A Topological Interpretation of the Atiyah-Patodi-Singer Invariant

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0. Let M be a closed, connected, oriented manifold of dimension $n = 2l - 1$ and let $\pi = \pi_1(M)$. For any unitary representation $\alpha : \pi \to U_k$, Atiyah, Patodi and Singer in [APS, **II**] define a numerical invariant $\rho_\alpha(M) \in \mathbb{R}$ as follows. Choose a Riemannian structure for M and then consider the self-adjoint elliptic differential operator B_α on the space of all differential forms of even degree with values in the flat bundle defined by α by the formula $B_\alpha = i^l(-1)^{p+1}(*d_\alpha - d_\alpha *)$ on forms of degree $2p$, where d_α is the covariant derivative of the flat bundle defined by α and $*$ is the duality operator defined by the Riemannian structure. They then consider the eta-function $\eta_\alpha(s) = \sum_{\lambda\neq 0}(\operatorname{sign}\lambda)|\lambda|^{-s}$, where λ runs over all nonzero eigenvalues of B_α counting multiplicities. Atiyah, Patodi and Singer show, in [**APS, I, II**], that $\eta_\alpha(s)$ defines an analytic function for $\Re(s)$ large, which can be analytically continued to have a finite value at $s = 0$. They then define $\rho_\alpha(M) = \eta_\alpha(0) - k\eta(0)$, where $\eta(s)$ is the eta-function of the trivial representation. It is an immediate consequence of their Index Theorem that $\rho_\alpha(M)$ is independent of the choice of metric and that the reduction of $\rho_\alpha(M)$ to $\mathbb{R}/\mathbb{Z}$ depends only on the oriented bordism class of M.

INDEX THEOREM ([**APS,II**]). *If M is the oriented boundary of an oriented compact Riemannian manifold V, such that the Riemannian structure on V is a product near M, and the representation α extends to a unitary representation β of $\pi_1(V)$, then*

$$\operatorname{sign}_\alpha(V) = k\int_V L(p) - \eta_\alpha(0).$$

In this formula, $L(p)$ is the Hirzebruch polynomial in the Pontriagin forms of V and $\operatorname{sign}_\alpha(V)$ is the signature of the intersection form on V over the twisted coefficient system defined by α.

1991 *Mathematics Subject Classification.* Primary 58G11; Secondary 57N65.
The research was partially supported by the USA - Israel Binational Science Foundation
The second author was supported in part by a NSF Grant
This paper is in final form and no version of it will be submitted for publication elsewhere.

COROLLARY. Given $M = \partial V$, with $\alpha = \beta|_M$, we have $\rho_\alpha(M) = k\operatorname{sign}(V) - \operatorname{sign}_\alpha(V)$, where $\operatorname{sign}(V)$ is the ordinary signature of V.

We address two problems in this work.

(1) Give an intrinsic topological definition of $\rho_\alpha(M)$.
(2) To what extent is $\rho_\alpha(M)$ an invariant of homotopy type?

Viewing $\rho_\alpha(M) = \rho(M) \cdot \alpha$ as a real-valued function on the variety of unitary representations of $\pi = \pi_1(M)$, our results consist of:

(a) a formula for the jumps in $\rho(M)$ at discontinuities, and
(b) a formula for the "differential" of $\rho(M)$, reduced $\bmod \mathbb{Z}$.

Both of these formulae depend only on the homotopy type of M and give an intrinsic homotopy invariant definition of $\rho(M)$, up to a locally constant function (which vanishes on the component of the trivial representation). It is known that homotopy invariance fails if π is finite [**W**] and in many cases when π has torsion [**We**].

Earlier work of Neumann [**N**] and Weinberger [**We**] showed that $\rho(M)$ is a homotopy invariant for a large class of $\pi_1(M)$.

1. For any group π we can consider the set of k-dimensional unitary representations of π, denoted $R_k(\pi)$. If π is finitely generated this set is, in a natural way, a real algebraic variety – any representation of π (with m generators) leads to an obvious manifestation of $R_k(\pi)$ as a subvariety of $U(k) \times \cdots U(k)$ (m times). It is not hard to see that this algebraic structure is independent of the presentation of π. In [**L**], [**L1**] the Atiyah-Patodi-Singer invariant $\rho_\alpha(M)$ is considered as a function $\rho(M): R_k(\pi) \to \mathbb{R}$, where $\rho(M) \cdot \alpha = \rho_\alpha(M)$ and $\pi = \pi_1(M)$. It is shown in [**L1**] that there is a stratification of $R_k(\pi)$ by subvarieties $R_k(\pi) = \Sigma_0 \supseteq \Sigma_1 \supseteq \cdots \supseteq \Sigma_i \supseteq \dots$ such that $\rho(M)|(\Sigma_i - \Sigma_{i+1})$ is continuous, for $i \geq 0$. Specifically, Σ_i is defined as follows. Define $d_\alpha = \sum_i \dim H_i(M;\alpha)$, where $H_i(M;\alpha)$ is homology with the twisted coefficient system defined by α. If $d = \min\{d_\alpha : \alpha \in R_k(\pi)\}$, then $\Sigma_i = \{\alpha : d_\alpha \geq d + i\}$. Note that this stratification depends only on the homotopy type of M.

We propose to study the discontinuities of $\rho(M)$. In [**APS-III**] it is shown that the reduction $\overline{\rho}_\alpha(M)$ of $\rho_\alpha(M)$ to $\mathbb{R}/\mathbb{Z}$ depends continuously on α. (In fact they give a (K-theoretic) formula for this reduction). This shows that the "jump" in $\rho(M)$ at a discontinuity is integral.

Set γ be an analytic curve in $R_k(\pi)$. Analyticity means that γ lies in some Σ_i so that it intersects Σ_{i+1} in a discrete set of points. We may assume that, for some $\epsilon > 0$ and $|t| < \epsilon$, the inclusion $\gamma(t) \in \Sigma_{i+1}$ holds if and only if $t = 0$. Then $\rho(M) \circ \gamma$ is continuous except for some (integer) jump at $t = 0$. More precisely, $\rho(M) \circ \gamma$ is continuous at $t \neq 0$ and, since $\overline{\rho}(M) \circ \gamma$ ($\overline{\rho}(M)$ is the reduction of $\rho(M)$ to $\mathbb{R}/\mathbb{Z}$) is continuous at $t = 0$, there is a well-defined limit of $\rho(M) \circ \gamma(t)$ as $t \to +0$, which agrees with $\rho(M) \circ \gamma(0) \bmod \mathbb{Z}$. We propose to find a formula for the difference.

We first interpret γ as a representation of π over P, the ring of power series with a positive radius of convergence, since the entries of the matrix $\gamma(t)$ are elements of P. We can then use γ to define, for example, a right action of π on the free P-module P^k of rank k, by regarding P^k as row vectors over P and using right multiplication by $\gamma(t)$. That each $\gamma(t)$ is unitary means that this action preserves the canonical P-valued Hermitian form on P^k. If P^k_γ denotes the right $\mathbb{C}\pi$-module defined by γ, then the conjugate left $\mathbb{C}\pi$-module $\overline{P}^k_\gamma$ (recall $\overline{P}^k_\gamma = P^k_\gamma$ with π-action defined by $g \cdot \alpha = \alpha \cdot g^{-1}$, for any $g \in \pi$, $\alpha \in P^k_\gamma$) is isomorphic to the module defined by regarding P^k as column vectors and using left multiplication by $\gamma(t)$.

We may use P^k_γ as a local coefficient system over $M(\pi = \pi_1(M))$ and define $H_*(M;\gamma) = H_*(P^k_\gamma \otimes_\pi C(\tilde{M}))$ and $H^*(M;\gamma) = H_*(\mathrm{Hom}_\pi(C(\tilde{M}), \overline{P}^k_\gamma))$. Now Poincaré duality applied to M shows that the intersection pairing over $\mathbb{Z}\pi$ on $\tilde{M}$ induces an isomorphism:

$$\overline{H_i(M;A)} \approx H^{n-i}(M;\bar{A}), \qquad 0 \le i \le n,$$

where A is any $(R,\mathbb{Z}\pi)$-bimodule (R any ring with involution) and, generally, $A \to \bar{A}$ denotes the usual passage from $(R,\mathbb{Z}\pi)$-bimodules to $(\mathbb{Z}\pi, R)$-bimodules. The duality isomorphism is one of right R-modules. We apply duality with $A = \bar{P}^k_\gamma$, $R = P$ with involution defined by complex conjugation.

Since P is a discrete valuation ring with fundamental ideal generated by t, we see that $H_i(M;\gamma)$ is determined by its rank r_i over P and its torsion submodule $T_i(M;\gamma)$. The universal coefficient theorem shows that $H^j(M;\gamma)$ has rank r_j and its torsion-module is the "dual module" of $T_{j-1}(M;\gamma)$ – if T is a (left) torsion P-module then the dual module is $T^* = \mathrm{Hom}_P(T; \hat{P}/P)$, where $\hat{P}$ is the quotient field of P, a (right) P-module. Now Poincaré duality tells us that $r_i = r_{n-i}$, $\bar{T}_i(M;\gamma) \approx T_{n-i-1}(M;\gamma)^*$ and, furthermore, there is a *non-singular*, sesquilinear, $\pm$Hermitian pairing $\langle\ ,\ \rangle$:

$$T_q(M;\gamma) \times T_q(M;\gamma) \to \hat{P}/P \quad (n = 2q+1) \quad \text{with}$$
$$\langle \lambda\alpha, \beta\rangle = \lambda\langle\alpha,\beta\rangle,\ \langle\alpha,\beta\rangle = \pm\overline{\langle\beta,\alpha\rangle}.$$

Now, non-singular, sesquilinear, $\pm$Hermitian pairings of a torsion P-module T can be classified by a collection of signatures. Specifically, let $\langle\ ,\ \rangle : T \times T \to \hat{P}/P$ be such a pairing and define $\Delta_i(T)$ to be quotient $\ker t^i/(t \ker t^{i+1} + \ker t^{i-1})$. It is easy to check that $\langle\ ,\ \rangle$ induces a non-singular, bilinear, $\pm$-Hermitian pairing (over $\mathbb{C}$) $\langle\ ,\ \rangle_i$: $\Delta_i(T) \times \Delta_i(T) \to \mathbb{C}$ by the formula:

$$t^i\langle\alpha,\beta\rangle_i \equiv \langle[\alpha],[\beta]\rangle \qquad \mathrm{mod}\, tP.$$

Note $\langle\alpha,\beta\rangle \in t^{-i}P/P$. Now we can define $\sigma_i(\langle\ ,\ \rangle) = \mathrm{sign}\,(\langle\ ,\ \rangle_i)$ and it is not hard to prove:

PROPOSITION 1. *Suppose that* $\langle\ ,\ \rangle$ *and* $\langle\ ,\ \rangle'$ *are two non-singular, sesquilinear, ϵ-Hermitian pairings* ($\epsilon = \pm 1$) *defined on the same torsion P-module T as above. Then* $\langle\ ,\ \rangle$ *and* $\langle\ ,\ \rangle'$ *are congruent if and only if* $\sigma_i(\langle\ ,\ \rangle) = \sigma_i(\langle\ ,\ \rangle')$ *for all i.*

Now, we go back to our analytic curve γ in $R_k(\pi)$ and define $\sigma_i(M;\gamma) = \sigma_i(\langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ is the pairing defined above on $T_q(M;\gamma)$.

Our main result is:

THEOREM 1.

$$\lim_{t\to+0} \rho(M)\cdot\gamma(t) - \rho(M)\cdot\gamma(0) = \sum_{i=1}^{\infty} \sigma_i(M;\gamma).$$

The first order part of the linking pairing described above was considered in [**KK**]; cf. also [**KK1**], where the first order part of Theorem 1 is proven in the case of manifolds with boundary. In [**KK1**] there is a discussion of an alternative version of the higher order part of Theorem 1 which would use cup products and higher Massey products.

We remark that it is a consequence of the *curve selection lemma* that the discontinuities of $\rho(M)$ along analytic curves determine $\rho(M)$ up to a continuous function on $R_k(\pi)$. More explicitly, suppose we define:

$$\sigma(M;\gamma) = \lim_{t\to+0} \rho(M)\cdot\gamma(t) - \rho(M)\cdot\gamma(0)$$

where γ is an analytic curve in $R_k(\pi)$ defined on a neighborhood of 0. In fact, if ρ is any real-valued function on $R_k(\pi)$ whose reduction mod $\mathbb{Z}$ is continuous and which is "piecewise-continuous" in the sense that, for some stratification of $R_k(\rho)$ by subvarieties $R_k(\pi) = \Sigma_0 \supseteq \Sigma_1 \supseteq \cdots \supseteq \Sigma_i \supseteq \ldots$, the function $\rho(M)|(\Sigma_i - \Sigma_{i+1})$ is continuous for all $i \geq 0$, then we can define $\sigma(\rho,\gamma)$ by the above formula. Then we have:

PROPOSITION 2. *If ρ_1, ρ_2 are two piecewise-continuous real-valued functions on $R_k(\pi)$ which are continuous* mod $\mathbb{Z}$, *then $\rho_1 - \rho_2$ is continuous if and only if* $\sigma(\rho_1.\gamma) = \sigma(\rho_2.\gamma)$ *for every analytic curve γ in $R_k(\pi)$.*

The proof of Theorem 1 begins with a general formula for the spectral jump at $t = 0$ of a path A_t of elliptic differential operators in terms of the associated signatures of a "linking" pairing on a torsion P-module, with values in $\hat{P}/P$, defined directly from A_t. Using a parametrized Hodge decomposition [K: Ch. VII, Th. 3.9] it is then shown that these signatures, for $A_t = B_{\gamma(t)}$, coincide with the signatures of the "topological" linking pairing $\langle\ ,\ \rangle$. The details will appear in a future paper [**FL**].

Originally we proved the formula of Theorem 1 under the rather stringent hypothesis that $\gamma(t)$ extends to an analytic path in $R_k(\pi_1(V))$ for some compact oriented manifold V bounded by M. We then use the Index Theorem to give a purely topological proof as follows. We put ourselves in a more general context.

Let (V, M) be an algebraic Poincaré (AP) pair of finite type over the ring P, in the sense of Miščenko [**M**], with dimension $V = 2q+2$. Then $H_q(M)$ supports a non-singular torsion pairing, on its torsion submodule, with value in $\hat{P}/P$, and we can define $\sigma_i(M)$ to be the signatures of the associated $\pm$Hermitian forms, as above. For any $\epsilon \geq 0$, let P_ϵ be the subring of P consisting of all power series with radius of convergence $> \epsilon$. We may assume that (V, M) comes from an AP-pair over P_ϵ, for some $\epsilon > 0$. Then we can define $(V_c, M_c) = (V, M) \otimes_c \mathbb{C}$, for $0 \leq c \leq \epsilon$, where $\mathbb{C}$ is regarded as a P_ϵ-module via the ring homomorphism $P_\epsilon \to \mathbb{C}$ defined by $f \mapsto f(c)$, and let $\sigma_c(V)$ be the signature of the intersection pairing on V_c. In fact, it is not hard to see that $\sigma_c(V)$ is constant for c in the interval $(0, \delta)$, for some $0 < \delta \leq \epsilon$ – let us denote this constant value by $\sigma_+(V)$. Then, our general result is $\sigma_+(V) - \sigma_0(V) = \sum_{i \geq 1} \sigma_i(M)$.

To prove this, first consider the special case in which $H_i(V) = 0$ for $i \leq q$. Then $H_{q+1}(V)$ is a free module and the intersection pairing is represented by a matrix B which can be decomposed into a block sum of matrices of the form $t^i B_i(t)$, where $B_i(0)$ is non-singular over $\mathbb{C}$, plus a 0 matrix. Thus for $0 < c \leq \epsilon$, $\sigma_c(V) = \sum_{i \geq 1} \operatorname{sign} B_i(c)$ and $\sigma_0(V) = \operatorname{sign} B_0(0)$. If $B_i(c)$ is non-singular in an interval $0 \leq c < \delta$, then, in this interval, $\sigma_c(V) = \sum_{i \geq 1} \operatorname{sign} B_i(0)$.

We now turn to the torsion pairing on $H_q(M)$. It is a standard fact, in this situation, that a matrix representative B of the intersection pairing on $H_{q+1}(V)$ is also a presentation matrix for $H_q(M)$ and the inverse of the non-degenerate part of B represents the torsion-pairing. Thus we see that $t^i B(t)$ presents the free P/t^i summand of $H_q(M)$ and, in addition, the associated Hermitian pairing on $\Delta_i(TH_q(M))$ is represented by the matrix $B_i(0)^{-1}$, for $i \geq 1$. Putting all these observations together gives the desired result.

For the general case we reduce to the special case by doing algebraic surgery on V as in [**M**]. It is not hard to see that we may kill all the homology of V of dimension $< q$ without changing M, but it is necessary to check that $\sigma_+(V)$ and $\sigma_0(V)$ are not changed by these surgeries. In fact, the only surgeries which change $H_{q+1}(V)$ are those to kill $H_q(V)$. When a class $\alpha \in H_q(V)$ is killed by a surgery then the effect on $H_{q+1}(V)$ and the intersection pairing are as follows:

Case 1. *α is a torsion class of $H_q(V)$ – say $t^r \alpha = 0$; then a rank 2 orthogonal summand is added to $H_q(V)$ with the intersection pairing on this new summand represented by the matrix* $\begin{bmatrix} 0 & t^r \\ \pm t^r & 0 \end{bmatrix}$.

Case 2. *α has "infinite order" in $H_q(V)$, but its image in $H_q(V, M)$ is torsion: then the rank of $H_{q+1}(V)$ is increased by one but the new element is totally isotropic, i.e. its intersection with all elements of $H_{q+1}(V)$ is zero.*

Case 3. *The image of α in $H_q(V, M)$ has infinite order: then $H_{q+1}(V)$ is unchanged.*

But in all three cases neither $\sigma_+(V)$ nor $\sigma_0(V)$ is changed and so the theorem follows.

2. We now study the reduction to $\mathbb{R}/\mathbb{Z}$ of $\rho(M)$. Denote by $\bar{\rho}(M): R_k(\pi) \to \mathbb{R}/\mathbb{Z}$ the function $\bar{\rho}(M)\cdot\alpha = \bar{\rho}_\alpha(M)$. As remarked above, it is proved in [**APS, III**] that $\bar{\rho}(M)$ is continuous. Moreover, they obtain a formula for $\bar{\rho}(M)$ as follows. For any unitary representation α of a discrete group Γ, there is associated an element $\beta(\alpha) \in K^{-1}(B_\Gamma; \mathbb{R}/\mathbb{Z})$, where B_Γ is the classifying space of Γ. Then, if $\alpha \in R_k(\pi)$ and $\phi: M \to B_\pi$ is the classifying map for $\pi \approx \pi_1(M)$, we have $\phi^*\beta(\alpha) \in K^{-1}(M; \mathbb{R}/\mathbb{Z})$. Now let $\sigma \in K^1(\tau M)$ be the "self-adjoint symbol" of the signature operator on M (τM is the Thom space of the cotangent bundle). The "index theorem for flat bundles" of [**APS, III**] then asserts that $\bar{\rho}_\alpha(M) = \phi^*\beta(\alpha)\cdot\sigma$, using the product

$$K^{-1}(M;\mathbb{R}/\mathbb{Z}) \times K^1(\tau M) \to K^0(\tau M;\mathbb{R}/\mathbb{Z}) \overset{\text{ind}}{\to} \mathbb{R}/\mathbb{Z}.$$

Using the ideas of this general result we can obtain a more explicit, though less definitive, determination of $\bar{\rho}(M)$. If $\alpha \in R_k(\pi)$ then let $\det\alpha \in R_1(\pi)$ be the obvious representation $(\det\alpha)(g) = \det\alpha(g)$. We can then define $\arg\det\alpha \in H^1(M;\mathbb{R}/\mathbb{Z})$ by the formula $\det\alpha(g) = \exp(2\pi i(\arg\det\alpha)(g))$, for any $g \in \pi$. We now define $\tilde{\rho}(M): R_k(\pi) \to \mathbb{R}/\mathbb{Z}$ by the cohomological formula:

$$\tilde{\rho}(M)\cdot\alpha = -(2(\arg\det\alpha) \cup L(M))[M]$$

where $L(M)$ is the Hirzebruch L-polynomial in the Pontriagin classes of M, defined by the generating series $x/\tanh(x)$. This makes sense, since the component of $L(M)$ in H^{2l-2} lifts to an integral class [**No**], but depends on the particular lift (here we assume that $\dim M = 2l-1$). However, we note that changing the lift will only change $\tilde{\rho}(M)$ by a locally constant $\mathbb{Q}/\mathbb{Z}$-valued function on $R_k(\pi)$. An alternative definition of $\tilde{\rho}(M)$ goes as follows. Chooose a basis $z'_1,\dots,z'_m$ of $H^1(M;\mathbb{Z})$ and $z_1,\dots,z_m \in H_1(M;\mathbb{Z})$ such that $z'_i\cdot z_j = \delta_{ij}$. Let τ_i be the signature of an oriented closed submanifold of M representing the Poincaré dual of z'_i in $H_{2l-2}(M;\mathbb{Z})$. Then, up to a locally constant function on $R_k(\pi)$:

$$\tilde{\rho}(M)\cdot\alpha = -2\sum_{i=1}^{m}\tau_i \arg\det\alpha(z_i)$$

See [**L1**] for a special case.

THEOREM 2. *$\bar{\rho}(M) - \tilde{\rho}(M)$ is constant on connected components of $R_k(\pi)$. For example, we have $\bar{\rho}(M) = \tilde{\rho}(M)$ on the component of the trivial representation.*

SKETCH OF THE PROOF. Suppose α_t $(0 \le t \le 1)$ is a path in $R_k(\pi)$. Then we can associate to α_t a "Hermitian" bundle ξ (i.e. unitary bundle with a connexion) over $I\times M$, such that on $t\times M$ the induced connexion is flat and has monodromy α_t. The curvature form ξ has the form $\Omega = dt\wedge\omega$, where ω is a 1-form on $I\times M$ with coefficients in $\mathrm{Hom}(\xi,\xi)$. Now the Index Theorem of [**APS, I**] applied to the generalized signature operator D_ξ on $I \times M$ with coefficient in ξ gives the formula:

$$\operatorname{Index} D_\xi = \int_{I\times M} 2^l \operatorname{ch}\xi \cdot \mathcal{L}(I \times M) - (\eta_{\alpha_1}(M) - \eta_{\alpha_0}(M))$$

where $\mathcal{L}$ is the Hirzebruch form with generating series: $\frac{x/2}{\tanh(x/2)}$, $\dim(I \times M) = 2l$ and $\operatorname{ch}\xi$ is the Chern character form of ξ. But this form reduces to $k + \frac{1}{2\pi i}\operatorname{Trace}(\Omega)$, since $\Omega = dt \wedge \omega$, and so we can derive from the Index Theorem the equation:

$$\rho_{\alpha_1}(M) - \rho_{\alpha_0}(M) \equiv \frac{2^{l-1}}{\pi i} \int_{I\times M} dt \wedge \operatorname{Trace}(\omega) \wedge \mathcal{L}(M) \qquad \operatorname{mod} \mathbb{Z}$$

Now the 1-form Trace(ω) on $I \times M$ defines a 1-parameter family of cohomology classes $(\operatorname{Tr}\omega)_t \in H^1(M;\mathbb{R})$ and so we have

$$\rho_{\alpha_1}(M) - \rho_{\alpha_0}(M) \equiv \frac{-2^{l-1}}{\pi i}\left[\left[\int_0^1 (\operatorname{Tr}\omega)_t dt\right] \cup \mathcal{L}(M)\right] \cdot [M] \quad \operatorname{mod} \quad \mathbb{Z}$$

where we use $\mathcal{L}(M)$ as above to denote the Hirzebruch polynomial in the Pontriagin classes of M. We now use the fact that

$$(\operatorname{Tr}\omega)_t(g) = \operatorname{Trace}\left(\frac{d\alpha_t}{dt}(g) \circ \alpha_t^{-1}(g)\right) = \frac{d}{dt}\left(\log\det\alpha_t(g)\right)$$

to obtain our final formula:

$$\rho_{\alpha_1}(M) - \rho_{\alpha_0}(M) \equiv -2((\arg\det\alpha_1(g) - \arg\det\alpha_0(g)) \cup L(M)) \cdot [M]$$

which implies the Theorem.

A formula for $\rho(M) \mod \mathbb{Q}$, in terms of Cheeger-Chern-Simons classes is given in [**CS**], Corollary 9.3.

3. We now discuss the implication of these Theorems for the question of the homotopy invariance of the ρ-invariant. Suppose M, M' are homotopy equivalent manifolds (odd-dimensional closed, oriented) and $\pi_1(M) \approx \pi_1(M')$ identified by the homotopy equivalence. Let $\Delta(M, M') = \rho(M) - \rho(M') : R_k(\pi) \to \mathbb{R}$. Now $\tilde{\rho}(M) = \tilde{\rho}(M')$ (by Novikov [**No**]) and so, by Theorem 2, $\Delta(M, M')$ reduced mod $\mathbb{Z}$ is constant on each component of $R_k(\pi)$. Furthermore, it is clear that $\sigma_i(M, \gamma) = \sigma_i(M', \gamma)$, for any analytic curve γ in $R_k(\pi)$, and so it follows from Theorem 1 and Proposition 2 that $\Delta(M, M')$ is continuous. Now putting these together we conclude that $\Delta(M, M')$ is constant on each component of $R_k(G)$ (0 at the trivial representation). (Weinberger [**We**]) prcves this under the assumption that M and M' are rationally cobordant over G or, alternatively, whenever G satisfies the Novikov conjecture).

The analysis of homotopy lens spaces in Wall [**W**] gives many examples where $\Delta(M, M')$ takes non-zero rational values when π is finite cyclic. Weinberger shows in [**We**] that $\Delta(M, M')$ is rational if π satisfies the Novikov conjecture, and has now announced a proof that $\Delta(M, M')$ is always rational, using the results of this paper.

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