The Borel Conjecture

F.T. Farrell*

Binghamton University, Department of Mathematical Sciences,
Binghamton, New York, USA

Lectures given at the
School on High-Dimensional Manifold Topology
Trieste, 21 May - 8 June 2001

LNS029002

*Farrell@math.binghampton.edu Supported in part by the National Science Foundation.

Contents

0 Preface

These lectures are about showing that homotopy equivalence implies homeomorphism for a large class of manifolds. About 50 years ago Borel conjectured that this class includes all closed manifolds with contractible universal covers. A more precise statement of his conjecture is the following.

Borel Conjecture. Let $f : M \to N$ be a homotopy equivalence where both M and N are closed aspherical manifolds. Then f is homotopic to a homeomorphism.

We explain in lectures 2-5 why this conjecture is true in the following special cases (due to Farrell and Jones);

1. M is a non-positively curved Riemannian manifold and $\dim(M) \neq 3, 4$.
2. Both M and N are complete affine flat manifolds.
3. $\pi_1(M)$ is isomorphic to a discrete subgroup of $GL_n(\mathbb{R})$ for some n, and $\dim(M) \neq 3, 4$.

The Borel Conjecture has the following (slightly weaker when $n \neq 3$) group theoretic interpretation in which $\mathrm{Top}(\mathbb{R}^n)$ denotes the group of all self-homeomorphisms of $\mathbb{R}^n$ equipped with the compact open topology.

(Topological) Strong Rigidity Conjecture. Let Γ_1 and Γ_2 be any pair of isomorphic subgroups of $\mathrm{Top}(\mathbb{R}^n)$. Suppose that the two naturally induced actions on $\mathbb{R}^n$ are free, properly discontinuous, and have compact fundamental domains. Then Γ_1 and Γ_2 are conjugate subgroups inside $\mathrm{Top}(\mathbb{R}^n)$ (with the isomorphism induced by the conjugation).

Special case 3 of the Borel Conjecture yields that the (Topological) Strong Rigidity Conjecture is true under the extra assumption that there exists a linear (virtually connected) Lie group G containing Γ_1 and contained in $\mathrm{Top}(\mathbb{R}^n)$, and $n \neq 3, 4$. This partial result is an analogue of Mostow's Strong Rigidity Theorem [57] in Lie group theory. In fact it was motivated by Mostow's result although the technique of proof, surgery theory, is very different from Mostow's.

Lecture 1 is an introduction to the general problem of classifying, up to homeomorphism, all manifolds homotopically equivalent to a given manifold M. This is the topic of surgery theory. After reading lecture 1, we recommend perusing W. Lueck's lectures on surgery theory before looking at lectures 2-5 which also depend on L.E. Jones' lectures and on the first two of A. Ranicki.

1 Introduction to high dimensional manifold topology

Throughout this talk M and N will denote (connected) closed n-manifolds by which I mean (as usual) compact Hausdorff spaces which are locally homeomorphic to $\mathbb{R}^n$.

Basic Problem. Find calculable invariants which imply that M and N are homeomorphic.

This problem is easy to solve when dim $M \leq 2$. For example the circle is the only such 1-manifold. And the following is a complete list of the orientable 2-manifolds:

Figure 1.1 genus = # of holes

To study this problem when dim $M > 2$, it is helpful to use the weaker notion of *homotopy equivalence* which is easier to study than homeomorphism because of the following result.

Theorem. *(J.H.C. Whitehead) A continuous map $f : M \to N$ is a homotopy equivalence iff it induces an isomorphism on π_n for all n.*

Caveat. There are examples where $\pi_n(M) \simeq \pi_n(N)$ for all n; but M is *not* homotopically equivalent to N. Whitehead requires that the isomorphism is induced by a continuous map! Here is an explicit example.

Let $M = S^2 \times S^2$ and $N = S(\eta^2 \oplus \theta^1)$ where η^2 is the canonical $\mathbb{C}$-line bundle over $\mathbb{C}P^1 = S^2$, θ^1 is the trivial $\mathbb{R}$-line bundle and $S(\eta^2 \oplus \theta^1)$ denotes the sphere bundle associated to the Whitney sum $\eta^2 \oplus \theta^1$. Since the fibration

$$S^2 \to N \to S^2$$

has a cross section

$$\pi_n(N) = \pi_n(S^2) \oplus \pi_n(S^2)$$

but the 2nd Stiefel-Whitney class $w_2(N) \neq 0$. And recall that w_2 is an invariant of homotopy equivalences but

$$w_2(S^2 \times S^2) = 0.$$

Remark. Note that $N = \mathbb{C}P^2 \# - \mathbb{C}P^2$ and hence its cup product pairing is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

while the pairing for M is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

And these are inequivalent bilinear forms over $\mathbb{Z}$.

However there is an important special case where this worry is unnecessary.

Definition. M is *aspherical* if $\pi_n(M) = 0$ for all $n \neq 1$. (This is equivalent to requiring that the universal cover $\tilde{M}$ of M is contractible; but not necessarily that $\tilde{M} = \mathbb{R}^m$ as M. Davis will show in his lecture today.)

Corollary. *(Hurewicz) If $\pi_1(M) \simeq \pi_1(N)$ and both M and N are aspherical, then M and N are homotopically equivalent.*

Historical Remark. Hurewicz proved this result before Whitehead proved his theorem.

Examples. Every orientable (connected) 2-manifold except the sphere is aspherical (as is the circle).

More generally every non-positively curved closed Riemannian m-manifold is aspherical because of Cartan's theorem that the universal cover of such a manifold is diffeomorphic to $\mathbb{R}^m$. On the other hand, the following homogeneous space G/Γ is an example of a closed aspherical 3-manifold which does not support a non-positively curved Riemannian metric. Here G is the matrix group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

and Γ is the discrete subgroup where x, y, z are all integers. Note that the universal cover of G/Γ is G which is diffeomorphic to $\mathbb{R}^3$; hence G/Γ is aspherical and is easily seen to be compact. Also

$$\pi_1(G/\Gamma) = \Gamma$$

which is *nilpotent* but *not abelian*. However Gromoll and Wolf [40] and Yau [71] independently proved that if M is a (closed) non-positively curved Riemannian manifold and $\pi_1(M)$ is nilpotent, then $\pi_1(M)$ is abelian (and if solvable, then virtually abelian).

Basic Question. *Are homotopically equivalent closed manifolds M and N homeomorphic?* More precisely*: Is every homotopy equivalence $f : M \to N$ homotopic to a homeomorphism?*

The answer is *Yes* for 1 and 2 dimensional manifolds. But Moise [55] showed that in general the answer is *No*. In fact it is No for 3-manifolds. Let me explain. Lens spaces were studied extensively in the 1930's. These are 3-manifolds whose universal covers are the 3-sphere and have cyclic π_1's of order > 2 (and all deck transformations in $SO(4)$).

Reidemeister gave examples of pairs of Lens spaces which are homotopically equivalent but *not diffeomorphic*. (See W. Lueck's lecture 2.4 for a detailed discussion of Lens spaces which includes this result.) In particular $M = L(7; 1, 1)$ and $N = L(7; 1, 2)$ are such a pair. Here the subgroups $\pi_1(M)$ and $\pi_1(N)$ of $SO(4)$ are described as follows. Note that $SU(2) \subseteq SO(4)$. Then $\pi_1(M)$ and $\pi_1(N)$ are the cyclic subgroups of order 7 in $SU(2)$ generated by the two diagonal matrices

$$\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta & 0 \\ 0 & \theta^2 \end{pmatrix}$$

respectively, where $\theta = e^{2\pi i/7}$. Since Moise [55] showed (1952) that *homeomorphic 3-manifolds are diffeomorphic*, it follows that *No* is the answer to the *Basic Question*.

In light of Moise's result, the Basic Question was refined at this time into 2 disjoint conjectures.

1. *Hurewicz Conjecture*. Homotopically equivalent closed manifolds with $\pi_1 = 0$ are homeomorphic.

2. *Borel Conjecture*. Homotopically equivalent aspherical manifolds (i.e. with $\pi_n = 0$ for $n \neq 1$) are homeomorphic.

Remark. In Moise's example both π_1 and $\pi_3 \neq 0$.

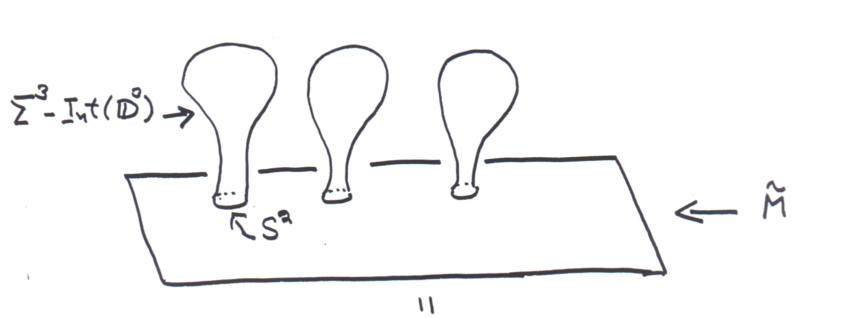
Remark. The Poincaré Conjecture is a special case of the Hurewicz Conjecture where $M = S^3$. It is still open; but the Hurewicz Conjecture has been proven when $M = S^m$, $m \neq 3$, by Smale ($m \geq 6$), Stallings ($m = 5$) and Freedman ($m = 4$).

Remark. Although S^3 is not aspherical, the Borel Conjecture also (indirectly) implies the Poincaré Conjecture. To see this we use the following two results.

Generalized Schoenflies Theorem. *(M. Brown [11]). Let $f : S^2 \to S^3$ be a bicollared embedding, then $f(S^2)$ bounds closed (topological) balls on both sides.*

Alexander Trick. *Let $h : S^n \to S^n$ be any homeomorphism. Then h extends to a homeomorphism $\bar{h} : \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ where $\mathbb{D}^{n+1}$ denotes the closed ball in $\mathbb{R}^{n+1}$ which bounds S^n.*

Now let Σ^3 be a closed 3-manifold homotopically equivalent to S^3 and consider the connected sum $M = T^3 \# \Sigma^3$, where T^3 denotes the 3-torus $S^1 \times S^1 \times S^1$. Since T^3 and M^3 have isomorphic fundamental groups and M is easily seen to be aspherical, T^3 and M^3 are homotopically equivalent because of the above mentioned Corollary due to Hurewicz. Hence Borel's Conjecture implies that $T^3 \# \Sigma^3$ is homeomorphic to T^3. And consequently $\tilde{M}$ = the universal cover of $T^3 \# \Sigma^3$ is homeomorphic to $\mathbb{R}^3$. Therefore the Generalized Schoenflies theorem shows that $\Sigma^3 - \mathrm{Int}(\mathbb{D}^3)$ is homeomorphic to $\mathbb{D}^3$.



$\mathbb{R}^3$

Figure 1.2

(This $\text{Int}(\mathbb{D}^3)$ is the interior of the 3-ball removed from Σ^3 in forming the connected sum with T^3.) Now applying the Alexander Trick we get that Σ^3 is homeomorphic to S^3. In this way, the Borel Conjecture implies the Poincaré Conjecture.

The Borel Conjecture is still open; but Novikov [58], [59] in 1966 showed that the Hurewicz Conjecture is *false*, in fact "generically" false.

Theorem. *(Novikov) Let M^m ($m \geq 5$ and $\not\equiv 2$ mod 4) be any smooth (closed) manifold such that*

1. $\pi_1(M) = 0$
2. $H_4(M, \mathbb{Q}) \neq 0$
3. $M - pt$ *is parallelizable*

(e.g. $M = S^4 \times S^5$). Then there exists a homotopically equivalent smooth (closed) manifold N which is not homeomorphic to M.

To understand this result, I need to recall the notions of tangent bundle and Pontryagin classes. Due to Whitney every smooth manifold M^m embeds in $\mathbb{R}^{2m+2}$.

$M^m \quad \subseteq \quad \mathbb{R}^{2m+2}$

Figure 1.3

To each point $x \in M$, let T_xM be the tangent space to M at x parallel translated to the origin $0 \in \mathbb{R}^{2m+2}$. This defines the *Gauss map* $TM : M \to G_m$ = the Grassman manifold of all m-planes (containing 0) in $\mathbb{R}^{2m+2}$. The Gauss map is continuous and well defined up to homotopy since any pair of embeddings are isotopic. Also if $f : M \to N$ is a diffeomorphism, then $TN \circ f$ is homotopic to TM.

The cohomology groups of G_m can be computed. In particular there are classes $p_i \in H^{4i}(G_m, \mathbb{Q})$ and their pullbacks under

$$(TM)^* : H^*(G_m, \mathbb{Q}) \to H^*(M, \mathbb{Q})$$

are called the (rational) *Pontryagin classes* of M and denoted by $p_i(M)$.

So if $f : M \to N$ is a diffeomorphism, then $f^*(p_i(N)) = p_i(M)$. But one can construct a smooth manifold N^9 which is homotopically equivalent to $M^9 = S^4 \times S^5$ where $p_1(N) \neq 0$. In fact N^9 is the total space of a fiber bundle over S^4 whose fibre is S^5 constructed using 24 times the generator of $\pi_3(O(6)) = \mathbb{Z}$. But $p_1(S^4 \times S^5) = 0$ since $S^4 \times S^5 \subseteq \mathbb{R}^{10}$ and hence

$$T(S^4 \times S^5) : S^4 \times S^5 \to G_9$$

factors through $\mathbb{R}P^9$ and $H^4(\mathbb{R}P^9, \mathbb{Q}) = 0$. So M and N are *not diffeomorphic.*

But, on the other hand, Milnor [53] had shown in 1956 that (in high dimensions) homeomorphism does *not* imply diffeomorphism as it does in dimensions 1, 2 and 3. Still Novikov [59] proved the following.

Theorem. *(Novikov 1966) If $f : M \to N$ is a homeomorphism between smooth manifolds, then $f^*(p_iN) = p_iM$.*

Corollary. *$S^4 \times S^5$ and N^9 are not homeomorphic thus disproving the Hurewicz Conjecture.*

Novikov had used the strong advances in Algebraic Topology made during the 1950's (e.g. Serre's mod-$\mathcal{C}$ Hurewicz Theorem) to reduce the proof of his Theorem to the following key lemma.

Lemma. *(Novikov) Let E be the total space of a real vector bundle η whose base space is S^m. If E is homeomorphic to $S^m \times \mathbb{R}^n$ and both $m \geq 5$ and $n \geq m + 2$, then η is the trivial bundle.*

We sketch a proof of Novikov's Key Lemma since it contains new ideas which allow the tools of differential topology to be used under a topological assumption. The strategy is to construct a smooth embedding $\sigma : S^m \to E$ homotopic to the 0-section embedding $\sigma_0 : S^m \to E$ and such that $v_\sigma =$ normal bundle of σ is trivial. This implies that η is trivial since $\eta = v_{\sigma_0}$ and $v_{\sigma_0} \simeq v_\sigma$ because of the Whitney Embedding Theorem which shows that σ is isotopic to σ_0. To construct σ, one first builds a sequence

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = E$$

of smooth submanifolds satisfying:

1. $\dim(M_i) = m + i$.

2. M_i is 2-sided in M_{i+1} $(i < n)$.

3. M_0 is homeomorphic to S^m.

4. The composite embedding $\tau : M_0 \to E$ is a homotopy equivalence.

Let us assume for the moment that this has been done. Now v_τ is clearly trivial. So if M_0 were diffeomorphic to S^m, we could set $\sigma = \tau$ and we'd be done. However M_0 may be an exotic sphere; i.e. a smooth manifold homeomorphic but *not* diffeomorphic to S^m. (See W. Lueck's lecture 6 for a detailed discussion of exotic spheres.) But Kervaire and Milnor [48], using a deep result of Adams [3], showed (for any exotic sphere M_0) that $M_0 \times \mathbb{R}^n$ is diffeomorphic to $S^m \times \mathbb{R}^n$. Identifying $M_0 \times \mathbb{R}^n$ with a tubular neighborhood of M_0 in E, we can set σ to be the composite

$$S^m = S^m \times 0 \subseteq S^m \times \mathbb{R}^n = M_0 \times \mathbb{R}^n \subseteq E.$$

Therefore it remains to indicate how the sequence of submanifolds M_i is constructed. Start by putting an "anchor ring" $T^{n-1} \times \mathbb{R}$ into $\mathbb{R}^n$ where T^s denotes the s-torus; i.e. $T^s = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{s-\text{copies}}$. This is easy to do; the case $n = 2$ is pictured below. The general construction of anchor rings proceeds by induction on n. In particular, cross

$$T^{n-1} \times \mathbb{R} \subseteq \mathbb{R}^n$$

with $\mathbb{R}$ to obtain

$$T^{n-1} \times \mathbb{R}^2 \subseteq \mathbb{R}^{n+1}$$

and note that

$$T^n \times \mathbb{R} = T^{n-1} \times (S^1 \times \mathbb{R}) \subseteq T^{n-1} \times \mathbb{R}^2$$

where $S^1 \times \mathbb{R} \subseteq \mathbb{R}^2$ is the case $n = 2$.

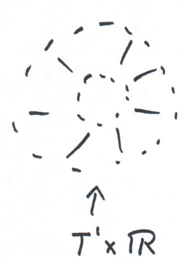


Figure 1.4

Consider the following diagram $*$ in which $f : E \to S^m \times \mathbb{R}^n$ is the given homeomorphism and $V = f^{-1}(S^m \times T^{n-1} \times \mathbb{R})$ which is a smooth manifold since it is an open subset of E.

$$
\begin{array}{ccccc}
E & \xrightarrow{f} & S^m \times \mathbb{R}^n & & \\
\cup & & \cup & & \\
V & \xrightarrow{f|_V} & S^m \times T^{n-1} \times \mathbb{R} & & \\
\cup & & \cup & & \\
M_{n-1} & \xrightarrow{f_{n-1}} & S^m \times T^{n-1} \times 0 & & = S^m \times T^{n-1} \\
\cup & & & & \cup \\
M_{n-2} & & \xrightarrow{f_{n-2}} & & S^m \times T^{n-2} \\
\cup & & & & \cup \\
\vdots & & & & \vdots \\
\cup & & & & \cup \\
M_1 & & \xrightarrow{f_1} & & S^m \times S^1 \\
\cup & & & & \cup \\
M_0 & & \xrightarrow{f_0} & & S^m \times 1 = S^m
\end{array}
$$

The manifolds M_i and homotopy equivalences $f_i : M_i \to S^m \times T^i$ are constructed (downwards) inductively by applying a codimension-one splitting theorem to $f_{i+1} : M_{i+1} \to S^m \times T^{i+1}$, when $i \leq n-2$, and to $f|_V : V \to S^m \times T^{n-1} \times \mathbb{R}$ when $i = n-1$.

The setup for such a theorem is the following:

Let $\phi : \mathcal{W} \to W$ be a (proper) homotopy equivalence between smooth manifolds and $T \subset W$ be a closed 2-sided codimension-one smooth submanifold.

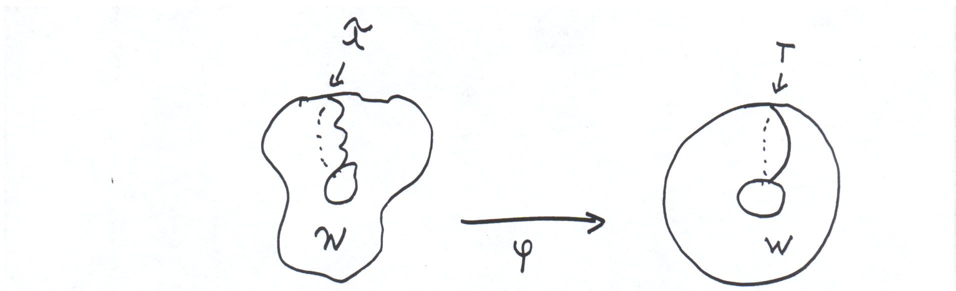


Figure 1.5

Question. Can $T \subset W$ be modeled in $\mathcal{W}$; i.e., can ϕ be (properly) homotoped to a map ψ which is transverse to T and such that

$$\psi|_{\mathcal{T}} : \mathcal{T} \to T$$

is a homotopy equivalence where $\mathcal{T} = \psi^{-1}(T)$.

In the situations occurring in Diagram $*$, the answer is Yes. Codimension-one splitting theorems of this sort have been proved by Browder, Novikov, Levine, Livesay, Siebenmann, Farrell, Hsiang, and Cappell. The two cases occuring in diagram $(*)$ are $W = T \times \mathbb{R}$ and $W = T \times S^1$. If $W = T \times \mathbb{R}$, this splitting theorem is due to Browder [9] when $\pi_1(T) = 0$, Novikov [59] when $\pi_1(T)$ is free abelian, and Siebenmann [67] in general. When $W = T \times S^1$ it is due to Browder and Levine [10] when $\pi_1(T) = 0$ and to Farrell [20] in general.

Notice that Novikov's Theorem also bears on the Borel Conjecture. In particular if the Borel Conjecture is true, then any homotopy equivalence $f : M \to N$ between smooth (closed) aspherical manifolds must preserve (rational) Pontryagin classes. Novikov formulated this explicitly as a conjecture and proved some partial results on it. Further partial results were obtained by Farrell, Hsiang, Kasparov, and Cappell using codimension-one splitting theorems. But perhaps the most interesting early result on this conjecture is due to Mishchenko [56] extending work of Lusztig [52]. (Their work uses a different technique; namely the extension of the Hirzebruck Index Theorem due to Atiyah and Singer.)

Theorem. *(Mischenko [56] 1974). Let $f : M \to N$ be a homotopy equivalence where M is a closed non-positive curved Riemannian manifold (and N is also closed), then $f^*(p_i(N)) = p_i(M)$.*

Let me finish this talk by briefly describing the three steps needed to replace a homotopy equivalence $f : M \to N$ by a homeomorphism. (This process is called surgery theory and is discussed in detail in W. Lueck's lectures.) For this purpose I now make a dimension assumption; namely, I assume that dim $M \geq 5$.

Step 1. Construct a *normal cobordism* W from M to N; i.e. a compact manifold W with boundary $\partial W = M \cup N$ together with a *tangential map* $F : (W, \partial W) \to (N \times [0,1], \partial)$ such that $F|_N = \mathrm{id}_N$ and $F|_M = f$.

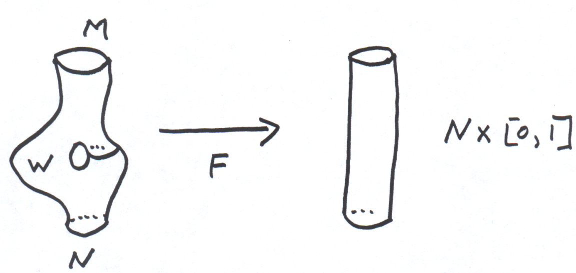


Figure 1.6

(A *tangential map* is a continuous function covered by a map of tangent bundles.) Notice that Step 1 implies that $f^*(p_i(N)) = p_i(M)$. In particular if the construction in Step 1 can be done whenever M is aspherical, then the above conjecture of Novikov is true.

Step 2. Modify some normal cobordism W from M to N, by cutting out (surgering) excess homology, to form a new normal cobordism $\bar{F} : \bar{W} \to N \times [0,1]$ so that $\bar{F}$ is a homotopy equivalence; i.e., $\bar{W}$ is a *h-cobordism.*

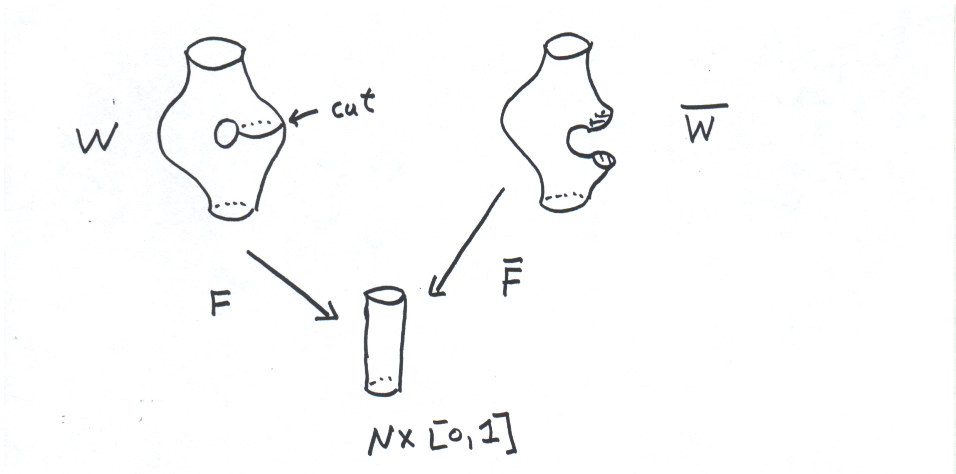


Figure 1.7

Note that the map F from Step 1 has degree one; hence it induces a split epimorphism on homology groups (even with twisted coefficients) because of Lefschetz duality. When there is one, there is usually many normal cobordisms W between M and N. To each of these is associated an element

$$\omega(W) \in L_{m+1}(\pi_1 M^m)$$

– an abelian group defined by C.T.C. Wall, cf. [70]. And $\omega(W) = 0$ iff the desired surgery can be done. Steps 1 and 2 involve calculating $L_{m+1}(\pi_1 M^m)$.
Step 3. Show that the h-cobordism $\bar{W}$ is a cylinder; i.e. $\bar{W} = N \times [0,1]$. Because of the (topological) s-cobordism Theorem, this step involves calculating J.H.C. Whitehead's group $\mathrm{Wh}(\pi_1 M)$. (See lecture 1 of W. Lueck for a discussion of Whitehead groups and the s-cobordism Theorem.)

In particular if the Borel Conjecture is true, then $\mathrm{Wh}(\pi_1 M) = 0$ for every compact aspherical manifold M. This is so even if $\partial M \neq \emptyset$ because Mike Davis has shown how to reduce this more general case to the special case where $\partial M = \emptyset$. (See M. Davis' lectures.) The Borel Conjecture also implies that $L_{m+1}(\pi_1 M^m)$ is finitely generated.

2 Splitting the surgery map under a geometric assumption

Throughout this lecture (unless otherwise stated) M (and N) will denote complete (connected) Riemannian manifolds. Furthermore Γ will denote the group of all deck transformations of the universal cover $\tilde{M} \to M$ and we identify Γ with $\pi_1(M)$. If v is a vector tangent to M (i.e. $v \in TM$ = tangent bundle of M) then

$$\alpha_v : \mathbb{R} \to M$$

denotes the unique geodesic such that $\dot{\alpha}_v(0) = v$.

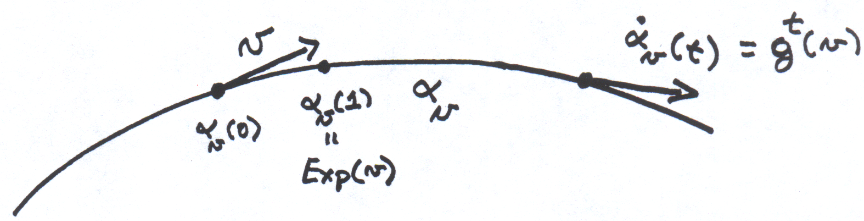


Figure 2.1

The function $\mathbb{R} \times TM \to TM$ defined by

$$g^t(v) = \dot{\alpha}_v(t)$$

for $t \in \mathbb{R}$ and $v \in TM$, is a flow on TM; i.e. it is smooth and satisfies the equation

$$g^s(g^t(v)) = g^{s+t}(v)$$

for all $s, t \in \mathbb{R}$ and $v \in TM$. This flow leaves invariant SM = unit sphere bundle of M and its restriction to SM is called the *geodesic flow.* Closely related to the geodesic flow is the *exponential function* Exp : $TM \to M$ defined by

$$\text{Exp}(v) = \alpha_v(1).$$

It is also a smooth function. If we fix a base point $x_0 \in M$, then the restriction of Exp to $T_{x_0}M$ = tangent space to M at x_0 is also called the exponential function and denoted by

$$\exp_{x_0} : T_{x_0}M \to M.$$

(Or more simply by exp when no ambiguity is possible.) Note that the vector space $T_{x_0}M$ considered as a smooth manifold $N = T_{x_0}M$ has a natural complete Riemannian metric; namely, if $u \in TN$, then $|u| = \sqrt{U \cdot U}$ where U is the parallel translate of u to 0.

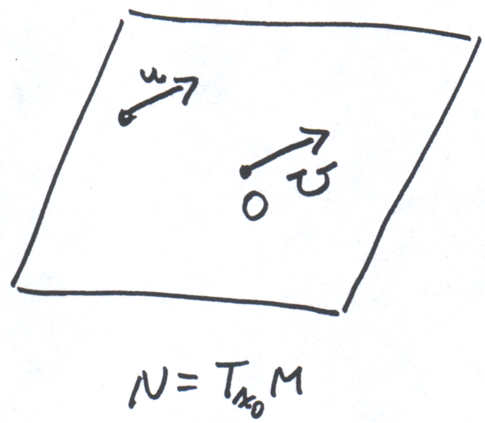


Figure 2.2

We say that M is *non-positively curved* (resp. *negatively curved*) if all its sectional curvatures are ≤ 0 (resp. < 0). And a negatively curved manifold is *pinched negatively curved* if its sectional curvatures are bounded away from 0 and $-\infty$. Note that a closed negatively curved manifold is pinched negatively curved.

Definition. A smooth map $f : M \to N$ is called (weakly) *expanding* if

$$|df(v)| \geq |v|$$

for all vectors $v \in TM$.

There is the following important result relative to these definitions.

Theorem. *(Cartan) Let M be non-positively curved and $x_0 \in M$ be a base point. Then* exp : $T_{x_0}M \to M$ *is an expanding map. Furthermore it is a covering projection and hence a diffeomorphism when $\pi_1(M) = 0$.*

Because of Cartan's theorem a non-positively curved (Riemannian) manifold M^m is aspherical since its universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^m$.

(See [42, p. 172] and [54, p. 102] for a discussion of Cartan's theorem.) It also leads to the following useful alternate description of TM as the bundle with fiber $\tilde{M}$ associated to the principal Γ-bundle $\tilde{M} \to M$; namely

$$\tilde{M} \times_{\Gamma} \tilde{M} \to M.$$

In fact this bundle is indentified with $TM \to M$ as $\mathrm{Diff}(\mathbb{R}^m)$-bundles via the Γ-equivariant diffeomorphism

$$T\tilde{M} \to \tilde{M} \times \tilde{M}$$

which sends $v \in T\tilde{M}$ to $(\alpha_v(0), \alpha_v(1))$. The 0-section of TM corresponds (under this identification) with the image of the diagonal Δ of $\tilde{M} \times \tilde{M}$ in $\tilde{M} \times_{\Gamma} \tilde{M}$.

There is also a natural *geodesic ray compactification* $\bar{M}$ of $\tilde{M}$ due to Eberlein and O'Neill [17] such that $(\bar{M}, \tilde{M})$ is homeomorphic to $(\mathbb{D}^m, \mathrm{Int}\, \mathbb{D}^m)$ where

$$\mathbb{D}^m = \{v \in \mathbb{R}^m \mid |v| \leq 1\}.$$

Let $M(\infty) = \bar{M} - \tilde{M}$ denote the points added; called *ideal points.* Each ideal point is an asymptoty class of geodesic rays in $\tilde{M}$. A *geodesic ray* is a subset of $\tilde{M}$ of the form

$$\{\alpha_v(t) \mid t \in [0, +\infty)\}$$

for some $v \in S\tilde{M}$. Two rays R_1 and R_2 are *asymptotic* if there exists a positive number b such that each point of R_1 is within distance b of some point of R_2 and vice-versa.

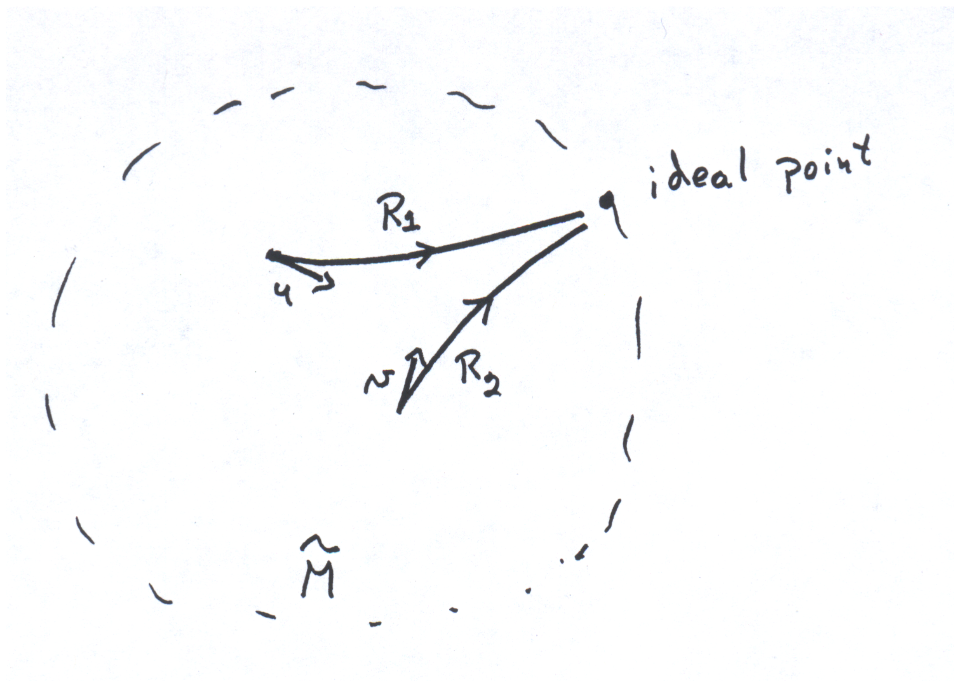


Figure 2.3

The deck transformation action of Γ on $\tilde{M}$ extends to an action on $\bar{M}$ since Γ acts via isometries on $\tilde{M}$ and isometries preserve both geodesic rays and the relation of being asymptotic.

W.C. Hsiang and I abstracted an additional key property possessed by the geodesic ray compactification in the following definition [24], see also lectures 6, 7, 8 in [21]. (For the rest of this lecture M denotes a closed topological manifold and *not necessarily a Riemannian manifold.*)

Definition. A closed manifold M^m satisfies condition $(*)$ provided there exists an action of $\Gamma = \pi_1(M^m)$ on $\mathbb{D}^m$ with the following two properties.

1. The restriction of this action to $\text{Int}(\mathbb{D}^m)$ is equivalent via a Γ-equivariant homeomorphism to the action of Γ by deck transformations on the universal cover $\tilde{M}$ of M^m.

2. Given any compact subset K of $\text{Int}(\mathbb{D}^m)$ and any $\epsilon > 0$, there exists a real number $\delta > 0$ such that the following is true for every $\gamma \in \Gamma$. If the distance between γK and $S^{m-1} = \partial \mathbb{D}^m$ is less than δ, then the diameter of γK is less than ϵ.

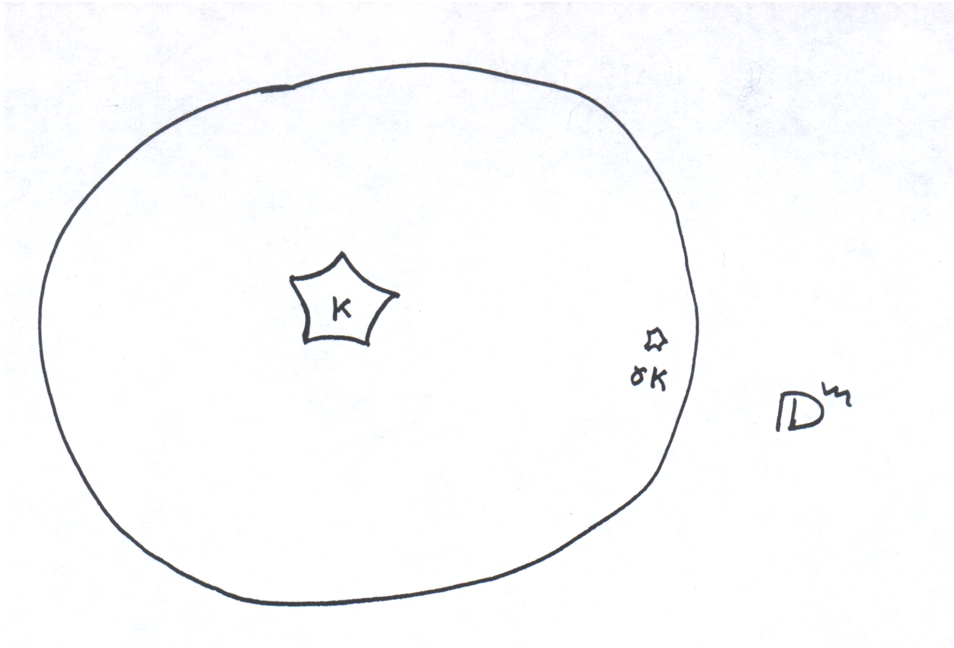


Figure 2.4

The above picture illustrates property 2 of condition $(*)$.

Remark. Hsiang and I showed that every closed (connected) non-positively curved Riemannian manifold M satisfies condition $(*)$ by using its geodesic ray compactification.

Remark. Any manifold satisfying condition $(*)$ is obviously aspherical. It was conceivable 20 years ago, when this condition was formulated, that every closed aspherical manifold M^m satisfies condition $(*)$. But then Mike Davis [14] constructed closed aspherical manifolds M^m where $\tilde{M} \neq \mathbb{R}^m$ contradicting property 1 of condition $(*)$.

On the other hand, $M^m \times S^1$ satisfies property 1 of condition $(*)$ whenever $\tilde{M} = \mathbb{R}^m$. This is seen as follows. Let $\mathbb{Z}$ denote the additive group of integers. Its natural action by translations on $\mathbb{R}$ extends to an action on $[-\infty, +\infty)$ where each group element fixes $-\infty$. We hence have a product action of $\pi_1(M \times S^1) = \pi_1(M) \times \mathbb{Z}$ on

$$\tilde{M} \times [-\infty, +\infty) = \mathbb{R}^m \times [0, +\infty)$$

which extends to its one point compactification $\mathbb{D}^{m+1}$. If we let this be the action posited in the above Definition, then it satisfies property 1 of condition $(*)$ but *not* property 2.

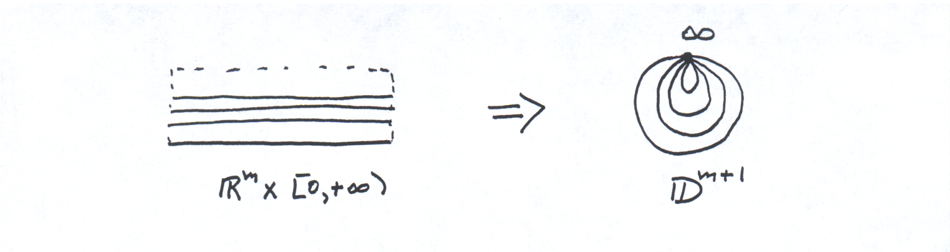


Figure 2.5

We also note that the universal cover X of $M^m \times S^1$ is $\mathbb{R}^{m+1}$ for any closed aspherical manifold M^m where $m \geq 5$ because X is contractible and simply connected at ∞. This is a result of Newman (1966).

Theorem. *(Farrell-Hsiang [24] 1981) Let M^m be a closed manifold satisfying condition $(*)$. Then the map in the (simple) surgery sequence*

$$\mathcal{S}^s(M^m \times \mathbb{D}^n, \partial) \to [M^m \times \mathbb{D}^n, \partial; G/\mathrm{Top}]$$

is identically zero when $n \geq 1$ and $n + m \geq 6$.

So as not to obscure the argument, we sketch the proof of this Theorem under the extra assumptions that M is triangulable and $n = 1$. (See also lectures 6, 7, 8 in [21].) Set

$$E^{2m} = \tilde{M} \times_\Gamma \tilde{M}$$

and let $p : E^{2m} \to M$ denote the bundle projection. Then the following square commutes:

$$\begin{array}{ccc} \mathcal{S}^s(\mathbb{D}^1 \times M, \partial) & \longrightarrow & [\mathbb{D}^1 \times M, \partial; G/\mathrm{Top}] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle (\mathrm{id}\times p)^*} \\ \mathcal{S}(\mathbb{D}^1 \times E, \partial) & \longrightarrow & [\mathbb{D}^1 \times E, \partial; G/\mathrm{Top}] \end{array}$$

where α is the transfer map defined as follows. Let the simple homotopy equivalence

$$h : (W, \partial W) \to (\mathbb{D}^1 \times M, \partial)$$

represent an element $b \in \mathcal{S}^s(\mathbb{D}^1 \times M, \partial)$. Then the proper homotopy equivalence

$$\hat{h} : (\mathcal{W}, \partial W) \to (\mathbb{D}^1 \times E, \partial)$$

represents $\alpha(b) \in \mathcal{S}(\mathbb{D}^1 \times E, \partial)$ where

$$\mathcal{W} = \{(x, y) \in W \times (\mathbb{D}^1 \times E) \mid h(x) = \mathrm{id} \times p(y)\}$$

and $\hat{h}(x, y) = y$. Since p is a homotopy equivalence, $(\mathrm{id} \times p)^*$ is an isomorphism. Hence the Theorem is a consequence of the following:

Assertion. *The map α is identically zero.*

We proceed to verify this. Note first that W is an s-cobordism and hence a cylinder because of the s-cobordism theorem. We may therefore assume that $W = [0, 1] \times M$ and that h is a homotopy between id_M and a self-homeomorphism $f : M \to M$. Furthermore, if f is pseudo-isotopic to id_M via a pseudo-isotopy homotopic to h rel ∂, then $b = 0$.

Let $\tilde{h}$ be the unique lift of h to $[0, 1] \times \tilde{M}$ such that $\tilde{h}$ is a proper homotopy between $\mathrm{id}_{\tilde{M}}$ and a self-homeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$, which is a lift of f. Then $\tilde{h} \times \mathrm{id}_{\tilde{M}}$ determines a proper homotopy

$$k : [0, 1] \times E \to [0, 1] \times E$$

between id_E and a self-homeomorphism $g : E \to E$ (which is also determined by $\tilde{f} \times \mathrm{id}_{\tilde{M}}$). Since

$$\hat{h} : (\mathcal{W}, \partial W) \to (\mathbb{D}^1 \times E, \partial)$$

can be identified with

$$k : ([0, 1] \times E, \partial) \to ([0, 1] \times E, \partial),$$

the Assertion is an immediate consequence of the following.

Lemma. *g is pseudo-isotopic to* id_E *via a pseudo-isotopy which is properly homotopic to k* rel ∂.

We now use our assumption that M^m satisfies condition $(*)$ to prove this lemma. Identify $\tilde{M}$ with $\mathbb{D}^m$ and define a manifold $\bar{E}$ by

$$\bar{E} = \mathbb{D}^m \times_\Gamma \tilde{M}.$$

Then $E = \text{Int}(\bar{E})$ and property 2 of condition $(*)$ implies that $\tilde{f}$ extends to a Γ-equivariant homeomorphism

$$\bar{f} : \mathbb{D}^m \to \mathbb{D}^m$$

by setting $\bar{f}|_{S^{m-1}} = \text{id}_{S^{m-1}}$. Consequently $\bar{f} \times \text{id}_{\tilde{M}}$ determines a self-homeomorphism

$$\bar{g} : \bar{E} \to \bar{E}$$

which extends $g : E \to E$ and satisfies $\bar{g}|_{\partial \bar{E}} = \text{id}_{\partial \bar{E}}$. We proceed to construct a pseudo-isotopy

$$\phi : \bar{E} \times [0,1] \to \bar{E} \times [0,1]$$

satisfying

1. $\phi|_{\bar{E} \times 0} = \bar{g}$;

2. $\phi|_{\bar{E} \times 1} = \text{id}_{\bar{E} \times 1}$;

3. $\phi|_{(\partial \bar{E}) \times [0,1]} = \text{id}_{(\partial \bar{E}) \times [0,1]}$.

Properties (1-3) define ϕ on $\partial(\bar{E} \times [0,1])$. To construct ϕ over $\text{Int}(\bar{E} \times [0,1])$ consider the natural fiber bundle

$$\bar{E} \times [0,1] \xrightarrow[q]{} M$$

with fiber $\mathbb{D}^m \times [0,1]$. And note the following. If Δ is an n-simplex in M, then $q^{-1}(\Delta)$ can be identified with $\mathbb{D}^{n+m+1}$.

The construction of ϕ proceeds by induction over the skeleta of M via a standard obstruction theory argument. And the obstructions encountered in extending ϕ from over the $(n-1)$-skeleton to over the n-skeleton are the problem of extending a self-homeomorphism of S^{n+m} to one of $\mathbb{D}^{n+m+1}$. But these obstructions all vanish because of the Alexander Trick. Recall that this Trick asserts that any self-homeomorphism η of S^n extends to a self-homeomorphism $\bar{\eta}$ of $\mathbb{D}^{n+1}$. In fact

$$\bar{\eta}(tx) = t\eta(x)$$

where $x \in S^n$ and $t \in [0,1]$ is an explicit extension.

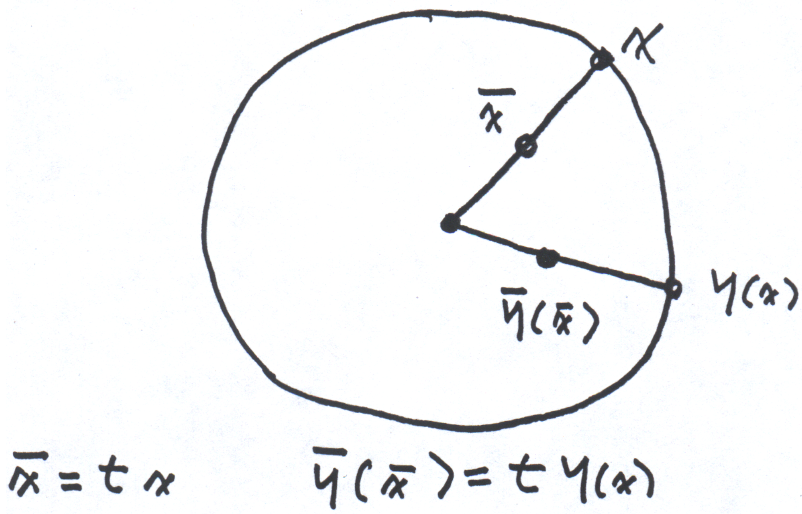


Figure 2.6

Now $\psi = \phi|_{E\times[0,1]}$ is the pseudo-isotopy from g to id_E posited in the Lemma. And a similar argument, which we omit, shows that ψ is properly homotopic to k rel ∂. Q.E.D.

Remark. It follows from results of Davis and Januszkiewicz [15] that PL non-positively curved closed manifolds also satisfy condition $(*)$. And Bizhong Hu showed that every non-positive curved finite complex K is a retract of such a manifold. Hu [45] (1995) deduced from this, using Ranicki's algebraic formulation of surgery theory, that the assembly map is split monic for such a K. Ferry-Weinberger [36], [12] and Carlsson-Pedersen [13] also obtained this in addition to many further results on the split injectivity of σ.

Corollary. *Let $f : N \to M$ be a homotopy equivalence between closed smooth manifolds such that M supports a non-positively curved Riemannian metric. Then N and M are stably homeomorphic; i.e.*

$$f \times \mathrm{id} : N \times \mathbb{R}^{m+4} \to M \times \mathbb{R}^{m+4}$$

is homotopic to a homeomorphism where $m = \dim(M)$.

Proof. Let $\phi : N \times S^1 \to M \times S^1 \times \mathbb{R}^{m+3}$ be an embedding homotopic to the composition

$$N \times S^1 \xrightarrow{f \times \mathrm{id}_{S^1}} M \times S^1 \times 0 \subseteq M \times S^1 \times \mathbb{R}^{m+3}.$$

Note that ϕ exists because of the Whitney Embedding Theorem. And let v denote the normal bundle to ϕ. We proceed to show that v is topologically trivial. Now Kwan and Szczarba [50] showed that $f \times \mathrm{id}_{S^1}$ is a simple homotopy equivalence and hence represents an element in $\mathcal{S}^s(M \times S^1)$. This element maps to 0 in $[M \times S^1; G/\mathrm{Top}]$ because of the Theorem and the 4-fold (semi) periodicity of the topological surgery exact sequence. But v (equipped with a specific homotopy trivialization) is this image element; in particular, v is topologically trivial.

Since the region outside an open tubular neighborhood of image(ϕ) is a (half open) h-cobordism, we can use the h-cobordism theorem to show that the total space E of v is diffeomorphic to $M \times S^1 \times \mathbb{R}^{m+3}$. But E can also be topologically identified with $N \times S^1 \times \mathbb{R}^{m+3}$ since v is topologically trivial. Hence there is a homeomorphism

$$\psi : N \times S^1 \times \mathbb{R}^{m+3} \to M \times S^1 \times \mathbb{R}^{m+3}$$

such that $\psi_{\#}(\pi_1 N) = \pi_1(M)$. The homeomorphism posited to exist in the Corollary is obtained by lifting ψ to the infinite cyclic covering spaces corresponding to $\pi_1(N)$ and $\pi_1(M)$, respectively. $\square$

3 The vanishing of $\mathbf{Wh}(\pi_1 M)$ for non-positively curved manifolds M

In my last lecture, I showed that step 1 in the program to replace a homotopy equivalence $f : N \to M$ between closed manifolds with a homeomorphism can be accomplished when M satisfies a certain geometric condition $(*)$. In particular, this can be done when M is a non-positively curved Riemannian manifold.

This lecture is about step 3 of the program; i.e., analyzing h-cobordisms with base M. Because of the s-cobordism theorem, this is equivalent to calculating $Wh(\pi_1 M)$ when $\dim(M) \geq 5$. The discussion will focus on the following vanishing result.

Vanishing Theorem. *(Farrell and Jones [31]) Let M be a closed non-positively curved Riemannian manifold. Then*

$$Wh(\pi_1 M) = 0.$$

Remark. The special cases of this theorem where M is the m-torus T^m was proven by Bass-Heller-Swan [6] (1964) and for arbitrary flat Riemannian manifolds M by Farrell-Hsiang [22] (1978).

We need to develop a few more geometric ideas before discussing the proof of the Vanishing Theorem. See [26] and [30, §3, §4] for more details. Throughout this lecture M will denote a closed (connected) non-positively curved Riemannian manifold and $\tilde{M}$ is its universal cover. And we keep the geometric notation from our last lecture; in particular

$$\begin{aligned} \Gamma &= \pi_1(M). \\ \bar{M} & \quad \text{is the geodesic ray compactification of } \tilde{M}. \\ M(\infty) &= \bar{M} - \tilde{M}. \\ \alpha_v & \quad \text{is the geodesic with } \dot{\alpha}_v(0) = v. \end{aligned}$$

We call a pair of vectors $u, v \in S\tilde{M}$ *asymptotic* if the two rays

$$\{\alpha_u(t) \mid t \geq 0\} \text{ and } \{\alpha_v(t) \mid t \geq 0\}$$

are asymptotic.

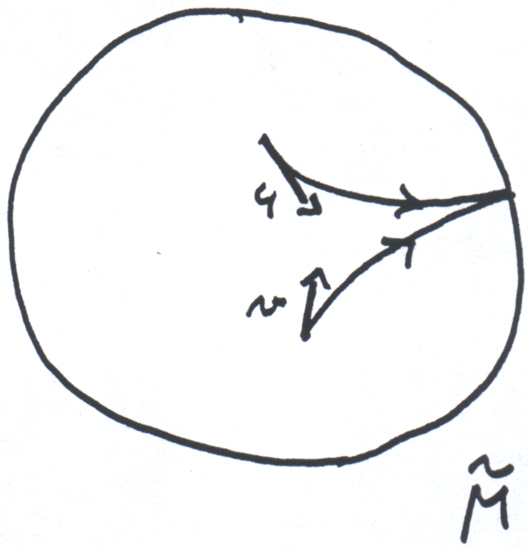


Figure 3.1
For each pair $v \in S\tilde{M}$ and $x \in \tilde{M}$, there is a unique asymptotic vector $v(x) \in S_x\tilde{M}$. ($S_x\tilde{M}$ = unit sphere in $T_x\tilde{M}$.)

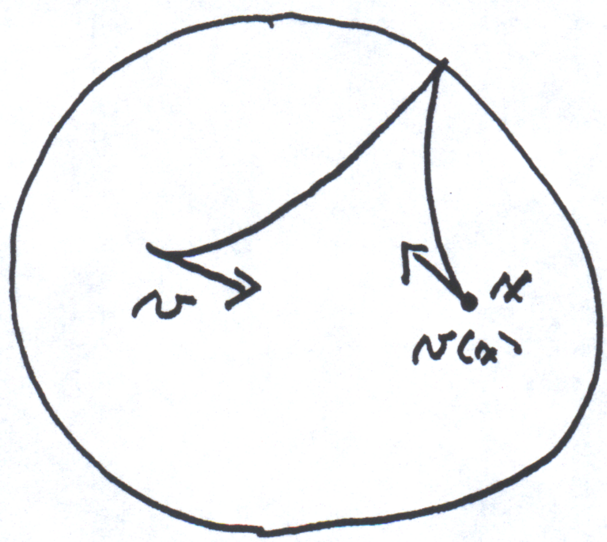


Figure 3.2

Furthermore the function $S\tilde{M} \times \tilde{M} \to S\tilde{M}$ defined by $(v, x) \to v(x)$ is continuous, C^1 in x, and its differential (in x) depends continuously on v. The (weakly) stable foliation of $S\tilde{M}$ has for its leaves the asymptoty classes of vectors. Note that under the bundle projection $S\tilde{M} \to \tilde{M}$ each leaf of this foliation maps diffeomorphically onto $\tilde{M}$. Since an isometry of $\tilde{M}$ sends asymptotic vectors to asymptotic vectors, this foliation induces a foliation of SM called its (*weakly*) *stable foliation*. Restriction of the bundle projection $SM \to M$ to any leaf L of this foliation is a covering space projection

$$L \to M.$$

And the geodesic flow $g^t : SM \to SM$ preserves the leaves of the (weakly) stable foliation.

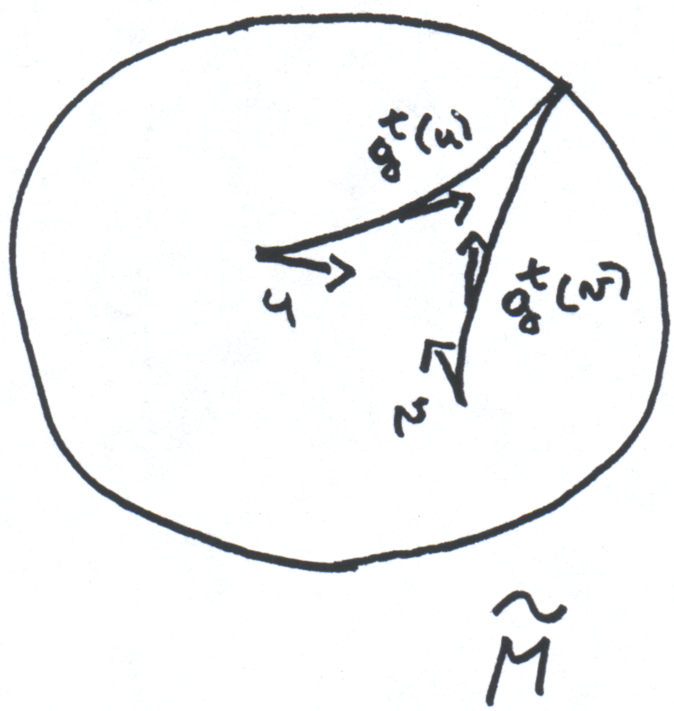


Figure 3.3

The total space SN of the unit sphere bundle of a Riemannian manifold N has a natural Riemannian metric defined as follows. Let $v(t)$ be a smooth curve in SN representing a tangent vector η to SN at $v(0)$; i.e., $v(t)$ is a unit length vector field along a smooth curve $\gamma(t)$ in N. Then

$$|\eta| = \sqrt{|\dot{\gamma}(0)|^2 + |u|^2}$$

where u is the covariant derivative of $v(t)$ at $t = 0$.

We next describe the *asymptotic transfer* of a path $\gamma : [0,1] \to M$ to a path $v\gamma$ in SM where $v \in S_{\gamma(0)}M$. The asymptotic transfer sits on top of γ in the sense that the composite path $p \circ (v\gamma)$ is γ; where

$$p : SM \to M$$

denotes the bundle projection. Let L be the leaf of the (weakly) stable foliation of SM containing v. Recall that

$$p|_L : L \to M$$

is a covering space. Then $v\gamma$ is defined to be the unique lift of γ starting at v.

The following are some of the properties of the asymptotic transfer.

1. If γ is a null homotopic loop, then so is $v\gamma$.

2. If γ is a constant loop, so is $v\gamma$.

3. If γ is a C^1-curve, so is $v\gamma$.

Furthermore, if $-a^2$ is any lower bound for the sectional curvatures of M, then

$$|\dot{v\gamma}(t)| \leq \sqrt{1 + a^2}\ |\dot{\gamma}(t)|$$

for each $t \in [0,1]$.

Let W be a smooth h-cobordism with base M equipped with a smooth deformation retraction h_t of W^{m+1} onto M^m. In particular $h_0 = \mathrm{id}_W$, and $r = h_1$ is a retraction of W onto M. Let $\mathcal{W}^{2m}$ be the total space of the pullback of $p : SM \to M$ via r; i.e.,

$$\mathcal{W} = \{(y,v) \in W \times SM \mid r(y) = p(v)\}.$$

Then $\mathcal{W}$ is an h-cobordism with base SM and the asymptotic transfer can be used to equip $\mathcal{W}$ with a useful C^1 deformation retraction k_t of $\mathcal{W}$ onto

SM defined as follows. First associate to h_t a family of paths $\{\gamma_y \mid y \in W\}$ in M called the *tracks* of h_t. These are given by the equation

$$\gamma_y(t) = r(h_t(y)).$$

Note that each track γ_y is a smooth null homotopic loop in M based at $r(x)$. Hence, for each vector $v \in S_{r(y)}M$, the asymptotic transfer $v\gamma_y$ of γ_y to SM is a C^1 null homotopic loop based at v. Now k_t is defined by the formula

$$k_t(y, v) = (h_t(y), v\gamma_y(t))$$

where $t \in [0, 1]$, $y \in W$ and $v \in S_{r(y)}M$. And notice that the retraction

$$k_1 : \mathcal{W} \to SM$$

is given by the formula

$$k_1(y, v) = v;$$

this follows from properties 1 and 2 of the asymptotic transfer together with the fact that each γ_y is a null homotopic loop. Consequently the tracks of k_t are

$$\{v\gamma_y \mid (y, v) \in \mathcal{W}\};$$

namely, they are all the asymptotic transfers of the tracks of h_t. Furthermore given a self-diffeomorphism $f : SM \to SM$ homotopic to id_{SM}, we can change k_t to a new C^1 deformation retraction of $\mathcal{W}$ onto SM whose tracks are

$$\{f \circ (v\gamma_y) \mid (y, v) \in \mathcal{W}\}.$$

This comment applies in particular when $f = g^{t_0}$ where g^t is the geodesic flow on SM and t_0 is a fixed (large) positive real number. Which is useful because of the following consequence of Anosov's analysis of the geodesic flow.

Key Property of $v\gamma$. The following is true when M is *negatively curved.* Given numbers β and ϵ in $(0, +\infty)$, there exists a number $t_0 \in (0, +\infty)$ satisfying the following. Let γ be any smooth path in M whose arc length is $\leq \beta$, and v be any vector in $S_{\gamma(0)}M$. Then, for any $t \geq t_0$, the composite path $g^t \circ (v\gamma)$ is (β, ϵ)-controlled in SM with respect to the 1-dimensional foliation by the orbits of the geodesic flow.

Figure 1.4 indicates why this property is true. In it $\tilde{\gamma}$ is a lift of γ to $\tilde{M}$; $u \in S_{\tilde{\gamma}(0)}\tilde{M}$ is the vector lying over v; $u\tilde{\gamma}$ is the lift of $v\gamma$ to $S\tilde{M}$ starting at u, and $u(\infty) \in M(\infty)$ is the ideal point corresponding to the ray $\{\alpha_u(t) \mid t \geq 0\}$. Also $\tilde{M}$ is identified with the (weakly) stable leaf L of $S\tilde{M}$ containing u. And the lines converging to $u(\infty)$ are the flow lines of the geodesic flow which are inside of L; while the $\perp$ codimension-one submanifolds abuting to $u(\infty)$ are the horospheres inside of L; i.e. the *strongly stable leaves.*

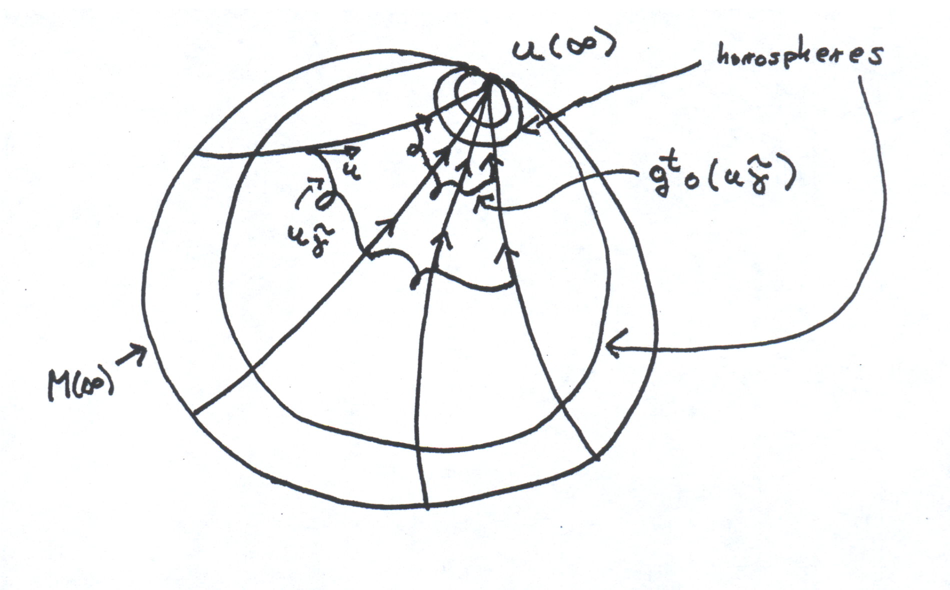


Figure 3.4

Each diffeomorphism g^t, $t > 0$, of the geodesic flow preserves the family of horospheres as well as the flow lines. It is (strongly) contracting on horospheres and is an isometry on flow lines.

Remark. This Key Property of the asymptotic transfer is *not* true (in general) when M is only non-positively curved. For example it doesn't hold when M is flat since asymptotic rays are parallel in Euclidean space.

Using the above construction of a deformation retraction of $\mathcal{W}$ onto SM relative to g^{t_0}, we see that $\mathcal{W}$ is a (β, ϵ)-controlled h-cobordism over SM for a fixed positive real number β but arbitrarily small positive numbers ϵ when M is negatively curved because of the Key Property of the asymptotic transfer. Hence the Foliated Control Theorem, Theorem 1.8 of L.E. Jones' lectures, shows that the Whitehead torsion $\tau(\mathcal{W}) = 0$.

Codicil. We must make the following minor addition to our setup in order to apply the Foliated Control Theorem. Let M^+ denote the "top" of the h-cobordism W; i.e., $\partial W = M^+ \coprod M$. And fix a second smooth deformation retraction h_t^+ of W onto M^+. Associate to h_t^+ a second family of tracks γ_y^+, $y \in W$, defined by the equation

$$\gamma_y^+(t) = r(h_t^+(y)).$$

Use these new tracks to define a second C^1 deformation retraction k_t^+ of the transferred h-cobordism $\mathcal{W}$ onto its "top" $\mathcal{M}^+$ where

$$\mathcal{M}^+ = \partial\mathcal{W} - SM.$$

Define k_t^+ by the formula

$$k_t^+(y, v) = (h_t^+(y), v\gamma_y^+(t))$$

where $t \in [0,1]$, $y \in W$ and $v \in S_{r(y)}M$. And note that the tracks of k_t^+ are

$$\{v\gamma_y^+ \mid (y, v) \in \mathcal{W}\};$$

namely, all the asymptotic transfers of the tracks of h_t^+. (Notice that the tracks γ_y^+ are *not* loops; but this is irrelevant since the retraction k_1 rather than k_1^+ is used in defining the tracks of k_t^+.) Furthermore we can change k_t^+ to a new C^1 deformation retraction of $\mathcal{W}$ onto $\mathcal{M}^+$ whose tracks are

$$\{g^{t_0} \circ (v\gamma_y^+) \mid (y, v) \in \mathcal{W}\}.$$

Clearly these tracks are also (β, ϵ)-controlled provided t_0 is sufficiently large, and hence the Foliated Control Theorem is applicable.

Since every element $x \in Wh(\pi_1 M)$ is the torsion $\tau(W)$ of some smooth h-cobordism with base M, the fact that $\tau(\mathcal{W}) = 0$ would show that $Wh(\pi_1 M)$ vanishes, when M is negatively curved, provided

$$\tau(\mathcal{W}) = \tau(W).$$

Unfortunately this equation is not true in general. In fact the following formula calculates $\tau(\mathcal{W})$ in terms of $\tau(W)$.

Theorem. *(D.R. Anderson [4] 1972). Let W and $\mathcal{W}$ be h-cobordisms with bases M and $\mathcal{M}$, respectively. And let $p : \mathcal{W} \to W$ be a smooth fiber bundle with $p^{-1}(M) = \mathcal{M}$ and $\dim M > 4$. Assume that $\pi_1(W)$ acts trivially on the integral homology groups of the fiber F of p, then*

$$p_*(\tau(\mathcal{W})) = \chi(F)\tau(W)$$

where $\chi(F)$ denotes the Euler characteristic of F and

$$p_* : Wh(\pi_1\mathcal{M}) \to Wh(\pi_1 M)$$

is the homomorphism induced by p.

Applying Anderson's theorem to the h-cobordism $\mathcal{W}$ constructed above, we see that

$$\tau(\mathcal{W}) = \begin{cases} 2\tau(W) & \text{if } m \text{ is odd} \\ 0\tau(W) = 0 & \text{if } m \text{ is even} \end{cases}$$

(provided M^m is orientable) since the fiber of $\mathcal{W} \to W$ is S^{m-1}.

To get around this difficulty we need a sub-bundle E of SM with fiber F satisfying

1. $\chi(F) = 1$;
2. E is invariant under g^t;
3. for each path γ in M and each vector $v \in E$ lying over $\gamma(0)$, $v\gamma$ is a path in E.

It unfortunately is impossible to find such a sub-bundle when M is closed because every orbit of the action of Γ on $M(\infty)$ is then dense. We are thus forced to consider a certain non-compact but complete and pinched negatively curved Riemannian manifold N^{m+1} called the *enlargement* of M^m. It is diffeomorphic to $\mathbb{R} \times M^m$ and contains M^m as a totally geodesic codimension-one subspace. In fact N is the warped product (defined by Bishop and O'Neill in [7])

$$N = \mathbb{R} \times_{\cosh(t)} M$$

and $0 \times M$ is the totally geodesic subspace identified with M.

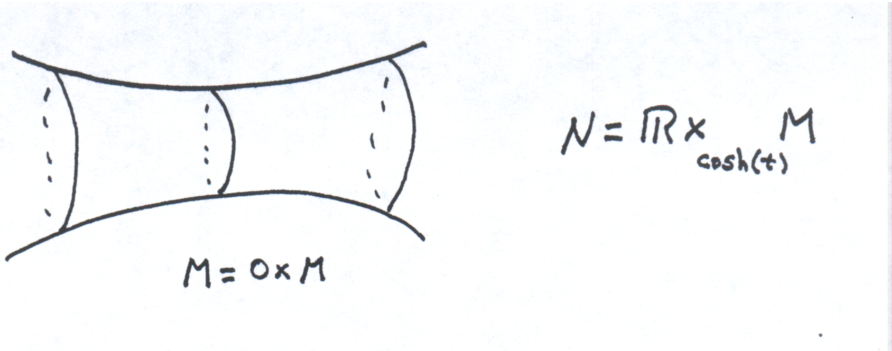


Figure 3.5

The Riemannian metric $|| \; ||$ on N is determined from the Riemannian metrics $| \; |$ on M and $| \; |$ on $\mathbb{R}$ by the properties

1. $\mathbb{R} \times x \perp t \times M$ for all $x \in M$, $t \in \mathbb{R}$.
2. $||v|| = \cosh(t)|v|$ if $v \in T(t \times \tilde{M})$.
3. $||v|| = |v|$ if $v \in T(\mathbb{R} \times x)$.

Let $q : N = \mathbb{R} \times M \to \mathbb{R}$ denote projection onto the first factor. Inside of SN is an *upper hemisphere* sub-bundle defined by $v \in S^+N$ iff the following set of real numbers is bounded below

$$\{q(\alpha_v(t)) \mid t \in [0, +\infty)\}.$$

(This lower bound depends on v.) That is $v \notin S^+N$ iff the geodesic $\alpha_v(t) \to$ "$-\infty$" as $t \to +\infty$. This sub-bundle satisfies the three conditions listed above; in particular its fiber is $\mathbb{D}^m$.

Now an arbitrary element $x \in Wh(\Gamma)$ can be realized as the Whitehead torsion $\tau(W)$ of a compactly supported h-cobordism with base N. And the associated h-cobordism $\mathcal{W}$ with base S^+N is (β, ϵ)-controlled for a fixed positive number β but arbitrarily small positive ϵ. Hence the Foliated Control Theorem (in one of its more sophisticated forms) together with Anderson's Theorem shows that

$$x = \tau(W) = \tau(\mathcal{W}) = 0$$

proving that $Wh(\pi_1 M) = 0$ when M is negatively curved.

To prove the general case of the Vanishing Theorem, where M is allowed to have some zero sectional curvature, we must replace the asymptotic transfer with a new *focal transfer*. It associates to each path $\gamma : [0,1] \to M$, each vector $v \in S_{\gamma(0)}M$, and every (large) positive number $d \in \mathbb{R}$ (called the *focal length* of the transfer) a path

$$v(\gamma, d) : [0,1] \to M.$$

The focal transfer satisfies properties 1-3 of the asymptotic transfer. And it satisfies the following analogue of the Key Property of $v\gamma$.

Key Property of $v(\gamma, d)$. Given M as well as numbers $\beta, \epsilon \in (0, +\infty)$, there exists a positive number t_0 ($t_0 > \beta$) satisfying the following statement for every smooth path γ in M whose arc length is $\leq \beta$ and every vector $v \in S_{\gamma(0)}M$. The composite path

$$g^d \circ v(\gamma, d)$$

is (β, ϵ)-controlled in SM with respect to the foliation given by the orbits of the geodesic flow provided $d \geq t_0$.

Remark. The focal transfer $v(\gamma, d)$ focuses when flowed a distance equal to its focal length d. When flowed farther, it gets out of focus.

To construct $v(\gamma, d)$ pick a lift $\tilde{\gamma}$ of γ to $\tilde{M}$ and let $u \in S_{\tilde{\gamma}(0)}\tilde{M}$ be the unique vector which maps to v via $d\rho$ where

$$\rho : \tilde{M} \to M$$

denotes the covering projection. Figure 6 illustrates the construction of the path $u(\tilde{\gamma}, d)$ in $S\tilde{M}$.

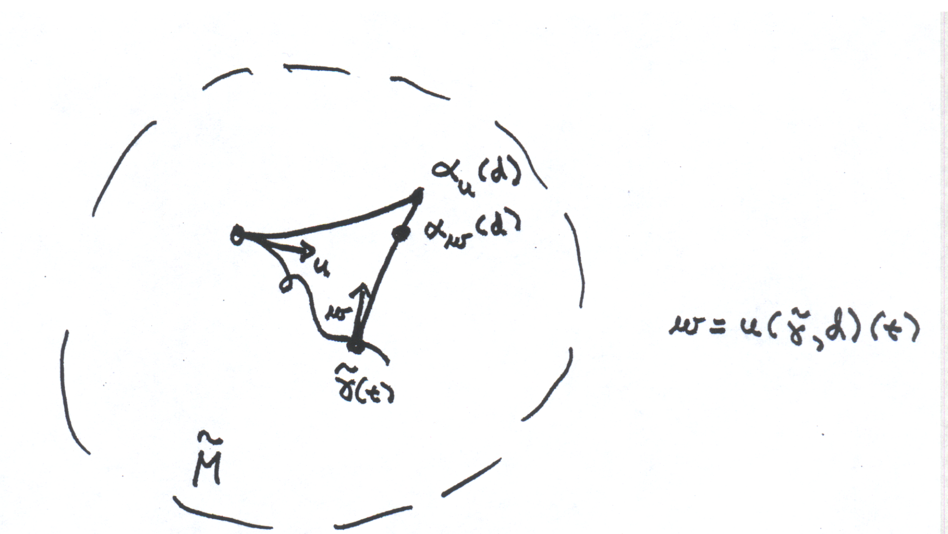


Figure 3.6

If w denotes the vector $u(\tilde{\gamma}, d)(t) \in S_{\tilde{\gamma}(t)}\tilde{M}$, then w is the unique vector such that the geodesic ray

$$\{\alpha_w(s) \mid s \geq 0\}$$

contains the point $\alpha_u(d)$. Note we must have that

$$d \geq \operatorname{diam}\{\gamma(t) \mid t \in [0,1]\}$$

for w to be necessarily defined. Since this construction is equivariant with respect to Γ, we can (and do) define the focal transfer $v(\gamma, d)$ by the equation

$$v(\gamma, d) = d\rho \circ u(\tilde{\gamma}, d).$$

The only problem with the focal transfer is that the bundle $S^+N \to N$ does not satisfy property 3 (see nine lines above Figure 1.5) with respect to it. But it does except near $\partial(S^+N)$ and so the construction is slightly modified near $\partial(S^+N)$. When this is done, then the argument given above proving the Vanishing Theorem in the special case where M is negatively curved works in general after the asymptotic transfer is replaced with the focal transfer. In fact a simplification can be made in the earlier argument by using N equal to the Riemannian product

$$\mathbb{R} \times M$$

instead of the warped product

$$\mathbb{R} \times_{\cosh(t)} M.$$

We can even set N equal to the Riemannian product $S^1 \times M$ and proceed as outlined in L.E. Jones' lecture 1. The advantage to this is that the basic Foliated Control Theorem (Theorem 1.8 of L.E. Jones' lectures) can then be used since $S^+(S^1 \times M)$ is compact. (See lecture 1 of L.E. Jones for the precise definition of $S^+(S^1 \times M)$.)

We end this lecture by discussing a generalization of the Vanishing Theorem to the case where M is complete but *not* necessarily compact. Needed for this purpose is an extra geometric condition on M; namely, that M is *A-regular*.

Definition. A Riemannian manifold N is A-regular if there exists a sequence of positive real numbers $A_0, A_1, A_2, \cdots$ with $|D^n(K)| \leq A_n$. Here K is the curvature tensor and D is covariant differentiation.

Remark. Every closed Riemannian manifold N is A-regular. This is a consequence of an elementary continuity argument.

Remark. Every locally symmetric space is A-regular since $DK \equiv 0$ is one of the definitions of a locally symmetric space.

Addendum. *(Farrell and Jones [35] 1998) Let N be any complete Riemannian manifold which is both non-positively curved and A-regular. Then $Wh(\pi_1 N) = 0$.*

Corollary 1. $Wh(\Gamma) = 0$ *for every discrete torsion-free subgroup* Γ *of* $GL_n(\mathbb{R})$.

Reason. Note that $\Gamma = \pi_1(N)$ where N is the double coset space

$$\Gamma \backslash GL_n(\mathbb{R})/O_n$$

which is a complete non-positively curved locally symmetric space and hence A-regular by Remark 2.

Corollary 2. *Let N be any complete and pinched negatively curved Riemannian manifold, then*

$$Wh(\pi_1 N) = 0.$$

Reason. Shi [66] and Abresch [2] show that the given Riemannian metric can be deformed to an A-regular one while keeping it negatively curved and complete.

The proof of the Addendum follows the same pattern as the proof of the Vanishing Theorem except that it uses the more difficult Foliated Control Theorem which Lowell Jones will discuss in his last lecture.

Let me also mention that Jones' former Ph.D. student B. Hu showed how to adapt the proof of the Vanishing Theorem to the language of Alexandroff PL-geometry thus obtaining the following result.

Theorem. *(Hu [44] 1993) Let K be a non-positively curved finite complex, then $Wh(\pi_1 K) = 0$.*

Remark. Hu's result does not obviously include the Vanishing Theorem since Davis, Okun and Zheng [16] have shown that *no* rank ≥ 2, irreducible, closed, non-positively curved locally symmetric space is also a non-positively curved PL-manifold.

4 The Borel Conjecture for non-positively curved manifolds

The focus of this lecture is Borel's Conjecture for closed non-positively curved Riemannian manifolds of dimension $\neq 3, 4$. It is an immediate consequence of the following result "TRT".

Topological Rigidity Theorem. *(Farrell and Jones [32]) Let M^m be a closed non-positively curved Riemannian manifold. Then the homotopy-topological structure set $\mathcal{S}(M^m \times \mathbb{D}^n, \partial)$ contains only one element when $m + n \geq 5$.*

Remark. TRT was proven for T^m $(m \geq 5)$ by Hsiang-Wall [43] (1969). And it was proven for all closed flat Riemannian manifolds M^m $(m \geq 5)$ by Farrell-Hsiang [25] (1983).

Corollary. *Let $f : N^m \to M^m$ be a homotopy equivalence between closed manifolds where $m \neq 3, 4$. If M^m is a non-positively curved Riemannian manifold, then f is homotopic to a homeomorphism.*

Proof. This result is classical when $m = 1$ or 2. When $m \geq 5$ set $n = 0$ in TRT to conclude that N and M are h-cobordant and hence homeomorphic by the s-cobordism since $Wh(\pi_1 M) = 0$ because of the Vanishing Theorem. □

Remark. Gabai [39] has a program for showing that the Borel Conjecture for closed hyperbolic 3-manifolds is equivalent to the Poincaré Conjecture.

Remark. The Borel Conjecture for closed non-positively curved 4-manifolds M^4 is an interesting open problem which is perhaps more accessible than the 3-dimensional case. The 5-dimensional s-cobordism Theorem of Freedman and Quinn [37] combined with TRT shows it is true when M^4 is a closed flat Riemannian manifold.

We now discuss the proof of the TRT. Throughout this lecture M^m denotes a closed (connected) non-positively curved m-dimensional Riemannian manifold. We also keep the notation from our last lecture; in particular

$\tilde{M}$	is the universal cover of M;
Γ	$= \pi_1(M)$;
α_v	is the geodesic with $\dot{\alpha}_v(0) = v$.

And we make the simplifying assumption that M^m is orientable so that our discussion is as transparent as possible. Note there are the following two identifications since $Wh(\Gamma) = 0$:

$$\begin{aligned} L_k^s(\Gamma) &= L_k(\Gamma) \quad \text{and} \\ \mathcal{S}^s(M^m \times \mathbb{D}^n, \partial) &= \mathcal{S}(M^m \times \mathbb{D}^n, \partial) \end{aligned}$$

where $\mathcal{S}^s(\ \ )$ denotes the simple homotopy-topological structure set.

The following result, used to reduce TRT to a special case, is a consequence of the codimension-one splitting theorems mentioned in my first lecture.

Lemma 0. *$\mathcal{S}(M^m \times \mathbb{D}^n, \partial)$ can be identified with a subset of $\mathcal{S}(M^m \times T^n)$ provided $m + n \geq 5$; and $\mathcal{S}(M^m)$ with a subset of $\mathcal{S}(M^m \times S^1)$ provided $m \geq 5$.*

Remark. Note that $\mathcal{S}^s(N \times [0,1], \partial)$ maps to $\mathcal{S}^s(N \times S^1)$ by sending the structure

$$f : (W, \partial_0 W \amalg \partial_1 W) \to (N \times [0,1], N \times 0 \amalg N \times 1)$$

to the structure

$$\mathcal{W} \to N \times S^1$$

where $\mathcal{W}$ results from W by glueing $\partial_0 W$ to $\partial_1 W$ via the composite homeomorphism $(f|_{\partial_1 W})^{-1} \circ (f|_{\partial_0 W})$. The first identification in Lemma 0 is a n-fold elaboration of this map using that $\mathbb{D}^n = \mathbb{D}^{n-1} \times [0,1]$. The second identification sends the structure $f : N \to M$ to the structure $f \times \text{id} : N \times S^1 \to M \times S^1$; which is shown in Lemma 3 (below) to be monic.

Lemma 0 together with the fact that $M^m \times T^n$ is also non-positively curved reduces the TRT to the special case where $n = 0$ and m is an *odd integer*.

Note that the main result of our second lecture, together with the (semi)-periodicity of the surgery exact sequence, yields the following short exact sequence of pointed sets

$$0 \to [M^m \times [0,1], \partial; G/\text{Top}] \xrightarrow{\ \sigma\ } L_{m+1}(\Gamma) \to \mathcal{S}(M^m) \to 0.$$

Remark. The techniques developed in this lecture (and the last) give an independent proof (via the focal transfer and the geodesic flow) that the surgery sequence is short exact for non-positively curved closed manifolds M^m. This alternate proof does not use the (semi)-periodicity of the surgery sequence.

Hence it remains to show that σ is an epimorphism; which is Step 2 in the program from Lecture 1 for replacing a homotopy equivalence $f : N \to M$ with a homeomorphism. This is the most complicated step in the program and was the last to be solved. The argument accomplishing it is modeled on the one used to solve Step 3 given in the last lecture. The s-cobordism theorem was used in that argument. It's surgery analogue is the algebraic classification of normal cobordisms over M due to Wall. Given a group π, Wall [70] algebraically defined a sequence of abelian groups $L_n(\pi)$ with $L_{n+4}(\pi) = L_n(\pi)$ for all $n \in \mathbb{Z}$. He then showed that there is a natural bijection between the equivalence classes of normal cobordisms W over $M^m \times \mathbb{D}^{n-1}$ and $L_{m+n}(\Gamma)$ with the trivial normal cobordism corresponding to 0. Denote this correspondence by

$$W \mapsto \omega(W) \in L_{m+n}(\Gamma).$$

Wall also proved the following product formula.

Let N^{4k} be a simply connected closed oriented manifold and W be a normal cobordism over $M^m \times \mathbb{D}^{n-1}$. Form a new normal cobordism $W \times N$ over $M^m \times \mathbb{D}^{n-1} \times N^{4k}$ by producting W with N, then

$$\omega(W \times N) = \mathrm{Index}(N)\omega(W).$$

Remark. Anderson's Theorem is an analogue of this result where $\chi(N)$ replaces $\mathrm{Index}(N)$.

This product formula has the following geometric consequence.

Proposition. *Let K^{4k} be a closed oriented simply connected manifold with* $\mathrm{Index}(K) = 1$. *Let $f : N \to M$ be a homotopy equivalence where N is also a closed manifold. If*

$$f \times \mathrm{id} : N \times K \to M \times K$$

is homotopic to a homeomorphism, then f is also homotopic to a homeomorphism.

Sketch of Proof. Arguing as in the proof of the main result of Lecture 2, we compare the surgery exact sequence for $\mathcal{S}(M)$ with that for $\mathcal{S}(M \times K)$. If $x \in \mathcal{S}(M)$ denotes the homotopy-topological structure $f : N \to M$, it goes to 0 in $\mathcal{S}(M \times K)$. And since the map $[M, G/\mathrm{Top}] \to [M \times K, G/\mathrm{Top}]$ is monic, x is the image of an element $\bar{x} \in L_{m+1}(\Gamma)$ which maps to an element

$\hat{x} \in L_{m+1+4k}(\Gamma)$ by producting the normal cobordism with K^{4k}. But the image of $\hat{x}$ in $\mathcal{S}(M \times K)$ is represented by

$$f \times \mathrm{id} : N \times K \to M \times K$$

and is hence zero. Therefore $\hat{x}$ is in the image of the Quinn assembly map in the surgery sequence for $M \times K$. But this map factors through the assembly map

$$[M^m \times \mathbb{D}^{4k+1}, \partial; G/\mathrm{Top}] \to L_{m+4k+1}(\Gamma)$$

which is periodic of period $4k$ with $\bar{x}$ going to $\hat{x}$. This factoring can be seen using Quinn's Δ-set description of the surgery sequence [60], [61] (cf. [70, §17A]) or Ranicki's algebraic formulation of it. (See Ranicki's 2nd lecture.) Hence $\bar{x}$ is in the image of σ, and therefore $x = 0$. □

The complex projective plane $\mathbb{C}P^2$ is the natural candidate for K when applying this Proposition. It is important for this purpose to have the following alternate description of $\mathbb{C}P^2$. Let C_2 denote the cyclic group of order 2. It has a natural action on $S^n \times S^n$ determined by the involution $(x, y) \mapsto (y, x)$ where $x, y \in S^n$. Denote the orbit space of this action by F_n; i.e.

$$F_n = S^n \times S^n / C_2.$$

Lemma 1. $\mathbb{C}P^2 = F_2$.

Proof. Let $sl_2(\mathbb{C})$ be the set of all 2×2 matrices with complex number entries and trace zero. Since $sl_2(\mathbb{C})$ is a 3-dimensional $\mathbb{C}$-vector space, $\mathbb{C}P^2$ can be identified as the set of all equivalence classes $[A]$ of non-zero matrices $A \in sl_2(\mathbb{C})$ where A is equivalent to B iff $A = zB$ for some $z \in \mathbb{C}$. The characteristic polynomial of $A \in sl_2(\mathbb{C})$ is $\lambda^2 + \det(A)$. Consequently, A has two distinct 1-dimensional eigenspaces if $\det(A) \neq 0$, and a single 1-dimensional eigenspace if $\det(A) = 0$ and $A \neq 0$. Also, A and zA have the same eigenspaces provided $z \neq 0$. These eigenspaces correspond to points in S^2 under the identification $S^2 = \mathbb{C}P^1$. The assignment

$$[A] \mapsto \text{ the eigenspaces of } A$$

determines a homeomorphism of $\mathbb{C}P^2$ to F_2. □

Remark. The TRT was first proved in the case where M^m is a hyperbolic 3-dimensional manifold by making use of Lemma 1. It was then realized that the general result for m odd could be proven using F_{m-1} once one could handle the technical complications arising from the fact that F_k is *not* a manifold when $k > 2$. The following result is used in overcoming these complications. It shows that F_k is "very close" to being a manifold of index equal to 1 when k is even.

Lemma 2. *Let n be an even positive integer. Then F_n has the following properties.*

1. *F_n is orientable $2n$-dimensional $\mathbb{Z}[\frac{1}{2}]$-homology manifold.*

2. *F_n is simply connected*

3. $$H_i(F_n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n, 2n \\ \mathbb{Z}_2 & \text{if } n < i < 2n \text{ and } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

4. $$H^i(F_n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n, 2n \\ \mathbb{Z}_2 & \text{if } n + 2 < i < 2n \text{ and } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

5. *The cup product pairing*

$$H^n(F_n) \otimes H^n(F_n) \to H^{2n}(F_n)$$

is unimodular and its signature is either 1 *or* -1.

Proof. There is a natural stratification of F_n consisting of two strata B and T. The bottom stratum B consists of all (agreeing) unordered pairs $\langle u, v\rangle$ where $u = v$; while the top stratum T consists of all (disagreeing) pairs $\langle u, v\rangle$ where $u \neq v$.

Note that B can be identified with S^n. Also real projective n-space $\mathbb{R}P^n$ can be identified with the set of all unordered pairs $\langle u, -u\rangle$ in F_n. It is seen that F_n is the union of "tubular neighborhoods" of S^n and $\mathbb{R}P^n$ intersecting in their boundaries. The first tubular neighborhood is a bundle over S^n with fiber the cone on $\mathbb{R}P^{n-1}$. The second tubular neighborhood is a bundle over $\mathbb{R}P^n$ with fiber $\mathbb{D}^n$. Furthermore, they intersect in the total space of the $\mathbb{R}P^{n-1}$-bundle associated to the tangent bundle of S^n. This description of F_n can be used to verify Lemma 2. See [29, p. 299] for more details. □

Caveat. The fundamental class of B represents twice a generator of $H_n(F_n)$. On the other hand, if we fix a point $y_0 \in S^n$, then the map $x \mapsto \langle x, y_0 \rangle$ is an embedding of S^n in F_n which represents a generator of $H_n(F_n)$.

Let $f : N \to M$ represent an element in $\mathcal{S}(M)$. Then $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ represents an element in $\mathcal{S}(M \times S^1)$. This defines a map $\mathcal{S}(M) \mapsto \mathcal{S}(M \times S^1)$.

Lemma 3. *The map $\mathcal{S}(M) \mapsto \mathcal{S}(M \times S^1)$ is monic.*

Proof. Suppose $f \times \mathrm{id}$ is homotopic to a homeomorphism g via a homotopy

$$h : N \times S^1 \times [0,1] \to M \times S^1 \times [0,1]$$

where $h|_{N\times S^1 \times 0} = f \times \mathrm{id}$ and $h|_{N \times S^1 \times 1} = g$. By one of the codimension-one splitting theorems mentioned in my first lecture [20], we can split h along $M \times 1 \times [0,1]$.

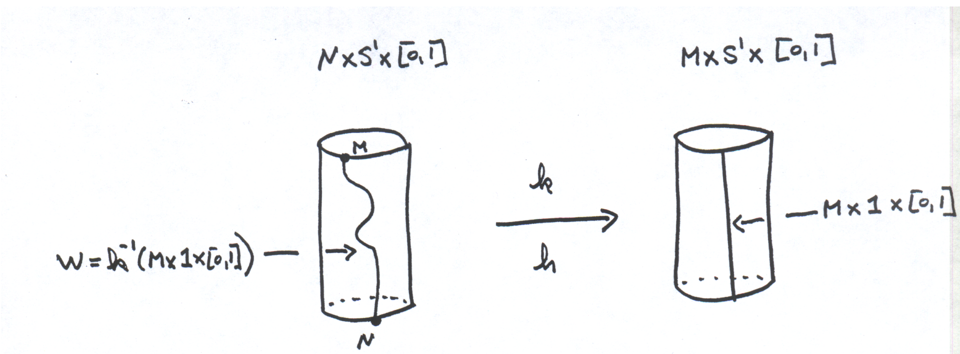


Figure 4.1

That is h is homotopic rel ∂ to a map k such that

$$k|_W : W \to M \times 1 \times [0,1]$$

is a homotopy equivalence where

$$W = k^{-1}(M \times 1 \times [0,1]).$$

We use that $Wh(\Gamma) = 0$ to get this. Now note that W is an h-cobordism between M and N. But W is a cylinder; again since $Wh(\Gamma) = 0$. ☐

Remark. In order to prove the TRT, it suffices to show that $f \times \mathrm{id}$ is homotopic to a homeomorphism because of Lemma 3.

We now formulate a variant of the Proposition. This variant is used in showing that

$$f \times \mathrm{id} : N \times S^1 \to M \times S^1$$

is homotopic to a homeomorphism. There is a bundle

$$p : \mathcal{F}M \to M \times S^1$$

whose fiber over a point $(x, \theta) \in M \times S^1$ consists of all unordered pairs of unit length vectors $\langle u, v \rangle$ tangent to $M \times S^1$ at (x, θ) and satisfying the following two constraints.

1. If $u \neq v$, then both u and v are tangent to the level surface $M \times \theta$.
2. If $u = v$, then the projection $\bar{u}$ of u onto $T_\theta S^1$ points in the counter-clockwise direction (or is 0).

The total space $\mathcal{F}M$ is stratified with three strata:

$$\begin{aligned} \mathbb{B} &= \{\langle u, u \rangle \mid \bar{u} = 0\} \\ \mathbb{A} &= \{\langle u, u \rangle \mid \bar{u} \neq 0\} \\ \mathbb{T} &= \{\langle u, v \rangle \mid u \neq v\}. \end{aligned}$$

Note that $\mathbb{B}$ is the bottom stratum and that $\mathcal{F}M - \mathbb{B}$ is the union of the two open sets $\mathbb{A}$ (auxiliary stratum) and $\mathbb{T}$ (top stratum). The restriction of p to each stratum is a sub-bundle. Let $\mathcal{F}_x$, B_x, A_x and T_x denote the fibers of these bundles over $x \in M \times S^1$; i.e.,

$$\mathcal{F}_x = p^{-1}(x),\ B_x = \mathcal{F}_x \cap \mathbb{B},\ A_x = \mathcal{F}_x \cap \mathbb{A},\ T_x = \mathcal{F}_x \cap \mathbb{T}.$$

Note that $B_x = S^{m-1}$, $A_x = \mathbb{D}^m$, $T_x \cup B_x = F_{m-1}$ and the bundle $p : \mathbb{B} \to M \times S^1$ is the pullback of the tangent unit sphere bundle of M under the projection $M \times S^1 \to M$.

The space F_{m-1} will play the role of the index one manifold K in our variant of the Proposition. Since it is unfortunately not a manifold when $m > 3$, we need to introduce the auxiliary fibers A_x. Hence the total fiber is homeomorphic to $F_{m-1} \cup \mathbb{D}^m$ where the subspace B in F_{m-1} is identified with $S^{m-1} = \partial \mathbb{D}^m$. Let

$$\mathcal{F}_f \to N \times S^1$$

denote the pullback of

$$\mathcal{F}M \to M \times S^1$$

along $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ and let

$$\hat{f} : \mathcal{F}_f \to \mathcal{F}M$$

be the induced bundle map. Note that the stratification of $\mathcal{F}M$ induces one on $\mathcal{F}_f$ and that $\hat{f}$ preserves strata.

We say that $\hat{f}$ is *admissibly homotopic to a split map* provided there exists a homotopy h_t, $t \in [0,1]$, with $h_0 = \hat{f}$ and satisfying the following four conditions.

1. Each h_t is strata preserving.

2. Over some closed "tubular neighborhood" $\mathcal{N}_0$ of $\mathbb{B}$ in $\mathbb{B} \cup \mathbb{T}$, each h_t is a bundle map; in particular, h_t maps fibers homeomorphically to fibers.

3. There is a larger closed "tubular neighborhood" $\mathcal{N}_1$ of $\mathbb{B}$ in $\mathbb{B} \cup \mathbb{T}$ such that h_1 is a homeomorphism over $\mathbb{B} \cup \mathbb{T} - \mathrm{Int}(\mathcal{N}_1)$ and over $\mathbb{B} \cup \mathbb{A}$.

4. Let $\rho : \mathcal{N}_1 \to M \times S^1$ denote the composition of the two bundle projections $\mathcal{N}_1 \to \mathbb{B}$ and $\mathbb{B} \to M \times S^1$. Then there is a triangulation K for $M \times S^1$ such that h_1 is transverse to $\rho^{-1}(\sigma)$ for each simplex σ of K. Furthermore

$$h_1 : h_1^{-1}(\rho^{-1}(\sigma)) \to \rho^{-1}(\sigma)$$

is a homotopy equivalence.

Remark. Conditions 3 and 4 should be heuristically replaced by the simpler and stronger condition that "h_1 is a homeomorphism". But for technical reasons we need to work instead with conditions 3 and 4.

The variant of the Proposition needed to prove the TRT is the following.

Proposition $(*)$**.** *The map $f : N \to M$ is homotopic to a homeomorphism provided $\hat{f} : \mathcal{F}_f \to \mathcal{F}M$ is admissibly homotopic to a split map.*

The proof of Proposition $(*)$ is basically an elaboration of the one sketched above for Proposition. (See [29, §4 and §9].) It in particular uses again Quinn's Δ-set approach to the surgery exact sequence and generalizes Wall's product formula to the stratified setting above by using Lemma 2.

Proposition $(*)$ is the surgery theory part of the proof of the TRT. The geometry of M (in particular, its non-positive curvature) is used to show that the hypothesis of Proposition (*) is satisfied; i.e, that $\hat{f}$ is admissibly homotopic to a split map. We now proceed to discuss how this is done.

It is a consequence of several applications of both ordinary and foliated topological control theory as discussed in Lowell Jones' lectures. Let $g : M \to N$ be a (strong) homotopy inverse to f and let h_t and k_t be (strong) homotopies of the composite $f \circ g$ to id_M and $g \circ f$ to id_N, respectively. Strong means base point preserving. It implies the following useful property.

Property (*)**.** For each point $x \in N$, the two paths

$$t \mapsto h_t(f(x)) \quad \text{and} \quad t \mapsto f(k_t(x))$$

are homotopic rel end points.

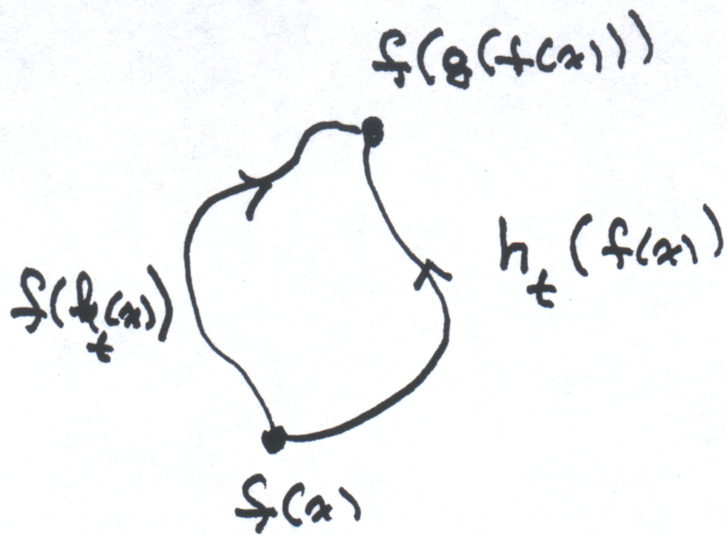


Figure 4.2

We may assume that N is a smooth manifold by using Kirby-Siebenmann smoothing theory [49]. For this we need only observe that the stable topological tangent bundle of N has a real vector bundle structure since it is the pull back of TM stabilized via f because $f : M \to N$ maps to 0 in $[M, G/\mathrm{Top}]$. Therefore we may also assume that f and g are smooth maps and that both h_t and k_t are smooth homotopies.

The crucial point is to construct "good" transfers of the map g and the homotopies h_t, k_t to a map $\hat{g} : \mathcal{F}M \to \mathcal{F}_f$ and homotopies $\hat{h}_t$, $\hat{k}_t$ from $\hat{f} \circ \hat{g}$ to $\mathrm{id}_{\mathcal{F}M}$, and $\hat{g} \circ \hat{f}$ to $\mathrm{id}_{\mathcal{F}_f}$, respectively, so that control theory can be applied to admissibly homotope $\hat{f}$ to a split map. We proceed to describe what a good transfer is and then indicate how to construct one. The first requirement is that $\hat{g}$, $\hat{h}_t$ and $\hat{k}_t$ be bundle maps covering $g \times \mathrm{id}$, $h_t \times \mathrm{id}$, $k_t \times \mathrm{id}$, respectively. (Here id is the identity map on S^1.) Second, each map $\hat{g}$, $\hat{h}_t$ and $\hat{k}_t$ should preserve strata. Finally, it is necessary that a certain family $\mathcal{T}$ of paths determined by the lift is sufficiently "shrinkable". A path $\alpha : [0,1] \to \mathcal{F}M$ is in $\mathcal{T}$ if either

$$\alpha(t) = \hat{h}_t(\omega) \qquad \text{for some } \omega \in \mathcal{F}M, \quad \text{or}$$
$$\alpha(t) = \hat{f}(\hat{k}_t(\omega)) \qquad \text{for some } \omega \in \mathcal{F}_f.$$

(The family $\mathcal{T}$ is called the tracks of the transfer.) Note that each track is contained in a single stratum of $\mathcal{F}M$.

We construct good transfers by constructing their tracks $\mathcal{T}$. Since this is easier to explain when *M is negatively curved, we now make this assumption.* The construction of $\mathcal{T}$ uses (mainly) the asymptotic transfer of paths discussed in lecture 3. (The general case uses the focal transfer which, although more elementary, requires greater technical details.) Let $\mathcal{T}_1$ be the tracks determined by f, g, h_t and k_t; i.e. a curve $\alpha : [0,1] \to M$ is in $\mathcal{T}_1$ if for all $t \in [0,1]$ either

$$\alpha(t) = h_t(x) \qquad \text{for some } x \in M; \quad \text{or}$$
$$\alpha(t) = f(k_t(y)) \qquad \text{for some } y \in N.$$

Given $\gamma \in \mathcal{T}_1$ and $\omega = \langle u, v \rangle \in \mathcal{F}M$ with foot $(\gamma(0), \theta) \in M \times S^1$, we associate a lift $\omega\gamma$ of γ to a path in $\mathcal{F}M$ covering γ_θ which is the path in $M \times S^1$ defined by

$$\gamma_\theta(t) = (\gamma(t), \theta).$$

When $\omega \in \mathbb{B} \cup \mathbb{T}$, $\omega\gamma$ is defined by

$$\omega\gamma(t) = \langle u\gamma_\theta(t), v\gamma_\theta(t) \rangle$$

where $u\gamma_\theta$ and $v\gamma_\theta$ are the asymptotic transfers defined in Lecture 3. When $\omega \in \mathbb{A}$ (and hence $u = v$) $\omega\gamma$ is defined by

$$\omega\gamma(t) = \langle u(\gamma_\theta, d)(t), u(\gamma_\theta, d)(t) \rangle$$

where $u(\gamma_\theta, d)$ is the focal transfer with focal length d and chosen so that $d \to \infty$ as the angle between u and the level surface $M \times \theta$ approaches 0. Using that the asymptotic and focal transfers both satisfy properties 1-3 of lecture 3 and that property $(*)$ is satisfied by g, f, h_t, k_t; there is a natural construction of a good transfer $\hat{g}$, $\hat{h}_t$, $\hat{k}_t$ whose tracks

$$\mathcal{T} = \{\omega\gamma \mid \gamma \in \mathcal{T}_1, \omega \in \mathcal{F}M\}.$$

We now address the problem of "shrinking" the paths $\omega\gamma \in \mathcal{T}$. Since the geodesic flow g^t is defined on $\mathbb{A} \cup \mathbb{B}$, applying it to $\omega\gamma$ gives a method for making $\omega\gamma$ skinny when $\omega \in \mathbb{A} \cup \mathbb{B}$; i.e. $g^t \circ (\omega\gamma)$ is (β, ϵ)-controlled with respect to the 1-dimensional foliation of the manifold $\mathbb{A} \cup \mathbb{B}$ by the flow lines of the geodesic flow.

But the situation is different when $\omega = \langle u, v \rangle \in \mathbb{T}$. We are tempted then to "flow ω" in the direction of its arithmetic average $\frac{u+v}{2}$. But this does nothing when $u = -v$. Fortunately a different method can be used on the top stratum $\mathbb{T}$. But to describe it we need some more geometric preliminaries. We start by defining the *core* $\mathbb{P}$ of $\mathbb{T}$ by

$$\mathbb{P} = \{\langle u, -u \rangle \in \mathbb{T}\}.$$

The core is naturally identified with the total space of the projective line bundle associated to $(TM) \times S^1$. In particular there is a natural 2-sheeted covering space

$$\mathbb{B} = (SM) \times S^1 \to (\mathbb{R}P^{m-1}M) \times S^1 = \mathbb{P}$$

and the image of the geodesic line foliation of $\mathbb{B}$ gives $\mathbb{P}$ a canonical 1-dimensional foliation denoted by $\mathcal{G}$. The top strata $\mathbb{T}$ also has an *asymptotic foliation* $\mathcal{A}$ by m-dimensional leaves where each leaf of $\mathcal{A}$ is an *asymptoty class* of elements in $\mathbb{T}$. We say that elements $\omega_1 = \langle u_1, v_1 \rangle$, $\omega_2 = \langle u_2, v_2 \rangle \in \mathbb{T}$ lying over $M \times \theta$ (for some $\theta \in S^1$) are *asymptotic* provided (up to interchanging u_1 and v_1) there exist points $x, y \in \tilde{M}$ together with vectors $\tilde{u}_1, \tilde{v}_1 \in S_{(x,\theta)}(\tilde{M} \times S^1)$ and $\tilde{u}_2, \tilde{v}_2 \in S_{(y,\theta)}(\tilde{M} \times S^1)$ lying over u_1, v_1, u_2, v_2, respectively, and satisfying:

$$\tilde{u}_1 \text{ is asymptotic to } \tilde{u}_2, \text{ and}$$
$$\tilde{v}_1 \text{ is asymptotic to } \tilde{v}_2.$$

Note that the restriction of the bundle map

$$\mathbb{T} \xrightarrow[p]{} M \times S^1 \xrightarrow[\text{proj}]{} M$$

to any leaf L of $\mathcal{A}$ is a covering space. This puts a flat structure on this bundle. And each leaf L of $\mathcal{A}$ inherits a negatively curved Riemannian metric from M via this covering projection. We call it the natural metric and note that it is compatible with the leaf topology on L.

The foliation $\mathcal{A}$ intersects the core $\mathbb{P}$ in its $\mathcal{G}$ foliation; i.e., there is a bijective correspondence between the leaves of $\mathcal{A}$ and $\mathcal{G}$ given by

$$L \mapsto L \cap \mathbb{P}, \quad L \in \mathcal{A}.$$

Also $L \cap \mathbb{P}$ is a closed subset of L in its leaf topology and is a (simple) geodesic of its natural metric. This geodesic $\mathbb{P} \cap L$ is called the *marking* of L. Furthermore, the inclusion map of $\mathbb{P} \cap L$ into L is a homotopy equivalence when L is given the leaf topology and $\mathbb{P} \cap L$ is given the subspace of L topology.

Now there is a bundle with fiber $\mathbb{R}^{m-1}$

$$\rho : \mathbb{T} \to \mathbb{P}$$

defined as follows.

For each $\omega \in \mathbb{T}$ let $L \in \mathcal{A}$ be the leaf containing ω and g be its marking. Then $\rho(\omega)$ denotes the (unique) closest point to ω on g measured inside L.

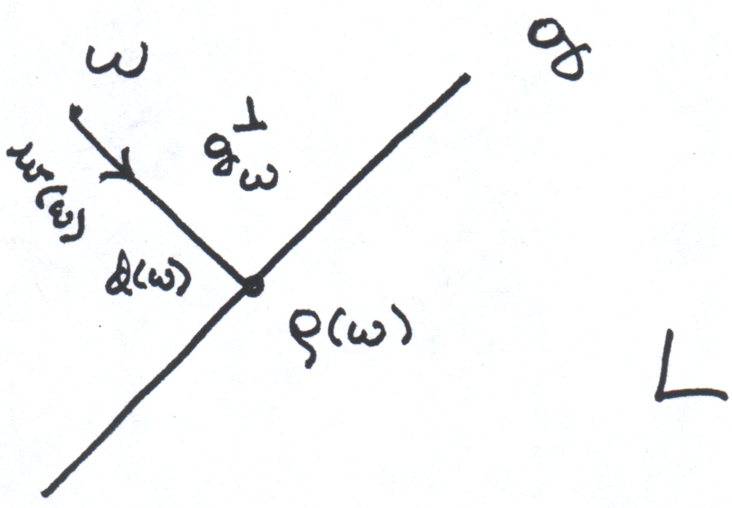


Figure 4.3

When $\omega \notin \mathbb{P}$, there is a unique geodesic segment $g_\omega^\perp$ in L connecting ω to $\rho(\omega)$. The unit length vector tangent to $g_\omega^\perp$ at ω which points towards $\rho(\omega)$ is denoted by $w(\omega)$. This defines a continuous vector field on $\mathbb{T} - \mathbb{P}$.

We denote the length of $g_\omega^\perp$ by $d(\omega)$. This extends to a continuous function $d : \mathbb{T} \cup \mathbb{B} \to [0, +\infty]$ when we set

$$d(\omega) = \begin{cases} 0 & \text{if } \omega \in \mathbb{P} \\ +\infty & \text{if } \omega \in \mathbb{B}. \end{cases}$$

There is also a bundle with fiber the open cone in $\mathbb{R}P^{m-1}$

$$\eta : (\mathbb{T} - \mathbb{P}) \cup \mathbb{B} \to \mathbb{B} = SM \times S^1$$

defined by

$$\eta(\omega) = \begin{cases} \omega & \text{if } \omega \in \mathbb{B} \\ dp(w(\omega)) & \text{if } \omega \in \mathbb{T} - \mathbb{P}. \end{cases}$$

Remark. We think of $\eta(\omega)$ as the *asymptotic average* of the two vectors u and v where $\omega = \langle u, v \rangle$ as opposed to their *arithmetic* average $\frac{1}{2}(u + v)$.

The vector field $w(\ )$ integrates to give an incomplete *radial flow* r^t on $\mathbb{T}$. In particular $r^t(\omega)$ is only defined for $t \in [0, d(w)]$. And there is the following important relation between r^t and g^t.

Intertwining Equation. $\eta(r^t(\omega)) = g^t(\eta(\omega))$ for all $\omega \in \mathbb{T} - \mathbb{P}$ and $t \in [0, d(\omega)]$.

We associate to each closed interval $J \subseteq [0, +\infty]$ a compact subspace W_J of $\mathbb{T} \cup \mathbb{B}$ defined by

$$W_J = d^{-1}(J).$$

If $+\infty \notin J$, then W_J is a codimension-0 submanifold of $\mathbb{T}$ with

$$\partial W_J = \begin{cases} d^{-1}(\partial J) & \text{if } 0 \notin J \\ d^{-1}(b) & \text{if } J = [0, b]. \end{cases}$$

Furthermore, we have the following:

1. If $0 \in J$ and $+\infty \notin J$, then $\rho : W_J \to \mathbb{P}$ is a fiber bundle with fiber $\mathbb{D}^m$.

2. If $+\infty \in J$ and $0 \notin J$, then $\eta : W_J \to \mathbb{B}$ is a fiber bundle with fiber the (closed) cone on $\mathbb{R}P^{m-1}$.

3. If neither 0 nor $+\infty$ is in J, then $\eta \times d : W_J \to \mathbb{B} \times J$ is a fiber bundle with fiber $\mathbb{R}P^{m-1}$.

Now fix a closed interval $I \subseteq (0, +\infty)$ containing 1 in its interior, and a very large positive real number σ together with a second closed interval R which contains $+\infty$ and is disjoint from σI. Then $[0, +\infty) - (\mathrm{Int}(R) \cup \mathrm{Int}(\sigma I))$ is the disjoint union of 2 closed intervals A and B denoted so that $0 \in A$.

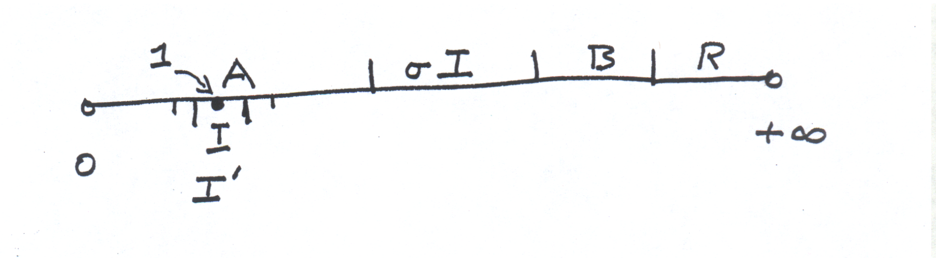


Figure 4.4

Fix another closed interval $I' \subseteq (0, +\infty)$ which contains I in its interior but is slightly larger and define a homeomorphism

$$\phi : W_{\sigma I'} \to W_{I'},$$

using the radial flow, by the formula

$$\phi(\omega) = r^t(\omega)$$

where $t = d(\omega) - \frac{1}{\sigma} d(\omega)$.

Note that ϕ becomes arbitrarily strongly contracting as we let $\sigma \to +\infty$. In particular if $\omega\gamma \in \mathcal{T}$ with $\omega \in W_{\sigma I}$, then $\phi \circ \omega\gamma$ is uniformly pointwise ϵ_σ-controlled in W_I with $\lim_{\sigma \to \infty} \epsilon_\sigma = 0$.

Therefore we can use the ordinary control theorem to homotope $\hat{f}$ over $W_{\sigma I}$ (i.e. homotope $\hat{f}|_{\hat{f}^{-1}(W_{\sigma I})}$), in a controlled way, to a homeomorphism provided σ is large enough. This begins our construction of the admissible homotopy of $\hat{f}$ to a split map. And it gives a slightly different collection of tracks $\mathcal{T}_1$. These new tracks differ only for some of those $\gamma \in \mathcal{T}_1$ which start in $W_{\sigma I}$. And for them $\phi \circ \gamma$ is pointwise close to $\phi \circ \bar{\gamma}$ where $\bar{\gamma} \in \mathcal{T}$ is the corresponding track.

We next extend this homotopy to a homotopy of $\hat{f}$ over W_A to a homeomorphism. This is done by using the fibered and foliated version of the control theorem with respect to the fiber bundle

$$\rho : W_A \to \mathbb{P}$$

and the foliation $\mathcal{G}$ of $\mathbb{P}$. It is applicable since the fiber of ρ is $\mathbb{D}^m$ and the structure set $\mathcal{S}(\mathbb{D}^m \times \mathbb{D}^k, \partial)$ contains only one element for each $k \geq 0$. We need only check the control condition. For this, note that there exists a positive number β such that

$$\rho \circ \omega\gamma$$

is $(\beta, 0)$-controlled for each $\omega\gamma \in \mathcal{T}$ such that $\omega \in A$. And hence the tracks of $\mathcal{T}_1$ which start in A are $(\beta + \epsilon, \epsilon)$-controlled where $\epsilon \to 0$ as $\sigma \to \infty$.

Independent of these two steps, we use the foliated control theorem with respect to the foliation of $\mathbb{B} \cup \mathbb{A}$ by the orbits of the geodesic flow and the control map

$$g^\sigma : \mathbb{B} \cup \mathbb{A} \to \mathbb{B} \cup \mathbb{A}$$

to homotope $\hat{f}$ over $\mathbb{B} \cup \mathbb{A}$ to a homeomorphism. And then use the covering homotopy theorem to extend this to a homotopy of bundle maps over W_R relative to the fiber bundle

$$\eta : W_R \to \mathbb{B}$$

to a homeomorphism over W_R.

Let $\tau : B \to [0, 1]$ denote the (unique) increasing linear homeomorphism, and fix a continuous function $\phi : B \to [\sigma, +\infty)$ such that

$$\phi(x) = \begin{cases} \sigma & \text{for all } x \text{ close to } B \cap R \\ (1 - \frac{1}{\sigma})x & \text{for } x \in \sigma I' \cap B. \end{cases}$$

Consider the fiber bundle

$$\xi : W_B \to \mathbb{B} \times [0, 1]$$

where ξ is the composite

$$W_B \xrightarrow{\eta \times d} \mathbb{B} \times B \xrightarrow{g^{\phi \circ d} \times \tau} \mathbb{B} \times [0, 1]$$

i.e.,

$$\xi(x) = (g^{\phi(d(x))}(\eta(x)), \tau(d(x))).$$

Finally, we use a foliated and fibered version of the control theorem with respect to the fiber bundle $\xi : W_B \to \mathbb{B} \times [0, 1]$ and the foliation of $\mathbb{B} \times [0, 1]$ by the flow lines of the geodesic flow in order to extend over W_B the homotopy defined in steps 1, 2 and 3 given above. And thus complete the construction of an admissible homotopy of $\hat{f}$ to a split map. The control condition is met provided σ is sufficiently large and R is contained in a sufficiently small neighborhood of $+\infty$. The intertwining equation is used to see this. But there is one extra point to observe. The fiber of ξ is $\mathbb{R}P^{m-1}$ and $\mathcal{S}(\mathbb{R}P^{m-1} \times \mathbb{D}^k, \partial)$ usually contains more than a single element. Consequently, the control theorem only yields the weaker conclusion that the result of the homotopy is a split map rather than a homeomorphism.

5 Some calculations of $\pi_n(\mathbf{Top}\ M)$, $\pi_n(\mathbf{Diff}\ M)$ and other applications

Recall (Lecture 3) that the Vanishing Theorem showing $Wh(\pi_1 M) = 0$ extends to complete, A-regular, non-positively curved Riemannian manifolds M. Likewise there is a version of the Topological Rigidity Theorem (TRT) valid for such manifolds which we proceed to formulate.

Let M be an arbitrary manifold; i.e. it can be non-compact and can have non-empty boundary. We say that M is *topologically rigid* if it has the following property. Let

$$h : (N, \partial N) \to (M, \partial M)$$

be any proper homotopy equivalence where N is another manifold. Suppose there exists a compact subset $C \subseteq N$ such that the restriction of h to $\partial N \cup (N - C)$ is a homeomorphism. Then there exists a proper homotopy

$$h_t : (N, \partial N) \to (M, \partial M)$$

from h to a homeomorphism and a perhaps larger compact subset K of N such that the restrictions of h_t and h to $\partial N \cup (N - K)$ agree for all $t \in [0, 1]$. (When M and N are closed, this just says that a homotopy equivalence $h : N \to M$ is homotopic to a homeomorphism.)

Addendum to TRT. *(Farrell and Jones [35] 1998). Let M^m be an arbitrary aspherical manifold with $m \geq 5$. Suppose $\pi_1(M)$ is isomorphic to the fundamental group of an A-regular complete non-positively curved Riemannian manifold. (This happens for example when $\pi_1(M)$ is isomorphic to a torsion-free discrete subgroup of $GL_n(\mathbb{R})$.) Then M is topologically rigid. In particular, every A-regular complete non-positively curved Riemannian manifold of* dim ≥ 5 *is topologically rigid.*

The special case of this Addendum where M is an A-regular complete non-positively curved Riemannian manifold is proved by an argument very close to that made in Lecture 4 for TRT. But stronger control theorems are needed when M is not closed; in particular when the injectivity radius at a point $x \in M$ goes to 0 as $x \to \infty$. These control theorems were discussed by Lowell Jones in his last lecture. The general case of the Addendum follows from this special case and the version of the surgery sequence for arbitrary spaces developed by Andrew Ranicki in his lectures; in particular that the assembly map in homology

$$A_* : H_*(M; \mathcal{L}) \to L_*(\pi_1 M, w)$$

is uniquely determined by the homotopy type of M and the orientation data $w : \pi_1(M) \to \mathbb{Z}_2$.

This Addendum even has (perhaps unexpectedly) consequences beyond, what follows from TRT, for closed manifolds. We now discuss some of these.

Corollary 1. *Let N and M be a pair of closed complete affine flat manifolds. If $\pi_1(N) \simeq \pi_1(M)$, then N and M are homeomorphic (via a homeomorphism inducing this isomorphism).*

Corollary 1 is an affine analogue of the classical Bieberbach Theorem valid for Riemannian flat manifolds. We note that Corollary 1 (when dim $(M) \geq 5$) does *not* follow from the TRT proved in Lecture 4 since there are closed complete affine flat manifolds M which *cannot* support a Riemannian metric of non-positive curvature. For example $M^3 = \mathbb{R}^3/\Gamma$ does *not* where Γ is the group generated by the three affine motions α, β and γ of $\mathbb{R}^3$ with

$$\begin{aligned} \alpha(x,y,z) &= (x+1,y,z) \\ \beta(x,\gamma,z) &= (x,y+1,z) \\ \gamma(x,y,z) &= (x+y,2x+3y,z+1). \end{aligned}$$

Since Γ is solvable but not virtually abelian, the result of Gromoll-Wolf [40] and Yau [71], quoted in lecture 1, shows that M cannot support a non-positively curved Riemannian metric. But Corollary 1 (when dim $(M) \geq 5$) does follow from the Addendum to TRT since M is aspherical and $\pi_1(M)$ is a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^m)$ which is a closed subgroup of $GL_{m+1}(\mathbb{R})$; namely

$$\mathrm{Aff}(\mathbb{R}^m) = \left\{ A \in GL_{m+1}(\mathbb{R}) \,\middle|\, A_{m+1,i} = \begin{cases} 0 & i \leq m \\ 1 & i = m+1 \end{cases} \right\}$$

Corollary 1 is a classical result when dim $(M) \leq 2$. And, when dim $(M) = 3$, Corollary 1 was proven by D. Fried and W.M. Goldman in [38]. Hence it remains to discuss the case when dim $(M) = 4$. In this case (in fact more generally when dim $(M) \leq 6$) H. Abels, G.A. Margulis and G.A. Soifer [1] proved that $\pi_1(M)$ is virtually polycyclic. And hence Corollary 1 follows from Farrell and Jones [28] when dim $(M) = 4$. A key ingredient in [28] is that M. Freedman and F. Quinn [37] have shown that topological surgery works in dimension 4 for manifolds with virtually poly-cyclic fundamental groups.

Corollary 1 suggests the following question.

Question. Are compact complete affine flat manifolds with isomorphic fundamental groups diffeomorphic?

Compare [34] where the analogous question for infrasolvmanifolds was affirmatively answered except in dimension 4.

We next use this Addendum to verify a special case of a well known conjecture of C.T.C. Wall; cf. [69].

Conjecture. (Wall) Let Γ be a torsion-free group which contains a subgroup of finite index isomorphic to the fundamental group of a closed aspherical manifold. Then Γ is the fundamental group of a closed aspherical manifold.

Corollary 2. *Let M^m be a closed (connected) non-positively curved Riemannian manifold and Γ be a torsion-free group which contains $\pi_1(M)$ as a subgroup with finite index. Assume that $m \neq 3, 4$, then the deck transformation action of $\pi_1(M)$ on the universal cover $\tilde{M}$ extends to a topological action of Γ on $\tilde{M}$. Consequently Wall's Conjecture is true in this case since $\tilde{M}/\Gamma$ is a closed apsherical manifold with $\pi_1(\tilde{M}/\Gamma) = \Gamma$.*

Remark. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional factors, Γ embeds in its isometry group $\mathrm{Iso}(\tilde{M})$ extending $\pi_1(M) \subseteq \mathrm{Iso}(\tilde{M})$; this is a consequence of Mostow's Strong Rigidity Theorem [57]. When $m = 2$, Corollary 2 is a consequence of a result due to Eckmann, Linnell and Muller [18], [19]; our proof only applies to the situation $m \geq 5$.

In proving Corollary 2 we can clearly make the simplifying assumptions that M is orientable and $\pi_1(M)$ is normal in Γ. We now use an important trick due to Serre [65] where he constructs a natural, properly discontinuous action of Γ via isometries on the Riemannian product $\mathcal{M}^{sm}$ of s-copies of $\tilde{M}$

$$\mathcal{M} = \tilde{M} \times \tilde{M} \times \cdots \times \tilde{M}$$

where $s = [\pi_1 M : \Gamma]$. (Serre's construction is a kind of geometric co-induced representation.) Note that $\mathcal{M}^{sm}$ is A-regular and non-positively curved since

M^m is. Hence $N^{sm} = \mathcal{M}/\Gamma$ is a complete (but *not* closed) A-regular non-positively curved manifold with $\pi_1(N) = \Gamma$. Thus the Addendum to TRT applies to $N^{sm} \times \mathbb{D}^k$ for all $k \geq 0$. From this we conclude that Ranicki's periodic assembly map

$$A_* : H_*(B\Gamma, \mathcal{L}) \to L_*(\Gamma)$$

is an isomorphism. Also the Vanishing Theorem applies showing that

$$Wh(\Gamma) = 0 = \tilde{K}_0(\mathbb{Z}\Gamma).$$

And Ranicki, by reworking the existence part of surgery theory, has shown that when this happens $B\Gamma$ is homotopically equivalent to a closed manifold K^m provided $B\pi$ is for some subgroup π of finite index in Γ; cf. [63, §13]. In this case, we can take $\pi = \pi_1(M)$.

Let $\hat{K}$ be the cover of K corresponding to $\pi_1(M)$. And note that $\hat{K}$ is homotopically equivalent to M since both are aspherical and have the same fundamental group. Therefore $\hat{K}$ is homeomorphic to M by the TRT. Consequently $\tilde{K} = \tilde{M}$ and the deck transformation action of $\Gamma = \pi_1(K)$ on $\tilde{M}$ is the desired extension of the action by $\pi_1(M)$. Q.E.D.

Corollary 2 can be applied to obtain positive information about the following generalization of the classical Nielsen Problem. Let Top(M) denote the group of all homeomorphisms of a manifold M and denote the group of all outer automorphisms of $\pi_1(M)$ by Out$(\pi_1 M)$.

Generalized Nielsen Problem. (GNP) Let M be a closed aspherical manifold and F be a finite subgroup of Out$(\pi_1 M)$. Does F split back to Top(M); i.e., does there exist a finite subgroup $\bar{F}$ of Top(M) which maps isomorphically onto F under the natural homomorphism

$$\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)?$$

Remark. There are cases where this is impossible. One necessary extra condition is that there exist an extension

$$1 \to \pi_1(M) \to \Gamma \to F \to 1$$

inducing the embedding $F \subseteq \mathrm{Out}(\pi_1 M)$. F. Raymond and L. Scott [64] gave an example where this condition is not satisfied. In their example M is a nilmanifold. There is a natural exact sequence

$$1 \to \mathrm{Center}(\Gamma) \to \Gamma \underset{\phi}{\longrightarrow} \mathrm{Aut}(\Gamma) \underset{\psi}{\longrightarrow} \mathrm{Out}(\Gamma) \to 1$$

where $\phi(\gamma)$ is conjugation by γ. Let $\Gamma_F = \psi^{-1}(F)$. When Center$(\Gamma) = 1$

$$1 \to \Gamma \to \Gamma_F \to F \to 1$$

is the necessary extension mentioned in this Remark.

Corollary 3. *The finite group F of the GNP splits back to* Top(M) *under the following extra assumptions:*

1. Center$(\pi_1 M) = 1$.
2. *M is a non-positively curved Riemannian manifold.*
3. $\dim(M) \neq 3, 4$.
4. *Γ_F is torsion-free.*

Remark. Conditions 1 and 2 are satisfied when M is negatively curved.

Remark. When $\dim(M) = 2$, this result is due to Eckmann, Linnell and Muller (1981).

Remark. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional metric factors, this result, due to Mostow [57], is true even with conditions 1, 3 and 4 dropped.

Remark. Corollary 3 remains true when condition 2 is replaced by the weaker condition that $\pi_1(M)$ is isomorphic to the fundamental group of a complete, A-regular non-positively curved Riemannian manifold. This is because Corollary 2 is also true under the same weakening of its hypotheses.

To prove Corollary 3, note that Γ_F satisfies the hypotheses for the group Γ in Corollary 2. Hence Γ_F acts on $\tilde{M}$ extending the action of $\pi_1(M)$ by deck transformations. The image of this action in Top(M) is the subgroup $\bar{F}$ asked for in GNP. Q.E.D.

There is also the related question of whether the natural homomorphism

$$\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)$$

is onto?

Corollary 4. *Let M^m be a closed aspherical manifold. Assume that $m \neq 3, 4$ and that $\pi_1(M)$ is isomorphic to the fundamental group of a complete, A-regular, non-positively curved Riemannian manifold. Then the natural homomorphism* $\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)$ *is a surjection.*

Corollary 4 is classical for $m = 2$ or 1. And for $m \geq 5$, it follows immediately from the Addendum to TRT since every outer automorphism of $\pi_1(M)$ is induced by a self homotopy equivalence of M; cf. Hurewicz's result mentioned in Lecture 1. Q.E.D.

Remark. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional metric factors, Corollary 4 is due to Mostow [57].

Give the group $\mathrm{Top}(M)$ the compact open topology and let its closed subgroup $\mathrm{Top}_0(M)$ be the kernel of the natural continuous homomorphism (analyzed in Corollary 4) to the discrete group $\mathrm{Out}(\pi_1 M)$. $\mathrm{Top}_0(M)$ is *not* in general the connected component of the identity element in $\mathrm{Top}(M)$. However the following was proved in [32].

Corollary 5. *Let M^m be a closed (connected) non-positively curved Riemannian manifold with $m > 10$. Then*

$$\begin{aligned} \pi_0(\mathrm{Top}_0\ M) &= \mathbb{Z}_2^\infty, \\ \pi_1(\mathrm{Top}\ M) \otimes \mathbb{Q} &= \mathrm{Center}(\pi_1 M) \otimes \mathbb{Q}, \quad \textit{and} \\ \pi_n(\mathrm{Top}\ M) \otimes \mathbb{Q} &= 0 \quad \textit{if} \ \ 1 < n \leq \frac{(m-7)}{3}. \end{aligned}$$

Remark. There is in particular the following exact sequence

$$1 \to \mathbb{Z}_2^\infty \to \pi_0(\mathrm{Top}\ M) \to \mathrm{Out}(\pi_1 M) \to 1.$$

And $\mathbb{Z}_2^\infty$ denotes the direct sum of a countably infinite number of copies of $\mathbb{Z}_2$.

The proof of Corollary 5 depends not only on the Addendum to TRT but also on the following result "PIT" concerning the stable topological pseudo-isotopy functor $\mathcal{P}(\ \ )$. Recall that this functor was defined and discussed earlier in lectures by Tom Goodwillie, Lowell Jones and Frank Quinn.

Pseudo Isotopy Theorem. *(Farrell and Jones [31]) Let M be a closed (connected) non-positively curved Riemannian manifold. Then, for all n,*

$$\begin{aligned} \pi_n(\mathcal{P}(M)) \otimes \mathbb{Q} &= 0 \quad \textit{and} \\ \pi_0(\mathcal{P}(M)) &= \mathbb{Z}_2^\infty. \end{aligned}$$

We will discuss the ideas behind the proof of PIT after first using it in proving Corollary 5.

For this we need to introduce the auxiliary spaces $G(M)$ and $\overline{\text{Top}}(M)$. Let $G(M)$ denote the H-space of all self-homotopy equivalences of M; note that $\text{Top}(M)$ is a subspace of $G(M)$. The semisimplicial group $\overline{\text{Top}}(M)$ of blocked homeomorphisms of M can be interpolated between $\text{Top}(M)$ and $G(M)$. A typical k-simplex of $\overline{\text{Top}}(M)$ consists of a homeomorphism

$$h : \Delta^k \times M \to \Delta^k \times M$$

such that $h(\Delta \times M) = \Delta \times M$ for each face Δ of Δ^k, where Δ^k is the standard k-simplex.

Let $G(M)/\text{Top}(M)$ and $\overline{\text{Top}}(M)/\text{Top}(M)$ denote the homotopy fiber of the map

$$B\ \text{Top}(M) \to BG(M) \quad \text{and} \quad B\ \text{Top}(M) \to B\ \overline{\text{Top}}(M),$$

respectively. Because of Frank Quinn's function space interpretation of the surgery exact sequence [60], [61], cf. [70, §17A]; the relative homotopy groups of the map

$$\overline{\text{Top}}(M) \to G(M)$$

can be identified with the groups

$$\mathcal{S}(M \times \mathbb{D}^n, \partial).$$

And these all vanish because of the TRT; consequently the following is true.

Fact 1. $G(M)/\text{Top}(M)$ and $\overline{\text{Top}}(M)/\text{Top}(M)$ have the same weak homotopy type.

Now the homotopy groups of $G(M)$ are easy to calculate. They are

Fact 2.

$$\pi_n(G(M)) = \begin{cases} \text{Out}(\pi_1 M) & \text{if } n = 0 \\ \text{Center}(\pi_1 M) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

Since the calculation for $n \geq 2$ is particularly easy to do, we sketch it. Let

$$f : S^n \times M \to M$$

represent an element in $\pi_n(G(M))$. To show this element is zero, we need to extend f to a map

$$\hat{f} : \mathbb{D}^{n+1} \times M \to M.$$

The construction of $\hat{f}$ is by an elementary obstruction theory argument. Fix a triangulation of M and assume $\hat{f}$ has already been defined over $\mathbb{D}^{n+1} \times \sigma$ for all simplices σ with $\dim(\sigma) < k$. Let σ be a k-simplex and identify $\mathbb{D}^{n+1} \times \sigma$ with $\mathbb{D}^{n+k+1}$. Then $\hat{f}|_{\partial \mathbb{D}^{n+k+1}}$ has already been defined and represents an element of $\pi_{n+k}(M)$ which vanishes since M is aspherical. Therefore $\hat{f}$ extends over $\mathbb{D}^{n+1} \times \sigma$. It is shown in this way that $\pi_n(G(M)) = 0$ when $n \geq 2$.

It therefore remains to analyze $\overline{\text{Top}}(M)/\text{Top}(M)$. Which can be done in terms of $\mathcal{P}(M)$ by using the following result of Hatcher [41].

Theorem. *(Hatcher) When $m > 10$ ($m = \dim M$) there is a spectral sequence converging to*

$$\pi_{p+q+1}(\overline{\text{Top}}(M)/\text{Top}(M))$$

with

$$E^2_{pq} = H_p(\mathbb{Z}_2; \pi_q(\mathcal{P}(M)))$$

in the stable range $q \leq \frac{(m+p-7)}{3}$.

Remark. This result depends on Igusa's Stability Theorem [47] for pseudo-isotopy spaces which Tom Goodwillie discussed in an earlier lecture.

Combining Hatcher's Theorem and PIT together with Facts 1 and 2 yields that

Fact 3.

$$\pi_n(\text{Top}(M)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } 1 \leq n \leq \frac{m-7}{3} \\ \text{Center}(\pi_1 M) \otimes \mathbb{Q} & \text{if } m = 1 \end{cases}$$

and the following exact sequence:

$$\text{Center}(\pi_1 M) \to H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty) \to \pi_0(\text{Top}(M)) \to \text{Out}(\pi_1 M).$$

Since the kernel of $\pi_0(\text{Top}(M)) \to \text{Out}(\pi_1 M)$ is $\pi_0(\text{Top}_0(M))$, this exact sequence can be rewritten as

Fact 4.

$$\text{Center}(\pi_1 M) \to H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty) \to \pi_0(\text{Top}_0(M)) \to 0.$$

Define a homomorphism $d : \mathbb{Z}_2^\infty \to \mathbb{Z}_2^\infty$ by

$$d(x) = x + \bar{x}$$

where $x \mapsto \bar{x}$ denotes the action of the generator of $\mathbb{Z}_2$ on $\mathbb{Z}_2^\infty$. Then the formula

$$H_0(\mathbb{Z}_2, \mathbb{Z}_2^\infty) = \mathbb{Z}_2^\infty/\text{image}(d)$$

is the definition of $H_0(\mathbb{Z}_2, \mathbb{Z}_2^\infty)$. We claim that $\mathbb{Z}_2^\infty/\text{image}(d)$ cannot be a finite group. If it were, then $\mathbb{Z}_2^\infty/\ker(d)$ would also be finite since

$$\ker(d) \supseteq \text{image}(d).$$

(Note that $d^2 = 0$ since $\mathbb{Z}_2^\infty$ has exponent 2.) But image(d) is isomorphic to $\mathbb{Z}_2^\infty/\ker(d)$. And the finiteness of both image(d) and $\mathbb{Z}_2^\infty/\text{image}(d)$ would imply that $\mathbb{Z}_2^\infty$ is also finite, which is a contradiction. Since $H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty)$ is thus a countable infinite group of exponent 2, it must be isomorphic to $\mathbb{Z}_2^\infty$. We therefore rewrite the sequence in Fact 4 as

Fact 5.

$$\text{Center}(\pi_1 M) \to \mathbb{Z}_2^\infty \to \pi_0(\text{Top}_0(M)) \to 0.$$

Now Lawson and Yau [51] showed that Center($\pi_1 M^m$) is finitely generated. (In fact it is isomorphic to $\mathbb{Z}^n$ where $n \leq m$.) Hence Fact 5 implies that $\pi_0(\text{Top}_0(M))$ is a countably infinite group of exponent 2, and therefore it is isomorphic to $\mathbb{Z}_2^\infty$. This result together with Fact 3 proves Corollary 5.

We now return to a discussion of PIT. Its proof follows the pattern established in proving the Vanishing Theorem (cf. Lecture 3). The main difference is that the corresponding foliated control theorem is obstructed since $\mathcal{P}(S^1)$ is *not* contractible. So we get a calculation instead of a vanishing theorem. Key ingredients for this calculation are ideas developed by Frank Quinn which were discussed in his and Lowell Jones' lectures.

We formulate a more precise result than PIT; namely a weak version of the Isomorphism Conjectures which Wolfgang Lueck talked about in one of his lectures. For the rest of this lecture M denotes a closed (connected) non-positively curved Riemannian manifold, $\tilde{M}$ its universal cover, and $\Gamma = \pi_1(M)$ its group of deck transformations. Fix a universal space $\mathcal{E}$ for Γ relative to the class $\mathcal{C}$ of all virtually cyclic subgroups of Γ.

Theorem. *(Farrell and Jones [31], also [33]) There exists a spectral sequence converging to $\pi_{p+q}(\underline{\mathcal{P}}(M))$ with $E^2_{pq} = H_p(\mathcal{E}/\Gamma; \pi_q(\underline{\mathcal{P}}(\tilde{M}/\Gamma_\sigma)))$.*

Remark. In this theorem Γ_σ denotes the subgroup of Γ fixing a cell σ of $\mathcal{E}$. And
$H_p(\mathcal{E}/\Gamma; \pi_q(\underline{\mathcal{P}}(\tilde{M}/\Gamma_\sigma)))$ is the p-th homology group of a chain complex whose p-th chain group is the direct sum of the groups $\pi_q(\underline{\mathcal{P}}(\tilde{M}/\Gamma_\sigma))$ where σ varies over a set S_p of p-cells of $\mathcal{E}$. The set S_p contains exactly one p-cell from each Γ-orbit of p-cells. Here $\underline{\mathcal{P}}(X)$ denotes the stable topological pseudo-isotopy Ω-spectrum of a space X; cf. [33, §1.1] for more details.

To deduce PIT from this result, we must analyze the spectral sequence. Note first that Γ_σ is either infinite cyclic or trivial since Γ is torsion-free. Therefore $\tilde{M}/\Gamma_\sigma$ is homotopically equivalent to either the circle S^1 or a point $*$ since $\tilde{M}/\Gamma_\sigma$ is aspherical. And there is the following important calculation:

Calculation 1. (a) $\pi_n(\mathcal{P}(*)) = 0$ for all n,

(b) $\pi_n(\mathcal{P}(S^1)) \otimes \mathbb{Q} = 0$ for all n,

(c) $\pi_0(\mathcal{P}(S^1)) = \mathbb{Z}_2^\infty$.

Calculation (a) is a consequence of Alexander's Trick discussed in Lecture 1. Calculation (b) is due to Waldhausen [68], and (c) is due to Waldhausen and Igusa [46]. Calculations (b) and (c) are deep results related to Tom Goodwillie's Lectures 1 and 2. Because of (a) and (b), $E^2_{pq} \otimes \mathbb{Q} = 0$. Hence the Theorem yields that $\pi_n(\mathcal{P}(M)) \otimes \mathbb{Q} = 0$; which is the first assertion of PIT.

Our Theorem also yields that

$$\pi_0(\mathcal{P}(M)) = H_0(\mathcal{E}/\Gamma; \pi_0(\mathcal{P}(\tilde{M}/\Gamma_\sigma))).$$

Since we can pick $\mathcal{E}$ to be a countable CW-complex (because Γ is countable) this equation together with Calculations (a) and (c) imply that $\pi_0(\mathcal{P}(M))$ is a quotient group of $\mathbb{Z}_2^\infty$; i.e. is a countable abelian group of exponent 2. To complete the proof of PIT, it remains to show that $\pi_0(\mathcal{P}(M))$ is an infinite group. We will only show this when M is negatively curved, since the general case depends on constructing a universal space $\mathcal{E}$ for Γ with better properties than the abstract construction. This geometric construction of $\mathcal{E}$ uses strongly the assumption that M is closed and non-positively curved. But, when M is negatively curved, the fact that $\pi_0(\mathcal{P}(M))$ is infinite is an immediate consequence of the following Assertion.

Assertion. *Assume M is negatively curved and let $\gamma : S^1 \to M$ represent a non-trivial element $[\gamma] \in \pi_1(M)$. Then*

$$\mathcal{P}(\gamma)_\# : \pi_0(\mathcal{P}(S^1)) \to \pi_0(\mathcal{P}(M))$$

is monic where $\mathcal{P}(\gamma) : \mathcal{P}(S^1) \to \mathcal{P}(M)$ is the functorially induced map.

We indicate the proof of this Assertion under the simplifying assumption that M is orientable. To do this we construct a transfer map

$$\tau : \mathcal{P}(M) \to \mathcal{P}(S^1)$$

such that $\tau \circ \mathcal{P}(\gamma)$ is homotopic to $\mathrm{id}_{\mathcal{P}(S^1)}$. The Assertion is clearly a consequence of this. Our construction uses ideas from Lecture 2. We first define a map

$$P(M) \to P(\bar{M}).$$

(Recall that $\bar{M} = \tilde{M} \cup M(\infty)$ is homeomorphic to $\mathbb{D}^m$.) This is done by sending the pseudo-isotopy f to the pseudo-isotopy $\bar{f}$ where

$$\bar{f}(x) = \begin{cases} x & \text{if } x \in M(\infty) \times [0,1] \\ \tilde{f}(x) & \text{if } x \in \tilde{M} \times [0,1] \end{cases}$$

and $\tilde{f}$ is the unique lift of f such that $\tilde{f}|_{\tilde{M}\times 0} = \mathrm{id}_{\tilde{M}\times 0}$. This pseudo-isotopy $\bar{f}$ is "well-defined" because Cartan's Theorem shows that property 2 of Condition $(*)$ holds (cf. Lecture 2). To be precise, $\bar{f}$ is only well defined after we collapse $x \times [0,1]$, $x \in M(\infty)$, to the single point x. But this quotient space can be identified with $\bar{M} \times [0,1]$. Note that $\bar{f}$ is Γ-equivariant. Let S be the infinite cyclic subgroup of Γ generated by $[\gamma]$. There are exactly two points S^+ and S^- on $M(\infty)$ fixed by S since $[\gamma]$ can be represented by a closed geodesic (because M is compact). Furthermore S acts freely and properly discontinuously on $\bar{M} - \{S^+, S^-\} = M_S$ and hence $\bar{f}$ induces a pseudo-isotopy

$$\hat{f} \in P(M_S/S).$$

But M_S/S is homeomorphic to $S^1 \times \mathbb{D}^{m-1}$ since M is orientable. The function $f \mapsto \hat{f}$ mapping

$$P(M) \to P(S^1 \times \mathbb{D}^{m-1})$$

stabilizes to give the desired transfer τ. (See [27, §2] for more details.) Q.E.D.

We end our lectures by giving an analogue of Corollary 5 true for $\mathrm{Diff}(M)$.

Corollary 6. *Suppose that M^m is orientable, $m > 10$ and $1 < n \leq \frac{(m-7)}{3}$. Then*

$$\pi_n(\mathrm{Diff}(M)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } m \text{ is even} \\ \bigoplus_{j=1}^{\infty} H_{(n+1)-4j}(M, \mathbb{Q}) & \text{if } m \text{ is odd.} \end{cases}$$

Furthermore, $\pi_1(\mathrm{Diff}(M)) \otimes \mathbb{Q} = \mathrm{Center}(\pi_1 M) \otimes \mathbb{Q}$.

Corollary 6 is an immediate consequence of the following result combined with TRT, PIT and the Vanishing Theorem.

Theorem. *(Farrell and Hsiang [23]) Let N^m be a closed aspherical manifold such that*

$$\begin{aligned} \mathcal{S}(N^m \times \mathbb{D}^k, \partial) &= 0 \qquad \text{for all } k \geq 0, \\ \pi_k(\mathcal{P}(N)) \otimes \mathbb{Q} &= 0 \qquad \text{for all } k \geq 0, \\ Wh(\pi_1(N) \times \mathbb{Z}^k) &= 0 \qquad \text{for all } k \geq 0. \end{aligned}$$

Then for $1 \leq n \leq (\frac{m-7}{3})$

$$\pi_n(\mathrm{Diff}(N)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } n > 1,\ n \text{ even} \\ \bigoplus_{j=1}^{\infty} H_{(n+1)-4j}(N, \mathbb{Q}) & \text{if } n > 1,\ n \text{ odd} \\ \mathrm{Center}(\pi_1 N) \otimes \mathbb{Q} & \text{if } n = 1. \end{cases}$$

6 Conclusion

Recall from lecture 1 that a basic problem in topology since the time of Poincaré has been to classify manifolds. Surgery theory gives an approach to doing this for all manifolds homotopy equivalent to a given manifold M. And this theory is effective provided the Wall L-groups $L_n(\pi_1 M)$ and the Whitehead group $Wh(\pi_1 M)$ can be calculated. Furthermore the calculation of $\pi_n(\mathrm{Top}(M))$ and $\pi_n(\mathrm{Diff}(M))$, through a stable range of dimensions n, can also be reduced to classical algebraic topology problems if additionally $\pi_n(\mathcal{P}(M))$ can be calculated.

In the above lectures, we've examined these problems for the case where M is aspherical. Let us now discuss the modifications needed to handle the general case; i.e., we no longer assume that M is aspherical. For this purpose, L.E. Jones and myself have formulated in [33] the "Isomorphism Conjectures" which can be thought of as a generalization of the Borel Conjecture (when dim $M \neq 3, 4$). There are separate conjectures for K-theory, L-theory and the stable topology pseudo-isotopy spectrum $\underline{\mathcal{P}}(X)$. For concreteness we focus attention here on the conjecture for $\underline{\mathcal{P}}(X)$.

F. Quinn in [62, appendix] constructed a covariant functor which associates to a continuous surjection $p : X \to B$ (of topological spaces) an Ω-spectrum $\underline{\mathcal{P}}(X, p)$–called the stable topological pseudo-isotopy spectrum of X with control relative to p. When p is the constant map to a point, $\underline{\mathcal{P}}(X, p)$ is the ordinary stable topological pseudo-isotopy spectrum $\underline{\mathcal{P}}(X)$. And there is a functorially induced forget control map from $\underline{\mathcal{P}}(X, p)$ to $\mathcal{P}(X)$, defined for arbitrary p, called the *Quinn assembly map.* The isomorphism conjecture for $\underline{\mathcal{P}}$ (and more generally the fibered isomorphism conjecture for $\underline{\mathcal{P}}$) is formulated in terms of the Quinn assembly map.

Let Γ be an arbitrary (discrete) group and $\mathcal{E}$ be a universal Γ-space for the class consisting of all virtually cyclic subgroups of Γ. (See [33, appendix] and W. Lueck's lectures for a detailed description of $\mathcal{E}$.) And let $\tilde{X} \to X$ be any regular covering space of a CW complex X with Γ for its group of deck transformations. Note that $\tilde{X} \to X$ is a principal Γ-bundle and form the associated bundle with fiber $\mathcal{E}$

$$\tilde{X} \times_\Gamma \mathcal{E} \underset{q}{\to} X.$$

Note that q is a homotopy equivalence, since $\mathcal{E}$ is contractible, and hence functorially induces an equivalence of the spectrum $\underline{\mathcal{P}}(\tilde{X} \times_\Gamma \mathcal{E})$ with the spectrum $\underline{\mathcal{P}}(X)$. Recall that $\tilde{X} \times_\Gamma \mathcal{E}$ is the quotient space of $\tilde{X} \times \mathcal{E}$ under the natural diagonal action of Γ. Let $\rho : \tilde{X} \times_\Gamma \mathcal{E} \to \mathcal{E}/\Gamma$ be the continuous map induced by projection of $\tilde{X} \times \mathcal{E}$ onto its second factor. The *fibered isomorphism conjecture* (FIC) *for* Γ states that the Quinn assembly map

$$\underline{\mathcal{P}}(\tilde{X} \times_\Gamma \mathcal{E}, \rho) \to \underline{\mathcal{P}}(\tilde{X} \times_\Gamma \mathcal{E}) = \mathcal{P}(X)$$

is an equivalence of Ω-spectra for every $\tilde{X} \to X$. And the *isomorphism conjecture* (IC) *for* Γ is the same statement made under the extra assumption that $\tilde{X} \to X$ is an arbitrary universal covering space, with $\pi_1(X) = \Gamma$. Two reasons why the FIC (and its special case the IC) is interesting are:

1. Quinn constructed in [62, appendix] a spectral sequence E^n_{pq} converging to $\pi_{p+q}(\underline{\mathcal{P}}(X))$ with

$$E^2_{pq} = H_p(\mathcal{E}/\Gamma; \pi_q(\underline{\mathcal{P}}(\tilde{X}/\Gamma_x)))$$

   where Γ_x denotes the virtually cyclic subgroup of Γ fixing $x \in \mathcal{E}$.

2. Anderson and Hsiang [5] showed that

$$\pi_q(\underline{\mathcal{P}}(X)) = \begin{cases} Wh(\pi_1(X)) & \text{if } q = -1 \\ \tilde{K}_0(\mathbb{Z}\pi_1(X)) & \text{if } q = -2 \\ K_{q+2}(\mathbb{Z}\pi_1(X)) & \text{if } q \le -3. \end{cases}$$

Evidence for the FIC is the following result proved by L.E. Jones and myself (1993) in [33, Th. 2.1 and Th. A.8].

Theorem. *The* FIC *is true for every discrete cocompact subgroup* Γ *of any (virtually connected) Lie group* G, *and (more generally) for every subgroup of such a group* Γ.

Final Remark. In order to study $\underline{\mathcal{P}}(M)$, let $\Gamma = \pi_1(M)$ and consider the universal covering space $\tilde{M} \to M$. If the IC is true for Γ, then the Ω-spectrum $\underline{\mathcal{P}}(M)$ is equivalent to $\underline{\mathcal{P}}(\tilde{M} \times_\Gamma \mathcal{E}, \rho)$ which can be analyzed using Quinn's spectral sequence. This analysis requires being able to calculate $\pi_q(\underline{\mathcal{P}}(\tilde{M}/\Gamma_x))$. For this note that $\tilde{M}/\Gamma_x$ is a manifold whose fundamental group is virtually cyclic. Now the Anderson-Hsiang result [5], mentioned above, is useful for $q < 0$ and [8] is useful for $q \ge 0$, at least when $\Gamma_x = 1$.

References

[1] H. Abels, G.A. Margulis and G.A. Soifer, Properly discontinuous groups of affine transformations with orthogonal linear part, *C.R. Acad. Sci. Paris Sér I Math.* **324** (1997), 253-258.

[2] U. Abresch, Über das Glätten Riemannischer Metriken, Habilitationsschrift, Rheinische Friedrich-Wilhelms-Universität Bonn, 1988.

[3] J.F. Adams, On the groups $J(X)$, *Topology* **5** (1966), 21-71.

[4] D.R. Anderson, The Whitehead torsion of the total space of a fiber bundle, *Topology* **11** (1972), 179-194.

[5] D.R. Anderson and W.C. Hsiang, The functors K_{-i} and pseudoisotopies of polyhedra, *Ann. of Math.* **105** (1977), 201-223.

[6] H. Bass, A. Heller and R. Swan, The Whitehead group of a polynomial extension, *Inst. Hautes Études Sci. Publ. Math.* **22** (1964), 64-79.

[7] R.L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.* **145** (1969), 1-49.

[8] M. Boekstedt, G. Carlsson, R. Cohen, T. Goodwillie, W.C. Hsiang and I. Madsen, On the algebraic K-theory of simply connected spaces, *Duke Math. J.* **84** (1996), 541-563.

[9] W. Browder, Structures on $M \times \mathbb{R}$, *Proc. Camb. Phil. Soc.* **61** (1965), 337-345.

[10] W. Browder and J. Levine, Fibering manifolds over a circle, *Comm. Math. Helv.* **40** (1966), 153-160.

[11] M. Brown, A proof of the generalized Shoenflies theorem, *Bull. Amer. Math. Soc.* **66** (1960), 74-76.

[12] J. Bryant, S. Ferry, W. Mio, S. Weinberger, *Bull. AMS* **28** (1993), 324-328.

[13] G. Carlsson and E.K. Pedersen, Controlled algebra and the Novikov conjecture for K- and L-theory, *Topology* **34** (1995), 731-758.

[14] M.W. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space, *Annals of Math.* **117** (1983), 293-325.

[15] M.W. Davis and T. Januszkiewicz, Hyperbolization of polyhedra, *J. Diff. Geom.* **34** (1991), 347-388.

[16] M.W. Davis, B. Okun and F. Zheng, Piecewise Euclidean structures and Eberlein's rigidity theorem in the singular case, *Geometric Topology* **3** (1999), 303-330.

[17] P. Eberlein and B. O'Neill, Visibility manifolds, *Pacific J. Math.* **46** (1973), 45-109.

[18] B. Eckmann and P. Linnell, Poincaré duality groups of dimension two. II, *Comment. Math. Helv.* **58** (1983), 111-114.

[19] B. Eckmann and H. Mueller, Poincaré duality groups of dimension two. *Comment. Math. Helv.* **55** (1980), 510-520.

[20] F.T. Farrell, The obstruction to fibering a manifold over a circle, *Indiana Univ. Math. Jour.* **21** (1971), 315-346.

[21] F.T. Farrell, Surgical Methods in Rigidity, Tata Institute of Fundamental Research Lecures on Mathematics and Physics, Vol. 86, Springer, Berlin, 1996.

[22] F.T. Farrell and W.C. Hsiang, The topological-Euclidean space form problem, *Invent. Math.* **45** (1978), 181-192.

[23] F.T. Farrell and W.C. Hsiang, On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds, *Proc. Sympos. Pure Math.*, Vol. 32, Amer. Math. Soc., Providence, RI, pp. 325-337.

[24] F.T. Farrell and W.C. Hsiang, On Novikov's conjecture for non-positively curved manifolds, I, *Annals of Math.* 113 (1981), 199-209.

[25] F.T. Farrell and W.C. Hsiang, Topological characterization of flat and almost flat Riemannian manifolds M^n $(n \neq 3, 4)$, *American Jour. of Math.* **105** (1983), 641-672.

[26] F.T. Farrell and L.E. Jones, K-theory and dynamics, I, *Annals of Math.* **124** (1986), 531-569.

[27] F.T. Farrell and L.E. Jones, K-theory and dynamics II, *Annals of Math.* **126** (1987), 451-493.

[28] F.T. Farrell and L.E. Jones, The surgery L-groups of poly-(finite or cyclic) groups, *Invent. Math.* **91** (1988), 559-586.

[29] F.T. Farrell and L.E. Jones, Topological analogue of Mostow's rigidity theorem, *Jour. of AMS* **2** (1989), 257-370.

[30] F.T. Farrell and L.E. Jones, Classical Aspherical Manifolds, CBMS Regional Conference Series, Vol. 75, AMS, Providence, RI, 1990.

[31] F.T. Farrell and L.E. Jones, Stable pseudoisotopy spaces of non-positively curved manifolds, *Jour. of Diff. Geom.* **34** (1991), 769-834.

[32] F.T. Farrell and L.E. Jones, Topological rigidity for compact non-positively curved manifolds, *Proc. Sympos. Pure Math.* **54** Part 3 (1993), 229-274.

[33] F.T. Farrell and L.E. Jones, Isomorphism conjectures in algebraic and geometric topology, *Jour. of AMS* **6** (1993), 249-297.

[34] F.T. Farrell and L.E. Jones, Compact infrasolvmanifolds are smoothly rigid, in Geometry from the Pacific Rim, edited by Berrick, Loo and Wang, Walter de Gruyter & Co., Berlin, 1997, 85-97.

[35] F.T. Farrell and L.E. Jones, Rigidity for aspherical manifolds with $\pi_1 \subset GL_m(\mathbb{R})$, *Asian Jour. of Math.* **2** (1998), 215-262.

[36] S. Ferry and S. Weinberger, Curvature, tangentiality and controlled topology, *Invent. Math.* **105** (1991), 401-414.

[37] M. Freedman and F. Quinn, Topology of four-manifolds, Princeton Univ. Press, Princeton, NJ, 1990.

[38] D. Fried and W.M. Goldman, Three-dimensional affine crystallographic groups, *Adv. in Math.* **47** (1983), 1-49.

[39] D. Gabai, On the geometric and topological rigidity of hyperbolic 3-manifolds, *Jour. Amer. Math. Soc.* **10** (1997), 37-74.

[40] D. Gromoll and J.A. Wolf, Some relations between the metric structure and the algebraic structure of the fundamental group in the manifolds of nonpositive curvature, *Bull. AMS* **77** (1971), 545-552.

[41] A.E. Hatcher, Concordance spaces, higher simple homotopy theory, and applications, *Proc. Sympos. Pure Math.* **32** (1978), 3-21.

[42] N.J. Hicks, *Notes on differential geometry*, van Nostrand, Princeton, 1965.

[43] W.C. Hsiang and C.T.C. Wall, On homotopy tori II, *Bull. London Math. Soc.* **1** (1969), 341-342.

[44] B. Hu, Whitehead groups of finite polyhedra with non-positive curvature, *Jour. of Diff. Geom.* **38** (1993), 501-517.

[45] B. Hu, Retractions of closed manifolds with nonpositive curvature, in Geometric Group Theory, de Gruyter, New York, 1995, 135-147.

[46] K. Igusa, What happens to Hatcher and Wagoner's formula for $\pi_0\mathcal{C}(M)$ when the first Postnikov invariant of M is nontrivial?, Lecture Notes in Math., vol. 1046, Springer, New York, 1984, pp. 104-177.

[47] K. Igusa, The stability theorem for pseudoisotopies, *K-theory* **2** (1988), 1-355.

[48] M. Kervaire and J. Milnor, Groups of homotopy spheres I, *Annals of Math.* **77** (1963), 504-537.

[49] R.C. Kirby and L.C. Siebenmann, Foundational essays on topological manifolds, smoothings, and triangulations, Ann. of Math. Studies, Vol. 88, Princeton Univ. Press, Princeton, NJ (1977).

[50] K.W. Kwun and R.H. Szczarba, Product and sum theorems for Whitehead torsion, *Ann. of Math.* **82** (1965), 183-190.

[51] B. Lawson and S.-T. Yau, Compact manifolds of nonpositive curvature, *Jour. Diff. Geom.* **7** (1972), 211-228.

[52] G. Lusztig, Novikov's higher signature and families of elliptic operators, *Jour. Diff. Geom.* **7** (1972), 229-256.

[53] J. Milnor, On manifolds homeomorphic to the 7-sphere, *Annals of Math.* **64** (1956), 399-405.

[54] J. Milnor, Morse Theory, Ann. of Math. Studies, Vol. 51, Princeton Univ. Press, Princeton, NJ (1963).

[55] E. Moise, Affine structures on 3-manifolds, *Ann. of Math.* **56** (1952), 96-114.

[56] A.S. Mishchenko, Infinite dimensional representations of discrete groups and higher signatures, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 81-106.

[57] G.D. Mostow, Strong Rigidity of Locally Symmetric Spaces, Ann. of Math. Studies, Vol. 78, Princeton Univ. Press, Princeton, 1973.

[58] S.P. Novikov, Homotopically equivalent smooth manifolds, *Izv. Akad. Nauk SSR Ser. Mat.* **28** (1964), 365-474.

[59] S.P. Novikov, On manifolds with free abelian fundamental group and their applications, *Izv. Akad. Nauk SSSR* **30** (1966), 207-246.

[60] F. Quinn, A geometric formulation of surgery, Ph.D. thesis, Princeton University, 1969.

[61] F. Quinn, A geometric formulation of surgery, Topology of manifolds, Markham, Chicago, 1970, pp. 500-511.

[62] F. Quinn, Ends of maps. II, *Invent. Math.* **68** (1982), 353-424.

[63] A. Ranicki, Algebraic L-theory and topological manifolds, Cambridge Tracts in Mathematics, Vol. 102, Cambridge University Press, 1992.

[64] F. Raymond and L.L. Scott, Failure of Neilson's theorem in higher dimensions, *Arch. Math.* (Basel) **29** (1977), 643-654.

[65] J.-P. Serre, Cohomologie des groupes discrets, *Annals of Math. Studies* **70** (1971), 77-169.

[66] W.-X. Shi, Deforming the metric on complete Riemannian manifolds, *Jour. Diff. Geom.* (1989), 223-301.

[67] L.C. Siebenmann, The obstruction to finding a boundary for an open manifold of dimension ≥ 5, Ph.D. thesis, Princeton University, 1965.

[68] F. Waldhausen, Algebraic K-theory of topological spaces I, Proc. Sympos. Pure Math., Vol. 32, Amer. Math. Soc., Providence, RI, 1978, pp. 35-60.

[69] C.T.C. Wall, The topological space-form problems, Topology of manifolds, Markham, Chicago, 1970, pp. 319-331.

[70] C.T.C. Wall, Surgery on compact manifolds, second edition edited by A.A. Ranicki, Mathematical Surveys and Monographs, 69, Amer. Math. Soc., Providence, RI, 1999.

[71] S.-T. Yau, On the fundamental group of compact manifolds of nonpositive curvature, *Ann. of Math.* **93** (1971), 579-585.