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Fundamental group of homotopy colimits

Emmanuel Dror Farjoun*

Department of Maths, Hebrew University of Jerusalem, Jerusalem, Israel

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Abstract

We give a general version of theorems due to Seifert–van Kampen and Brown about the fundamental group of topological spaces. We consider here the fundamental group of a general homotopy colimit of spaces. This includes unions, direct limits and quotient spaces as special cases. The fundamental group of the homotopy colimit is determined by the induced diagram of fundamental *groupoids* via a simple commutation formula. We use this framework to discuss homotopy (co-)limits of groups and groupoids as well as the useful Classification Lemma 6.4. Immediate consequences include the fundamental group of a quotient spaces by a group action $\tilde{\pi}_1(K/G)$ and of more general colimits. The Bass–Serre and Haefliger's decompositions of groups acting on simplicial complexes is shown to follow effortlessly. An algebraic notion of the homotopy colimit of a diagram of groups is treated in some detail.

Keywords: Fundamental groups; Groupoids; Pushouts; Homotopy colimits; van Kampen-Seifert theorem

1. Introduction and main results

In the present note we reformulate the Seifert–van Kampen theorem, concerning the fundamental group of a union of spaces, and extend it to a tautology about the fundamental groupoid, $\tilde{\pi}_1 \operatorname{hocolim}_I \mathbf{X}$, of the homotopy colimit of an arbitrary diagram of (arbitrary) spaces. By a diagram of spaces $\mathbf{X}$ over a small category I, which serves as an indexing category, we mean a functor $\mathbf{X} : I \to \mathscr{S}$ from I to a convenient category of topological spaces, simplicial complexes, or CW-complexes. If the indexing category is a discrete group considered as a category with a single

*Fax: +972-256-30702.
E-mail address: farjoun@huji.ac.il.

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object, a diagram is just a space with a given action of that group. Brown and others wrote extensively on this subject, using groupoids, see [4–8,23], the first work being a survey. In a sense, some of present material is "folklore" implicit in the work of Anderson [1] and later work of Thomason and coauthors about homotopy (co-)limits and the homotopy theory of small categories. Here we relate their work to that of Brown, Higgins and coauthors, which also contains, at least implicitly, wide swathes of the present work.

Rather than dealing with the representation of a space Y as a union of subspaces we consider direct limits of an arbitrary small diagram of spaces $Y = \mathrm{colim}_{i \in I}\, Y(i) = \mathrm{colim}_I\, \mathbf{Y}$. This covers both group actions and unions as special cases. In fact, as homotopy theory realized long ago, it is more efficient and correct to work with *homotopy (co)limits* of spaces. This allows one, as explained below, to modify diagrams, locally at every index $i \in I$, by "weak equivalences" without changing the resulting homotopy colimit, greatly facilitating computations and comprehension.

Notice that given an I-diagram of spaces $\mathbf{X}$—e.g. a space with an action by a group G, thinking of G as I, a category with one object—we cannot, in general, assume the existence of base points preserved by all the maps in the diagram which would allow associating with $\mathbf{X}$ the corresponding diagram of the fundamental groups $\pi_1(\mathbf{X}, *)$. Nor can we assume that the fixed point spaces for group actions are connected or even non-empty. Therefore, we are forced to work with fundamental groupoids in order to get a closed formula for the fundamental group $\pi_1(\mathrm{hocolim}_I\, \mathbf{X})$ of a homotopy colimit or of a strict colimit, e.g. an orbit space. Recall that a groupoid is simply a small category in which any arrow is (uniquely) invertible. In fact, as a category any groupoid is equivalent to a (possibly empty) disjoint union of groups—which are the most commonly known groupoids. Once inside this framework neither local nor global connectivity assumptions are needed on the spaces or the diagrams. As an illustration consider the very special case of connected and pointed spaces, which already illuminates the problem one faces:

Question. Is there a functor $\Phi : \{I - \text{diagrams of Groups}\} \to \{\text{Groups}\}$ with the property that, for a small (indexing) category, I, and an I-diagram of *connected and pointed* spaces, $\mathbf{X}_*$, its value on the induced diagram of fundamental groups, gives the fundamental group of the homotopy colimit: $\Phi(\pi_1 \mathbf{X}_*) \cong \pi_1\, \mathrm{hocolim}_I\, \mathbf{X}_*$.

Such a functor would compute the fundamental group of any such homotopy direct limit from the corresponding diagram of fundamental groups, which here we assume for a moment to exist.

This kind of question arises naturally in, say, the presentation of groups acting on trees and, more generally, simplicial complexes. A combinatorial viewpoint discussed below of this case gives a quick and natural approach to well-known results of Bass–Serre and Haefliger concerning decomposition of groups acting on trees and other complexes and the fundamental group of an orbit space for non-free action. See Sections 4.1–4.3 below.

The above question is "nearly meaningful", and the answer is conditionally affirmative, only if the diagram I is "connected". The question relates to the unpointed homotopy colimit of pointed spaces; such a limit might be a disconnected disjoint union. Further, the colimit is not naturally pointed—thus it has no well-defined fundamental group. An appropriate functor Φ would give, for a connected I, a connected groupoid, and so only an "isomorphism type" of a group. It is not the usual colimit functor for groups, but rather something that can be properly called the *homotopy colimit of a diagram of groups* of which HNN-extensions and semi-direct products are special cases (see 3.5 and 3.7 below). Given the resulting groupoid, one can obtain a particular group by choosing a base point, i.e. an object in the groupoid. Even under the strong assumptions in the question above, the associated diagram of fundamental groups does not "determine" the fundamental group of the homotopy colimit via a purely group theoretical construction, i.e. inside the category of groups via some universal property such as the colimit. Rather it is necessary to enlarge the category of groups to that of groupoids and consider (co-)limits and homotopy (co-)limits there.

From the correct, abstract, point of view, the homotopy colimit of groups is obtained by embedding the category of groups in the larger category of groupoids where there is a natural notion of *weak equivalence which is not an isomorphism*; see 3.1 below or compare with [1] and the appendix in [3]. This notion of weak equivalence leads to a corresponding notion of homotopy limit and colimit, see [1, Section 2]. Then one notices that in the above situation a group can be extracted from the homotopy colimit of such a diagram of groups. In anticipation, we can say that in the above situation Φ can be written as $\pi_1(\operatorname{hocolim} K(\mathbf{G}, 1), *)$, where $K(G, 1)$ is any construction of the corresponding pointed aspherical space which is functorial in G and $*$ is some base point. The general construction is combinatorial-categorical and leads to groupoids, see 5.4.

1.1. The main theorem

In the following formulation, the Seifert–van Kampen theorem is given as an equivalence of groupoids. There is no connectivity assumption, in fact some or all spaces in the diagram may well be empty. The theorem uses the notion of a homotopy colimit of groupoids as explained below. It is related to groups proper since the equivalence 1.1 says that *the two disjoint unions of groups encoded by the two sides of the equation-equivalence are strictly isomorphic*. A diagram of spaces $\mathbf{X}$ is simply a functor $\mathbf{X} : I \to \mathscr{S}$ from a small category I, the indexing category, to the relevant category of spaces; in particular commutativity inside diagrams $\mathbf{X}$ is strict. For convenience we work in the category of simplicial sets, simplicial complexes, or CW-complexes, but this is not essential for the results or arguments.

The fundamental groupoid of a complex (K, K_0), where K_0 is a choice of zeroth skeleton for K, denoted here by $\tilde{\pi}_1 K = \tilde{\pi}_1(K, K_0)$, is taken to be the small category whose set of "objects" is K_0—the 0-cells (= vertices) of K and whose "morphisms" are the homotopy classes of paths, i.e. maps $[0, 1] = \mathbb{I} \to K$ between any two of these 0-cells, relative to their end-points. Thus $\tilde{\pi}_1 K$ contains the full information about the set of components $\pi_0 K$ as well as the isomorphism type of the fundamental group of each component, and this in a natural way. Clearly, the fundamental group of a space does not change its isomorphism type when one changes the base point within its own path component. It is similarly not hard to show that the, equivalence type of $\tilde{\pi}_1 Y$ *as a category* does not change when we replace given vertex point $\bigstar \in K_0$ by any other non-empty set of base points, all in the component of the given point $\bigstar$.

The construction of $\tilde{\pi}_1$ is clearly functorial on maps of complexes. Given a diagram $\mathbf{X}$ of simplicial sets (or of simplicial complexes and simplicial maps) it thus induces a diagram of groupoids $\tilde{\pi}_1\mathbf{X}$, since maps are assumed to preserve vertices. The general Seifert–van Kampen–Brown Theorem is now given by a commutation formula:

Theorem 1.1. *For any I-diagram of spaces* $\mathbf{X}$ *there is a natural equivalence of groupoids*:

$$\tilde{\pi}_1 \operatorname{hocolim}_I \mathbf{X} \xrightarrow{\cong} \operatorname{hocolim}_I \tilde{\pi}_1\mathbf{X}.$$

If the homotopy colimit is a connected space, this gives a corresponding isomorphism of groups. Further, if the given diagram of spaces has free 0*-skeleton* $\mathbf{X}_0$ (*see Section* 3.3 *below*), *then we can replace the right-hand side by the strict colimit of groupoids*: $\operatorname{colim}_I \tilde{\pi}_1\mathbf{X}$.

1.2. Remarks

The proof is straightforward and appears in the last section below. The theorem is not surprising in view of the equivalence between the model categories of groupoids and that of aspherical spaces. But note that the analogous formula for a homotopy (inverse) limit of spaces fails completely without strong assumptions on the spaces involved; compare 6.5 below. As it stands, it is impossible to replace homotopy colimit by strict colimit (= direct limit) in this theorem; see however the discussion in Sections 2, 4.3 and 4.4 below. This equivalence is entirely analogous to other "folklore", tautological equivalences of sets of path components and of chain complexes:

$$\pi_0 \operatorname{hocolim} \mathbf{X} \xrightarrow{\cong} \operatorname{hocolim}_{Sets} \pi_0\mathbf{X},$$

$$C_* \operatorname{hocolim} \mathbf{X} \xrightarrow{\cong} \operatorname{hocolim}_{Chains} C_*\mathbf{X},$$

where hocolim of sets is just the usual colimit (i.e. direct limit) of the diagram of sets and hocolim in the second equivalence is inside the category of chain complexes.

The above results refer to *unpointed* homotopy colimits. A corresponding result for the pointed homotopy colimit of a diagram of pointed spaces follows easily.

1.3. Organization of the paper

The rest of the paper is organized as follows: In Section 2 we illustrate with simple examples the topological difference on the π_1-level, between the colimit and homotopy colimit. In Section 3 we give the necessary combinatorial material that allows one to define in an elementary way the homotopy (co-)limit of a diagram of groups and groupoids and state some basic properties thereof. We then turn to our main examples and applications in Section 4. The special case of pointed diagrams and pointed homotopy limits is treated in Section 5. Section 6 ends the paper with a proof of the main result 1.1, using the tools introduce earlier.

2. Fundamental group of colimits and homotopy colimits

To motivate further the introduction of fundamental groupoid and homotopy limits, consider the problem of giving a formula for the fundamental group of a quotient space $\pi_1(K/G,*)$ in terms of $\pi_1 K$, the group G and some relation between them. One problem is that to define $\pi_1 K$ we need a base point in K and this base point is not always fixed under the action of G; so that there is no induced action of G on $\pi_1(K,*)$. Another problem is that $\pi_1 K$ might well be trivial as for $K = S^2$, the 2-sphere with the antipodal action by $G = \mathbb{Z}/2\mathbb{Z}$, while $\pi_1(K/G,*) \cong \mathbb{Z}/2\mathbb{Z}$ is not trivial. Even when we consider an appropriate set of fundamental groups, say $\{\pi_1(K, g(*))\}_{g\in G}$, in the sphere case it consists of a collection of trivial groups. Similar problems arise for a general diagram of spaces over some small category I. This problem has little to do with the assumption of connectivity which hinders the direct use of the usual Seifert–van Kampen theorem for unions. But they share a common resolution discovered and elaborated on in the case of unions by Brown, namely, we replace the fundamental group by the fundamental groupoid $\tilde{\pi}_1 Y$ which is associated with a simplicial or combinatorial complex Y. This comes at a price (or with a bonus): We allow Y to possess an arbitrary (non-empty) set of base points. Hence, the main extra structure we must assume on our spaces is a 0-skeleton Y_0 that, in fact, might be quite an arbitrary collection of points in Y. However, here we consider Y_0 to be a discrete subset coming from an actual 0-skeleton of a simplicial structure. For convenience of discussion we take spaces to be simplicial sets or complexes, but CW spaces will do just as well. An implicite assumption used in the main results is that every path component of Y contains at least one vertex of Y_0. Further, when considering an I-diagram of spaces, $\mathbf{X}$, or a space with a group action, we mean of course that $\mathbf{X}_0$ is invariant under the action of the group or diagram, namely, that $\mathbf{X}_0$ has an induced action of the group or, more generally, of a small category I.

For example, when considering the sphere with the antipodal action by $G = \mathbb{Z}/2\mathbb{Z}$, we need S^2 to have either no base point (where we get the empty groupoid for both S^2 and S^2/G) or at least two antipodal base points. In case of two points, we get as $\tilde{\pi}_1(S^2, \{x, y\})$ the groupoid: $J = [x \rightleftarrows y]$ denoted by J, with two objects and two mutually inverse arrows between them. This antipodal action on the 2-sphere is free as an action of group on a spaces, hence, as is well known (see Sections 3.3 and 3.5), the homotopy colimit is equivalent to the usual strict colimit, namely, the projective plane $\mathbb{R}P^2$. Further, the induced action of our G on both objects and morphisms of J is also free. The main result says in this case that $\tilde{\pi}_1(S^2/G)$ is equivalent to the quotient category J/G, which is a category with a single object and a single self-inverse non-identity arrow. In other words, the groupoid J/G is just $\mathbb{Z}/2\mathbb{Z}$ as expected for $\tilde{\pi}_1(S^2/G) = \tilde{\pi}_1\mathbb{R}P^2$; compare Corollary 4.2 below.

2.1. Counter-example

As it stands one cannot replace in Theorem 1.1 the homotopy colimits by strict colimit. For example, consider the action of $G = \mathbb{Z}/2\mathbb{Z}$ on the unit interval $\mathbb{I} = [0, 1]$ sending x to $1 - x$. The unit interval $\mathbb{I}$, taken with $\{0, 1\}$ as its vertices, clearly has the groupoid J above as its fundamental groupoid. However, in $\mathbb{I}/G$ there is a single base point and $\tilde{\pi}_1(\mathbb{I}/G) = \{*\}$, the trivial groupoid. But as we said above $(\tilde{\pi}_1\mathbb{I})/G = J/G = \mathbb{Z}/2\mathbb{Z}$. So there is no commutation with respect to strict colimit. On the other hand, the *homotopy colimit* of the space $\mathbb{I}$ with respect to the G-action is in fact equivalent to the infinite-dimensional projective $\mathbb{R}P^\infty$ because the unit interval with this G-action is weakly equivalent via $\mathbb{I} \to *$ to a single point with the trivial action. Therefore, the claim of Theorem 1.1 holds here since, J being free over G, the colimit and homotopy colimit of J over G coincide by Section 3.3. Notice that the space $\mathbb{I}$ is a legitimate example of simplicial space with group action, but its fixed point subspace does not form a sub-complex; nor is the map to the orbit space simplicial. A simple examination shows that the introduction of a middle base point to $\mathbb{I}$, via a subdivision, to render the fixed points simplicial sub-complex, corrects the problem we had here with the strict colimit since the colimit or quotient groupoid of $* \rightleftarrows * \rightleftarrows *$ is just the groupoid J above. The homotopy limit approach will allow us to examine the same problem for a general group action and I-diagrams of simplicial complexes or, indeed, spaces; see Section 4.3. The basic fact behind this counter example is the lack of a good notion of strict colimits for geometrical simplicial complexes.

3. Free resolutions and homotopy colimits of diagrams of groupoids

The Seifert–van Kampen Theorem 1.1 identifies the fundamental groupoid up to equivalence. Recall that by a weak equivalence of categories one means a functor $F : \mathscr{A} \to \mathscr{B}$ between small categories that induces a weak equivalence on their nerves. This is a weaker demand than *equivalence* of categories which requires the existence of two functors, mutually inverse up to natural *equivalences* ($=$ isomorphisms) of functors. In the special case of groupoids the existence of a weak equivalence F implies that the two categories are actually equivalent via F in the usual sense that F has an inverse up to a natural equivalence:

3.1. Nerves, weak equivalence of categories and groupoids

The *nerve* (or classifying space) of a category C is a simplicial set denote by $|C|$, whose k-simplices are sequences of length k of composable arrows: $c_0 \to c_1 \to \cdots \to c_k$. For a recent and detailed exposition see [15, Sections 4.10, 5.11]. Another important source which gives the general axiomatic approach and some basic definitions is [1, Sections 2 and 5]. The boundary and degeneracy maps $d_i, s_i : |C|_k \to |C|_{k \pm 1}$ are obtained by composition and insertion of identities. Thus, the nerve can be thought of as the simplicial set of singular simplices in C. While the nerve of a general category might be a quite arbitrary simplicial set, a basic property of the classifying space or nerve of a *groupoid* is that it is always a *Kan complex* and in addition it is homotopy equivalent to a disjoint union of $K(\Pi, 1)$'s. See [15, Section (5.10)], [22, p. 91]. This gives, e.g. using Theorem 6.4, a version of a Whitehead theorem namely, that a weak equivalence of groupoids is always an equivalence of them as categories. This will also mean that the fundamental group of $|G|$ for a groupoid G can be more easily approached by combinatorial devices, cf. Section 6.2. Notice, moreover, that the nerve $|-|$ is right adjoint, up to homotopy, to $\tilde{\pi}_1$ defined above. This, by itself, makes one expect the commutation equivalence in Theorem 1.1. Clearly, $\eta : G \to H$ is an equivalence of groupoids if and only if it induces an isomorphism on $\pi_0(\eta) : \pi_0 G \xrightarrow{\cong} \pi_0 H \equiv \pi_0|H|$ and for each object $x \in G$ an isomorphism $G(x,x) \equiv \operatorname{aut} x \xrightarrow{\cong} \operatorname{aut} \eta(x) \equiv H(x,x)$, where we denote as usual by $G(x,y)$ the set of morphisms in the category G from x to y. In the following we consider the set obj-Γ for a groupoid Γ as a discrete sub-groupoid of Γ. The above remarks imply the straightforward and well known:

Lemma 3.1. *Given an equivalence of groupoids $\eta : G \xrightarrow{\cong} H$ and a map of groupoids $f : \Gamma \to H$ together with a lift $\tilde{f}_0 : obj\text{-}\Gamma \to G$ on the objects of Γ, the map $\tilde{f}_0$ can be extended* uniquely *to a lift $\tilde{f} : \Gamma \to G$ of f. Therefore, if, in addition, the equivalence η is an isomorphism on objects then it is an isomorphism of groupoids.*

We now need a concept of homotopy colimit for diagram of groupoids (and, in particular, of groups) which is invariant under weak equivalence $F : \mathscr{P} \to \mathscr{P}'$ between two I-diagrams of groupoids. An equivariant map of diagrams of categories, F, is a *weak equivalence* if, for each $i \in I$, the functor $F(i)$ is a weak equivalence of categories. Notice that even if $F(i)$ is an equivalence for all $i \in I$ it does not mean that there is an inverse, up to natural equivalence, to F. For example, the weak equivalence $\eta : J \to *$ of groupoids with $\mathbb{Z}/2\mathbb{Z}$-action as in Section 2.1 above has no equivariant inverse, in fact there is no equivariant map whatsoever $* \to J$.

As is well known, in general, the usual notion of limit and colimit of categories are not (even weakly) invariant under weak equivalence of diagrams. Notice that the colimit includes in particular the notion of a free product of groups. Therefore, in the colimit there are new morphisms $[g]\circ[f]$ coming as compositions from old ones f, g that in the colimit have range$[f]$ = domain$[g]$.

To define the homotopy colimit of a diagram of groupoids, we consider below *free diagrams* of groupoids. (In fact, in what follows, we give a certain skewed presentation of the basics in the model category of groupoids and diagrams thereof, but no knowledge of that model category is needed for a formal understanding of the definitions and proofs.)

By the groupoid *associated* with or *generated* by a (small) category, we mean the groupoid associated in the evident way with a category via its nerve $\pi C \equiv \tilde{\pi}_1|C|$, where it is understood that objects do not change: obj $C =$ obj πC. Equivalently, πC is obtained from C by formally inverting all the morphisms in C. It is clear that π is left adjoint to the forgetful (inclusion) functor $\mathscr{G}pd \to \mathscr{C}at$ from groupoids to (small) categories. More formally:

Definition 3.2. Given a category C we say that the groupoid Gd is generated by or associated with C if Gd comes with a functor (i.e. a map of categories) $f : C \to Gd$ which is initial for functors $C \to H$ of C to any groupoid H.

3.2. Existence, example

Using π as defined above, there is a localization map $C \to \pi C$. If C is a groupoid, then by universality (or by construction) $C \to \pi C$ is an isomorphism. Therefore, π is an idempotent functor $\mathscr{C}at \to \mathscr{C}at$; in fact π is the localization L_j with respect to the functor that inverts one arrow in $\mathscr{C}at$:

$$j : [\to] \to [\leftrightarrows].$$

If $C \to C'$ is a weak equivalence of categories, then it induces an equivalence of groupoids $\pi C \to \pi C'$.

Example. Consider the category with two objects and two morphisms:

$$\begin{tikzcd} * \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & * \end{tikzcd}$$

The associated groupoid is denoted here by $\tilde{\mathbb{Z}}$ and is equivalent to the group of integers $\mathbb{Z}$. It has infinitely many morphisms but can be pictured by two morphisms and their inverses since compositions are implied.

$$\begin{tikzcd} * \arrow[r, bend left, leftrightarrow, "a^{-1}, a"] \arrow[r, bend right, leftrightarrow, "b^{-1}, b"'] & * \end{tikzcd}$$

3.3. Diagrams and free diagram of sets and groupoids

Given a diagram of small categories $\mathbf{C} : I \to Cat$ we associate with it the evident diagram of groupoids $\pi\mathbf{C}$. We say, as usual, that a map of two diagrams of (small) categories $\mathbf{C}, \mathbf{C}'$, namely natural transformation between the two functors $e: \mathbf{C} \to \mathbf{C}'$ is a weak equivalence if, when restricted to each place in the diagram $C(i) \to C'(i)$, it is a weak equivalence of categories.

Note: While every groupoid is equivalent to a disjoint union of groups, in general an I-diagram of groupoids *is not weakly equivalent* to any I-diagram of (disjoint union of) groups. For example, it is not hard to check directly that the groupoid $\tilde{\mathbb{Z}}$ defined above together with the group action by $I = G = \mathbb{Z}/2\mathbb{Z}$, which switches the two objects and sends the morphism a to b^{-1} is not weakly equivalent to any group with a G-action. It cannot be related by an equivariant G-map $e: \tilde{\mathbb{Z}} \to \mathbb{Z}$ or a map $e: \mathbb{Z} \to \tilde{\mathbb{Z}}$ to the group of integers with whatever action by the same group G, via a map e which is a weak equivalence of groupoids.

To define free diagram of groupoids we start with discrete groupoids—i.e. diagrams of sets—and use a generalization of the usual concepts of free G-orbit and of a set with a free action of G, for a group G. To define a free I-diagram of sets we start with special case. The *elementary* free I-diagram of sets generated by an object $i \in I$, denoted here by F^i, is the functor $F^i : I \to \mathscr{S}ets$ defined by $F^i(d) = \hom(i, d)$ where the hom-set is the set of morphisms in I and where compositions induce $F^d \to F^{d'}$ for any $d \to d'$ in I. In other words, F^i is the functor represented by $i \in I$, cf. Appendix of [10]. A free I-diagram of sets is just any disjoint union of elementary free diagrams, i.e. of the form $\coprod_{i \in T} F^i$. Thus, a free diagram of sets is always associated with a diagram of sets T over the discrete version of I, namely I^δ. There is a pair of adjoint functors where a free diagram is one of the form $\mathrm{Free}(T)$, for some discrete I^δ-diagram T:

$$\{\text{Discrete } I^\delta\text{-diagrams}\} \underset{Forget}{\overset{Free}{\rightleftarrows}} \{\text{I-diagrams}\}$$

The basic property of these free diagrams of sets that makes them useful is the Yoneda lemma. A map $\mathbf{F} \to \mathbf{S}$ of a free diagram of sets $\mathbf{F}$ to any diagram of sets $\mathbf{S}$ is uniquely determined by its values on the generators of $\mathbf{F} = \coprod F^d$, a generator of F^d can be assigned an arbitrary value in the set $\mathbf{S}(d)$. This allows good control over maps from free diagrams and is, of course, implied by the above adjunction.

Example. Over the pushout category $\bigstar \leftarrow \bigstar \to \bigstar$, a free diagram of sets is a diagram of sets $S_1 \leftarrow S_2 \to S_3$ in which both arrows are injective. Over the opposite, pullback, category $\bigstar \to \bigstar \leftarrow \bigstar$, a free diagram of sets is a corresponding diagram of injections whose images intersect in the empty set. The usual infinite inverse limit diagram $\cdots \to S_n \to \cdots \to S_2 \to S_1$ of sets is free if all maps in it are injective and that any given $s \in S_1$ cannot be lifted beyond some $S_N \to S_1$ $(N = N(s))$, namely, it is a tower of injections with empty inverse limit lim $S_i = \emptyset$. Lastly an important example of free diagrams is associated with group actions on sets and is used below in 4.3. If the discrete group acts on a set S one associates with it a diagram of sets over the opposite orbit category of G namely, $\mathcal{O} = \{G/H\}^{op}_{H \subseteq G}$. This is the functor $S^{\mathcal{O}}$ that assigns to G/H the fixed points set in S of the subgroup H i.e. $S^{\mathcal{O}}(G/H) = Map_G(G/H, S) \subseteq S$. The diagram $S^{\mathcal{O}}$ breaks up as disjoint union of free diagram indexed by S/G. If $S \cong \coprod_i G/H_i$ as a G-set than $S^{\mathcal{O}} = \coprod_i F^{G/H_i}$, cf. [11]. Notice that the passage to $S^{\mathcal{O}}$ converts an arbitrary G-set S into a free $\mathcal{O}$-diagram of sets from which S can be recovered as a G-set.

Using the concept of free diagram of sets one defines and uses free diagrams of other structures given by sets and maps between them. We say that a diagram of simplicial sets $\mathbf{X}$ is free if in each dimension $\mathbf{X}_n$ it gives a free diagram of sets. A diagram of categories is free if both objects and morphisms are free as diagrams of sets, see Section 6.2. But in the definition of a free diagram of groupoids we mind only the objects.

Definition 3.3. A diagram of groupoids $\mathbf{G}$ is called a *free diagram* of groupoids if the corresponding diagram of objects obj-$\mathbf{G}$ is free as a diagram of sets.

Thus, free diagram of groupoids is not, in general, free as a diagram of categories since only its underlying diagram of objects is assumed to be free. Nothing is assumed about the diagram of morphisms. An important example of a free diagram of groupoids is $\pi\mathbf{X}$ where $\mathbf{X}$ is a free diagram of spaces. The main property of free diagrams of groupoids we need says that for them weak equivalence implies equivalence:

Proposition 3.4. *Let $f: \mathbf{G} \to \mathbf{G}'$ be a weak equivalence between free I-diagrams of groupoids. Then f is an equivalence i.e. f has an inverse up to natural equivalence.*

Proof. We need to show the existence of homotopy inverse to f. More generally, we show that given f, u in a triangle of diagrams of groupoids

$$\begin{tikzcd} & \mathbf{G} \arrow[d, "f"] \\ \mathbf{F} \arrow[ur, "\tilde{u}"] \arrow[r, "u"'] & \mathbf{G}' \end{tikzcd}$$

with $\mathbf{F}$ free and f a weak equivalence, the map u always admits a lift $\tilde{u}$ up to natural equivalence and any two lifts differ by a natural transformation. (In the language of Section 6.2 below, the function complexes $\mathrm{Nat}(\mathbf{F}, \mathbf{G})$ and $\mathrm{Nat}(\mathbf{F}, \mathbf{G}')$ are equivalent as groupoids.) The uniqueness up to homotopy follows from Lemma 3.1 since the diagram $J \times \mathbf{F}$ is free for a free $\mathbf{F}$, where J is the unit interval groupoid from Section 2. To obtain the lift $\tilde{u}$, we proceed, as usual, by factoring the map f into a embedding $e: \mathbf{G} \to \bar{\mathbf{G}}$ via an equivalence, in fact deformation retract; followed by a surjection $\bar{f}: \bar{\mathbf{G}} \to \mathbf{G}', f = \bar{f} \circ e$. The diagram $\bar{\mathbf{G}}$ is obtained by adding a free I-set F^i_x to obj-$\mathbf{G}$ for every element $x \in \mathbf{G}'(i)$ not in the image of $\mathbf{G}(i)$; then connecting the generator with a single copy of $J \times F_x^i$, gluing the generator to any object in the component lying over that of $\mathbf{G}'(i)$. We define $\bar{f}$ on generator of F_x^i sending it to x while the new added morphism is mapped to any morphism in $\mathbf{G}'$ connecting x to an object in the right path component (this can be done naturally by taking one copy of F_x^i for each path). Again since $F^i \times J$ is free as a diagram of categories, this defines uniquely a map on $\bar{\mathbf{G}}$. We use the assumed isomorphism of the diagrams of sets of components $\pi_0 f : \pi_0\mathbf{G} \xrightarrow{\cong} \pi_0\mathbf{G}'$. A contraction of this added J gives the deformation claimed. Therefore we can assume, that f is surjective. Now one first constructs a lifting up to natural transformation $\tilde{u}_0 :$ obj-$\mathbf{F} \to \mathbf{G}$ of the diagram of objects considered as a diagram of discrete groupoids. This can done by the decomposition of the domain obj-$\mathbf{F}$ as a disjoint union of elementary free I-sets. Then one proceeds to extend $\tilde{u}_0$ by observing that the weak equivalence implies that at each object $x \in \mathbf{G}(i)$ the map f induces an isomorphism $\operatorname{aut} x \to \operatorname{aut} f(x)$ so using the proof Lemma 3.1 there is a unique extension $\tilde{u}$ of $\tilde{u}_0$ to all of $\mathbf{F}$. □

3.4. Example (cf. [1, Section 5], and further [2,20], appendix)

Consider the category $I = [* \to *]$ with two object and one non-identity map. Any map of groups $\phi : H \to H'$ (that is, of groupoids with one object each: o, o') is a diagram over I. Note that ϕ gives a free diagram of singletons–objects $o \to o'$. Thus ϕ is a free diagram of groupoids over I. In [1] ϕ is regarded as a special case of cofibrations of groupoids, the point being that cofibrations are maps injective on objects. It is moreover of the form $\pi\mathbf{F}$ for $\mathbf{F}$ a free diagram of categories, where both objects and morphisms give free diagrams of sets. However, a double arrow diagram of groups $G \rightrightarrows H$ is never free since it is not free on objects. This is the root cause of the special care taken in constructing HNN-extensions as in Section 4.2 below.

3.5. Hocolim of a diagram of groupoids G

One important advantage of groupoids over groups is the existence of free resolutions: any I-diagram of groupoids $\mathbf{G}$ can be resolved by a free diagram of groupoids via a weak equivalence which we take surjective on the objects for each $i \in I : \phi :\ \mathbf{G}^{\text{free}} \to \mathbf{G}$. Even when $\mathbf{G}$ is a diagram of groups its resolution will be diagram of groupoids. This can be done by first resolving the diagram of spaces $|\mathbf{G}|$ by a free diagram of spaces via a weak equivalences: $e :\ \mathbf{W} \to |\mathbf{G}|$ and then taking the free resolution to be

$$\mathbf{G}^{\text{free}} = \tilde{\pi}_1 \mathbf{W};$$

with the map ϕ coming from the adjunction $\tilde{\pi}_1|\mathbf{G}| \to \mathbf{G}$. We have used the fact that one can always resolve a diagram $\mathbf{X}$ of spaces by a free diagram $F\mathbf{X} \to \mathbf{X}$ via a weak equivalence as is done for simplicial complexes and sets e.g. in [9]. Another, more canonical and intrinsic approach to resolution is to construct the co-localization map $CW_{\mathscr{F}}\mathbf{G}\to\mathbf{G}$ with respect to the set $\mathscr{F}=\{F^i\colon\ i\in I\}$ of free orbits as in [10,19]. In the present case, this construction is comprised of three steps only where we built the necessary objects, generators and relations in the free diagram of groupoids $\mathbf{G}^{\text{free}}$. We remark that even if $\mathbf{G}$ is composed of discrete groupoids its resolution will not be discrete, see below.

Given any I-diagram $\mathbf{G}$ of groupoids $\mathbf{G}: I\to\mathscr{G}pd$ there is now an evident construction of $\text{hocolim}_I\,\mathbf{G}$: we resolve it by a free diagram of groupoids $r:\ \mathbf{G}^{\text{free}}\to\mathbf{G}$ as above, (taking care for convenience that the map r is surjective on objects,) and then define the homotopy colimit by taking the usual strict colimit:

$$\text{hocolim}_I\,\mathbf{G}\equiv\text{colim}_I\,\mathbf{G}^{\text{free}}$$

which will be a groupoid. This can be done naturally by taking natural resolutions starting with the map $\coprod F^d\to\mathbf{G}$ of a disjoint union in which now we regards each F^d as a diagram of discrete groupoids and the union ranges over all objects in all the groupoids in $\mathbf{G}$. Another approach to $\text{hocolim}_I\,\mathbf{G}$ which avoids the notion of free diagram is to define the homotopy colimit as the fundamental groupoid $\pi Gr(\mathbf{G})$ of the Thomason construction of the diagram of groupoids $\mathbf{G}$; see [15,25, Section 5.15]. In abstract existential form this is considered in [1, Section 2] especially Theorem 2.9. We shall see in the examples and in 4.2 below that the notion of free diagram of groupoids helps in the recognition of certain strict colimits as homotopy colimits.

3.6. Uniqueness of resolutions and hocolim

We note that the homotopy colimit as defined above is independent of the chosen resolution by a free diagram of groupoids up to equivalence of groupoids. This is true since by the proof of Propositions 3.4 and 3.1 any two free resolutions $\mathbf{Y},\mathbf{Y}'$ of a diagram of groupoids $\mathbf{G}$:

$$\mathbf{Y}\xrightarrow{f}\mathbf{G}\xleftarrow{f'}\mathbf{Y}'$$

are equivalent via a map of diagram invertible up to a natural equivalence. Such an equivalence induces an equivalence on the strict colimits of these diagrams of groupoids. The equivalence is easier to see if both f,f' are surjective on the objects. However, surjectivity is not essential, we assume it only for convenience.

3.7. Examples

- Consider the trivial groupoid $*$ with a action of a group G. A resolution of $*$ is given by the map $EG=[\text{Objects},\text{Morphisms}]=[G,G\times G]\to *$ from the free and contractible groupoid EG with one arrow for each ordered pair of elements in G—which simply the “one-skeleton” of the simplex spanned by the underlying set of G. Thus to get the homotopy colimit of $*=[*,*]$ we simply divide EG by the action of G: $\text{hocolim}_G\,\{*\}=G$. More generally, if $*$ is the trivial groupoid over any small category I namely, it is the I-diagram of singletons then clearly $\mathrm{hocolim}_I\,\{*\} \cong \pi I \cong \tilde{\pi}_1\,\mathrm{hocolim}\,|*| \cong \tilde{\pi}_1|I|$—e.g. as a consequence of Theorem 1.1 above using $* = \tilde{\pi}_1|*|$.

- Another simple example is obtained by considering a group G acting on any other group L as a diagram consisting of a single groupoid denoted by $\tilde{L} = [*, L]$ over the small category G. The resolving groupoid is the free G-groupoid $\tilde{L} \times EG = [G, L \times G \times G]$ with the diagonal action. It is weakly equivalent to $\tilde{L}$ since they differ by a contractible factor. The resulting homotopy colimit is isomorphic to the usual semi-direct product of groups $L \tilde{\times} G$ as also follows from 5.4 below.
- Consider the usual presentation of the circle as a homotopy colimit of two points P_i:

$$*_1 \leftarrow \{P_1, P_2\} \rightarrow *_2\ .$$

If we think of this as a diagram of discrete groupoids, it is not free; to turn it into a free diagram we replace each object denoted by $\{*_i\}$ by the contractible groupoid: $J = * \leftrightarrows *$, formed by two mutually inverse arrows, see Section 2.1. Now the colimit of the new free diagram will give us the groupoid $\tilde{\tilde{\mathbb{Z}}}$ from Section 3.2; up to equivalence, the group of integers $\mathbb{Z}$. Diagrammatically, we get the equivalence of groupoids:

$$\widetilde{\pi}_1\ \mathrm{hocolim}\,(* \longleftarrow \{P_1, P_2\} \longrightarrow *) \cong P_1 \rightleftarrows P_2 \cong \mathbb{Z}.$$

- Consider the homotopy colimit of the following two parallel arrows between two copies of the integers.

$$\mathrm{hocolim}(\mathbb{Z} \underset{\times 2}{\overset{\times 3}{\rightrightarrows}} \mathbb{Z}) \cong \mathbb{Z}.$$

The colimit is the zero group (0), but the homotopy colimit is equivalent to the integers. Additional parallel arrows of whatever value will increase the rank of the hocolim—they do not change the value of the (strict) colimit. This can be seen directly by adding objects to the range.

- Here is an evident corollary, which includes, of course, the usual formulation of the van Kampen theorem for the category $* \leftarrow * \rightarrow *$:

Corollary 3.5.
An I-diagram of groups is a free diagram of groupoids if and only if the category I has an initial object.

Proof.
For such a small category the orbit generated at the initial object is free. □

In such a case we can replace the homotopy colimit of a diagram of groups by the usual colimit as is done in the usual formulation of the Seifert–van Kampen theorem.

4. Applications to orbits spaces and unions

All the known cases of the van Kampen theorems from Seifert's to Brown's formulation for certain unions of spaces follow easily from the present formulation, see Theorem 4.3 below. Notice that if the given I-diagram of spaces $\mathbf{X}$ consists of single points $X_i = pt$, then the equivalence in Theorem 1.1 becomes a presentation of the fundamental group of the nerve $|I|$ as homotopy colimit of singletons groupoids. For the category of low-dimensional simplices of a simplicial complex, with inclusions as morphisms, it gives the canonical presentation of the fundamental group of a simplicial complex with the 1-simplices as generators and 2-simplices as relations.

4.1. Bass–Serre presentation of groups acting on trees

Consideration of the Bass–Serre tree of groups is natural here as was noticed by Higgins, as well as its extension to higher dimension cases as was done by Haefliger using different, more geometric, methods [17,18,24]. It should be noticed that the well known concept of HNN-extension is a special case of the notion of homotopy limit of groups and groupoids explained here, cf. [24, Sections 1.4 and 5.1] Example 3 with examples in Section 3.7 and the discussion in Section 4.2.

Bass–Serre theory of group action on trees writes a "colimit presentation" of a group G acting on a contractible 1-complex, i.e. a tree L, in terms of the stabilizers of various simplices in the colimit (= orbit space L/G) of the action. Here it is natural to use homotopy colimit since this way we can use efficiently the information that the tree is homotopy equivalent to a point via an G-equivariant map $L \to pt$, this collapse map induces a homotopy equivalence $\mathrm{hocolim}_G\, L \cong \mathrm{hocolim}_G\,(pt) = K(G,1)$ and thus the desired group G appears as the fundamental group of $\mathrm{hocolim}_G\, L$ with any base point. So we are looking for a decomposition of $\pi_1\, \mathrm{hocolim}_G\, L$. The version of Seifert–van Kampen theorem above 1.1 gives just the Bass–Serre presentation when we present L as a G-cell complex in the usual way. To proceed we need to take sub-complexes of fixed points. Hence we assume that any element $g \in G$ that sends a simplex $\sigma \subseteq L$ to itself does this by the identity map. This is always achieved after a single subdivision, see [24, Section 3.1].

We now outline a decomposition of the (space level) Borel construction $EG \times_G L = \mathrm{hocolim}_G\, L$ for a simplicial G-action as above on any complex L, using sub-complexes of fixed points as further explained in Section 4.3 below:

$$
\begin{aligned}
\mathrm{hocolim}_G\, L &\cong \mathrm{hocolim}_G\, \mathrm{hocoend}_{G/H \in \mathcal{O}}\, (L^H \times G/H) \\
&\cong \mathrm{hocoend}_{G/H \in \mathcal{O}}\, \mathrm{hocolim}_G\, (L^H \times G/H) \\
&\cong \mathrm{hocoend}_{G/H \in \mathcal{O}}\, \{L^H \times BH\} \cong \mathrm{hocolim}_{\sigma \in L/G}\, BH_\sigma.
\end{aligned}
$$

Only the last step needs extra explanation, but by naturality it is sufficient to check it for a single G-orbit $L = G/H = L^H$ where it is clear. Compare [9] and appendix of

[10] where the notion of homotopy coend is discussed. For contractible L and in particular for a tree L the homotopy colimit is just $BG = K(G,1)$. Taking the fundamental groupoids, by Theorem 1.1 we get an equivalence of connected groupoids:

$$G \cong \mathrm{hocolim}_{\sigma \in L/G} H_\sigma.$$

It is straightforward then to check that the hocolim in the category of groupoids, which in this case gives a group up to equivalence, is exactly the usual group theoretical decomposition given by Bass–Serre [24, Section 5, Theorem 13]. In the same way, we get an analogous presentation when the action is on any 1-complex and not only on a tree. In higher dimensions one gets a presentation of a group acting on any simplicial complex, in terms of the stabilizers.

In general, one has a fibration

$$K \to \mathrm{hocolim}_G K \to BG.$$

When K is a connected space one gets an exact sequence:

$$1 \to \pi_1 K \to \mathrm{hocolim}_{\sigma \in K/G} H_\sigma \to G \to 1.$$

4.2. HNN-extensions

Compare [24, Sections 1.4 and 5.1]. Consider the two group maps $f, h : A \rightrightarrows G$ associated to loop-diagram of groups. Regard now these two maps as a diagram Φ of groupoids over the indexing category $* \rightrightarrows *$.

Claim. *For the special case when the two maps are inclusions, the homotopy colimit of Φ is equivalent to the well-known HNN-extension.*

To see this we can use Theorem 1.1 and Proposition 6.6 as follows. The homotopy colimit of the diagram of nerves, namely hocolim $|\Phi|$, can be identified as the double mapping torus $(|A| \times [0,1] \cup |G|)/\{f(a) \sim (a,0); h(a) \sim (a,1)\}$ for $a \in |A|$, the nerve or classifying space of A. This mapping torus is the strict colimit of a free diagram of double cylinder which resolves $|\Phi|$: explicitly, to present the homotopy colimit as a usual colimit we have replaced here $|G|$ by its equivalent "double hat" $|A| \times [0, 0.5] \cup_f |G| \cup_h |A| \times [0.5, 1]$ so that the given double map is now a free diagram by the example in 3.3 above since the two maps f, h become two injections with *disjoint images*: $A \times 0$ and $A \times 1$. By standard consideration (namely, taking the pull back of the universal cover $\mathbb{R} \to S^1$) the homotopy fiber of the map $|\mathrm{hocolim}\, \Phi| \to |\mathbb{Z}| = S^1$, i.e. its $\mathbb{Z}$-cover, is just (homeomorphic to) the homotopy colimit (obtained by gluing cylinders as usual) of the classifying spaces (= nerves) of the infinite "zigzag" amalgam diagram of groups in [24].

4.3. *The fundamental group of space of orbits K/G and of* $\operatorname{colim}_I \mathbf{X}$

Given an action of a group G on a space K, we ask for the fundamental group (or groupoid) of space of orbits $\pi_1(K/G, *)$ with respect to some base point: under what condition taking the orbit space commutes with taking $\tilde{\pi}_1$, up to equivalence. The orbit space is not directly a homotopy colimit so we do not apply the main theorem directly. One way to get a general formula is to rewrite this space as a homotopy colimit of an associated diagram. Let $K^{\mathcal{O}}$ be the diagram of fixed points set $\{K^H\}_{H\subseteq G} = \{\hom_G(G/H, K)\}$, where H runs over all subgroups of G. This is a diagram over $\mathcal{O}$, the small (opposite) category of all G-orbits and G-maps between them. One can restrict attention to the orbit types G/H_σ actually appearing in K. From the definition we get an isomorphism of the strict colimits $K/G \cong \operatorname{colim}_{\mathcal{O}} K^{\mathcal{O}}$. But $K^{\mathcal{O}}$ is always a free diagram, by the example in Section 3.3 and compare [9], so its colimit is equivalent to its homotopy colimit, compare (4.C.4-5) in [11]. Therefore, we have

$$K/G \cong \operatorname{hocolim}_{\mathcal{O}} K^{\mathcal{O}}.$$

In order to be in a position to use Theorem 1.1 to rewrite $\tilde{\pi}_1$ of the right-hand side, we need to assume that the fixed point sets are sub-complexes so that $K^{\mathcal{O}}$ is a diagram of simplicial (sub-)complexes. Compare with examples in Section 2.1. We say that the action of G on K is *proper* if every simplex which is sent to itself by a group element is fixed point-wise by that element. This assumption appeared already in Section 4.1 above. This means that the fixed point set K^H of any subgroup inherit the simplicial structure from K. In particular, one has the equality of zero skeletons: $K^H \cap K_0 = (K^H)_0 = (K_0)^H$, which is the only condition really needed. The induced action on the first barycentric subdivision is always proper. Under these conditions one gets the following two results, the first being immediate form Theorem 1.1.

Proposition 4.1. *Let G be a discrete group acting on a simplicial complex K by a proper action. Then there is an equivalence of groupoids*

$$\tilde{\pi}_1(K/G) \cong \operatorname{hocolim} \tilde{\pi}_1(K^{\mathcal{O}}) \equiv \operatorname{hocolim}_{H\subseteq G} \tilde{\pi}_1(K^H).$$

Notice that the proposition implies, not surprisingly, that if all the spaces K^H are simply connected then so is the quotient space K/G. This is so because the diagram of groupoids is equivalent to a free $\mathcal{O}$-diagram of trivial *groups*, say $\pi_1(X^H, pt)$, where pt is any point in X^G which is assumed to be 1-connected and in particular non-empty. Now use Corollary 3.5. From this one can deduce a commutation formula which, in a different context, appears in [5, Theorem 1.11].

Corollary 4.2. *Let G be a discrete group acting on a simplicial complex K by a proper action. Then there is a natural commutation equivalence*:

$$\tilde{\pi}_1(K/G) \cong (\tilde{\pi}_1 K)/G.$$

Proof. We use the definition of a free diagram of groupoids in order to compute the right-hand side of Proposition 4.1 as a strict colimit. To evaluate $\mathrm{hocolim}_{\mathcal{O}}\, \tilde{\pi}_1 K^{\mathcal{O}}$ ($= \mathrm{hocolim}_{H \subseteq G}\, \tilde{\pi}_1 K^H$) we refer to Definition 3.3. The sub-diagram of objects in the diagram of groupoids $\{\tilde{\pi}_1(K^H)\}_H$ is $(K^H)_0$ which is equal by our assumption of proper action to fixed points of the zero skeleton of K namely, $(K_0)^H$. We saw above—example in Section 3.3—that this last diagram is always free as $\mathcal{O}$-diagram of sets. From this we get that the $\mathcal{O}$-diagram of groupoids $\tilde{\pi}_1(K^{\mathcal{O}})$ is free. Thus we can use the last part of Theorem 1.1 or Definition 3.3 above and take colimit, rather than homotopy colimit of this diagram of groupoids.

This gives that the right-hand side of 4.1 is equivalent to $\mathrm{colim}_{\mathcal{O}}\, \tilde{\pi}_1(K^{\mathcal{O}})$ but all the fixed point subspaces $K^{\mathcal{O}}$ map to the simplicial complex K itself. All groupoids in the diagram $\tilde{\pi}_1(K^{\mathcal{O}})$ are mapped to $\tilde{\pi}_1 K$. In fact, the colimit of any diagram $\mathbf{Y}$ over $\mathcal{O}$ is isomorphic to quotient of the value of $\mathbf{Y}$ on the free orbit $G/\{e\} = G$, with respect to the action of G which is implicit in the very diagram $\mathbf{Y}$. So we can compute the colimit $\mathrm{colim}_{\mathcal{O}}\, \tilde{\pi}_1(K^{\mathcal{O}})$ by taking the colimit of the diagram restricted to $\tilde{\pi}_1 K = \tilde{\pi}_1 K^e$ itself. That colimit is just the quotient $(\tilde{\pi}_1 K)/G$ as claimed. $\square$

4.4. *Remark: strict colimits*

We expect that similar considerations using [9,13] will show that the same holds for the strict colimit of the barycentric subdivision of an arbitrary diagram $\mathbf{X}$ of simplicial complexes and simplicial maps over any indexing category I. Thus, the fundamental groupoid $\tilde{\pi}_1\, \mathrm{colim}_I\, \mathbf{X}'$ of the colimit of the barycentric subdivision $\mathbf{X}'$ is expected to be equivalent to $\mathrm{colim}_I\, \tilde{\pi}_1 \mathbf{X}'$, the colimit of the induced diagram of groupoids. The formula above for quotient space K/G is a special case. Similarly, the commutation with colimit will hold for any diagram of simplicial sets. The main point being that after subdivision the diagram has good strict colimit.

4.5. *Fundamental group of a union*

Here we show how to recover quickly some of the results concerning the fundamental group and groupoid of a union of spaces. Union of spaces is not on the face of it a homotopy colimit of the subspaces involved, thus some adjustment is needed before applying the main result about the fundamental groupoid of a homotopy colimit. The main point here is the conversion of a union $X = \bigcup X_i = \mathrm{colim}_I\, \mathbf{X}$ into a homotopy colimit. A good recent reference is [14]. The following is the direct analog of the evident formula: $\pi_0(\bigcup X_i) = \mathrm{colim}_{\{i,j\}}\, \pi_0(X_i \cap X_j)$.

Theorem 4.3. *Let $X = \bigcup X_i = \mathrm{colim}_I\, \mathbf{X}$ be a union of open subspaces or simplicial sub-complexes X_i. Let $\mathbf{X}(3)$ be the diagram of the spaces $X_i \cap X_j \cap X_k$ for all $i, j, k \in I$ and their inclusions. Then the general formula for the fundamental* *groupoid reduces to*

$$\tilde{\pi}_1 X \cong \mathrm{hocolim}\, \tilde{\pi}_1 \mathbf{X}(3).$$

For the special case of pointed spaces one gets:

Corollary 4.4. *In particular, if the diagram* $\mathbf{X}(3)$ *consists of connected and pointed spaces then* $\pi_1 X \cong \mathrm{colim}\, \pi_1 \mathbf{X}(3)$, *where the colimit is over the diagram of the triple intersections as above. In that case one can write the group as a quotient of the free product*: $\pi_1 X \cong (\bigstar_I \pi_1 X_i)/(r_{ij})$ *were* r_{ij} *are the obvious relations coming from the* (*pointed*) *inclusions* $X_i \cap X_j \subseteq X_i$.

Proof. Since here X is given as a colimit of X_i's, in order to apply the main result one needs to rewrite it as a homotopy colimit as in [14]. To do that, a priori we need to take all intersections $X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n}$ or in fact $\bigcap_{j \in J \subseteq I} X_j$ for all $J \subseteq I$, so as to form a free diagram of spaces with the same limit, cf. [9]. For a free diagram the homotopy limit coincides with the strict colimit which is here the union. However, the main point of the theorem is that here we need take only *triple* intersections, which include, of course, the double intersections, etc. Thus, we use the following result for $n = 1$, which is proven in [14]: For any $n \geqslant 0$, the Postnikov approximation $P_n \bigcup X_i$ is n-equivalent to the homotopy colimit over the diagram $\mathbf{X}(n+2)$ of all $(n+2)$-intersections of the spaces X_i. This is just a dimension argument: the higher intersections do not come into the computation of the second skeleton of the colimit of diagram of small categories or groupoids, on which the fundamental groupoid depends. Since we are interested in $\tilde{\pi}_1 \bigcup X_i \cong \tilde{\pi}_1 P_1 \bigcup X_i$ taking the homotopy colimit over $\mathbf{X}(3)$ gives the desired result. The last claim of the corollary follows directly from the definition of colimits of groups. $\square$

4.6. Covering

Here is an alternative view of the basic commutation result, Theorem 1.1, via covering spaces. Given a space X let $\mathrm{Cov}(X, \cong) = \mathrm{Cov}(X)$ be the category of all the covering spaces of X with *equivalences*, *i.e. isomorphisms of covers* as morphisms. Clearly, $\mathrm{Cov}(-, \cong)$ is a contra-variant functor from spaces to (large!) groupoids. This is another canonical non-pointed way to look at the fundamental group. The following statement uses the notion of homotopy (co-)limits of a (small) diagram of (possibly large) categories; compare Definition 6.2 below and [12].

Theorem 4.5. *For any* I*-diagram of spaces* $\mathbf{X}$ *we have an equivalence of groupoids*:

$$\mathrm{Cov}(\mathrm{hocolim}_I\, \mathbf{X}) \cong \mathrm{holim}_I\, \mathrm{Cov}(\mathbf{X}).$$

Proof. This is just a dual formulation to the former one. It is the basic result of covering space theory that $\mathrm{Cov}(X)$ as a category is equivalent to the category of functors to the category $\mathcal{S}ets$ of sets:

$$\mathrm{Cov}(X) \cong \mathrm{Nat}(\tilde{\pi}_1 X, \mathcal{S}ets). \qquad \square$$

4.7. The second homotopy group

A parallel formulation for the second homotopy groups of homotopy colimits is called for. It should not be hard to see that this is possible using the notion of crossed modules and its variations, see for example [21], thus extending two-dimensional versions of van Kampen theorems of Brown, Higgins and Loday concerning the second homotopy group as a crossed module. Of course, the second homotopy, in the form of a crossed module, of a homotopy colimit will depend on the corresponding first and second homotopy groupoids of the spaces in the diagram together with the additional "gluing information" encoded in the crossed modules.

5. Pointed diagrams, diagrams of groups

The above discussion can be adjusted to diagrams of pointed spaces where the maps in the diagram preserve the base points of the spaces involved. These give rise to diagrams of groups obtained by taking fundamental groups of the pointed spaces. We deduce the following from Corollary 3.5 in Section 3.7.

Corollary 5.1. *Let I be a category with initial object and let $\mathbf{X} : I \to S_*$ be a diagram of pointed and connected spaces. Then the fundamental group of the homotopy colimit is given as a colimit of groups*:

$$\pi_1 \operatorname{hocolim}_I \mathbf{X} \cong \operatorname{colim}_I \pi_1 \mathbf{X}.$$

Proof. For pointed connected spaces the fundamental groupoid can be taken to be a group. The fact that I has an initial object implies that the diagram of singletons $\{*\}$ over I is the elementary free diagram generated at that initial object. We obtain by Corollary 3.5 a free diagram of groups $\pi_1 \mathbf{X}$, hence its homotopy colimit is equivalent to its usual colimit. $\square$

For a pointed diagram of pointed spaces one might well be interested in a pointed homotopy colimit of the pointed diagram $\mathbf{X}$. This, by definition, is the cofiber in the sequence: $\mathrm{nerve}(I) = |I| = \operatorname{hocolim}_I \{*\}$:

$$\mathrm{nerve}(I) \to \operatorname{hocolim} \mathbf{X} \to \operatorname{hocolim}_* \mathbf{X}.$$

One defines the homotopy colimit for a pointed diagram of groupoids $\mathbf{G}$ to be the quotient groupoid $(\operatorname{hocolim}_I \mathbf{G})/\pi I$ via the natural map that is induced by the coherent base points map $* \to \mathbf{G}$ as in the first example in Sections 3.7 and 3.1.

The quotient in the above cofiber sequence is equivalent to the homotopy quotient (i.e. homotopy colimit or mapping cone), and the same goes for the quotient in the definition of the pointed $\mathrm{hocolim}_* \mathbf{G}$. So we can apply Theorem 1.1 to get its pointed version:

Theorem 5.2. *For any pointed diagram of pointed spaces $\mathbf{X} = \mathbf{X}_*$ there is a natural equivalence of groupoids*: $\tilde{\pi}_1 \,\mathrm{hocolim}_* \mathbf{X} \cong \mathrm{hocolim}_* \tilde{\pi}_1 \mathbf{X}$. *If the homotopy colimit is a connected space, this gives a corresponding isomorphism of groups.*

If the diagram $\mathbf{X}$ consists of pointed and connected spaces, then our diagram is equivalent to a diagram of groups. In turns out, however, that for the special case of groups the corresponding notion of *pointed* homotopy colimit of a diagram of groups collapses to the usual notion of colimit.

Proposition 5.3. *The pointed homotopy colimit of a diagram of* groups *taken as pointed groupoids is equivalent to the strict colimit of the said diagram inside the category of groups.*

Proof. We have a map of pointed groupoids: $\mathrm{hocolim}_* \mathbf{G} \to \mathrm{colim}\, \mathbf{G}$. To prove the equivalence we demonstrate that the pointed function categories to any pointed groupoid are equivalent. Consider the functor category

$$\mathrm{Nat}_*(\mathrm{hocolim}_* \mathbf{G}, H) = \mathrm{holim}_i \,\mathrm{Nat}_*(G_i, H)$$

for the pointed groupoid H. Since $\mathbf{G}$ is a diagram of groups the pointed homotopy colimit is a connected groupoid. Since we consider pointed functors we can replace H by the component of the base point and therefore, up to equivalence by the automorphisms of the base point H_*, namely, the group at the base point, without changing the type of the left-hand side. So $\mathrm{Nat}_*(G_i, H) \cong \mathrm{Nat}_*(G_i, H_*)$. But the groupoid of pointed functors between two groups is a discrete groupoid, that is, a set with identity maps, since the base point preserving natural transformation must assign the identity to the single object in the domain group which is our base point. So we can replace the right-hand side by the homotopy limit of a diagram of hom-sets $\mathrm{holim}_i \,\mathrm{Hom}_{\mathrm{gps}}(G_i, H_*) = \lim_I \mathrm{Hom}_{\mathrm{gps}}(\mathbf{G}, H_*)$. For a diagram of sets the homotopy limit is the same as the limit.

We can conclude with the required equivalence, since H_* is an arbitrary group:

$$\begin{aligned}\mathrm{Nat}_*(\mathrm{hocolim}_* \mathbf{G}, H_*) &= \lim_i \mathrm{Hom}_{\mathrm{gps}}(G_i, H_*)\\ &= \mathrm{Hom}_{\mathrm{gps}}(\mathrm{colim}_I \mathbf{G}, H_*) = \mathrm{Nat}_*(\mathrm{colim}_* \mathbf{G}, H_*). \qquad \square\end{aligned}$$

From the definition of pointed homotopy colimit and the last result follows:

Corollary 5.4. *For a diagram* $\mathbf{G} : I \to \{Groups\}$ *of groups, there is a pushout diagram of groupoids*:

$$\begin{array}{ccc} \pi I & \longrightarrow & \mathrm{hocolim}_I\ \mathbf{G} \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{colim}_I\ \mathbf{G} \end{array}$$

where πI *is the groupoid associated to the category* I. *If* I *is connected, then, up to equivalence, this is a pushout of groups.*

6. Proof of the Main Theorem 1.1

6.1. Discussion

First, note that by Definition 3.3 and Section 3.6 above the last assertion of the theorem follows from the main assertion. To prove the main assertion, we start with a basic formula, which holds generally for spaces, connected or not. For any space Y let $P_1 Y$ denote the first, aspherical, Postnikov approximation to Y; this is a canonical space whose higher homotopy groups vanish but whose set of components π_0 agrees with that of Y and π_1 of each component again agrees with that of the corresponding component in Y. Now take a diagram of spaces $\mathbf{X}$; then we have an equivalence:

$$P_1\ \mathrm{hocolim}\ \mathbf{X} \cong P_1\ \mathrm{hocolim}\ P_1\mathbf{X}.$$

This is a special case of [10, Theorem 1.D.3 and 1.A.1.1]. It can be verified directly using the universal property of the map $Y \to P_1 Y$ as initial, to homotopy, among all maps of a space Y to aspherical spaces. To do so we only need to prove that the function complexes of both sides of the above equivalence, *before the application of* P_1, into any aspherical space are equivalent, which is evident from universality. Now we will see soon that the first Postnikov approximation to any space Y depends directly and simply on the fundamental groupoid $\tilde{\pi}_1 Y$; in fact they contain nearly the same information on Y. Thus in some precise sense these properties of P_1 with respect to hocolim already give the main result, taking into account that in the proper context $\tilde{\pi}_1$ is left adjoint to the nerve $|-|$ up to homotopy. Notice that a similar commutation for P_1 fails for homotopy inverse limit. Let us spell out the details since they pass by some useful constructions and lemmas. We use the following evident result where we need crucially that $\pi_0 Y_0 \to \pi_0 Y$ is surjective, in other words there is at least one vertex in each path component of Y; this is of course given by the simplicial or even CW structure.

Lemma 6.1. *For any simplicial space or complex* Y *one has a weak equivalence*:

$$P_1 Y \xrightarrow{\cong} |\tilde{\pi}_1 Y|$$

which is natural up to homotopy.

Proof. For a simplicial space there is an obvious map $Y \to |\tilde{\pi}_1 Y|$. Using the fact that the range is a Kan complex 3.1 we send each vertex to itself and each higher simplex to the appropriate composable chain of paths coming from its 1-dimensional faces. This map factors uniquely up to homotopy through $P_1 Y$. To conclude that it is a weak equivalence we recall that both sides are aspherical and the map above clearly induces an isomorphism on path components and on the fundamental group for each component. □

The above equivalence shows that the "fundamental groupoid" of a homotopy colimit depends only on the diagram of the "fundamental groupoids" of the spaces in the given diagram. This formula is true for the non-connected version of Postnikov P_1—applied to non-connected spaces. Without assuming connected and pointed spaces we cannot work with groups and must employ groupoids—since if one has no consistent base point then we cannot go from $P_1\mathbf{X}$ to a diagram of groups.

Outline of the proof of Theorem 1.1: We will show using adjunction and homotopy (inverse) limits as considered below that the nerves of the two sides of the desired equivalence are equivalent as simplicial sets. To prove the desired equivalence of groupoids it is sufficient (3.1) that after applying the nerve functor to both sides we get an equivalence of spaces. But the nerve functor reduces the desired equivalence to the equivalence of Postnikov approximations given above. One must be careful since:

Warning. The homotopy colimit, inside the category of groupoids, may not commute with the nerve functor $|-|$; this is in contrast to the beautiful property of the Thomason construction [15]. This is because the internal hocolim in the category of groupoids loses information about higher homotopy groups. To see this, take a pushout diagram of non-injection of groups. Then, on the level of the nerve spaces, we do not get a $K(\Pi, 1)$ as a pushout. However, if we first take a pushout on the level of groups—before taking nerves—we get a group whose nerve is a $K(\Pi, 1)$. So there is no commutation in general. Of course, if we apply to the diagram of groups the Thomason construction, we do not, in general, get a group as the homotopy pushout—in the sense of Thomason—and so its nerve is not aspherical, so there is no contradiction.

This concludes our outline. We now continue by recalling:

6.2. Internal mapping categories, homotopy (inverse) limit of a diagram of groupoids

We consider the *homotopy* (*inverse*) *limit* of a diagram of categories—in fact groupoids—$\operatorname{holim} \mathscr{P}$. We will also use the pointed case for groupoids with a base object. The classification Lemma 6.4 assures the commutation of taking nerves with taking homotopy limits. The homotopy limit is in fact a slightly extended version of the *internal hom* construction that for any two categories $\mathscr{C}, \mathscr{D}$, with $\mathscr{C}$ small, gives the category of all functors and natural transformations $\mathrm{Nat}(\mathscr{C}, \mathscr{D})$ whose objects are functors $\mathscr{C} \to \mathscr{D}$ and whose morphisms are natural transformation of functors. This is carefully explained in [16, pp. 91–100]. This is the exact analog of the Bousfield–Kan construction of homotopy limits of diagram of spaces.

Example. Here is a standard example of the internal hom construction for groupoids, [1, Section 5]. Let G, H are two groups taken as categories with one object each. Then $\mathrm{Nat}(G, H)$ is a category, in fact a groupoid, whose objects are group maps, considered as functors, $\phi : G \to H$, and whose morphisms $T : \phi \to \phi'$ are given by natural transformations. Each of these assigns to the single object in G a map $T(*)$ in H namely a group element $h \in H$ with the right commutation property for each morphism in G. This amounts to the equation $h\phi(g) = \phi'(g)h$ for all $g \in G$. In other words, two group maps ϕ, ϕ' are in the same component of $\mathrm{Nat}(G, H)$ if and only if the differ by an inner automorphism in determined by $h \in H$. Thus the set of components is $\mathrm{Hom}(G, H)/\mathrm{inner}(H)$. To specify the groupoid up to equivalence we can easily check that the isomorphism type of each component $[\phi]$ is $C_\phi(H)$, the centralizer in H of the image of ϕ. This gives via Theorem 6.4 below (recalling that for constant diagrams the homotopy limit is just the internal function object $\mathrm{Nat}(-,-)$) the usual computation of the homotopy type of the topological function complex $\mathrm{map}(K(G,1), K(H,1)) \cong \mathrm{map}(|G|, |H|) \cong |\mathrm{Nat}(G, H)|$.

Given two I-diagrams of small categories $\mathscr{P}, \mathscr{Q} : I \to \mathscr{C}at$ we may consider the category of all "equivariant functors" $\mathscr{P} \to \mathscr{Q}$ as objects and natural transformation between them as morphisms. We get the category $\mathrm{Nat}_I(\mathscr{P}, \mathscr{Q})$. For example, if $\mathscr{P} = *$ then the resulting category is just the inverse limit category of the given diagram of categories $\mathscr{Q}$. The problem with the $\mathrm{Nat}_I(-,-)$-construction is a familiar one in homotopy and category theory: if one changes locally the categories in the diagram to equivalent ones, the resulting new equivariant mapping category will not necessarily be equivalent to the old one. But one is often interested in exactly such an invariance. Technically speaking, for this the range and domain must be cofibrant and fibrant correspondingly. Recall that a diagram of categories is free if the corresponding diagram of objects and morphisms are free as diagrams of set. The main example we use of free diagram of categories is the diagram of over-categories $EI = -/I$ defined and used by Bousfield and Kan [3]. We note, without explicitly using it, that a free diagram of categories is cofibrant while a diagram of groupoids is fibrant. This motivates the definition the homotopy limit $\mathrm{holim}_I\, \mathscr{P}$ of an I-diagram of groupoids $\mathscr{P}$. A more general definition of homotopy limit of diagram of categories is implicit here:

Definition 6.2. An I-diagram of categories $\mathscr{E}$ is *locally contractible* if the nerve $|\mathscr{E}(i)|$ is weakly equivalent to a point for all $i \in I$. The(EI-) homotopy limit of a diagram of groupoids $\mathscr{P}$ for a fixed EI, is defined to be the category of functors $\mathrm{holim}_I\, P \coloneqq \mathrm{Nat}_I(EI, \mathscr{P})$, where EI is a (chosen and fixed once and for all) free and locally contractible diagram of categories.

A basic result here is:

Proposition 6.3. *If* $\mathbf{G}$ *is a* I*-diagram of groupoids and* $\mathscr{E}, \mathscr{E}'$ *are two free contractible* $\mathbf{I}$*-diagrams of categories then the two categories* $\mathrm{Nat}(\mathscr{E}, \mathbf{G}), \mathrm{Nat}(\mathscr{E}', \mathbf{G})$ *are equivalent as categories.*

Proof. It is enough to prove weak equivalence, see 3.1. For a functor $\mathscr{A} \to \mathscr{B}$ to be a weak equivalence of categories it means that the functor induces a weak equivalence on their nerves $|\mathscr{A}| \to |\mathscr{B}|$. Since $|\mathbf{G}(i)|$ is a Kan complex, for weakly equivalent spaces X, Y the function complexes $\mathrm{map}(X, \mathbf{G}(i))$ and $\mathrm{map}(Y, \mathbf{G}(i))$ are equivalent. □

Now the proposition follows immediately from Section 3.1 and the following basic commutation formula 6.4, which in turn is a slight generalization of the commutation discussed in [16, pp. 91–100].

Theorem 6.4. *Let* $\mathscr{P} : I \to Cat$ *be a diagram of categories and functors between them. There is a natural weak equivalence of simplicial sets:*

$$|\mathrm{holim}_I\, \mathscr{P}| \xrightarrow{\simeq} \mathrm{holim}_I\, |\mathscr{P}|.$$

This classification formula gives immediately and effortlessly many known classification results e.g. the classification of various fibrations, Mislin's genus, Wilkerson's Postnikov conjugates. The set of components of the left-hand side is the desired unknown set of types while on the right-hand side we have the topological homotopy limit over spaces which form the "building blocks" of the final structure. See [11].

Note the following analog of 1.1 for homotopy limits of spaces, which is implicit in the work of Brown and coauthors: it follows directly from the commutation above using the fact that aspherical spaces are weakly equivalent to $|G|$ for some groupoid G.

Theorem 6.5. *For any diagram of aspherical spaces* $\mathbf{X}$ *there is a natural equivalence of groupoids:*

$$\tilde{\pi}_1\, \mathrm{holim}\, \mathbf{X} \xrightarrow{\simeq} \mathrm{holim}\, \tilde{\pi}_1 \mathbf{X}.$$

If the homotopy limit is a connected space, this gives a corresponding isomorphism of groups.

Proof of Theorem 1.1. We are now ready to put the various pieces together. The proof proceeds by showing that there is a homotopy equivalence between the two sides after taking their nerves. This establishes a weak equivalence of groupoids which we saw in 3.1 is always an equivalence. We start with a lemma which is in fact the heart of the Seifert commutation formula: □

Lemma 6.6. *For any diagram of groupoids* $\mathbf{G}$ *there is a natural equivalence of groupoids*:

$$\text{hocolim}\,\mathbf{G} \to \tilde{\pi}_1 \text{ hocolim}\,|\mathbf{G}|.$$

Proof. We proceed by considering the "dual" of the claimed equivalence: it is enough to show that for any groupoid H, the categories of functors from both sides into the groupoid H are equivalent. So we are going to show that the nerves of the mapping categories to H, of both sides of the desired equivalence, are equivalent. Consider the left-hand side. By the basic property of homotopy colimit we have

$$\text{Nat}(\text{hocolim}\,\mathbf{G}, H) \cong \text{holim}\,\text{Nat}(\mathbf{G}, H).$$

(The internal mapping construction in $\mathscr{C}at$ is denoted by Nat, see 6.2 above, we have suppressed the indexing category I from the notation.) Using the classification formula 6.4 we get

$$|\text{Nat}(\text{hocolim}\,\mathbf{G}, H)| \cong \text{map}(\text{hocolim}\,|\mathbf{G}|, |H|).$$

Now we check the right-hand side of 6.6 using 6.1: Using classification 6.4 and the equivalence: $|\tilde{\pi}_1 Y| \cong P_1 Y$, we take the nerve of the right-hand side in the lemma and we have:

$$\text{map}(|\tilde{\pi}_1 \text{ hocolim}|\mathbf{G}||, |H|) \cong \text{map}(P_1(\text{hocolim}\,|\mathbf{G}|), |H|).$$

But since the range is aspherical the right-hand side is equivalent, using $\text{map}(P_1 Y, |H|) \cong \text{map}(Y, |H|)$, to the function complex $\text{map}(\text{hocolim}\,|\mathbf{G}|, |H|)$. But this again, by the basic property of homotopy colimits and by commutation 6.4, is equivalent to the nerve $|\text{holim}\,\text{Nat}(\mathbf{G}, H)|$ of the homotopy inverse limit above, as needed. □

To continue with the proof we circumvent the non-commutation of hocolim and the nerve functor: The proof of 1.1 is completed by proving the following lemma, which by Lemma 6.1 is the desired equivalence of the groupoids after taking nerves of both sides.

Lemma 6.7. *For any diagram of spaces* $\mathbf{X}$ *there is a natural equivalence of aspherical spaces*:

$$P_1(\text{hocolim}\,\mathbf{X}) \xrightarrow{\cong} |\text{hocolim}\,\tilde{\pi}_1\,\mathbf{X}|.$$

Proof. We first notice that by the equation above

$$P_1 \text{ hocolim}\,\mathbf{X} \cong P_1 \text{ hocolim}\,P_1\mathbf{X}.$$

We also have by Lemma 6.1: $|\tilde{\pi}_1 X| \cong P_1 X$, and: $\tilde{\pi}_1 |\mathbf{G}| \cong \mathbf{G}$. Therefore we have the following chain of equivalences using 6.6 in the last equivalence:

$$P_1 \text{ hocolim } \mathbf{X} \cong P_1 \text{ hocolim } P_1 \mathbf{X} \cong P_1 \text{ hocolim } |\tilde{\pi}_1 \mathbf{X}|$$

$$\cong |(\tilde{\pi}_1 \text{ hocolim } |\tilde{\pi}_1 \mathbf{X}|)| \cong |\text{hocolim } \tilde{\pi}_1 \mathbf{X}|. \qquad \square$$

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