May 20, 1978

Dear Andrew,

This letter is about the surgery map $\sigma : [M^n, G/O] \to L_n(\pi_1 M^n)$. (I'll write to you later about your Nil groups and codim-1 splitting problems where the submanifold is 1-sided.)

The map $\sigma$ is not (in general) additive if we give $G/O$ the H-space structure coming from Whitney sum of ~~bundles~~ bundles; let me illustrate this with an example where $M^n = S^4 \times S^4$.

If $\alpha \in [M^n, G/O]$, denote its image in $[M, BO]$ by $\hat{\alpha}$; this correspondence is additive, i.e.,

$\widehat{\alpha + \gamma} = \hat{\alpha} \oplus \hat{\gamma}$ (Whitney sum of bundles).

For any $\alpha \in [S^4 \times S^4, G/O]$, $\sigma(\alpha)$ is given by the formula

$$\cancel{\sigma(\alpha) = [L_2(\hat{\alpha}), [}$$

(1) $$\sigma(\alpha) = \langle L_2(\hat{\alpha}), [S^4 \times S^4] \rangle ;$$

i.e., the Kronecker product of the 2nd L-genus of the bundle $\hat{\alpha}$ ~~...~~ with the fundamental class of $S^4 \times S^4$. (Here $L_8(0)$ is identified with $8\mathbb{Z}$.)

It is easy to construct fiber homotopically trivial bundles $Y_i$ $(i = 1, 2)$ over $S^4 \times S^4$ such that

(2) $$p_1(Y_1) = x_1 \times 1,\ p_1(Y_2) = 1 \times x_2 \quad \text{and}$$
$$p_2(Y_i) = 0 \qquad (i = 1 \text{ and } 2)$$

where $x_i \in H^4(S^4)$ and both are non-zero. (Here, $p_1, p_2$ denote the 1st and 2nd Pontryagin classes.) Clearly, there are elements $\alpha_i \in [S^4 \times S^4, G/O]$ such that $\hat{\alpha}_i = Y_i$. Consider the following calculation using formula (1)

(3) $$\sigma(\alpha_1 + \alpha_2) = \langle L_2(\widehat{\alpha_1 + \alpha_2}), [S^4 \times S^4] \rangle$$
$$= \langle L_2(\hat{\alpha}_1 \oplus \hat{\alpha}_2), [S^4 \times S^4] \rangle$$

$$= \langle L_2(\gamma_1 \oplus \gamma_2), [S^4 \times S^4] \rangle$$

$$= \langle L_2(\gamma_1) + L_1(\gamma_1)L_1(\gamma_2) + L_2(\gamma_2), [S^4 \times S^4] \rangle.$$

Recall the formulas from page 12 of Hirzebruch's book "Top. Methods in Alg. Ge."

(4) $\qquad L_1 = \frac{1}{3} p_1 \quad$ and $\quad L_2 = \frac{1}{45}(7p_2 - p_1^2).$

Substituting (2) into (4), we obtain

(5) $\qquad L_1(\gamma_1) = \frac{1}{3} x_1 \times 1, \; L_1(\gamma_2) = 1 \times \frac{1}{3} x_2 \quad$ and

$\qquad L_2(\gamma_i) = 0 \quad$ (for $i = 1$ and $2$).

Substituting (5) into (3), we obtain

(6) $\qquad \sigma(\alpha_1 + \alpha_2) = \langle \frac{1}{9} x_1 \times x_2, [S^4 \times S^4] \rangle \neq 0$;

but, by (1) and (5), we have $\sigma(\alpha_i) = 0$ (for $i = 1$ and $2$)

Hence, $\sigma(\alpha_1 + \alpha_2) \neq \sigma(\alpha_1) + \sigma(\alpha_2)$.

Best wishes,

Tom

June 15, 1978

Dear Andrew,

I don't have a copy of Rourke's exposition of Casson-Sullivan periodicity (Hsiang does); but, I'll try to recall what Hsiang and I were thinking in March, 1977 when we went over that paper. We wish to show that the diagram on page 278 of Kirby-Siebenmann's book doesn't commute. Assume it does and try to deduce a contradiction. First, observe that the group structure on $[I^4 \times M^4, G/Top]$ (replace $T^8$ by an arbitrary closed manifold and $L_8(0)$ by $L_4(\pi_1 M^4)$) is independent of the H-space structure put on $G/Top$. Hence $\Gamma$ induces a unique group structure on $[M^4, G/Top]$. Second, the way $\Gamma$

(over)

is constructed (in Rourke's exposition) shows that this group structure (on $[M^4, G/Top]$) is the same as the one obtained by using the Whitney sum H-space structure on $G/Top$. (As I recall, $\Gamma : [M, G/Top] \to [I^4 \times M, G/Top]$ is constructed by taking a normal map $f : N \to M$ representing an element $\omega \in [M, G/Top]$ and sending it to $f \times id : N \times \mathbb{C}P^2 \to M \times \mathbb{C}P^2$ representing an element $\alpha \in [M \times \mathbb{C}P^2, G/Top]$. This element is modified by subtracting an appropriate element $\gamma$ using the Whitney sum H-space structure on $G/Top$ so that the new element maps to $0$ in $[M \times S^2, G/Top]$ and hence pulls back to an element $\Gamma(\omega) \in [I^4 \times M, G/Top]$. This, I believe, is very roughly what is done. ~~by~~ If you have access to it, could you send me a copy of

Rourke's exposition?)

Now the surgery map $\Theta : [I^4 \times M^n, G/Top]$ $\longrightarrow L_{n+4}(\pi_1(M^n))$ is additive; as is $\Gamma$ by definition. Hence, $\Theta : [M^n, G/Top] \to L_n(\pi_1(M^n))$ must be additive using the Whitney sum H-space structure on G/Top. But, the example in my previous letter contradicts this; consequently, the diagram does not, in general, commute.

Best regards,

Tom

P.S. I'll write to you about the relation of your Nil group to a codim-1 splitting problem shortly.