THE MORSE-NOVIKOV THEORY OF CIRCLE-VALUED FUNCTIONS AND NONCOMMUTATIVE LOCALIZATION

MICHAEL FARBER AND ANDREW RANICKI

ABSTRACT. We use noncommutative localization to construct a chain complex which counts the critical points of a circle-valued Morse function on a manifold, generalizing the Novikov complex. As a consequence we obtain new topological lower bounds on the minimum number of critical points of a circle-valued Morse function within a homotopy class, generalizing the Novikov inequalities.

1. Introduction.

Let us start by recalling the way in which chain complexes are used to count the critical points of a real-valued Morse function.

A Morse function $f: M \to \mathbb{R}$ on a compact (differentiable) *m*-dimensional manifold M with $c_i(f)$ critical points of index i determines a handlebody decomposition of M with $c_i(f)$ *i*-handles. The *Morse-Smale complex* is the cellular chain complex of the corresponding CW decomposition of the universal cover \widetilde{M} of M, a based f.g. free $\mathbb{Z}[\pi_1(M)]$ -module chain complex $C(\widetilde{M})$ with

$$\operatorname{rank}_{\mathbb{Z}[\pi_1(M)]} C_i(\widetilde{M}) = c_i(f) .$$

For any ring morphism $\rho : \mathbb{Z}[\pi_1(M)] \to R$ there is induced a based f.g. free *R*-module chain complex

$$C(M;R) = R \otimes_{\mathbb{Z}[\pi_1(M)]} C(M)$$

with homology *R*-modules $H_*(M; R) = H_*(C(M; R))$ (which in general depend on ρ as well as *R*). The number $c_i(f)$ of critical points of index *i* is bounded from below by the minimum number $\mu_i(M; R)$ of generators in degree *i* of a finite f.g. free *R*-module chain complex which is chain equivalent to C(M; R)

$$c_i(f) \ge \mu_i(M; R)$$
.

For a principal ideal domain R and a ring morphism ρ : $\mathbb{Z}[\pi_1(M)] \to R$ the *R*-coefficient *Betti numbers* of M are defined as usual by

$$b_i(M; R) = \operatorname{rank}_R(H_i(M; R)/T_i(M; R)),$$

 $q_i(M; R) = \operatorname{minimum number of generators of } T_i(M; R)$

The research was supported by a grant from the Israel Academy of Sciences and Humanities, by the Hermann Minkowski Center for Geometry and by EPSRC grant GR/M20563

with $T_i(M; R) \subseteq H_i(M; R)$ the torsion submodule, and

$$\mu_i(M;R) = b_i(M;R) + q_i(M;R) + q_{i-1}(M;R)$$

The Morse inequalities

$$c_i(f) \ge b_i(M; R) + q_i(M; R) + q_{i-1}(M; R)$$

are thus an algebraic consequence of the existence of the Morse-Smale complex.

Now for circle-valued functions. Given a Morse function $f: M \to S^1$ let $c_i(f)$ denote the number of critical points of index *i*. The Novikov complex of [N] is a based f.g. free chain complex $C^{Nov}(M, f)$ over the principal ideal domain

$$\mathbb{Z}((z)) = \mathbb{Z}[[z]][z^{-1}],$$

such that

(i)
$$\operatorname{rank}_{\mathbb{Z}(z)} C_i^{Nov}(M, f) = c_i(f)$$

(ii) $C^{Nov}(M, f)$ is chain equivalent to $C(M; \mathbb{Z}((z)))$, with

$$\rho: \mathbb{Z}[\pi_1(M)] \xrightarrow{f_*} \mathbb{Z}[\pi_1(S^1)] = \mathbb{Z}[z, z^{-1}] \to \mathbb{Z}((z)) .$$

The chain complex $C^{Nov}(M, f)$ is constructed geometrically using the gradient flow. The Novikov inequalities

$$c_i(f) \ge \mu_i \big(M; \mathbb{Z}((z)) \big) = b_i \big(M; \mathbb{Z}((z)) \big) + q_i \big(M; \mathbb{Z}((z)) \big) + q_{i-1} \big(M; \mathbb{Z}((z)) \big)$$

are an algebraic consequence of the existence of a chain complex $C^{Nov}(M, f)$ satisfying (i) and (ii). The $\mathbb{Z}((z))$ -coefficient Betti numbers are called the *Novikov* numbers of M. The Novikov numbers depend only on the cohomology class $\xi = f^*(1) \in H^1(M)$, and so may be denoted by

$$b_i ig(M; \mathbb{Z}((z)) ig) \; = \; b_i(\xi) \; , \; q_i ig(M; \mathbb{Z}((z)) ig) \; = \; q_i(\xi) \; .$$

A map $f: M \to S^1$ classifies an infinite cyclic cover $\overline{M} = f^* \mathbb{R}$ of M. We shall assume that M and \overline{M} are connected, so that there is defined a short exact sequence

$$0 \to \pi \to \pi_1(M) \xrightarrow{f_*} \pi_1(S^1) = \mathbb{Z} \to 0$$

with $\pi = \pi_1(\overline{M})$. Let $z \in \pi_1(M)$ be such that $f_*(z) = 1$, so that

$$\pi_1(M) = \pi \times_{\alpha} \mathbb{Z} , \ \mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi]_{\alpha}[z, z^{-1}] .$$

with $\alpha: \pi \to \pi; g \mapsto z^{-1}gz$ the monodromy automorphism.

Let Σ denote the set of square matrices with entries in $\mathbb{Z}[\pi_1(M)]$ having the form 1-ze where e is a square matrix with entries in $\mathbb{Z}[\pi]$. A ring morphism $\mathbb{Z}[\pi_1(M)] \to R$ is called Σ -inverting if it sends matrices in Σ to invertible matrices over the ring R. There exists a noncommutative localization in the sense of P. M. Cohn [C], a

ring $\Sigma^{-1}\mathbb{Z}[\pi_1(M)$ together with a ring morphism $\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ which has the universal property that every Σ -inverting homomorphism $\mathbb{Z}[\pi_1(M)] \to R$ has a unique factorization

$$\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to R$$
.

In particular, the inclusion of the group ring $\mathbb{Z}[\pi_1(M)]$ in the Novikov completion

$$\mathbb{Z}[\widehat{\pi_1(M)}] = \mathbb{Z}[\pi][[z]]_{\alpha}[z^{-1}]$$

is Σ -inverting, so that there is a factorization

$$\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}]$$
.

Pazhitnov [P1] extended the geometric construction of the Novikov complex to a based f.g. free chain complex $C^{Nov}(M, f)$ over $\mathbb{Z}[\widehat{\pi_1(M)}]$, such that

- (i) $\operatorname{rank}_{\mathbb{Z}[\widehat{\pi_1(M)}]} C_i^{Nov}(M, f) = c_i(f),$
- (ii) $C^{Nov}(M, f)$ is chain equivalent to $C(M; \mathbb{Z}[\pi_1(M)])$, with

$$\rho = \text{inclusion} : \mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}] .$$

Moreover, Pazhitnov [P2],[P3] showed that the geometric construction of $C^{Nov}(M, f)$ can be adjusted so that the differentials are rational, in the sense that the entries of their matrices belong to $\operatorname{im}(\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}])$. If $\pi_1(M)$ is abelian and $\alpha = 1$ the localization is a subring of the completion

$$\Sigma^{-1}\mathbb{Z}[\pi_1(M)] = (1 + z\mathbb{Z}[\pi])^{-1}\mathbb{Z}[\pi][z, z^{-1}] \subset \mathbb{Z}[\widehat{\pi_1(M)}] = \mathbb{Z}[\pi][[z]][z^{-1}]$$

so that $C^{Nov}(M, f)$ is induced from a chain complex defined over $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$.

Our main result is a direct construction of such a rational lift of the Novikov complex, which is valid for arbitrary $\pi_1(M)$ (not necessary abelian) and arbitrary α :

Main Theorem. For every Morse function $f: M \to S^1$ there exists a based f.g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complex $\widehat{C}(M, f)$ such that

- (i) $\operatorname{rank}_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]}\widehat{C}_i(M, f) = c_i(f),$
- (ii) $\widehat{C}(M, f)$ is chain equivalent to $\Sigma^{-1}C(\widetilde{M})$.

The Main Theorem is proved in section 3 by cutting M at the inverse image $N = f^{-1}(x) \subset M$ of a regular value $x \in S^1$ of f, to obtain a fundamental domain $(M_N; N, zN)$ for \overline{M} , and considering the chain homotopy theoretic properties of the handle decomposition of the cobordism, with $c_i(f)$ *i*-handles.

The existence of the chain complex $\widehat{C}(M, f)$ has as immediate consequence:

Corollary (Generalized Novikov inequalities). For any Morse function $f : M \to S^1$ and any Σ -inverting ring morphism $\rho : \mathbb{Z}[\pi_1(M)] \to R$

$$c_i(f) \ge \mu_i(M; R)$$
.

Recall that the number $\mu_i(M; R)$ is defined as the minimum number of generators in degree *i* of any f.g. free *R*-module chain complex, which is chain homotopy equivalent to C(M; R). The numbers $\mu_i(M; R)$ are homotopy invariants of *M* as a space over S^1 (via *f*); they depend on the homotopy class of *f* and on the ring homomorphism ρ .

As an example consider the following ring of rational functions

$$\mathcal{R} = (1 + z\mathbb{Z}[z])^{-1}\mathbb{Z}[z, z^{-1}]$$

introduced in [F]. This ring is a principal ideal domain [F]. The induced homomorphism $f_*: \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ determines a ring homomorphism

$$\rho: \mathbb{Z}[\pi_1(M)] \to \mathcal{R} \; ; \; g \mapsto z^{f_*(g)} \quad (g \in \pi_1(M))$$

It is easy to see that ρ is Σ -inverting. The corollary implies the inequalities

$$c_i(f) \ge \mu_i(M; \mathcal{R}), \quad i = 0, 1, 2, \dots$$

These inequalities coincide with the classical Novikov inequalities since

$$\mu_i(M; \mathcal{R}) = \mu_i(M; \mathbb{Z}((z))) = b_i(\xi) + q_i(\xi) + q_{i-1}(\xi) .$$

The equivalence between this approach (using the ring \mathcal{R} of rational functions) and the original approach of Novikov (which used the formal power series ring $\mathbb{Z}((z))$) was proved in [F].

2. The endomorphism localization.

This section describes the endomorphism localization of a Laurent polynomial extension, and provides the algebraic machinery required for the construction of the chain complex $\widehat{C}(M, f)$ in section 3.

Given a ring A and an automorphism $\alpha : A \to A$ let $A_{\alpha}[z], A_{\alpha}[[z]], A_{\alpha}[z, z^{-1}], A_{\alpha}((z))$ be the α -twisted polynomial extension rings of A, with z an indeterminate over A with $az = z\alpha(a)$ $(a \in A)$; for the record, $A_{\alpha}[z]$ is the ring of finite polynomials $\sum_{j=0}^{\infty} a_j z^j, A_{\alpha}[[z]]$ is the ring of power series $\sum_{j=0}^{\infty} a_j z^j, A_{\alpha}[z, z^{-1}]$ is the ring of finite Laurent polynomials $\sum_{j=-\infty}^{\infty} a_j z^j$, and $A_{\alpha}((z)) = A_{\alpha}[[z]][z^{-1}]$ is the Novikov ring of power series $\sum_{j=-\infty}^{\infty} a_j z^j$ with only a finite number of non-zero coefficients $a_j \in A$ for j < 0.

Definition 2.1 Let Σ be the set of square matrices in $A_{\alpha}[z, z^{-1}]$ of the form 1 - zewith e a square matrix in A. The *endomorphism localization* $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ is the noncommutative localization of $A_{\alpha}[z, z^{-1}]$ inverting Σ in the sense of Cohn [C]. \Box By construction, $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ is the ring obtained from $A_{\alpha}[z, z^{-1}]$ by adjoining generators corresponding to the entries in formal inverses $(1 - ze)^{-1}$ of elements $1 - ze \in \Sigma$, and the relations given by the matrix equations

$$(1-ze)^{-1}(1-ze) = (1-ze)(1-ze)^{-1} = 1$$
.

Given an A-module B let zB be the A-module with elements $zx \ (x \in B)$ and A acting by

$$A imes zB o zB$$
; $(a, zx) \mapsto azx = z\alpha(a)x$

If B is a f.g. free A-module with basis $\{b_1, b_2, \ldots, b_r\}$ then zB is a f.g. free A-module with basis $\{zb_1, zb_2, \ldots, zb_r\}$. For f.g. free A-modules B, B' an $A_{\alpha}[z, z^{-1}]$ -module morphism $f: B_{\alpha}[z, z^{-1}] \to B'_{\alpha}[z, z^{-1}]$ is a finite Laurent polynomial

$$f = \sum_{j=-\infty}^{\infty} z^j f_j : B_{\alpha}[z, z^{-1}] \to B'_{\alpha}[z, z^{-1}]$$

with coefficients A-module morphisms $f_j : z^j B \to B'$. Note that in the special case $\alpha = 1 : A \to A$ there is defined a natural A-module isomorphism

$$B \to zB$$
; $x \mapsto zx$.

Proposition 2.2. (i) A ring morphism $A_{\alpha}[z, z^{-1}] \to R$ which sends every $1 - ze \in \Sigma$ to an invertible matrix in R has a unique factorization

$$A_{\alpha}[z, z^{-1}] \to \Sigma^{-1} A_{\alpha}[z, z^{-1}] \to R$$

(ii) If E is a f.g. free A-module then for any A-module morphism $e: zE \to E$ there is defined an automorphism of a f.g. free $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ -module

$$1 - ze : \Sigma^{-1} E_{\alpha}[z, z^{-1}] \to \Sigma^{-1} E_{\alpha}[z, z^{-1}]$$

(iii) The inclusion $A_{\alpha}[z, z^{-1}] \to A_{\alpha}((z))$ factorizes through $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$

$$A_{\alpha}[z, z^{-1}] \to \Sigma^{-1} A_{\alpha}[z, z^{-1}] \to A_{\alpha}((z))$$

with $A_{\alpha}[z, z^{-1}] \to \Sigma^{-1} A_{\alpha}[z, z^{-1}]$ an injection.

Proof. (i) This is the universal property of noncommutative localization. (ii) By construction.

(iii) Every matrix of the type 1 - ze is invertible in $A_{\alpha}((z))$, with

 $(1-ze)^{-1} = 1+ze+z^2e^2+\dots$

so that (i) applies. \Box

For commutative A and $\alpha = 1 : A \to A$

$$\Sigma^{-1}A[z, z^{-1}] = (1 + zA[z])^{-1}A[z, z^{-1}]$$

is just the usual commutative localization inverting the multiplicative subset

$$1 + zA[z] \subset A[z, z^{-1}]$$

We shall need the following

2.3. Deformation Lemma. Let C be an A-module chain complex of the form

$$d_C = \begin{pmatrix} d_D & a & c \\ 0 & d_F & b \\ 0 & 0 & d_{D'} \end{pmatrix} : C_i = D_i \oplus F_i \oplus D'_i \to C_{i-1} = D_{i-1} \oplus F_{i-1} \oplus D'_{i-1}$$

where $a: F_i \to D_{i-1}, b: D'_i \to F_{i-1}$ and $c: D'_i \to D_{i-1}$. Suppose that the morphism $c: D'_i \to D_{i-1}$ is an isomorphism for all *i*. The formula

$$\widehat{d}_C = d_F - bc^{-1}a : F_i \to F_{i-1}$$

defines a "deformed differential" on F_i (i.e. $(\hat{d}_C)^2 = 0$), and the chain complex \hat{C} defined by

$$\widehat{d}_C$$
 : $\widehat{C}_i = F_i \to \widehat{C}_{i-1} = F_{i-1}$

is chain equivalent to C.

Proof. Note that $(d_C)^2 = 0$ implies

$$(d_D)^2 = 0, (d_F)^2 = 0, (d_{D'})^2 = 0,$$

and also

(1)
$$d_D a + a d_F = 0 ,$$
$$d_F b + b d_{D'} = 0 ,$$
$$d_D c + c d_{D'} + a b = 0 .$$

Using (1) we obtain

$$(\widehat{d}_C)^2 = (d_F - bc^{-1}a) \cdot (d_F - bc^{-1}a) = -d_F bc^{-1}a - bc^{-1}ad_F + bc^{-1}a \cdot bc^{-1}a = -d_F bc^{-1}a - bc^{-1}ad_F - bc^{-1}[d_D c + cd_{D'}]c^{-1}a = 0.$$

Now we define two chain maps:

$$u = (-bc^{-1} \quad 1 \quad 0): C \to \widehat{C}, \text{ and } v = \begin{pmatrix} 0\\ 1\\ -c^{-1}a \end{pmatrix}: \widehat{C} \to C.$$

One checks that

$$ud_C = \hat{d}_C u, \quad d_C v = v\hat{d}_C, \quad uv = 1_{\widehat{C}}$$

and

$$vu = 1_C - d_C w - w d_C$$
, where $w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^{-1} & 0 & 0 \end{pmatrix} : C \to C.$

Hence u and v are mutually inverse chain equivalences. \Box

Theorem 2.4. Let C be a finite based f.g. free $A_{\alpha}[z, z^{-1}]$ -module chain complex of the form

$$C = \mathcal{C}(g - zh : D_{\alpha}[z, z^{-1}] \to E_{\alpha}[z, z^{-1}])$$

where $g: D \to E$, $h: zD \to E$ are chain maps of finite based f.g. free A-module chain complexes, and C denotes the algebraic mapping cone. If each $g: D_i \to E_i$ is a split injection sending basis elements to basis elements, then there is defined a $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ -module chain complex \widehat{C} such that

(i) each \widehat{C}_i is a based f.g. free $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ -module with

$$\operatorname{rank}_{\Sigma^{-1}A_{\alpha}[z,z^{-1}]}\widehat{C}_{i} = \operatorname{rank}_{A}E_{i} - \operatorname{rank}_{A}D_{i} ,$$

(ii) there is defined a chain equivalence $\Sigma^{-1}C \to \widehat{C}$.

Proof. Let $F_i \subseteq E_i$ be the submodule generated by the basis elements not coming from D_i , so that

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_{i} \to E_{i} = D_{i} \oplus F_{i} ,$$

$$h = \begin{pmatrix} e \\ f \end{pmatrix} : zD_{i} \to E_{i} = D_{i} \oplus F_{i} ,$$

$$d_{E} = \begin{pmatrix} d_{D} & a \\ 0 & d_{F} \end{pmatrix} : E_{i} = D_{i} \oplus F_{i} \to E_{i-1} = D_{i-1} \oplus F_{i-1} ,$$

$$d_{C} = \begin{pmatrix} d_{D} & a & 1 - ze \\ 0 & d_{F} & -zf \\ 0 & 0 & d_{D} \end{pmatrix} :$$

$$C_{i} = (D_{i} \oplus F_{i} \oplus D_{i-1})_{\alpha}[z, z^{-1}] \to C_{i-1} = (D_{i-1} \oplus F_{i-1} \oplus D_{i-2})_{\alpha}[z, z^{-1}] .$$

Now apply Lemma 2.3 to the induced chain complex $\Sigma^{-1}C$ over $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ with

$$d_{\Sigma^{-1}C} = \begin{pmatrix} d_D & a & 1-ze \\ 0 & d_F & -zf \\ 0 & 0 & d_D \end{pmatrix} :$$

$$\Sigma^{-1}C_i = \Sigma^{-1}(D_i \oplus F_i \oplus D_{i-1})_{\alpha}[z, z^{-1}]$$

$$\to \Sigma^{-1}C_{i-1} = \Sigma^{-1}(D_{i-1} \oplus F_{i-1} \oplus D_{i-2})_{\alpha}[z, z^{-1}]$$

where each

$$c = 1 - ze : \Sigma^{-1}(D_{i-1})_{\alpha}[z, z^{-1}] \to \Sigma^{-1}(D_{i-1})_{\alpha}[z, z^{-1}]$$

is an automorphism. Explicitly, Lemma 2.3 gives a based f.g. free $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ -module chain complex \widehat{C} with

$$\widehat{d}_{C} = d_{F} + (zf)(1-ze)^{-1}a : \widehat{C}_{i} = \Sigma^{-1}(F_{i})_{\alpha}[z, z^{-1}] \to \widehat{C}_{i-1} = \Sigma^{-1}(F_{i-1})_{\alpha}[z, z^{-1}]$$

which satisfies (i) and (ii). \Box

3. The chain complex $\widehat{C}(M, f)$.

Given a Morse function $f: M \to S^1$ we now construct a chain complex $\widehat{C}(M, f)$ over $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ satisfying the conditions of the Main Theorem.

Choose a regular value $x \in S^1$ for $f: M \to S^1$, and cut M along the codimension 1 framed submanifold

$$N^{m-1} = f^{-1}(x) \subset M^m$$

to obtain a fundamental domain $(M_N; N, zN)$ for the infinite cyclic cover

$$\overline{M} = \bigcup_{j=-\infty}^{\infty} z^j M_N$$

of M, with a Morse function

$$f_N$$
 : $(M_N; N, zN) \to ([0, 1]; \{0\}, \{1\})$

such that f_N has exactly as many critical points of index i as f

$$c_i(f_N) = c_i(f) .$$

Let \widetilde{N} , \widetilde{M}_N be the covers of N, M_N obtained from the universal cover \widetilde{M} of M by pullback along the inclusions $N \to \overline{M}$, $M_N \to \overline{M}$. The cobordism $(M_N; N, zN)$ has a handle decomposition

$$M_N = N \times I \cup \bigcup_{i=0}^m \bigcup_{c_i(f)} h^i$$

with $c_i(f)$ *i*-handles $h^i = D^i \times D^{m-i}$. Let $\pi_1(\overline{M}) = \pi$ with monodromy automorphism $\alpha : \pi \to \pi$, so that $\pi_1(M) = \pi \times_{\alpha} \mathbb{Z}$, $\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi]_{\alpha}[z, z^{-1}]$ as in the Introduction. The relative cellular chain complex $C(\widetilde{M}_N, \widetilde{N})$ is a based f.g. free $\mathbb{Z}[\pi]$ -module chain complex with

$$\operatorname{rank}_{\mathbb{Z}[\pi]} C_i(\widetilde{M}_N, \widetilde{N}) = c_i(f) .$$

Choose an arbitrary CW structure for N, and let $c_i(N)$ be the number of *i*-cells. Let M_N have the CW structure with $c_i(N) + c_i(f)$ *i*-cells: for each *i*-cell $e^i \subset N$ there is defined an *i*-cell $e^i \times I \subset M_N$, and for each *i*-handle h^i there is defined an *i*-cell $h^i \subset M_N$. Then $M = M_N/(N = zN)$ has a CW complex structure with $c_i(f) + c_i(N) + c_{i-1}(N)$ *i*-cells. The universal cover \widetilde{M} has cellular $\mathbb{Z}[\pi_1(M)]$ -module chain complex

$$C(\widetilde{M}) = \mathcal{C}(g - zh : C(\widetilde{N})_{\alpha}[z, z^{-1}] \to C(\widetilde{M}_N)_{\alpha}[z, z^{-1}])$$

where

$$g : C(\widetilde{N}) \to C(\widetilde{M}_N) , h : zC(\widetilde{N}) \to C(\widetilde{M}_N)$$

are the $\mathbb{Z}[\pi]$ -module chain maps induced by the inclusions

$$g : N \to M_N$$
, $h : zN \to M_N$.

Since $g: N \to M_N$ is the inclusion of a subcomplex

$$g = \begin{pmatrix} 1\\ 0 \end{pmatrix}$$
 : $C_i(\widetilde{N}) \to C_i(\widetilde{M}_N) = C_i(\widetilde{N}) \oplus C_i(\widetilde{M}_N, \widetilde{N})$

is a split injection. Now apply Theorem 2.4 to the based f.g. free $\mathbb{Z}[\pi_1(M)]$ -module chain complex

$$C(\widetilde{M}) = \mathcal{C}(g - zh : C(\widetilde{N})_{\alpha}[z, z^{-1}] \to C(\widetilde{M}_N)_{\alpha}[z, z^{-1}])$$

with $D = C(\widetilde{N}), E = C(\widetilde{M}_N).$

References

- [C] P.M. Cohn, Free rings and their relations, Academic Press, 1971.
- [F] M. Farber, *Exactness of the Novikov inequalities*, Functional Anal. Appl. **19** (1985), 40–48.
- [N] S. P. Novikov, The hamiltonian formalism and a multi-valued analogue of Morse theory, Russian Math. Surveys 37:5 (1982), 1–56.
- [P1] A.V. Pazhitnov, On the Novikov complex for rational Morse forms, Annales de la Faculté de Sciences de Toulouse 4 (1995), 297–338.
- [P2] _____, Incidence coefficients in the Novikov complex for Morse forms: rationality and exponential growth properties, preprint dg-ga/9604004 (1996).
- [P3] _____, Simple homotopy type of Novikov complex for closed 1-forms and Lefschetz ζ -function of the gradient flow, preprint dg-ga/9706014 (1997).

SCHOOL OF MATHEMATICAL SCIENCES, TEL-AVIV UNIVERSITY, RAMAT-AVIV 69978, ISRAEL

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF EDINBURGH, EDINBURGH EH9 3JZ, SCOTLAND, UK

E-mail address: farber@math.tau.ac.il, aar@maths.ed.ac.uk