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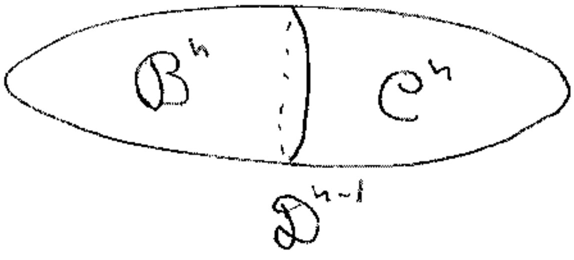
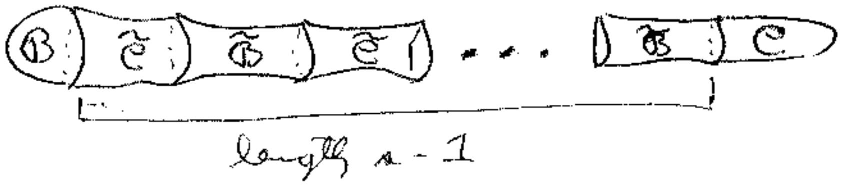
Dear Andrew,

Thanks for your letter and the copy of Rourke's notes. Let me explain the relationship of my UNil-paper to the results of Browder and Levine; actually, to be more precise, I use my fibering theorem and Siebenmann's splitting theorem. Consider a manifold M^n containing a codimension-1 submanifold $\mathcal{D}^{n-1}$ separating it into 2 halves $\mathcal{B}^n$ and $\mathcal{C}^n$ as illustrated below.



Assume the fundamental groups all inject so that $\pi_1 M = B \ast_D C$ where $B = \pi_1\mathcal{B}$, $C = \pi_1 \mathcal{C}$ and $D = \pi_1 \mathcal{D}$. Also, assume $[B : D] = 2 = [C : D]$, then D is normal in $\Gamma = \pi_1 M$ and there is an epimorphism $\varphi : \Gamma \to T_2 * T_2 = T \rtimes$ inducing this amalgamated free product structure where T denotes the ∞-cyclic group and T_s the cyclic group of order s; ...

We can define epimorphisms $\varphi_s : \Gamma \to T_s \supset T_2$ (finite dihedral group of order $2s$) for each $s \in \mathbb{Z}^+$. We wish to analyse the s-sheeted irregular covering space $\tilde{M}_s$ of M corresponding to $\Gamma_s = \varphi_s^{-1}(T_2) \subset \Gamma$. Let $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ be the 2-sheeted covers of $\mathcal{B}$, $\mathcal{C}$ corresponding to $D \subset B$, $D \subset C$, respectively. Then the picture below illustrates $\tilde{M}_s$ when s is <u>odd</u>.

$\tilde{M}_s =$ 

When s is even, there are 2 cases corresponding to the 2 conjugacy classes of elements of order 2 in Γ_s.

$\tilde{M}_s$ 

and

$\tilde{M}_s$  .

Also, the covers corresponding to $D \subset \Gamma$ and $\varphi^{-1}(T) \subset \Gamma$ are respectively

and

 .

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If $f: N \to M$ is a simple homotopy equivalence, then using Siebenmann's thesis and mine, respectively, f splits when lifted to each of the last 2 covers. Then, using the compactness of $\mathcal{D}$, we see that all of the first 3 covers also split for all s sufficiently large. Let $\sigma(f) \in UNil(D; B, C)$ be Cappell's splitting obstruction, then the above argument shows that $\sigma(f)$ vanishes under many transfer (restriction) maps. Then, applying Frobenius induction using a careful analysis of Dress's ring $GW(\mathbb{Z}, T_s \downarrow T_2)$ when s is a <u>power</u> of 2, it can be deduced that UNil has exponent 4. (One must algebraically reduce the general case to the case above ~~when~~ where $[C:D] = [B:D] = 2$.) Perhaps I should modify my paper, as you suggest, to includ its geometric origin as I described above. It should make it more readable. But, I'm getting a bit discouraged since I submitted the paper to Topology 2 years ago and have yet to hear anything from the editor Bott concerning a referee's report.

Your LNil group is related to Wall's algebraic formula (his surgery book) for the codimension - 1 splitting problem in the case of 1-sided submanifolds (Browder - Livesay situation). I promise to write you later details about this relationship.

Best regards,

Tom

9/20/79

Dear Andrew,

I believe there is a geometric motivation for your formula

$$(1) \qquad L_n(A_\alpha[x,x^{-1}]) = L_n(A) \oplus L_n(A,\alpha) \oplus \widetilde{LNil}_n(A,\alpha) \oplus \widetilde{LNil}_n(A$$

$$(\text{where } \bar{x} = x)$$

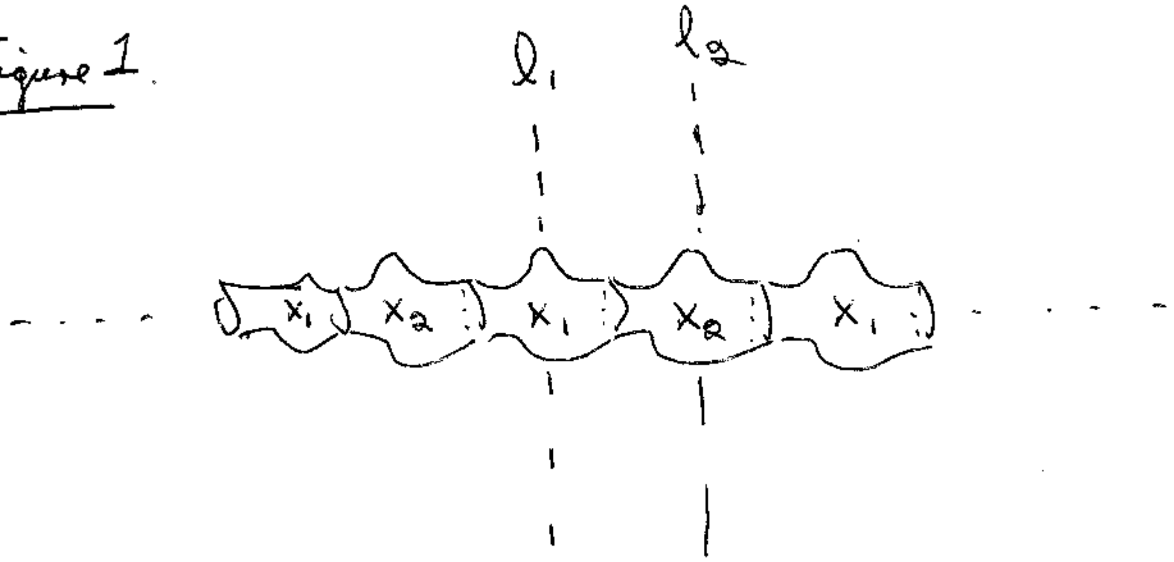
at least in the case when $\alpha = id$. (I'll have to think more about the general case $\alpha \neq id$; but I think there is a good change that it can be geometrically motivated also.) Of course, I'm assuming $A = \mathbb{Z}(\pi)$ where π is a finitely presented group. All three types of codimension-1 splitting theorems (namely; Wall's book Ch 12 parts A, B and C) are used in the geometric motivation (over)

First, we introduce some notation and make a few constructions. Let P^3 denote real projective 3-space and consider the connected sum of 2 copies of P^3 denoted P_1^3 and P_2^3, respectively, i.e., consider $P_1^3 \# P_2^3$. Let a_i denote the generator of $\pi_1(P_i)$, then $\pi_1(P_1^3 \# P_2^3)$ is free product $\mathbb{Z}_2 * \mathbb{Z}_2$ (∞-dihedral group) generated by $\{a_1, a_2\}$. Let $b = a_1 a_2$; then b generates an ∞-cyclic subgroup $\mathbb{Z}$ of $\mathbb{Z}_2 * \mathbb{Z}_2$ having index 2 in $\mathbb{Z}_2 * \mathbb{Z}_2$. Note the universal cover of $P_1 \# P_2$ is $S^2 \times \mathbb{R}$ and the intermediate cover corresponding to $\mathbb{Z} \subseteq \mathbb{Z}_2 * \mathbb{Z}_2$ is $S^2 \times S^1$. This can be seen as follows. Let

$X_i = \widetilde{P_i - D^3}$; i.e., X_i is the universal cover of $P_i - D^3$ which (of course) is $S^3 - 2$ discs $= S^2 \times [-1, 1]$

Figure 1 illustrates the universal cover of $P_1 \# P_2$

Figure 1.



Then a_i considered as a covering transformation of $\widetilde{P_1 \# P_2}$ is "reflection about l_i" and $b = a_1 a_2$ is "translation to the left by 2 unit"; from this we see that the covering space corresponding to $\mathbb{Z} \subseteq \mathbb{Z}_2 * \mathbb{Z}_2$ is ~~$S^2\times$~~ $S^2 \times S^1$. Let $p: S^2 \times S^1 \to P_1 \# P_2$ denote the covering projection and L denote the mapping cylinder of p; L is a ~~I~~ $[-1,1]$-bundle over $P_1 \# P_2$ ~~so~~ with associated sphere bundle $p: S^2 \times S^1 \to P_1 \# P_2$. Denote the projection map $L \to P_1 \# P_2$ also by p. Let M^{n-3} be an arbitrary (closed)

(over)

$(n-3)$-manifold with $\pi_1 M = \pi$. Consider the $[-1,1]$-bundle $p \times id : L \times M \to P_1 \# P_2 \times M$ and denote ~~all~~ this projection also by p. Let $\mathcal{P}_i = p^{-1}((P_i - D^3) \times M)$ and $\mathcal{S} = p^{-1}(S^2 \times M)$, then $\mathcal{S}$ is a codim-1 submanifold of $L \times M$ dividing it into 2 components $\mathcal{P}_1$ and $\mathcal{P}_2$. Also $(P_1 \# P_2) \times M$ is a codimension-1 non-separating submanifold of $L \times M$ (namely the 0-section of $p : L \times M \to (P_1 \# P_2) \times$

By examining Page 150 of Wall's book, we see that $L_n(A[x, x^{-1}])$ is isomorphic to $LN_n(\pi \times \mathbb{Z} \to \pi \times (\mathbb{Z}_2 * \mathbb{Z}_2))$ where $A = \mathbb{Z}\pi$, $\bar{x} = x$ and $\bar{g} = \omega(g) g^{-1}$ for $g \in \pi$.

We now start to geometrically motivate your decomposition formula (1). Let $\gamma \in L_n(A[x, x^{-1}])$ and represent it by a splitting problem

(2) $$f : V^{n+1} \longrightarrow L^4 \times M^{n-3} \underset{\cup\!|}{} (P_1 \# P_2) \times M$$

where f and $\partial f = f|\partial V$ are simple homotopy equivalences. (We should allow ~~M~~ $(P^3 \# P^3) \times M$ to be a manifold with boundary in ~~ord~~ order to realize all elements in $LN_n(\pi \times \mathbb{Z} \to \pi \times (\mathbb{Z}_2 * \mathbb{Z}_2))$; but I believe this is just a technical complication.)

Note that $\mathcal{S} = S^2 \times [-1, 1] \times M$ and

$\partial \mathcal{S} = S^2 \times \{-1, 1\} \times M$ ~~Consider the~~ $\subseteq \partial(L \times M)$.

Consider the splitting problem

(3) $$\partial f : \partial V \longrightarrow \partial(L \times M) = (\partial L) \times M \underset{\cup\!|}{} (S^2 \times \{-1, 1\}) \times M$$

(over)

This is a problem of the type considered in Ch 12, B of Wall's book and can be split since ∂f is a simple homotopy equivalence. But there is an obstruction to extending this splitting over δ; i.e. to solve the splitting problem

(4) $$f : V^{n+1} \longrightarrow L^4 \times M^{n-3}$$
$$\cup\!\!|$$
$$\delta = (S^2 \times [-1,1]) \times M^{n-3}.$$

Since (4) is the type of problem in Wall, Ch 12 A where δ divides $L^4 \times M^{n-3}$ into 2 components C_1 and C_2 leading to the fundamental group diagram

(5)
$$\begin{tikzcd} & \pi \arrow[dl] \arrow[dr] & \\ \pi \times \mathbb{Z}/2 \arrow[dr] & & \pi \times \mathbb{Z}/2 \arrow[dl] \\ & \pi \times (\mathbb{Z}/2 * \mathbb{Z}/2) & \end{tikzcd}$$

we obtain a Cappell UNil obstruction

in $UNil_n(\mathbb{Z}\pi ; \overline{\mathbb{Z}(\pi\times\mathbb{Z}_2)}, \overline{\mathbb{Z}(\pi\times\mathbb{Z}_2)})$,

<u>Question.</u> Can you algebraically identify this group of Cappell with ~~of the direct~~ your $\widetilde{LNil}$-groups occurring in (1); namely, to

$$\widetilde{LNil}_n(A) \oplus \widetilde{LNil}_n(A) \ ?$$

~~Pa~~ If this UNil-obstruction vanishes, then we can solve problem (4); namely, split f along $\mathcal{S}$; let $W^{n} = f^{-1}(\mathcal{S})$ in this case and $V_i = f^{-1}(P_i)$. Also by π-π type splitting ~~then~~ theorem of Browder-Wall, we can solve the splitting problem

(over)

(6)
$$f : W^n \longrightarrow \mathcal{S} = (S^2 \times [-1,1]) \times M^{n-3} \\ \cup\!| \\ (S^2 \times 0) \times M^{n-3} \, .$$

let $f^{-1}((S^2 \times 0) \times M^{n-3})$ be denoted by U^{n-1}.
Cutting ~~V~~ V^{n+1} ~~of~~ apart along W^n reduces the splitting problem (2) to two splitting problems

(7.i)
$$f : V_i^{n+1} \longrightarrow P_i^{n+1} \\ \cup\!| \\ (P_i - D^3) \times M^{n-3} \quad ;$$

note in each case ~~∂f~~ $f \mid \partial V_i = f \mid W$ is (6) and hence already split. Each of i ~~The~~ two problems ~~(7.i)~~ (7.1) and (7.2) determine an~~d~~ obstruction $\gamma_i \in LN_n(\pi \to \pi \times \mathbb{Z}_2^-)$ $= ~~L~~\; L_n(\pi)$ by ~~Th~~ Wall, Corollary 12.9.1 (page 151).

If both γ_1 and γ_2 vanish, then (2) is solvable and hence $\gamma = 0$. By the above type argument, I believe you can see an exact sequence of the form

$$(8) \qquad L_n(\pi) \oplus L_n(\pi) = L_n(A) \oplus L_n(A) \xrightarrow{\varphi}$$

$$LN_n(\pi \times \mathbb{Z} \to \pi \times (\mathbb{Z}_2 * \mathbb{Z}_2)) = L_n(A[x, x^{-1}]) \text{ (where } x$$

$$\xrightarrow{\psi} UNil_n(\cancel{\pi}\, \mathbb{Z}\pi; \overline{\mathbb{Z}(\pi \times \mathbb{Z}_2)}, \overline{\mathbb{Z}(\pi \times \mathbb{Z}_2)})$$

$$\overset{?}{=} \widetilde{LNil}_n(A) \oplus \widetilde{LNil}_n(A) .$$

I think one can also see that φ is a naturally split epimorphism, but I don't see yet why φ is monic.

Best wishes,

Tom