomorphism is onto.

The torsion functor t_q is an additive functor from $\mathbf{S}$ to $\mathbf{Mod}(\mathbb{Z})_{finite}$, the category of finite abelian groups which is compatible with the dualities D_{2q} of $\mathbf{A}$ and $Hom_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ of $\mathbf{Mod}(\mathbb{Z})_{finite}$, by Alexander duality in S^{2q+2}, and which gives rise to the Farber-Levine pairing on $t_q(\tilde{X}(K))$. Theorem 2 implies that the Witt class of the Farber-Levine pairing of a doubly null concordant $2q$-knot must be trivial. If $q > 2$ every $(-1)^{q+1}$-symmetric Farber-Levine pairing may be realized by some simple $2q$-knot and so the homomorphism is onto. □

6. Simple Knots

If a duality preserving additive functor from a subcategory of $\mathbf{LA}$ to an additive category with duality is fully faithful then it determines a complete invariant for stable knots with Seifert manifolds in the given subcategory, and Theorem 2 implies that the invariant of a doubly null concordant knot must be hyperbolic.

For instance, simple $(2q-1)$-knots have Seifert manifolds which are homotopy equivalent to wedges of q-spheres; the q^{th} Betti functor B_q is a complete invariant for such spaces and maps between them. A simple odd-dimensional knot is doubly null concordant if and only if its Blanchfield pairing is hyperbolic [16, 26]. Moreover a Blanchfield pairing is hyperbolic if and only if its Witt class in $CH^{\epsilon}(\mathbb{Z}) = WQ_0^{\epsilon}(\mathbf{M}, *)$ is trivial [2].

Simple $2q$-knots have Seifert manifolds which are $(q-1)$-connected and have homological dimension $q+1$, i.e., which are so-called A_q^1 spaces. Such spaces are determined by their homology [1: p. 287], but to detect maps between them other invariants are needed. These are described and used in [7] to construct a complete invariant for simple $2q$-knots; the invariant is reformulated as an "L-quintuple" in [8]. (For both odd and even-dimensional simple knots there are realization theorems characterizing the possible values of the invariants.)

In the remainder of this section we shall assume that K is a simple $2q$-knot. Let $A = H_q(\tilde{X}(K))$, $B = H_{q+1}(\tilde{X}(K))$, $C = H_{q+1}(\tilde{X}(K); \mathbb{Q}/\mathbb{Z})$, $\Pi_{q+1} = \pi_{q+1}^{st}(\tilde{X}(K)) \otimes \mathbb{Z}/2\mathbb{Z}$ and $\Pi = \pi_{q+2}^{st}(\tilde{X}(K))$. (Note that if $q \geq 4$ the latter two groups agree with the corresponding unstable homotopy groups, i.e., $\Pi_{q+1} = \pi_{q+1}(X(K)) \otimes \mathbb{Z}/2\mathbb{Z}$ if $q \geq 3$ and $\Pi = \pi_{q+2}(X(K))$ if $q \geq 4$.) The maps $\eta : A/2A \to \Pi_{q+1}$ and $\alpha : A/2A \to \Pi$ given by composition with the stable Hopf map and with its square, respectively, are monomorphisms. These groups have natural L-module structures, determined by the action of the covering group $Aut(\tilde{X}(K)/X(K)) \cong \mathbb{Z}$, and the homomorphisms η and α are L-linear. Let ℓ be the Farber-Levine pairing on the $\mathbb{Z}$-torsion submodule tA and let $\psi : \Pi \times \Pi \to \pi_3^{st}$ be the nondegenerate $(-1)^{q+1}$-symmetric pairing determined by SW-duality, as in [6, 7]. The pairing ψ determines a homomorphism $\beta : \Pi \to Hom_{\mathbb{Z}}(A, \mathbb{Z}/2\mathbb{Z})$ by $\beta(b)(a) = \psi(b, \alpha(a))$ for all a in A and b in Π. This homomorphism is onto, its kernel is the image of α and if $\beta(b)(x)) = \ell(a,x)$ for all x in tA then $2b = \alpha(a)$. (This property is established in [6].) If N is an L-module, let $e^iN = \overline{Ext^i_L}(N, L)$, where the overbar denotes conjugation with respect to the involution of L determined by $\bar{z} = 1 - z$.

The L-quintuple is the quintuple $(A, \Pi, \alpha, \ell, \psi)$. It is a complete invariant for simple $2q$-knots with $q \geq 4$ [8], and subsumes both the Farber-Levine pairing ℓ, which is a complete invariant for simple $2q$-knots with $q \geq 4$ and $H_q(\tilde{X}(K))$ finite of odd order, and the F-form, which is a complete invariant for simple $2q$-knots with $q \geq 3$ and $t_q(\tilde{X}(K)) = 0$ [14, 17]. (When there is no 2-torsion the L-quintuple is determined by the module A together with the Farber-Levine pairing on tA and an F-form based on $A/tA = B_q(\tilde{X}(K))$.) Since the interrelations between the two pairings ℓ and ψ are somewhat involved, we may attempt to reformulate the L-quintuple as a self dual object of an additive category with duality (cf. [21]).

We shall give such reformulations in two important special cases: when the knot is fibred (i.e., $A = H_q(\tilde{X}(K))$ is finitely generated as an abelian group) and when it is torsion free (i.e., $tA = t_q(\tilde{X}(K)) = 0$).

In the fibred case we may obtain such a formulation on applying Pontrjagin duality to the invariants of [7]. We take as our invariants the sextuple $(A, C, \Pi, \alpha, \delta, \gamma)$, where $\delta : C \to A$ is the homology Bockstein and $\gamma : \Pi \to C$ is the map α_{34} of [7]. There is a natural isomorphism from $B \otimes \mathbb{Q}/\mathbb{Z}$ to $ker(\delta)$ and so the standard topology on $\mathbb{Q}/\mathbb{Z}$ (as the torsion subgroup of S^1) determines a topology on C. The composite $\eta\alpha\delta\gamma$ is multiplication by 2 on Π and $2\gamma = 0$. Morphisms between such sextuples are systems of homomorphisms of the underlying modules such that all relevant diagrams commute. If M is an abelian topological group which is also an L-module, let M^* be the group of continuous homomorphisms from M to S^1, with the L-module structure determined by $zf(m) = f(\bar{z}m)$ for all f in M^* and m in M. We define the dual sextuple by $(A, C, \Pi, \alpha, \delta, \gamma)^* = (C^*, Hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}), \Pi^*, \hat{\gamma}, \hat{\delta}, \hat{\alpha})$. (The maps are given by $\hat{\gamma}(f + 2C^*) = f\gamma$ for f in C^*, $\hat{\delta}(g) = g\delta$ for $g : A \to \mathbb{Q}/\mathbb{Z}$ and $\hat{\alpha}(h) = h\alpha p$ for h in Π^*, where p is the projection of A onto $A/2A$.)

When $B_q(\tilde{X}(K)) = 0$ then $A = tA$ and is finite, $B = 0$ and γ is an isomorphism, and this invariant simplifies to a triple (A, Π, E) where E is an exact sequence $0 \to A/2A \to \Pi \to_2 A = ker(2 : A \to A) \to 0$ such that the composition $\Pi \to_2 A \to A/2A \to \Pi$ is multiplication by 2, together with an isomorphism ϕ to the dual triple. (The dual is given by $(A, \Pi, E)^* = (e^2A, e^2\Pi, e^2E)$, for if M is finite then $M^* = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ and is naturally isomorphic to e^2M.) In this case, stably hyperbolic implies hyperbolic and the corresponding Witt group is infinitely generated and of exponent 4 [13]. If moreover the finite group A has odd order then the invariant simplifies further to be just the Farber-Levine pairing.

If $t_q(\tilde{X}(K)) = 0$ we may take as our invariant the F-form, which can be formulated as the quintuple $(A, B, \Pi_{q+1}, \eta, \omega)$ where $\omega : \Pi_{q+1} \to B/2B$ is the Hurewicz homomorphism, together with an isomorphism ϕ to the dual quintuple. (The dual is given by $(A, B, \Pi_{q+1}, \eta, \omega)^* = (e^1B, e^1A, e^2\Pi_{q+1}, e^2\omega, e^2\eta)$. Note that if M has no 2-torsion multiplication by 2 gives a short exact sequence $0 \to M \to M \to M \otimes \mathbb{Z}/2\mathbb{Z} \to 0$, and so $e^2(M \otimes \mathbb{Z}/2\mathbb{Z}) \cong (e^1 M) \otimes \mathbb{Z}/2\mathbb{Z})$. Here we need not assume that K is fibred, i.e., that A is finitely generated as an abelian group. To relate the invariants in the fibred case, note that if $tA = 0$ then $\delta = 0$ and $C = B \otimes \mathbb{Q}/\mathbb{Z}$, and the following lemma implies that composition with the stable Hopf map from Π_{q+1} to Π is onto if $q \geq 4$ and A has no 2-torsion. Since these two groups have the same order this map must be an isomorphism.

Lemma 3. *Let X be a $(q-1)$-connected complex such that $A = H_q(X;\mathbb{Z})$ has no 2-torsion and $H_{q+2}(X;\mathbb{Z}) = 0$. Then composition with the stable Hopf map η maps $\pi^{st}_{q+1}(X)$ onto $\pi^{st}_{q+2}(X)$.*

Proof. After suspending if necessary, we may assume that $q \geq 4$, i.e., that we are in the stable range. Let $X^{[q+1]}$ be the $(q+1)$-skeleton of X. The pair $(X, X^{[q+1]})$ is $(q+1)$-connected, so the Hurewicz homomorphism from $\pi_{q+2}(X, X^{[q+1]})$ to $H_{q+2}(X, X^{[q+1]};\mathbb{Z})$ is an isomorphism. Since $H_{q+2}(X;\mathbb{Z}) = 0$ the connecting homomorphism from $H_{q+2}(X, X^{[q+1]};\mathbb{Z})$ to $H_{q+1}(X^{[q+1]};\mathbb{Z})$ is a monomorphism. Therefore the connecting homomorphism from $\pi_{q+2}(X, X^{[q+1]})$ to $\pi_{q+1}(X^{[q+1]})$ is also a monomorphism, and so the natural map from $\pi_{q+2}(X^{[q+1]})$ to $\pi_{q+2}(X)$ is onto. Since $X^{[q+1]} \simeq M(A,q) \vee \bigvee S^{q+1}$ [1: p. 287], the Hilton-Milnor theorem implies that $\pi_{q+2}(X^{[q+1]})$ is generated by the images of $\pi_{q+2}(M(A,q))$ and $\pi_{q+2}S^{q+1}$, for as $2q-1 > q+2$ there are no nontrivial Whitehead products. As $A * \mathbb{Z}/2\mathbb{Z} = 0$ the composition $\eta^2 : A \to \pi_{q+2}(M(A,q))$ is onto, by $3a.7$ of [1: p. 269]. Thus $\eta : \pi_{q+1}(X) \to \pi_{q+2}(X)$ is onto. □

This lemma can also be proven by means of the Atiyah-Hirzebruch spectral sequence for stable homotopy.

In all these cases, Theorem 2 implies that if such a knot is doubly null concordant its invariant is hyperbolic. Kearton has shown that a torsion free simple $2q$-knot is doubly null concordant if and only if its F-form is hyperbolic ([18] - see also [23]).

7. $\mathbb{Q}$-Acyclic Stable Knots

We shall say that a 1-simple knot K is $\mathbb{Q}$-*acyclic* if $\tilde{H}_*(\tilde{X};\mathbb{Q}) = 0$ or equivalently if $H_i(\tilde{X};\mathbb{Z})$ is finite for $2 \leq i \leq [(n+1)/2]$. If K is any $\mathbb{Q}$-acyclic stable knot then $2K = K\sharp K$ is $-$amphicheiral [15]; it follows that $4K$ is doubly null concordant, and hence the subgroup of DNC_n represented by $\mathbb{Q}$-acyclic stable n-knots is a countable group of exponent 4. On the other hand there are countably many $(q-1)$-simple $2q$-knots with $H_q(\tilde{X};\mathbb{Z})$ finite of odd order (which are thus $\mathbb{Q}$-acyclic) whose Farber-Levine pairings are independent elements of order 4 in the appropriate Witt group [13].

If A is an abelian group and $j \geq 2$ then $M(A,j)$ shall denote the Moore space with homology A in degree j, which is well defined up to homotopy equivalence. In particular, if $A = \mathbb{Z}/q\mathbb{Z}$ then $M(A,j)$ is the cofibre $S^j \cup_q e^{j+1}$ of the degree q self-map of S^j. (Cf. Chapter V.3a of [1] or Chapter 2.1 of [4].) Since $M(A_1 \oplus A_2, j) \simeq$

$M(A_1, j) \vee M(A_2, j)$ the general Moore space is a wedge of such cofibres. (Note also that the suspension $SM(A, j)$ is an $M(A, j+1)$-space.)

Lemma 4. *Let A and B be finite abelian groups. Then if $j > 2$*
(a) $[M(A,i); M(B,j)] = 0$ if $1 < i < j-1$;
(b) $[M(A,j-1); M(B,j)] \cong Ext(A,B)$;
(c) $[M(A,j); M(B,j)] \cong Hom(A,B)$;
(d) $[M(A,k); M(B,j)] = 0$ if $j < k < 2j-3$ *and the p-primary summand of B is 0 for primes $p \leq j/2$.*

Proof. We may assume that A is cyclic. Since in all cases $M(A,k)$ has dimension at most $2(j-1)$ (i.e., $k \leq 2j-3$) the homotopy classes are stable and so we may also assume that B is cyclic, of prime power order. We shall prove only (d). (The other cases may be proven in a similar fashion. See also p. 268 of [1].)

Associated to the cofibration $S^j \to S^j \to M(\mathbb{Z}/q\mathbb{Z}, j)$ there is a homotopy sequence $\pi_i(S^j) \to \pi_i(S^j) \to \pi_i(M(\mathbb{Z}/q\mathbb{Z}, j)) \to \pi_{i-1}(S^j) \to \ldots$ which is exact for $i \leq 2j-3$ (Theorem 1.6 of [4]), and where the maps between the homotopy groups of S^j are given by multiplication by q. Since multiplication by a prime p is invertible on $\pi_i(S^j)$ for $j < i < j+2p-3$ by Corollary 9.7.13 of [24], it follows that $\pi_i(M(\mathbb{Z}/p^e\mathbb{Z}, j)) = 0$ for $j < i < min\{2j-2, j+2p-3\}$. We can now use the Barratt-Puppe sequence for the cofibration $S^k \to S^k \to M(\mathbb{Z}/t\mathbb{Z}, k)$ to show that $[M(\mathbb{Z}/t\mathbb{Z}, k); M(\mathbb{Z}/p^e\mathbb{Z}, j)] = 0$ if $j < k < min\{2j-3, j+2p-4\}$. This proves (d). □

Theorem 8. *Let K be a $\mathbb{Q}$-acyclic n-knot which is r-simple for some $r > n/3$ and such that the p-primary component of $H_*(\tilde{X}; \mathbb{Z})$ is trivial for $p \leq (n-2r+3)/2$ and $H_i(\tilde{X}; \mathbb{Z}) = 0$ for $i = [n/2]$ or $[(n+1)/2]$. Then K is doubly null concordant.*

Proof. By the remarks above K is a stable fibred knot. The fibre V is an r-connected $(n+1)$-manifold with boundary S^n and so has the homotopy type of a finite complex of dimension $n-r$. Since it is $\mathbb{Q}$-acyclic and has homotopy length $(n-r)-r = n-2r$ which is less than its connectivity it decomposes up to homotopy as a wedge of its p-primary components for primes p, and this decomposition is respected by the maps z and ϕ. (Cf. Theorem 4.40 and Corollary 4.29 of [4].) As this decomposition corresponds to knot sum we may henceforth assume that V is p-primary for some prime $p > (n-2r+3)/2$. As the homotopy length of V is less than $2p-3$, V is homotopy equivalent to a wedge of Moore spaces (cf. Proposition 1.1 of [12]). In particular, we may write $V \simeq V(-) \vee V(+)$, where $V(-)$ has dimension $\leq [n/2]$ and $V(+)$ is $[(n+1)/2]$-connected.

Lemma 4 implies that $[V(-); V(+)] = [V(+); V(-)] = 0$. Therefore the carving map z has matrix $\begin{pmatrix} z(-) & 0 \\ 0 & z(+) \end{pmatrix}$ with respect to the decomposition $V = V(-) \vee V(+)$. Similarly $S^{n+1}D(V)$ is homotopy equivalent to $S^{n+1}D(V(-)) \vee S^{n+1}D(V(+))$ where the summands are again wedges of p-primary Moore spaces, the first summand is $[(n+1)/2]$-connected and the second summand has dimension at most $[n/2]$, and $D(z)$ has matrix $\begin{pmatrix} D(z(-)) & 0 \\ 0 & D(z(+)) \end{pmatrix}$ with respect to this decomposition. The lemma also implies that $[V(-);S^{n+1}D(V(-))] = [V(+);S^{n+1}D(V(+))] = 0$. Therefore the duality map ϕ must have matrix $\begin{pmatrix} 0 & \psi \\ \theta & 0 \end{pmatrix}$ with respect to these decompositions, and since ϕ is $(-1)^{n+1}$-hermitean $\theta = (-1)^{n+1}D(\psi)$. It now follows easily that the map from $V(-) \vee V(+)$ to $V(-) \vee S^{n+1}D(V(-))$ with matrix $\begin{pmatrix} 1 & 0 \\ 0 & \psi \end{pmatrix}$ determines an isometry from the stable isometry structure (V, z, ϕ) of K to the hyperbolic stable isometry structure $(V(-) \vee S^{n+1}D(V(-)), \begin{pmatrix} z(-) & 0 \\ 0 & D(z(-)) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ (-1)^{n+1} & 0 \end{pmatrix})$. The result now follows from the Corollary to Theorem 6 above. □

The lowest dimension in which there are nontrivial examples satisfying the hypotheses of the theorem is $n = 14$. The class of knots to which the theorem applies includes all antisimple $\mathbb{Q}$-acyclic stable n-knots with trivial p-primary torsion for all primes $p < (n+10)/6$. (An n-knot is ***antisimple*** if its exterior has a handlebody decomposition with no handles in dimensions $q+1$ (if $n = 2q$) or $q+1$ and $q+2$ (if $n = 2q+1$). A 1-simple n-knot is antisimple if and only if $H_i(\tilde{X};\mathbb{Z}) = 0$ for $[n/2] \leq i \leq [(n+3)/2]$ [11]. The lowest dimension in which there are nontrivial antisimple examples is $n = 20$.) In all cases the homology of $\tilde{X}$ has odd order. Can the condition on the middle dimensional homology be relaxed to "the Farber-Levine pairing on $H_{n/2}(\tilde{X};\mathbb{Z})$ is hyperbolic" if n is even, and dropped completely if n is odd? In particular, are $\mathbb{Q}$-acyclic $(q-1)$-simple $(2q+1)$-knots with $H_q(\tilde{X};\mathbb{Z})$ of odd order always doubly null concordant? (Such knots are stable if $q \geq 4$.)

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References

[1] Baues, H.J. *Algebraic Homotopy Theory*, Cambridge Studies in Advanced Mathematics 15, Cambridge University Press, Cambridge-New York-Melbourne (1989).

[2] Bayer-Fluckiger, E. and Stoltzfus, N.W. Stably hyperbolic ϵ-hermitian forms and doubly sliced knots, J. Reine Angew. Math. 366 (1986), 129–141.

[3] Browder, W. and Levine, J. Fibering manifolds over a circle, Comment. Math. Helvetici 40 (1966), 153–160.

[4] Cohen, J.M. *Stable Homotopy*, Lecture Notes in Mathematics 165, Springer-Verlag, Berlin-Heidelberg-New York (1970).

[5] Farber, M.S. Isotopy types of knots of codimension two, Trans. Amer. Math. Soc. 261 (1980), 185–209.

[6] Farber, M.S. Classification of stable fibred knots, Mat. Sbornik 115 (1981), 223–262.

[7] Farber, M.S. The classification of simple knots, Russian Math. Surveys 38:5 (1983), 63–117.

[8] Farber, M.S. An algebraic classification of some even-dimensional knots I, Trans. Amer. Math. Soc. 281 (1984), 507-527; II, *ibid.*, 529–570.

[9] Farber, M.S. Knots and stable homotopy, in *Topology Conference, Leningrad 1982*, Lecture Notes in Mathematics 1060, Springer-Verlag, Berlin - Heidelberg - New York (1984), 140–150.
[10] Farber, M.S. Minimal Seifert manifolds and the knot finiteness theorem, Israel J. Math. 66 (1989), 179–215.
[11] Hausmann, J.-C. Noeuds antisimples, in *Knot Theory*, Proceedings, Plans-sur-Bex, Switzerland 1977 (edited by J.-C.Hausmann), Lecture Notes in Mathematics 685, Springer-Verlag, Berlin - Heidelberg - New York (1978), 171–202.
[12] Henn, H.-W. Classification of p-local low-dimensional spectra, J. Pure Appl. Algebra 45 (1987), 45–71.
[13] Hillman, J.A. Finite simple even-dimensional knots, J. London Math. Soc. 34 (1986), 369–374.
[14] Hillman, J.A. and Kearton, C. Seifert matrices and 6-knots, Trans. Amer. Math. Soc. 309 (1988), 843–857.
[15] Hillman, J.A. and Kearton, C. Stable isometry structures and the factorization of $\mathbb{Q}$-acyclic stable knots, J. London Math. Soc. 39 (1989), 175–182.
[16] Kearton, C. Cobordism of knots and Blanchfield duality, J. London Math. Soc. 10 (1975), 406–408.
[17] Kearton, C. An algebraic classification of certain simple even-dimensional knots, Trans. Amer. Math. Soc. 276 (1983), 1–53.
[18] Kearton, C. Doubly-null-concordant simple even-dimensional knots, Proc. Roy. Soc. Edinburgh 96A (1984), 163–174.
[19] Levine, J. Knot modules. I, Trans. Amer. Math. Soc. 229 (1977), 1–50.
[20] Levine, J. Doubly slice knots and doubled disc knots, Michigan J. Math. 30 (1983), 249–256.
[21] Quebbemann, H.G., Scharlau, W. and Schulte, R. Quadratic and hermitean forms in additive and abelian categories, J. Algebra 59 (1979), 264–289.
[22] Ranicki, A. The algebraic theory of surgery II. Applications to topology, Proc. London Math. Soc. 40 (1980), 193–283.
[23] Silver, D. On Aitchison's construction by isotopy, Trans. Amer. Math. Soc. 305 (1988), 641–652.
[24] Spanier, E.H. *Algebraic Topology*, McGraw-Hill, New York (1966).
[25] Stoltzfus, N.W. Algebraic computations of the integral concordance and double null concordance group of knots, in *Knot Theory, Proceedings, Plans-sur-Bex, Switzerland (1977)* (edited by J.C.Hausmann), Lecture Notes in Mathematics 685, Springer-Verlag, Berlin - Heidelberg - New York (1978), 274–290.
[26] Sumners, D.W. Invertible knot cobordisms, Comment. Math. Helv. 46 (1971), 240–256.
[27] Wall, C.T.C. Thickenings, Topology 5 (1966), 73–94.