697 Topics in Topology

F. T. Farrell 1978, Fall

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X is <u>aspherical</u> if X is connected and $\pi_i X = 0$ for $i > 1$.

<u>Thm</u> If K and L are two aspherical CW complexes with isomor fundamental groups, then K and L are homotopy equivalent.

<u>Conjecture I</u> Two asph. closed mfds with isomorphic π_1's are homeomorphic.

[Includes Poincaré conjecture of dim 3 and 4, see [3]]

More hopeful (smooth case)

? Are two asph. closed C^∞-mfds with isomorphic π_1's diffeomorphic?

e.g. $T^n = S^1 \times \cdots \times S^1$, Σ^n = h'ty sphere. Then T^n and $T^n \# \Sigma^n$ are diffeomorphic iff Σ^n is diffeomorphic to S^n ($n > 4$).

<u>Pf</u> of Thm. We may suppose K and L have finitely many cell with one 0-cell. [When K has finite number of 0-cells, constract them to a point using 1-cells joining the 0-cells]

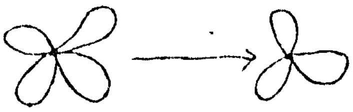
Define $f: K \longrightarrow L$ as follows from $\varphi: \pi_1 K \xrightarrow{\cong} \pi_1 L$.

0-cell obvious

1-cell Use

(not iso)

$$\begin{tikzcd} F_n \arrow[r, dashed, "\hat{\varphi}"] \arrow[d] & F_m \arrow[d, two heads] \\ \pi_1 K \arrow[r, "\varphi", "\cong"'] & \pi_1 L \end{tikzcd}$$



($F_n \to F_m$)

g_1 (a leaf of K) $\xmapsto{\hat{\varphi}}$ $\hat{\varphi}(g_1)$ = word in F_m.

2-cells.

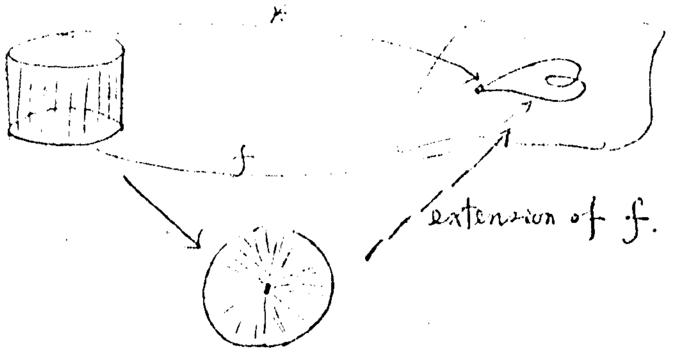
$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
F_r \arrow[r, dashed] \arrow[d] & F_t \arrow[d] \\
F_n \arrow[r, "\hat{\varphi}"] \arrow[d] & F_m \arrow[d] \\
\pi_1 K \arrow[r, "\varphi", "\cong"'] \arrow[d] & \pi_1 L \arrow[d] \\
0 & 0
\end{tikzcd}$$

F_r, F_t : relator group of $\pi_1 K, \pi_1 L$.

∂(Any 2-cell of K) $\in F_r$ since K is asph., (so it is $0 \in \pi_1 K$.)

$$\underset{\overset{\|}{f : S^1 \longrightarrow L}}{\underline{\hat{\varphi}\ \partial(\text{2-cell})}} = 0 \in \pi_1 L$$

null htpc $\Rightarrow$ f extends to $D \longrightarrow L$



We get $f_\# = \varphi : \pi_1 K \longrightarrow \pi_1 L$ Now can define $g : L \longrightarrow K$ using $\varphi^{-1} : \pi_1 L \longrightarrow \pi_1 K$ so that $g_\# = \varphi^{-1}$! ▨

Defn. (J.H.C. Whitehead) Let K, L be two finite CW complexes. K and L have the same <u>simple homotopy type</u> if $\exists$ finite sequen of finite CW complexes $K_1, K_2, \cdots, K_n$ with

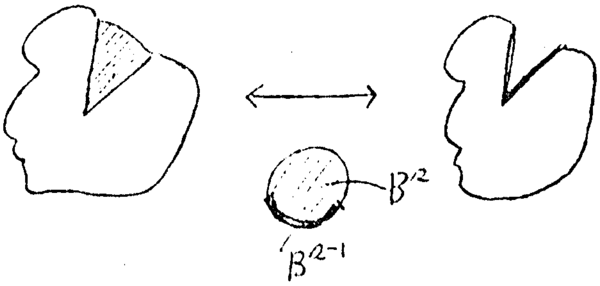
1° $K_1 = K$, $K_n = L$

2° K_i and K_{i+1} are related via either elementary expansion or elementary collapse.

Elementary expansion $A \nearrow B$ (A is a subcomplex of B)
if $B = A \cup_{B^{n-1}} B^n$

[B^{n-1} is (n-1) dim'l ball in $S^{n-1} = \partial B^n$.]

elementary collapse $B \searrow A$ if $A \nearrow B$.



(Ref: M. Cohen, J. Milnor)

So

$K \xrightarrow{f} L$ is SHE $(K \searrow\nearrow L)$ if

$$\begin{tikzcd} K \arrow[rrr, "f"] \arrow[d, equal] & & & L \arrow[d, equal] \\ K_1 \arrow[r] & K_2 \arrow[r] & \cdots \arrow[r] & K_n \end{tikzcd}$$

(Chapmann) a homeomorphism (between finite CW complexes) is a SHE.

V ? ∀ asph. closed mfd has finite SH type? (Kirby & Siebenmann)

Conjecture II Two finite aspherical CW complexes with isomorphic have the same simple homotopy type?

[Weaker than conjecture I by Chapman's result]

Algebraic Fomulation of SHE — Whitehead torsion

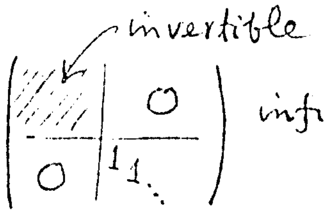
$R =$ ring with 1.

$GL_n(R) =$ gp of all invertible $n \times n$ matrices with entries in R

$$GL_n(R) \hookrightarrow GL_{n+1}(R)$$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

let $GL(R) = \bigcup_{n \geq 1} GL_n(R)$

matrices of the form 

infinite matrix.

$$\boxed{K_1 R} = GL(R)^{ab}$$

$$= GL(R) \big/ [GL(R), GL(R)] \quad \text{abelianize.}$$

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & I \end{pmatrix} \text{ in } K_1($$

$$\boxed{Wh(\overset{\text{group}}{G})} = K_1(\mathbb{Z}G) \big/ \pm G$$

$$\pm G \subset \text{Units of } \mathbb{Z}G = GL_1(\mathbb{Z}G) \subset GL(\mathbb{Z}G)$$

$$\left(\begin{array}{l} g \cdot g^{-1} = 1 \\ -g \cdot -g = 1 \end{array} \right.$$

<u>Conjecture III</u> If Γ is a torsion-free group, then $Wh\,\Gamma = 0$

$$\begin{tikzcd} \text{Conj I} \arrow[dr, "(\text{Chapmann})"'] & & \text{Conj III} \arrow[dl] \\ & \text{Conj II} & \end{tikzcd}$$

(π_1(asph. mfd) is torsionfree.
(~~$Wh(\pi_1 M)$~~ $\Rightarrow f : M \xrightarrow[HE]{} N$ If $\tau(f) = 0 \in Wh\pi$, then f is SHE.

<u>Example</u> $Wh\,\Gamma \neq 0$ where Γ has torsion.

T_5 (cyclic group of order 5) $\underset{\sim\sim\sim}{Wh(T_5) \neq 0}$

Since T_5 is abelian, $\mathbb{Z}T_5$ is a commutative ring so that we can talk about determinant.

Let γ be a generator of $T_5 = \{\gamma, \gamma^2, \cdots, \gamma^5 = 1\}$

$$GL_n(\mathbb{Z}T_5) \xrightarrow{det} \text{Units}(\mathbb{Z}T_5)$$

commutators $\longmapsto 1$ since det. is group homo.

$\therefore [GL_n, GL_n] \subset \ker(\det)$.

$$\begin{tikzcd}
GL(\mathbb{Z}T_5) \arrow[d] \arrow[dr] & \\
K_1(\mathbb{Z}T_5) \arrow[r, dashed] \arrow[d] & \text{Units}(\mathbb{Z}T_5) \arrow[d] \\
Wh\,T_5 \arrow[r, dashed] & \text{Units}(\mathbb{Z}T_5)/\pm T_5
\end{tikzcd}$$

Note that $1-\gamma+\gamma^2 \in \text{Units}(\mathbb{Z}T_5)$ with inverse $\gamma+\gamma^2-\gamma^4$

$1-\gamma+\gamma^2 \neq 0 \in \text{Units}(\mathbb{Z}T_5)/\pm T_5$

$Wh\,T_5 \longrightarrow \text{Units}(\mathbb{Z}T_5)/\pm T_5$ is onto

Thus, $Wh\,T_5 \neq 0$. ∎

* If G is an abelian group which contains an element of order $q \neq 1, 2, 3, 4, 6$, then $Wh\,G \neq 0$. [Cohen p44]

an $n \times n$ matrix A is <u>elementary</u> if

$$A = I + aE_{ij} \quad (i \neq j) \qquad [a \text{ may be } 0]$$

E_{ij} is the matrix which ~~non-zero~~ has only one non-zero entry at (i,j).

$E(R)$ = subgp of $GL(R)$ generated by elementary matrices.

Note : product of elementary matrices is not elementary.

<u>Lemma</u> $E(R) = [GL(R), GL(R)]$

$\therefore$
$$\begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & -a \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix}$$

Conversely,

$$A \sim B \overset{\text{def.}}{\longleftrightarrow} \exists\, E_1, E_2 \in E(R) \text{ s.t. } A = E_1 B E_2$$

$$I_R : \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \sim \begin{pmatrix} P_1 + XP_2 \\ P_2 \end{pmatrix} \text{ by } \text{~~right~~ left multipl. by } \begin{pmatrix} 1 & & \textcircled{1} \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$II_R : \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \sim \begin{pmatrix} P_1 + P_2 \\ P_2 \end{pmatrix} \sim \begin{pmatrix} P_1 + P_2 \\ -P_1 \end{pmatrix} \sim \begin{pmatrix} P_2 \\ -P_1 \end{pmatrix}.$$

For $\forall A, B \in GL(R)$,

$$AB = \begin{pmatrix} AB & \\ & I \end{pmatrix} \sim \begin{pmatrix} AB & A \\ & I \end{pmatrix} \sim \begin{pmatrix} 0 & A \\ -B & I \end{pmatrix} \sim \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix}$$

$$BA \sim \begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \underset{II_c}{\sim} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \underset{II_R}{\sim} \begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix} \qquad \therefore AB \sim BA.$$

If $E \in E(R)$, $X \in GL(R)$ then $(XE)X^{-1} \sim X^{-1}(XE) = E$

$$\therefore XEX^{-1} = E_1 E E_2 \in E(R).$$

$$(AB)(A^{-1}B^{-1}) = [E_1 BA E_2](BA)^{-1} = E_1((BA)E_2(BA)^{-1}) \in E(R). \quad ▨$$

$K_1(R) = GL(R) / E(R)$.

✓ $K_1(\text{field}) = 0$ field $-\{0\}$

$K_1(\mathbb{Z}1) = \{\pm 1\}$ $Wh(1) = K_1(\mathbb{Z}1) / \pm 1 = 0$

↑ any invertible matrix is diagonalizable via elem. matr

Thm π_1(finite asph. CW cplx) is torsion-free → Poincare duality group

pf. Assume $\exists g \in \pi_1 X$ with $\text{ord}(g) = p$ prime. Let $\tilde{X} \to X$ be the universal cover. Then $\pi_1 X$ acts on $\tilde{X}$ as covering transfor Let

$$T_p = \langle g \rangle \subset \pi_1 X.$$

T_p acts freely and cellularly on $\tilde{X}$. [cell-structure of $\tilde{X}$ is given by that of X]. Let

$$c_i = i\text{-th cellular chain group of } \tilde{X}.$$
$$= H_i(\tilde{X}^i, \tilde{X}^{i-1}) \qquad = \tilde{H}_i(\tilde{X}^i / \tilde{X}^{i-1}).$$

claim C_i is a free $\mathbb{Z}T_p$-module [C_i is a free $\mathbb{Z}[\pi_1 X]$ module and $T_p < \pi_1 X$]

Since $\tilde{X}$ is univ. cover of X = asph., $\pi_i \tilde{X} = 0$ all i. By Whiteho $\tilde{X}$ is contractible $\therefore 0 = H_i(\tilde{X}) = H_i(C_*)$. ~~Thus~~ for $i > 0$. Thus

$$C_n \to C_{n-1} \to \cdots \to C_i \xrightarrow{d} C_{i-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

is exact. so it is a free $\mathbb{Z}T_p$-resolution of $\mathbb{Z}$ = trivial $\mathbb{Z}T_p$-modul

$$\therefore H_i(T_p; \mathbb{Z}) = 0 \quad \text{for large } i.$$

But $H_i(T_p, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}_p, & i \text{ odd} \\ 0, & \text{otherwise} \end{cases}$

~~ef $H_i(T_p; \mathbb{Z}) = H_i(L(p,q); \mathbb{Z})$~~ Get a contradiction. ▨

Calculation of $H_i(T_p, \mathbb{Z})$

augmentation

$$\rho = \cdots \xrightarrow{\nu} \mathbb{Z}T_p \xrightarrow{1-g} \mathbb{Z}T_p \underset{\nu = g+g^2+\cdots+g^p}{\xrightarrow{(1-g)^{-1}}} \mathbb{Z}T_p \xrightarrow{1-g} \mathbb{Z}T_p \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

trivial $\mathbb{Z}T_p$-module

is a free $\mathbb{Z}T_p$ resolution of $\mathbb{Z}$. Apply $\underset{\mathbb{Z}T_p}{\otimes}(\mathbb{Z})$ to get

$$Q = \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to 0$$

$$\therefore\ H_*(T_p, \mathbb{Z}) = H_*(\rho \underset{\mathbb{Z}T_p}{\otimes} \mathbb{Z}) = H_*(Q).$$

★ How to associate a torsion elt to a homotopy equivalence

(milnor's defn) $C_* =$ finite f.g. acyclic free chain complex with given basis over $\Lambda = \mathbb{Z}\pi$.

$$\begin{tikzcd} 0 \arrow[r] & B_i \arrow[r] & C_i \arrow[r, "\partial"] & B_{i-1} \arrow[r] \arrow[l, dashed, bend left, "\delta"] & 0 \end{tikzcd}$$

$B_i = Z_i$

let c_i be the given base for C_i choose any bases b_i for

Then $b_i b_{i-1} = b_i \delta(b_{i-1})$ is a base for C_i let

$$\tau(C_*) = \sum (-1)^i [c_i | b_i b_{i-1}] \in Wh(\pi)$$

This does not depend on the choice of b_i $([a|b] = [a|b'] + [b'|b])$

(Cohen) $$C_{even} = \sum C_{2i} \xrightarrow{\partial + \delta} \sum C_{2i-1} = C_{odd}$$

$\tau(C_*) =$ matrix of $C_{even} \to C_{odd}$ wrt the preassigned bases.

For homotopy equivalences

let $L \xrightarrow{f} K$ be cellular HE. Then $L \longrightarrow M_f \overset{SHE}{\simeq} K$

finite CW cx

Let $C_*(f) = C_*(\tilde{M}_f, \tilde{L})$ be the acyclic f.g. free chai
complex over $\Lambda = \mathbb{Z}\pi_1 K$

$$\tau(f) \overset{\text{def}}{=} \tau(C_*(f))$$

<u>special case</u>

$K \underset{i}{\hookrightarrow} L$

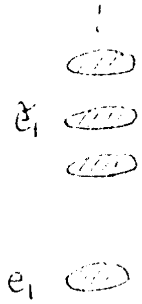
✓ Suppose K is a deformation retract of L. (
and L−K consists of cells in 2-dimensions $s, s+1$
(by trading cells)

$\tilde{K} \subset \tilde{L}$, $\tilde{K}$ is a deformation retract of $\tilde{L}$.

$$0 \longrightarrow C_{s+1}(\tilde{L}, \tilde{K}) \xrightarrow[\cong]{d} C_s(\tilde{L}, \tilde{K}) \longrightarrow 0$$

and these are free $\mathbb{Z}\pi_1 L$-modules, d is $\mathbb{Z}\pi_1 L$-linear

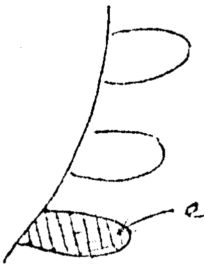
Let $e_1, \cdots, e_n$; $\sigma_1, \cdots, \sigma_n$ be s; $(s+1)$-cells in L−K



let $\tilde{e}_1, \cdots, \tilde{e}_n$; $\tilde{\sigma}_1, \cdots, \tilde{\sigma}_n$ be a particular choice
of cells in $\tilde{L} - \tilde{K}$ covering the above ones.

With these Basis of $C_s(\tilde{L}, \tilde{K})$ and $C_{s+1}(\tilde{L}, \tilde{K}$
the matrix of d determines an elt of Wh $\pi_1 L$

change of $\tilde{e}_i, \tilde{\sigma}_i$ corresponds to $\pm \pi_1(L)$ in $Wh(\pi_1 L) = K_1 / \pm \pi_1 L$.



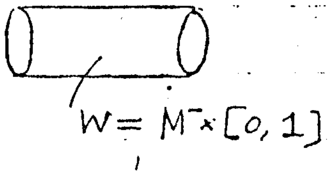
elementary expansion corresponds to $A \longmapsto \begin{pmatrix} A & * \\ 0 & 1 \end{pmatrix}$

* ∀ elt of $Wh(\pi_1 L)$ can be realized in this fashion "deformation
retract".

Cobordism

Let W^{n+1} be a cpt (smooth) mfd with 2 ∂-componts $M^- \& M$

(W, M^-, M^+) is an <u>h-cobordism</u> if both of M^- and M^+ are deformation retract of W. (topological or smooth).
↖ Kirby & Siebenman



* M^+ is a deformation retract of W and $\pi_1 M^- \longrightarrow \pi_1 W$ injective then (W, M^-, M^+) is an h-cobordism.

pf. Use Lefschetz Duality with compact support, & Whitehead thm ($\pi_* \to H_*$).

h-cobordism Thm

~~Let $\partial W^{n+1} = V \cup V'$,~~ Let (W^{n+1}, V, V') be an h-cobordism wi V and V' simply connected. If $n \geq 5$, then it is product and hence $V \cong V'$ diffeomorphic.

An h-cobordism (W, M^-, M^+) is called an <u>s-cobordism</u> if $\tau(M^- \subset W) = 0 \in Wh\, \pi_1 W$.

s-cobordism Thm

An h-cobordism (W, M^-, M^+) is product (diffeomorphic to $M^- \times [0,1]$) iff $\tau(M^- \subset W) = 0 \in Wh\, \pi_1 W$. (for dim $W > 5$)

* For known asph M^+, construct another M^- using non-zero elt of $Wh\, \pi_1 M^+$. Then get $M^+ \underset{she}{\not\cong} M^-$ non SHE asph. mfd

§§ Wh ($\overset{\text{f.g.}}{\text{free}}$ abelian group) = 0

Let R be a ring with 1. "Grothendieck construction" for the class $\mathcal{S}$ of all f.g. projective R-modules is called $K_0(R)$.

$F(\mathcal{S})$ = free abelian group generated by isomorphism classes in $\mathcal{S}$.

$H(\mathcal{S})$ = subgroup of $F(\mathcal{S})$ generated by $\{[M_2]-[M_1]-[M_3$
where $0 \to M_1 \to M_2 \to M_3 \to 0$ exact.

Define $K_0(R) = F/H$.

1) Since $0 \to P \to P\oplus Q \to Q \to 0$ is exact, $[P\oplus Q]-[P]-[Q]$
in $K_0(R)$. i.e, $[P]+[Q] = [P\oplus Q]$

2) Universal property.

$$\begin{tikzcd} \text{iso class } [\mathcal{S}] \arrow[r] \arrow[dr, "x"'] & K_0(R) = K_0(\mathcal{S}) \arrow[d, dashed, "\exists!\, \tilde{x}"] \\ x(M_2)=x(M_1)+x(M_3) & G \end{tikzcd}$$

any abelian gp

3) $K_0(\text{field}) \xrightarrow[\cong]{\dim} \mathbb{Z}$

Def $\tilde{K}_0(R) \overset{\text{def}}{=} K_0(R)/$subgp gen'td by the free modules

projective class group

elts of $\tilde{K}_0(R)$ are weakly stable isomorphism classes of f.g. projective R-modules. i.e,

$[P]=[Q]$ in $\tilde{K}_0(R)$ iff $P\oplus F_0 \cong Q\oplus F_1$ for some free F_0, F_1.

cf. $[P]=[Q]$ in K_0R iff $P\oplus F \cong Q+F$ for some free F.

I. (Bass–Heller–Swan) $Wh(\mathbb{Z}\oplus\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}) = 0$

II. (Serre) Let k be a field, $k[x_1, \cdots, x_n]$ be the polynomial ring in n variables. Then

$$\tilde{K}_0(k[x_1, \cdots, x_n]) = 0$$

i.e, for any projective f.g. $k[x_1 \cdots x_n]$-module, $\exists$ a free module F such that $P\oplus F$ is free.

[Due to Sullivan and Quillen, P is in fact free].

Another discription of $K_0(R)$

$K_0(R)$ = idempotent matrices $A^2 = A$

proj module $\overset{1-1}{\longleftrightarrow}$ idempt matrix

- Given proj P let $P\oplus Q = R^n$. Let A = matrix of $R^n \twoheadrightarrow P \hookrightarrow R^n$. Then A is idempt.
- Given idempt matrix A, let $R^n \xrightarrow{A} R^n$. Then $\mathrm{im}(A)$ is a proj module $\quad 1-A$

$$A\oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Nil $R = \{A \mid A^r = 0 \text{ for some } r > 0\}$

= iso class of nilpt endo of f.g free R-module

Free abelian gp generated by $\{(F, \varphi) \mid F = $ f.g. free $\varphi : F \circlearrowleft$ is nilpt endo$\} \,/\!\sim$

$(F_1, \varphi_1) \sim (F_2, \varphi_2)\oplus(F_3, \varphi_3)$ if

$$\begin{tikzcd} 0 \arrow[r] & F_2 \arrow[r] \arrow[d, "\varphi_2"'] & F_1 \arrow[r] \arrow[d, "\varphi_1"'] & F_3 \arrow[r] \arrow[d, "\varphi_3"'] \arrow[l, dashed, bend right] & 0 \\ 0 \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[dd] & F_3 \arrow[r] \arrow[d] \arrow[l, dashed, bend left] & 0 \\ & A & & B & \\ & \begin{pmatrix} A & \\ & n \end{pmatrix} & \begin{pmatrix} A & C \\ & \end{pmatrix} & = (0_B) & \end{tikzcd}$$

splitting do not commute w/ φ_1 & φ_3

$(F, \varphi) \longmapsto$ matrix of φ = nilpt.

$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \sim \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ inverse $\searrow$ p. 20

Thm 1 $K_1(R[x]) \cong K_1 R \oplus \mathrm{Nil}\, R$

$R[x, x^{-1}]$ = finite Laurent series

T = infinite cyclic group $\Rightarrow Z[T] \simeq Z[x, x^{-1}]$

✓ $\mathbb{Z}[G \oplus T] \simeq \mathbb{Z}[G][x, x^{-1}]$

$\Sigma n_i (g_i, \lambda_i) \mapsto \Sigma (n_i g_i) x^{\lambda_i}$

Thm 2 $K_1(R[x, x^{-1}]) \simeq K_1 R \oplus \mathrm{Nil}\, R \oplus \mathrm{Nil}\, R \oplus K_0 R$

(Atiyah-Bott pf of Bott Periodicity Thm)

$\bar{X} \longrightarrow \mathbb{C}(\bar{X})$

$K_0(\mathbb{C}(\bar{X})) \simeq K^0(X)$ — topological K^0

Thm 3 If R is a <u>regular</u> ring, then $\mathrm{Nil}\, R = 0$

① R is Noetherian ($\forall$ submodule is f.g)

② $\forall$ R-module M has a proj. resol. of finite length.

Ex. $\mathbb{Z}$[free abelian group] is a regular ring.

Use Hilbert basis and Syzigy Thm. $\mathbb{Z}[G \times T] = \mathbb{Z}[G][x, x^{-1}]$

✓ Γ is free abelian in n variables $\Rightarrow \mathbb{Z}\Gamma = \mathbb{Z}[x_1, \cdots x_n, x_1^{-1}, \cdots x_n^{-1}]$

Thm 4 If $\mathrm{Nil}\, R \neq 0$, then $\mathrm{Nil}\, R$ is not f.g.

✓ ex. of $R \ni \mathrm{Nil}\, R \neq 0$ $\exists\, G$ = f.g. abelian gp $\ni \mathrm{Nil}(\mathbb{Z}G) = 0$

proc. AMS 1977.

① ② are corollaries of Thm 1 and 2.

In particular,

G = infinite cyclic $\oplus$ cyclic of order 4

$\mathrm{Nil}(\mathbb{Z}(T \oplus T_4))$ is not f.g.

$$f : R \longrightarrow S$$

induces a group homo

$$f_* : K_1R \to K_1S$$

$$\underset{\text{invertible matrix}}{[A]} \longmapsto f[A] = [f(a_{ij})]$$

$$\underset{?}{AB} \mapsto \left(f((AB)_{ij})\right) \overset{\shortparallel}{} (f(a_{ij}))(f(b_{ij}))!$$

$$f_* : K_0R \longrightarrow K_0S$$

$$\underset{\text{idempotent}}{[A]} \longmapsto [f(a_{ij})]$$

For projective modules, this corresponds to the tensor product construction.

$$[P] \longmapsto f_*[P] = [S \underset{R}{\otimes} P]$$

$$f_* : \mathrm{Nil}\,R \longrightarrow \mathrm{Nil}\,S$$

$$\underset{\text{nilpt}}{[A]} \longmapsto [f(a_{ij})]$$

For K_0, suppose $P \oplus Q = R^n$ and let a matrix A corresponds to the map $R^n \twoheadrightarrow P \rightarrowtail R^n$. Then

$$\begin{tikzcd} S^n \arrow[r, phantom, "\simeq"] \arrow[rrrr, bend right, "f(A)"'] & S \underset{R}{\otimes} R^n \arrow[rr, "id \otimes A", "\shortparallel \; f(A)"'] & & S \underset{R}{\otimes} R^n \arrow[r, phantom, "\simeq"] & S^n \end{tikzcd}$$

$$(1 \otimes r \longmapsto f(r))$$

If $A = (a_{ij})$, then $(id \otimes A) = (1_S \otimes a_{ij})$

$$\therefore f(A) = (f(a_{ij})).$$

Pf of Thm 1

$$\begin{tikzcd} R[x] \arrow[r, shift left, "\varepsilon"] & R \arrow[l, shift left, dashed, "i"] \arrow[r] & 0 \end{tikzcd}$$

i : inclusion

$\varepsilon : p(x) \longmapsto p(0)$

$$\varepsilon \circ i = 1_R \quad \text{(splits)}$$

$\therefore$ $R[x] = R \oplus \ker \varepsilon$.

so $K_i R$ is a direct summand of $K_i R[x]$. abelian gp.

($\because$ K_i preserves direct sum).

$$\mathrm{Nil}\, R \longrightarrow K_1 R[x].$$

$$N \longmapsto I - Nx$$

$(I-Nx)^{-1} = I + Nx + N^2x^2 + \cdots + N^{s-1}x^{s-1}$ (if $N^s = 0$) so that $I - Nx$ is invertible.

Group homo ; $[N]+[M] = \left[\begin{smallmatrix} N & 0 \\ 0 & M \end{smallmatrix}\right) \longmapsto \cancel{(I-Nx)+(I-Mx)} \rightarrow \begin{pmatrix} I-Nx & \\ & I-Mx \end{pmatrix} = (I-Nx)(I-$

So we have embedded $K_1 R$ and $\mathrm{Nil}\, R$ into $K_1 R[x]$.

Want to show $\mathrm{Nil}\, R$ and $K_1 R$ generates $K_1 R[x]$.

For any $[A] \in K_1 R[x]$, where A is an invertible matrix with entries in $R[x]$,

$$A = A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n, \quad (A_i = n \times n \text{ matrix in } R)$$

Use Higman's Trick :

$$A \sim \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \sim \begin{pmatrix} A & A_n x \\ 0 & I \end{pmatrix} \sim \begin{pmatrix} A - A_n x^n & A_n x \\ -x^{n-1} & I \end{pmatrix} = \begin{pmatrix} A_0 + A_1 x + \cdots + A_{n-1}x^{n-1} & A_n x \\ -x^{n-1} & I \end{pmatrix}$$

Thus, if $A \in GL_m(R[x])$ of degree n, then rhs $\in GL_{2m} R[x]$ of degree $n-1$.

$$A = A_0 + A_1 x + \cdots + A_n x^n \sim \underbrace{\begin{pmatrix} A_0 & \\ & I \end{pmatrix} + \begin{pmatrix} A_1 & A_n \\ 0 & 0 \end{pmatrix} x}_{\text{usual sum}} + \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix} x^2 + \cdots + \begin{pmatrix} A_{n-2} & 0 \\ 0 & 0 \end{pmatrix} x^{n-2} + \begin{pmatrix} A_{n-1} & 0 \\ -1 & 0 \end{pmatrix}$$

$$\sim \cdots \sim \begin{pmatrix} A_0 & & \\ & I & \\ & & \ddots \; I \end{pmatrix} \overset{\oplus}{+} \begin{pmatrix} A_1 & A_n \cdots A_2 \\ & 0 & \end{pmatrix} x * = \left(\begin{array}{c|c} A_0 + A_1 x & A_n x \;\; \cdots \;\; A_2 x \\ \hline 0 & \begin{matrix} I & & \\ & I & \\ & & \ddots \; I \end{matrix} \end{array}\right) \sim \left(\begin{array}{c|} A_0 + A_1 x \\ \hline 0 \end{array}\right.$$

$\therefore [A] = [A_0 + A_1 x]$ with $A_0 \in GL(n, R)$.

$\left((A(x) \cdot B(x) = I(x) \Rightarrow A(0) \cdot B(0) = I \Rightarrow A_0 B_0 = I \Rightarrow A_0\right.$ invertibl

$[A_0 + A_1 x] = [A_0(I - B_1 x)], \quad B_1 = -A_0^{-1} A_1.$

$= [A_0] \oplus [I - B_1 x] \in K_1 R + \text{im Nil} R$

$\Big(I - B_1 x \in GL\, R[x] \subset GL\, R[[x]]$ — power series.

$GL\, R[x] \ni (I - B_1 x)^{(-1)} \underset{\text{in } GL\,R[[x]]}{=} I + B_1 x + B_1^2 x^2 + \cdots$

This power series should terminate so that B_1 is nilpt

<u>Pf</u> of Thm 2

$$\begin{tikzcd} R \arrow[r, shift right, "i"'] & R[x, x^{-1}] \arrow[l, shift right, "\sigma"'] \end{tikzcd} \qquad \sigma : p(x) \longmapsto p(1). \quad \sigma \circ i = 1_R$$

$\text{Nil } R \longrightarrow K_1 R[x, x^{-1}]$

$N \longmapsto I - Nx$

$N \longmapsto I - Nx^{-1}$

$K_0 R \longrightarrow K_1 R[x, x^{-1}]$

$[A] \longmapsto xA + (I - A)$

idempotent

invertible with inverse $x^{-1}A + (I - A)$

Construct maps the other way.

$K_1 R[x, x^{-1}] \longrightarrow K_0 R$

$\underbrace{\left[\sum_{i=s}^{t} A_i x^i\right]}_{A}$

$\exists\ s > 0$ integer such that $x^s A = \underbrace{\left[\sum_0^m A_i x^i\right]}_{B}$ $n \times n$ matrix in $R[x]$.

e.g. $(xI)(A_{-1}x^{-1} + A_0 + A_1 x + \cdots) = A_{-1} + A_0 x + A_1 x^2 + \cdots$

B is not invertible (in R[x]) but determines an endomorphism of free R[x]-module with n generators

$$B : (R[x])^n \longrightarrow (R[x])^n.$$

claim $(R[x])^n / \mathrm{im}\, B = \mathrm{coker}\, B$ is a f.g. projective R-module (has a nilpt action of x).

(lemma) M = R-module, $f, g : M \circlearrowleft$ R-endos. and f is 1-1. Then

$$0 \longrightarrow \mathrm{cok}\, g \longrightarrow \mathrm{cok}\, f \circ g \longrightarrow \mathrm{cok}\, f \longrightarrow 0 \quad (R\text{-exact})$$

$\because$ $(fg)(M) \subset f(M) \subset M$.

$$\begin{tikzcd} 0 \arrow[r] & \frac{f(M)}{fg(M)} \arrow[r] \arrow[d, "\cong \text{ since } f \text{ is } 1\text{-}1"] & \frac{M}{fg(M)} \arrow[r] & \frac{M}{f(M)} \arrow[r] & 0 \\ & \frac{M}{g(M)} & & & \end{tikzcd}$$

$\square$

Now, let $C = x^t B^{-1}$ $(B^{-1} \in R[x, x^{-1}])$, where t is sufficiently large so that C has entries in $R[x]$. Then

$$CB = x^t I = BC$$

✓ Let $M = (R[x])^n$, $f = B$, $g = C$. $\qquad B$ is 1-1 ?

$$0 \longrightarrow \mathrm{cok}\, C \longrightarrow \mathrm{cok}\, x^t I \longrightarrow \mathrm{cok}\, B \longrightarrow 0$$

$\therefore$ cok B is f.g.

$$0 \longrightarrow (R[x])^n \xrightarrow{C} (R[x])^n \longrightarrow \mathrm{cok}\, C \longrightarrow 0$$

This is a free R-resolution (not f.g.) of cok C, so that $\dim_R(\mathrm{cok}\, C) = 0$ or 1. $\therefore$ cok B is R-projective.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{cok}\, B \arrow[r] & \mathrm{cok}\, x^t I \arrow[r] \arrow[d, equal] & \mathrm{cok}\, C \arrow[r] & 0 \\ & & R^{nt} & & \end{tikzcd}$$

$\square$

Now define

$$K_1 R[x, x^{-1}] \longrightarrow K_0 R$$
$$[A] \longmapsto [\mathrm{cok}\, B] - [R^{ns}]$$

To show K_0R, $Nil\,R$, $Nil\,R$, K_1R generates $K_1R[x,x^{-1}]$

Let $a=[A]\in K_1R[x,x^{-1}]$, $A\in GL(R[x,x^{-1}])$

$B=x^2A$ has entries in $R[x]$.

$\hat{x}I\in K_0R$

$B\sim B_0+B_1x$ (B_0, B_1 matrix in R)

$B(x,x^{-1})\cdot C(x,x^{-1})=I$

B_0+B_1 is invertible over R.

inv. in Laurent series ring.

$$\left(\because \quad \begin{aligned} R[x,x^{-1}] &\longrightarrow R \\ p(x) &\longmapsto p(1) \end{aligned} \right)$$

$n\times n$ $R[x,x^{-1}]$

$$(B_0+B_1)^{-1}B = C + (I-C)x$$

$R[x,x^{-1}] \to R$

$x \to 1$

B_0+B_1

RHS is invertible on $R[x,x^{-1}]$

$D+Ex$

Hard : $\exists m$ integer. $\ni$ $C^m(I-C)^m\equiv 0$.

$D+E=I$

$R^n=\ker C^m \oplus \ker (I-C)^m$

$\circlearrowright \qquad \circlearrowright$

auto of R^n

K_1R

$R^n \xrightarrow{(I-C)^m} R^n \xrightarrow{C^m} R^n \to 0$

$C^m x = 0 \to (I-C)^m y = x$

$\mathbb{Z}[x]$

$$p(x)\,x^m + q(x)(1-x)^m = 1$$

$$p(C)\,C^m + q(C)(I-C)^m = I$$

Thm 3 If R is regular, then $Nil\,R=0$.

Pf. $a\in Nil\,R$. $a=[(F,\varphi)]$

f.g. free module. $\varphi^r=0$ for some r.

Filterate

$$0=M_r\subset M_{r-1}\subset\cdots\subset M_1\subset M_0=F$$

$$M_i=\operatorname{im}(\varphi^i)$$

Clearly, $\varphi(M_i)\subseteq M_{i+1}$.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\hat{\varphi}=\varphi|_{M_1}"] & F \arrow[r] \arrow[d, "\varphi"] & F/M_1 \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & F \arrow[r] & F/M_1 \arrow[r] & 0
\end{tikzcd}$$

If M_1, F/M_1 are f.g. free, then we would have

$$(F,\varphi) \sim (M_1,\hat\varphi) \oplus (F/M_1, 0) = (M_1,\tilde\varphi)$$

So $(F,\varphi) \sim (M_1,\tilde\varphi) \sim (M_2,\hat\varphi) \sim \cdots \sim (0,0)$.

~~But~~

We modify as follows for $\left(M_1, F/M_1 \text{ f.g. free}\right)$

Case $\ell = 2$ $\quad \varphi^2 = 0$.

$$0 \subset M = \operatorname{im}\varphi \subset F.$$

Let $F/M = N$.

Since M, N are f.g., $\exists\ \hat M, \hat N$ $\boxed{\text{f.g.}}$ free $\ni$

$$\begin{tikzcd}
\hat M \arrow[dd, "f"'] & & \hat N \text{-free} \arrow[dl, "\bar g"'] \arrow[dd, "g"] & \\
& F \arrow[dr] & & \\
M \arrow[d] & & N \arrow[r] \arrow[d] & 0 \\
0 & & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
& F_1 = \hat M \oplus \hat N \arrow[r, dashed, "\hat\varphi"] \arrow[d, "p = f \oplus \bar g"'] & \hat M \oplus \hat N \arrow[d] \\
F_0 & F \arrow[r, "\varphi"] \arrow[d] & F \\
& 0 &
\end{tikzcd}$$

$\to \ker \longrightarrow F_1 \xrightarrow{\ell'} F_0 \longrightarrow 0$

$F_1 = \ker \oplus F_0$

F_2

Can show F_2 is free

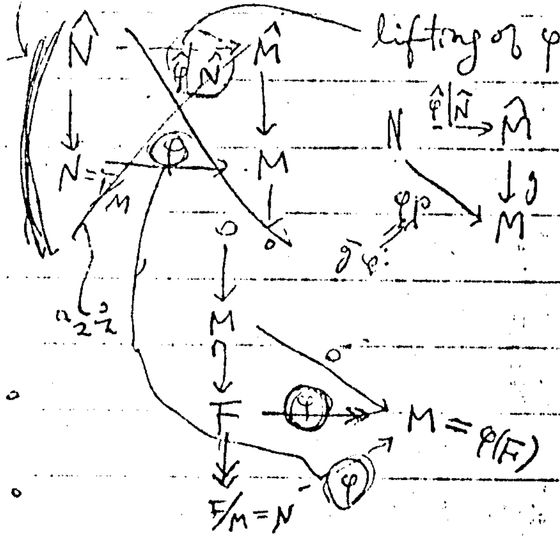
$\hat\varphi$ is nice nilpt $(\hat\varphi^2 = 0)$

$(\hat M \oplus \hat N, \hat\varphi) = ? \in \operatorname{Nil} R$

$$\therefore \begin{tikzcd}
0 \arrow[r, dashed] & \hat M \arrow[r] \arrow[d, "0"] & \hat M \oplus \hat N \arrow[r] \arrow[d, "\hat\varphi"] & \hat N \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \hat M \arrow[r] & \hat M \oplus \hat N \arrow[r] & \hat N \arrow[r] & 0
\end{tikzcd}$$

$\hat\varphi|_{\hat M} = 0$

$\hat\varphi|_{\hat N}$ = how define?



Repeat the process, end up with $\hat M$, $\hat N$ free.

$$\hat\varphi = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

Recall : inverse of (F_0, φ_0) in Nil R :

start with (F_0, φ_0), make

$$0 \longrightarrow (F_2, \varphi_2) \longrightarrow (F_1, \varphi_1) \longrightarrow (F_0, \varphi_0) \longrightarrow 0$$

with $[(F_1, \varphi_1)]$ upper triangle $(\Rightarrow (F_1, \varphi_1) \equiv 0)$

$$\begin{tikzcd}
0 \arrow[r] & (F_2, \varphi_2) \arrow[r] & (F_1, \varphi_1) \arrow[r] \arrow[d, phantom, "\parallel"] & (F_0, \varphi_0) \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
 & & (\hat{M} \oplus \hat{N}, \hat{\varphi}) & (F, \varphi) &
\end{tikzcd}$$

(F_2, φ_2) is simpler than (F_0, φ_0). That is,

claim ∃ a submodule of (F_2, φ_2) M' such that $0 < M' \subset F_2$

$\varphi_2(F_2) \subset M'$

$\varphi_2(M') = 0$

and $\dim M' = \dim M - 1$

$\dim F_2/M' = \dim F/M - 1.$

∴

$$\begin{tikzcd}
F_2 = \ker p \arrow[d] \\
F_1 = \hat{M} \oplus \hat{N} \arrow[d, "p"] \\
F_0 = F \arrow[d] \\
0
\end{tikzcd}$$

$M' = \ker p \cap M'$

$\varphi_2 = \hat{\varphi} | F_2$

$$\begin{tikzcd}
0 \arrow[r, shift left] & M' \arrow[l, shift left, "\varphi"] \arrow[r, phantom, "\subset"] & F_2 \arrow[l, bend left, "\hat{\varphi}"]
\end{tikzcd}$$

$$0 \longrightarrow M' \longrightarrow \hat{M} \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow F_2/M' \longrightarrow \hat{N} \longrightarrow N \longrightarrow 0$$

Since $\hat{M}, \hat{N}$ are free, $\dim M' = \dim M - 1.$

M' is f g since the ring is regular.

Corol (Bass-Heller-Swan, Serre, Grothendieck)

If R is a regular ring, then

$$K_0R \simeq K_0R[x] \simeq K_0R[x,x^{-1}]$$

$$\begin{tikzcd}
0 & K_1R[t,t^{-1}] \arrow[l] & K_1(R[t,t^{-1}][x]) \arrow[l, "\sigma_*"'] \arrow[d, equal] & \mathrm{Nil}\, R[t,t^{-1}] \arrow[l] \arrow[d, equal] & 0 \arrow[l] \\
 & & K_1(R[x][t,t^{-1}]) & 0 & \\
 & K_0R \arrow[uu, "f", "1-1"'] & K_0R[x] \arrow[l, twoheadrightarrow, "\varepsilon_*"'] \arrow[u, "g", "1-1"'] & & \\
 & p(0) & p(x) \arrow[l, mapsto, "\varepsilon"] & &
\end{tikzcd}$$

since $R[t,t^{-1}]$ is regular if so is R by Hilbert Thm.

$\therefore K_0R \xleftarrow[\simeq]{\varepsilon_*} K_0R[x]$. if commutes.

Need to check commutativity of above diagram.

Let $a = [A] \in K_0R[x]$, $A^2 = A$ matrix with entries in $R[x]$

$$A \overset{\text{let}}{=} A_0 + A_1x + \cdots + A_nx^n$$

$\varepsilon_*[A] = [A_0]$

$f[A_0] = [tA_0 + (I - A_0)]$

$g[A] = [tA + (I - A)]$

$\quad = [tA_0 + tA_1x + \cdots + tA_nx^n + (I - A_0) + -xA_1 - x^2A_2 - \cdots - x^nA_n]$

$\sigma_x[\;\swarrow\;] = [tA_0 + (I - A_0)]$

Thus, $\quad K_0R \xleftarrow{\simeq} K_0R[x]$.

Corr $\widetilde{K}_0 k[x_1 \cdots x_n] = 0$ (k field.)

$\because k[x_1, \cdots x_n] = k[x_1, \cdots x_{n-1}][x_n]$

$K_0k[x_1, \cdots x_n] \simeq K_0k[x_1 \cdots x_{n-1}]$

$\simeq K_0k$

$= \mathbb{Z}$.

$$\begin{tikzcd}
R \arrow[r] & R[x] \arrow[r] & R[x,x^{-1}] \arrow[ll, bend left, "1 \mapsfrom x"]
\end{tikzcd}$$

$$\begin{tikzcd}
K_0R \arrow[r] & K_0R[x] \arrow[r] & K_0R[x,x^{-1}] \arrow[r] \arrow[ll, bend left, dashed, "\text{splitting}"] & 0
\end{tikzcd}$$

$k \subset k[x_1, \cdots x_n]$

$i_* : K_0k \xrightarrow{\simeq} K_0k[x_1, \cdots x_n]$

$\quad p \longmapsto p \otimes_k k[x_1, \cdots x_n]$

For any $\overset{b}{\searrow} a \in K_0k[x_1 \cdots x_n]$

if $b = [F]$, then

Cor 1. If R is regular, then

$$K_0 R \simeq \underline{K_0 R[x] \simeq K_0 R[x, x^{-1}]}$$

$\because$ It remains to show that $\sigma : R[x] \hookrightarrow R[x,x^{-1}]$ induces

$$\text{epi} : K_0 R[x] \longrightarrow K_0 R[x,x^{-1}]$$

Let $a = [P] \in K_0 R[x,x^{-1}]$ where P is f.g. proj $\underline{R[x,x^{-1}]\text{-mod}}$

Can find F_i = free $R[x,x^{-1}]$-module $\ni$

$$\begin{tikzcd} & F_1 \arrow[r, "A"] & F_0 \arrow[r] & P \arrow[r] & 0 \\ & K_0 \arrow[ur] & & & \\ 0 \arrow[ur] & & & & \end{tikzcd}$$

K_0 f.g. since $R[x,x^{-1}]$ is Noetherian. P.11 —— ?

A = $n \times m$ matrix over $R[x,x^{-1}]$

$B = x^s A$ for ~~matrix~~ s sufficiently large, $\underline{B \text{ has entries in}}$

$\operatorname{im} B = \operatorname{im} A$ [$\because$ multpl. by x^s is an iso.]

$$F_1 \xrightarrow{B} F_0 \longrightarrow P \longrightarrow 0$$

F_i = free $R[x,x^{-1}]$-module, B

Let $\hat{F}_i$ be free $R[x]$-module $\ni$

$$F_i = \hat{F}_i \otimes_{R[x]} R[x,x^{-1}]$$

$\operatorname{rk} F_i = \operatorname{rk} \hat{F}_i$. Then

$$\hat{F}_1 \xrightarrow{B} \hat{F}_0 \longrightarrow M \longrightarrow 0$$

($- \otimes_{R[x]} R[x,x^{-1}]$ is right exa

$M = \operatorname{cok} B$ is f.g.

Claim $M \otimes_{R[x]} R[x,x^{-1}] \cong P$ $\angle$

Then, $\exists$ resolution

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M by f.g. proj. $\underline{R[x]\text{-modules}}$

Claim $0 \longrightarrow P_n \otimes_{R[x]} R[x,x^{-1}] \longrightarrow \cdots \longrightarrow P_0 \otimes R[x,x^{-1}] \longrightarrow M \otimes R[x,x^{-1}] \longrightarrow 0$
is still exact.

(pf. of claim) $R[x,x^{-1}]$ is a flat $R[x]$-module.

$$\therefore \quad N_0 = R[x] \subset R[x,x^{-1}]$$

$$\begin{aligned} &\cap\!\!\!\mid \\ N_1 &= \left\{ \frac{a_{-1}}{x} + a_0 + a_1 x + \cdots \right\} \\ &\cap\!\!\!\mid \end{aligned}$$

$N_2 =$ all Laurent series with lowest term of degree -2

$\cap\!\!\!\mid$

$\vdots$

$$R[x,x^{-1}] \simeq \bigcup_{i=0}^{\infty} N_i \quad \checkmark$$

$\sigma_x[P] = [P \otimes_{R[x]} R[x,x^{-1}]]$

From $(*)$,

$$[P] = \sum (-1)^i \left[P_i \otimes_{R[x]} R[x,x^{-1}] \right] = \sum (-1)^i \sigma_x [P_i]$$

QED of cor

Cor. 2 $\quad Wh(\text{free ab. gp}) = 0.$

✓ Pf. Suffices to show for Γ is f.g. ?

assume $\Gamma =$ free ab. gp on n generators.

$\Gamma = \Gamma_0 \times T,$ $\qquad \Gamma_0 =$ free of rk $n-1$,
$T =$ infinite cyclic.

$$\mathbb{Z}\Gamma = \mathbb{Z}(\Gamma_0 \times T) \overset{p.41}{=} (\mathbb{Z}[\Gamma_0])[x,x^{-1}]$$

$$K_1 \mathbb{Z}\Gamma = K_1 (\mathbb{Z}[\Gamma_0])[x,x^{-1}] = K_1 \mathbb{Z}[\Gamma_0] \oplus K_0 \mathbb{Z}[\Gamma_0] \oplus Nil\, \mathbb{Z}\Gamma_0 \oplus Nil\, \mathbb{Z}\Gamma_0.$$

By Hilbert Basis & Syzygy Thm, $\mathbb{Z}\Gamma_0$ is regular $(\Rightarrow Nil\, \mathbb{Z}\Gamma_0 = 0)$
$\qquad \| \; \mathbb{Z}[x_1, \cdots x_{n-1}, x_1^{-1}, \cdots x_{n-1}^{-1}]$

✓ $K_0 \mathbb{Z}\Gamma_0 = \infty$-cyclic.

$$\therefore K_1 \mathbb{Z}\underbrace{\Gamma}_{\text{free ab. of } n\text{-gen.}} = K_1 \mathbb{Z}\underbrace{\Gamma_0}_{\text{free ab. of } (n-1) \text{ gen.}} \oplus T. \qquad \text{- reduction formula}$$

$$\begin{aligned} K_1 \mathbb{Z}\Gamma &= K_1 \mathbb{Z} \oplus T^n \qquad K_1\mathbb{Z} = \{\pm 1\}. \\ &= K_1 \mathbb{Z} \oplus \Gamma = \pm \Gamma \quad (\Gamma \oplus \Gamma) \end{aligned}$$

$$\therefore \quad Wh(\Gamma) = K_1 \mathbb{Z}\Gamma / \pm\Gamma = 0.$$

Thm4 If R is a ring with 1, then either Nil $R=0$ or not f.g.

(By analyzing transfer maps)

$\sigma: R \subset S$

$K_1 R \xleftarrow{\sigma^*} K_1 S$

Setting $U = R[x^n]$, $S = R[x]$
$n \geq 1$

Have

$$\begin{tikzcd} R[x^n] \arrow[r, hook, "\text{inclusion}"] \arrow[dr, "\sigma_n", "\varepsilon_n"'] & R[x] \arrow[d, "\varepsilon"] & \varepsilon: p(x) \mapsto p(0) \arrow[l, dashed] \\ & R & \end{tikzcd}$$

$$\begin{tikzcd} K_1 R[x^n] \arrow[r, shift right, "\sigma_{n*}"'] \arrow[d, "(\varepsilon_n)_*"] & K_1 R[x] \arrow[l, shift right, "\sigma_n^*"'] \arrow[d, "\varepsilon_*"] \\ K_1 R \arrow[r, equal] & K_1 R \end{tikzcd}$$

(Fact 1°) $(\sigma_n)_*(\ker \varepsilon_n) \subseteq \overset{= \text{Nil} R}{\ker \varepsilon_*}$ (Swan) (In fact, ker is nil g.p. ?)

(Fact 2°) $(\sigma_n)^*(\sigma_n)_*$ is multipl. by n

(Fact 3°) Given $a \in \operatorname{Ker} \varepsilon_*$, $\exists\, N_a > 0 \ni (\sigma_n)^*(a) = 0$ for all $n \geq N_a$
Special fact. (AMS. paper, use matrix identity)

Pf of Thm. Assume Nil R is f.g. and show Nil $R=0$,
$\exists\, N > 0 \ni (\sigma_n)^*(\ker \varepsilon_*) = 0$ provided ~~N~~ $n \geq N$.
So, if $n > N$, $\ker(\varepsilon_n)_* \circlearrowleft$ multpl. by n is 0-map.

$$\left(\begin{tikzcd} \ker(\varepsilon_n)_* \arrow[r, "(\sigma_n)_*"] \arrow[rr, bend right, "\text{mult by } n"'] & \ker \varepsilon_* \arrow[r, "(\sigma_n)^*", "\text{0-map}"'] & \ker(\varepsilon_n)_* \end{tikzcd} \right.$$

Exponent $\ker(\varepsilon_n)_*$ devides n. $(n \geq N) \Rightarrow$ Nil $R = 0$.

* Nil $(\mathbb{Z}(T \oplus T_4)) \neq 0$

Nil ($\mathbb{Z}T_4$) not known

~~Nil~~ Wh (f.g gp) is not f.g. $\xrightarrow{Wh(\mathbb{Z}\times T\times T_4)}$ Bass-Murthy Annals of Math 196

$Wh(\pi_1 T^n) = 0$ $\quad T^n = S^1\times\cdots\times S^1$ n-torus (asph mfd)

$\therefore$ asph. mfd with $\pi_1 \simeq$ to $\pi_1 T^n$ is homeo. $n > 4$

Thm. (Wall, Hsiang - Shaneson) Surgery

a closed asph. mfd with abelian fund. gp and $n>4$, then M^n is homeo to T^n.

(Not true for diffeo, PL) $\leftarrow$ $T^n \# \Sigma^n$.

π_1(asph) = torsion free $\Downarrow$ free $\therefore$ $\therefore M \underset{he}{\simeq} T^n$.

$\mathcal{S}(M^n)$ mfd set of equiv. classes of the following objects.

object: a homotopy structure on M is a mfd N^n together with a homotopy equivalence $f : N \to M$ $\quad$ (N, f)

equiv. relation: Two structures $N \xrightarrow{f} M$ & $K \xrightarrow{g} M$ are equivalent if $\exists$ $\boxed{\text{homeo}}$ $F : N \to K$ $\ni$

$$\begin{tikzcd} N \arrow[dr, "f"] \arrow[d, "F"'] & \\ K \arrow[r, "g"'] & M \end{tikzcd}$$

Ω htpy comm

If M has ∂,

object $\quad N \xrightarrow{f} M$, (HE) also requires $f|\partial N$ is a homeo onto ∂

relation $\quad g\circ F|\partial N \ \underset{\text{homeo}}{} \ f|\partial N$.

Thm says $\mathcal{S}(T^n)$ has only one elt $T^n \xrightarrow{1} T^n$.

Milnor's exmpl

1°. Alexander trick (good for top. pL - category but fails for

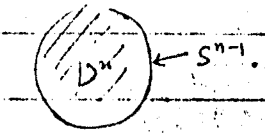
2°. poincare conjecture (difficulty dim 3 & 4)

3°. Fibering thm. (" dim 3, 4, 5).

Alexander trick

$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$

$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$



Any homeo $S^{n-1} \circlearrowleft$ extends to a homeo $D^n \circlearrowleft$ (also true

poincare conjecture

$\mathcal{S}(D^n)$ has only one elt. ← Alexander trick / poincare conj.

⇓

a homotopy S^n is homeo to S^n fails for smooth. ~~even for high~~ (true also for pl) True if $n>4$.

Fibering thm

$(n>5)$ unkown. $n=4,5$.

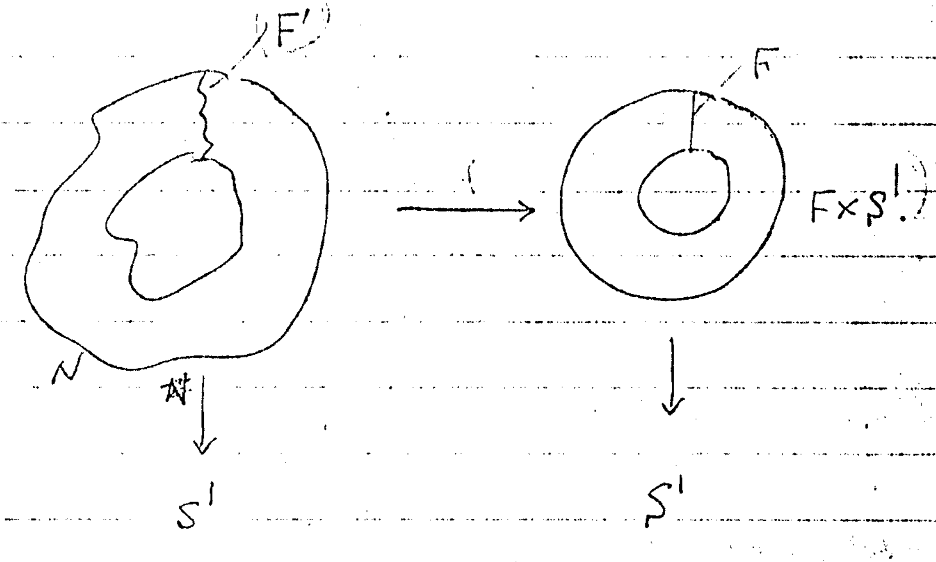
mfd

$N \xrightarrow{f} F^{n-1} \times S^1$

he.

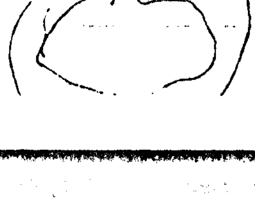
Then $N = F' \times S^1$? No. in general

$\Rightarrow$ N fibers over S^1?



When N is cut by some ~~[illegible]~~ codim 1 submfd, result is $F' \times [0,1]$

True if f is a simple hty equiv.



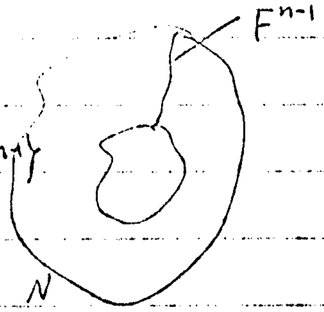
pf

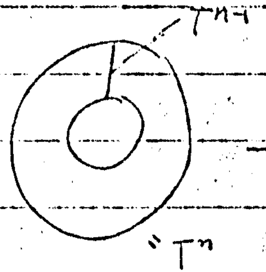
$$\begin{tikzcd}
F^{n-1} \arrow[d, dashed] \arrow[rr, "he"] & & T^{n-1} \arrow[d] \\
N^n \arrow[r, "f", "he"'] \arrow[d] & T^n = & T^{n-1}\times S^1 \arrow[d] \\
S^1 & & S^1
\end{tikzcd}$$

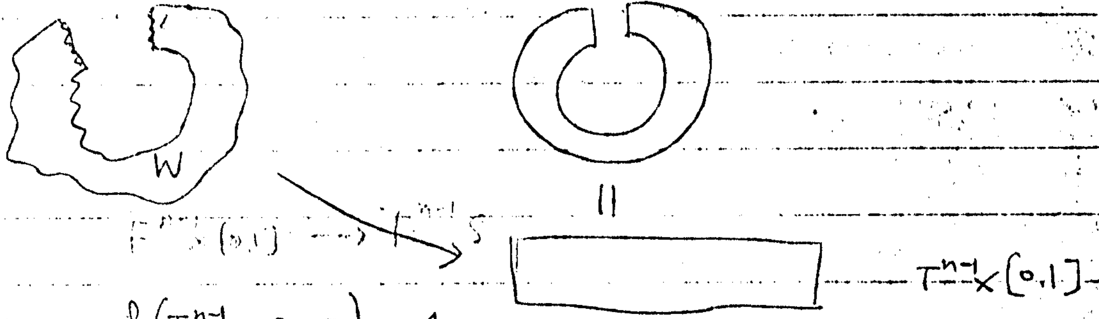
By induction hypo,
(assume $\mathcal{S}(T^{n-1}) = \{T^{n-1} \xrightarrow{1} T^{n-1}\}$)

$F^{n-1} \xrightarrow{f'} T^{n-1}$

is htpic to a homeo. $\hat{f}: F^{n-1} \to T^{n-1}$







Need $\mathcal{S}(T^{n-1}\times[0,1]) = 1.$

$(\text{Statement})_n$: $\mathcal{S}(T^n \times D^m)$ has only one elt.

Exercise Using 1°, 2°, 3°, prove above statement (use surgery theo

$F^{n-1}\times[0,1] \xrightarrow{f'\times 1} T^{n-1}\times[0,1]$ is htpic to a homeo $F^{n-1}\times[0,1] \to T^{n-1}\times[0,1]$
h.e. with ∂-homeo.

$\Downarrow$ ∂-homeo

$F^n \longrightarrow T^n$
is $\simeq$ to homeo.

(Proc. Cambridge Phil. Soc.) Matumoto - Siebenman.

Top'l s-cobordism is false in either 4 or ~~5~~ with base a mfd-∂.

$Wh(T \times T_2) = \cancel{Wh}(T_2) + \cancel{\tilde{K}_0}(\mathbb{Z}T_2) + 2\,\mathrm{Nil}(\mathbb{Z}T_2)$

Any $\mathbb{Z}_2$-module is $\mathbb{Z}[\mathbb{Z}_2]$, $\mathbb{Z}$,

If G is finite abelian of
order divisible by a sq
$0 \neq \mathrm{Nil}_0(\mathbb{Z}G) \subseteq \mathrm{Nil}\,\mathbb{Z}(G \times \mathbb{Z}_C)$

$1°$ is true for PL. So $2°$ or $3°$ is false for $n = 4, 5$.

Thm′ $\mathcal{S}(T^n \times D^m)$ has only one elt. if $m > 4$. [by induction on n

R^{ring}; $\forall n$, can define $K_n R$.

Def (Bass) $n < 0$; $K_{-1}R = \operatorname{coker}\left(K_0R[x] \oplus K_0R[x^{-1}] \longrightarrow K_0R[x, x^{-1}]\right)$

analogue of Bass-Heller-Swan Thm.

$K_nR = \operatorname{coker}.\left(K_{n+1}R[x] \oplus K_{n+1}R[x^{-1}] \longrightarrow K_{n+1}(R[x,x^{-1}])\right)$: $n < 0$.

$$\begin{tikzcd}
K_1R[x] \oplus K_1R[x] \arrow[r] & K_1R[x\,x^{-1}] \arrow[r] \arrow[d, equal] & K_0R \arrow[r] & 0 \\
(K_1R \oplus \mathrm{Nil}R) \quad (K_1R \oplus \mathrm{Nil}R) \arrow[r, dashed] & K_1R \oplus 2\,\mathrm{Nil}R \oplus K_0R & &
\end{tikzcd}$$

Def (Topological)

$K_{-1}X = K_0\Sigma X$

$K_{-n}X = K_0(\Sigma\Sigma \cdots \Sigma X)$

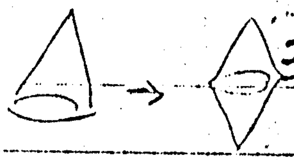
Difficult to define higher K_i's

$K_0R \simeq K_1(\Sigma R)$

$K_{-n}R \cong K_0(\Sigma\Sigma \cdots \Sigma R)$

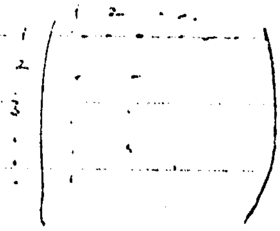
Suspension of R

top'l suspension. $X \subset CX \longrightarrow CX/_X = \Sigma X$



<u>Susp'n of Rings</u>

$M_\infty(R)$ = infinite matrices



$\cup$

CR = matrixes locally-finite. ($\forall$ row & column has finite non-
entries)

$\cup$

<u>Imbedding of R (as an ideal) in CR.</u>

$M(R)$ = consists matrices with only a finite # of non-zero entrie

$M(R)$ is a two sided ideal in $\underset{\uparrow}{CR}$
$\searrow \not\ni 1$. ring with 1

$$\Sigma R = CR/_{MR}.$$

<u>Thm</u> $K_0R \simeq K_1(\Sigma R)$

Can define $K_{-n}R = K_0(\Sigma\cdots\Sigma R)$.

<u>Quillen's Defn</u>

$K_iR \quad (i > 1)$

$R \xrightarrow{\text{associate}} GL(R)$

trivial

$$K_1 = GL(R)/_{[GL(R), GL(R)]} = H_1(GL(R); \mathbb{Z})$$

(K_i looks like higher hty grps)

$\pi_1(BGL(R)) = 0$

X: top'l space $[\pi_1(X), \pi_1(X)]$ is <u>perfect</u> (A: perfect — $[A, A] = A$)

Can construct a space X^+ and a map $X \xrightarrow{\varphi} X^+ \ni$

① $\pi_1 X^+ = \pi_1 X^{ab}$

② φ induces an iso on homology (~~with $\mathbb{Z}$ & twisted coeff~~ with $\forall$ local system of coeff. coming from $\pi_1 X^+$.

constr. $X \to (X^+)$

attaching 2-cells to kill commutator, and add 3-cells (same # of 2-cells ← perfect) and kill the appearing 3 homology when adding 2-cells.

Note $[GL(R), GL(R)] = E(R)$ is perfect.

$$\pi_1 BGL(R) = GL(R)$$

Define $K_i R = \pi_i (BGL(R))^+$

↖ Wagoner — Topology 1972 & [illegible]

$K_1 \Sigma R = K_0 R$. This is true for higher — $K_{i+1} \Sigma R = K_i R$.

$$\Omega BGL(\Sigma R)^+ = BGL(R)^+ \times K_0 R$$ Delooping prop

$GL(CR) \simeq$ unit CR

↗ can show $H_i(\text{unit } CR, \mathbb{Z}) = 0$, $i > 0$

Cone ring is contractable.

Poly $\mathbb{Z}$-gps.

topological $\overset{s}{h}$-cobordism on either $S^1\times P^2$ or $S^1\times S^1\times P^2$ which is not a product

$\mathrm{Nil}\ \mathbb{Z}(\pi_2)=0$? geometric meaning ?

Thm (Wall, Hsiang–Shaneson) $\mathcal{S}_{Top}(T^n)=0$ if $n>4$. false for PL.

Thm′ $\mathcal{S}_{Top}(T^n\times D^m)=0$ if $m>4$ (without surgery.) — geometrically Smale's Handlebody., S

Using Alex, Poi., Fiberig.

(Thm′ + Bott periodicity) $\Rightarrow$ Thm. — true for PL. ∴ one or two of — is false.

Periodicity Thm. — false for PL.

Surgery (i) If M is a mfd-∂, connected, then

$$\mathcal{S}_{Top}(M\times D^4)\xleftrightarrow[1\text{-}1]{\ \ \overset{\ast}{\phi}\ \ }\text{~~Topological~~}\ \mathcal{S}_{Top}(M)$$

topological structure — (dim M > 4

(ii) If M is closed, dim $M>4$, then

$$|\mathcal{S}_{Top}(M)|\le\mathcal{S}_{Top}(M\times D^4).$$

(Kirby & Siebenmann 의 논문 mis-state

False for PL-top.

Meaning of Periodicity Thm. (Wall, Sullivan)

Characteristic Variety.

(sketch of periodicity)

$\mathcal{S}(M)$ is ab. gp.

∃ exact seq. of ab. gps.

$$[M\times[0,1],G/Top]\to L_{n+1}(\pi_1M)\to\mathcal{S}(M)^n\to[M,G/Top]\to L_n(\pi_1M)$$

$\partial(\ )\to\ast$ (Wall) Hermitian   sort of H-sp. $\partial M\to\ast$ period 4 by def.

(Sullivan) has period 4 alg. variety.

$=[M\times D^4,G/Top]$

map preserve periodicity (ha

(Sull., Casson, R

$$[M, G/Top] \otimes Q = \bigoplus_{\substack{i \equiv \dim M \\ \bmod 4}} H^i(M, Q)$$

<u>Sullivan's char variety</u>. Theorem — Not for C^ω. — simply connected.

Let M^n be a closed mfd ($n > 4$). Then $\exists$ <u>a collection of</u> "submfds" $\mathcal{S}$ of M (called <u>characteristic variety for M</u>) such that for $\forall$ hty equivalence $f : N^n \to M^n$, <u>if f can be split al</u> (and only if) each elt of $\mathcal{S}$, then

$f|(N - \text{small open disk}) \overset{\text{homotopic}}{\simeq}$ to a <u>homeo to (M− open disk</u>

$$L_n(1) = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z}_2 & 4n+2 \\ \mathbb{Z} & 4n. \end{cases}$$

"split" $\quad N \xrightarrow{f} M, \quad \Sigma \in \mathcal{S}$ closed submfd of M.

$$\begin{tikzcd} N \arrow[r, "f"] & M \\ f^{-1}(\Sigma) \arrow[u, phantom, "\cup"] \arrow[r, "f|f^{-1}(\Sigma) \text{ is h.e.}"'] & \Sigma \arrow[u, phantom, "\cup"] \end{tikzcd}$$

don't need ambient
need transversality.

Since $\quad L_i(\pi_1\Sigma) \xrightarrow{\exists \text{ map}} L_i(1) = \begin{cases} - \\ - \end{cases}$

surgery obstr. of Σ

$\{S^n \times *, * \times S^m, S^n \times S^m, *\}$ is a char. var. of $S^n \times S^m$.

$\{*, CP^1, CP^2, \ldots CP^{n-1}\}$ $\qquad CP^n$

Poly Z sp $\quad$ Johnson - Auslander

<u>Amsterdam Conf.</u> 1970 $\quad$ Spring LN ~~[illegible]~~ 197

Conjecture Suppose Γ is π_1 of a closed asph. mfd. and Γ' is torsion-
gp containing Γ as a subgp of finite index, then Γ' is the π_1 of
a closed asph. mfd.

$$\underset{\text{closed asph. mfd}}{\pi_1(M)} = \Gamma \underset{\text{finite}}{<} \underset{\text{torsion free}}{\Gamma'} \Longrightarrow \Gamma' = \pi_1(\underset{\text{for some c}}{M'})$$ 존재

Special case Γ is π_1 of 2-mfd.

problem in Fuchsian gp. $SL_2(\mathbb{R})$

Next approach doesn't work for special case.

$$M^2 \xrightarrow[f]{} B\Gamma'$$ poincare mfd 2 dim coh. $\mathbb{Z}$

f induces $\cong$ on H^2.

By surj. on high dim. can get a

$$M^2 \times S^1 \times S^1 \times S^1 \xrightarrow{f \times id} B\Gamma' \times S^1 \times S^1 \times S^1$$ co-cyclic

$L_5(\Gamma' \times T \times T \times T)$

End up with $\exists$ closed 5-mfd $N^5 \underset{he}{\cong}$ to $B\Gamma' \times S^1 \times S^1 \times S^1$.

$F^4 \to N^5 - S^1$, $F^4 \sim B\Gamma' \times S^1 \times S^1$ No

Indication of special case.
(Wilder)

$S \sim B\Gamma' \times S^1$
(Stallings)

$\widetilde{B\Gamma'} \doteqdot$ one pt cptification. conjecture may be true

Teichmuller space

Torsion free case is o.k. (Raymond says)

~~Teichmuller?~~

Given fuchsian gp Γ, extn of Γ by finite gp

$$\begin{tikzcd} 1 \arrow[r] & \Gamma \arrow[r] & \Gamma' \arrow[r] & F \arrow[r] \arrow[d, "\varphi"', "\text{finite mono}"] & 1 \\ & & & \mathrm{Out}\,\Gamma & \end{tikzcd}$$

$\Rightarrow \Gamma'$ fuchsian ?

defines extn.

Zieschang, Heiner
Math. Z 151 (1976)
No 2 165-188

poly $\mathbb{Z}$-group.

Γ is a <u>poly-$\mathbb{Z}$-gp</u> if it has a filtration

$$1 = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_n = \Gamma$$

such that ① $\Gamma_i \triangleleft \Gamma_{i+1}$

② Γ_{i+1}/Γ_i is infinite cyclic.

a <u>virtual poly $\mathbb{Z}$-group</u> is a gp containing a poly-$\mathbb{Z}$ subgp of fi

index.

1° $\Gamma = \text{poly } \mathbb{Z} \Rightarrow Wh(\Gamma) = 0.$

2° (likely) Γ: torsion free virtual poly-$\mathbb{Z} \Rightarrow Wh(\Gamma) = 0$

eg. torsion free virtual poly $\mathbb{Z}$ which is not poly

π_1 of some 3-mfd.

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow \Gamma \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1$$

$$H_1(\Gamma) = \mathbb{Z}_4 \oplus \mathbb{Z}_4$$

infra

<u>Thm1</u> poly (finite or cyclic) is same as (virtual) poly-$\mathbb{Z}$.

def. $\Updownarrow$ Γ_{i+1}/Γ_i is finite or -cyclic

<u>Thm2</u> (Tits) If Γ is a f.g. discrete subgp of $GL_n(\mathbb{C})$ [or Lie gp with fini

many components ?], then either Γ is vertually poly $\mathbb{Z}$ or

it contains ~~an~~ non-abelian free subgp. (J. Alg 72)

Indication of Pf of 1°.

(Th 1) $Wh\,\Gamma = 0$ if Γ is poly $\mathbb{Z}$.

(Th 2) F.E.A (Johnson & ~~Anderson~~ Auslander) For each virtual poly-$\mathbb{Z}$ gp Γ, $\exists$ closed

smooth ~~top~~ asph. mfd with $\pi_1 = \Gamma$

(Th3) Two closed asph. mfds ~~with~~ (dim > 4) with $\cong \pi_1$ ~~(?)~~ which are ~~with~~ poly-

Then the mfds are homeo (

(Wall)

(Thm 3') $\mathcal{S}(M \times D^{m})$, $m>4)$ consists of only one elt if M is a closed as

with poly-$\mathbb{Z}$ π_1.

[∵ By induction on dim M]

$$\mathcal{S}(M) \subset \mathcal{S}(M \times D^4) \leftarrow \underline{\text{periodicity Thm}} \text{ for } \forall M$$
$$\subset \mathcal{S}(M \times D^8)$$

Wall.

(Thm 2') If Γ is poly-$\mathbb{Z}$ gp then $\exists$ a closed asph. mfd M

with $\pi_1 M = \Gamma$.

assume Thm 2' true if dimΓ=5 cd: and let Γ' have dim

$$\cdots \cdots \subseteq \Gamma_5 \subseteq \Gamma_6 = \Gamma'$$
Γ.

Thm 3' 가정

$\mathcal{S}(N)$ has only 1-elt

$\pi_1(N^5) = \Gamma$ by hyp.

closed asph.

$$\begin{tikzcd} N \arrow[r, "\varphi"] \arrow[dr, "f"'] & N \\ & N \arrow[u, "id"'] \end{tikzcd}$$

If φ is self-hty equiv.

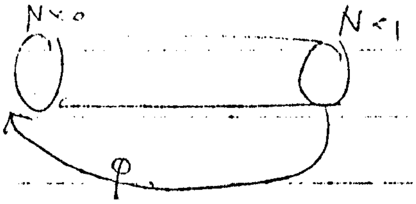
φ self hty eqv.

$\Gamma' = \mathbb{Z} \underset{(\alpha)}{\rtimes} \Gamma$

$((n)(m), \gamma) \sim (n \cdot \alpha(m)\gamma)$

$\underline{\alpha : \mathbb{Z} \to \Pi}$

$B\Gamma = N \times_{\varphi}$



Zieschang (Bull AMS 1974)

Solved If Γ is a torsion free gp containing $\pi_1 M^2$ as a subgp of finite index $\Rightarrow \Gamma = \pi_1 N^2$ for some N^2.

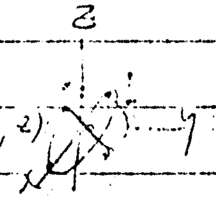
Example $1 \to \mathbb{Z}^3 \to \Gamma \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1$ composite of transl. & orth. rota

Γ = subgp of Rigid motions of $\mathbb{R}^3 = \{(x,y,z)\}$

generated by α, β, γ

torsion-free

$$\begin{cases} \alpha(x,y,z) = (\frac{1}{2}+x, -y, -z) \\ \beta(x,y,z) = (-x, \frac{1}{2}+y, \frac{1}{2}-z) \dashrightarrow (x, 1+y, z) \\ \gamma(x,y,z) = (\frac{1}{2}-x, \frac{1}{2}-y, \frac{1}{2}+z). \end{cases}$$



(Lyndon, Hochschield-Serre ; using spectral seq) can calculate cohomolgy.

$\mathbb{Z}^3 \subset \Gamma$ is translation

$\underline{(1,0,0)} \to \alpha^2$ $\quad (x,y,z) \mapsto (x+1, y, z)$ $\quad \begin{cases} (1,0,0) \mapsto \alpha^2 \\ (0,1,0) \mapsto \beta^2 \\ (0,0,1) \mapsto \gamma^2 \end{cases}$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is rotational part.

$$\alpha = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad \gamma = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

α, β, γ together with I is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts on $\mathbb{Z}^3$

$$E^2_{pq} \Rightarrow H_*(\Gamma, \mathbb{Q})$$

$$E^2_{pq} = H_p(\mathbb{Z}_2 \oplus \mathbb{Z}_2 ; \underbrace{H_q(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} ; \mathbb{Q})}_{\text{rational vector sp}})$$

$= 0$ if $p > 0$.

$* = -\frac{*}{2}(\alpha+\beta+\gamma) - ($

So $E^2_{0q} \cong H_q(\Gamma ; \mathbb{Q})$

$$H_1(\Gamma, \mathbb{Q}) \overset{E^2_{01}}{\simeq} H_0(\mathbb{Z}_2 \oplus \mathbb{Z}_2 ; H_1(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} ; \mathbb{Q})).$$

$H_0 = (\mathbb{Q} \oplus \mathbb{Q} + \mathbb{Q})_{\mathbb{Z}_2 \oplus \mathbb{Z}_2} = 0$ $\quad = H_0(\overset{\mathbb{Z}_2 \oplus \mathbb{Z}_2}{\cancel{\text{Finite}}} ; \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})$ — twisted action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

[illegible]

✓ $H_1(\Gamma, \mathbb{Q}) = 0 \Rightarrow \Gamma$ is not poly$\mathbb{Z}$ (f.g.)

~~Suppose Γ is poly Z~~. ... commutator ...

$0 = H_1(\Gamma, \mathbb{Q}) = H_1(\Gamma; \mathbb{Z}) \otimes \mathbb{Q} = \Gamma/[\Gamma,\Gamma] \otimes$

$\therefore$ $\Gamma/[\Gamma,\Gamma]$ is finite gp.

If Γ is poly $\mathbb{Z}$; $0 \subset \Gamma_n \subset \cdots \subset \Gamma_1 \subset \Gamma_0 =$

then $[\Gamma,\Gamma] \subseteq \Gamma_1$

$|\Gamma/[\Gamma,\Gamma]| \geq \Gamma/\Gamma_1 \simeq \mathbb{Z}$. not finite contra

Th 1 $\mathrm{Wh}\,\Gamma = 0$ if Γ is poly-$\mathbb{Z}$

$R_\alpha[x]$ = twisted polynomial ring

Let $\alpha : R \circlearrowleft$ be an auto of the ring R (ring with 1, and $\alpha(1) = 1$.)

$R_\alpha[x] = \{ a_0 + a_1 x + a_2 x^2 + \cdots$ polynomials $\}$

if $a \in R$, $ax = x\alpha(a)$ ← multipl. conv. $ax = x\alpha^{-1}(a)$

$ax^{-1} = x^{-1}\alpha^{-1}(a)$

$(a_0 + a_1 x + \cdots)\cdot(b_0 + b_1 x + \cdots)$

$\alpha^{-1}(a) \cdot x = x a$

$a_1 x \cdot b_2 x^2 = a_1 \alpha^{-1}(b_2) x^3$

Γ gp & $\alpha \in \mathrm{Aut}(\Gamma)$.

$\mathbb{Z}\Gamma \; \exists \alpha$

Suppose Γ is a poly-$\mathbb{Z}$ $\quad \Gamma \twoheadrightarrow \mathbb{Z} \quad \cdots < \Gamma_1 < \Gamma$

$\Gamma \twoheadrightarrow \Gamma/\Gamma_1 \cong \mathbb{Z}$

$$1 \longrightarrow \underset{\text{kernel}}{\pi} \longrightarrow \Gamma \twoheadrightarrow \underset{T}{\mathbb{Z}} \longrightarrow 1$$

$\Gamma = T \underset{\alpha}{\times} \pi$ semi-direct product.

$\mathbb{Z}\Gamma = R_\alpha[x, x^{-1}]$, where $R = \mathbb{Z}\pi$. & $\alpha : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ induced by

(Thm1) $K_1 R_\alpha[x] = K_1 R \oplus \mathrm{Nil}_\alpha(R)$

$\mathrm{Nil}_\alpha(R) = \{(F, \varphi)\}$ $\quad$ F: f.g. free R-module

$\varphi : F \circlearrowleft$ ~~R-homo~~ nilpt α-semi linear R homo

$\varphi(a+b) = \varphi(a) + \varphi(b)$

$\varphi(ra) = \alpha(r)\varphi(a)$.

(Thm2) If R is a regular ring. $\mathrm{Nil}_\alpha(R) = 0$ for any α.

(Thm3) $K_1 R_\alpha[x, x^{-1}] = \mathrm{Nil}_\alpha(R) \oplus \mathrm{Nil}_{\alpha^{-1}}(R) \oplus X$

where X fits into an exact seq

$$K_1R \xrightarrow{1-\alpha_*} K_1R \longrightarrow X \longrightarrow K_0R \xrightarrow{1-\alpha_*} K_0R$$

application.

(Farrell & Hsiang) Symposia Pure Math. 1970

cor. $Wh\,\pi_1 M^2 = 0$ if M is 2-mfd.

pf. (Thm 1, 2, 3) with Stallings " $Wh(\text{free gp}) = 0$ ",
Bass " $\tilde{K}_0(\mathbb{Z}[\text{free gp}]) = 0$ "

π_1(open 2-mfd) = free. ✓

S^2, -

For higher genus closed 2-mfd, $\chi(M^2) \leq 0$

$H^1(M:\mathbb{Z}) \neq 0$

$\parallel$ $\mathrm{Hom}(\pi_1 M, \mathbb{Z})$

$\pi_1 M = T \times_\alpha F_1$ free but not f.g.

inf. cycl.

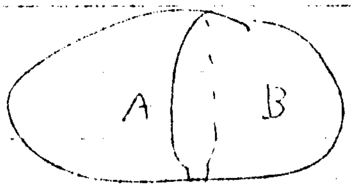
$$1 \to F \to \pi_1 M \longrightarrow T \longrightarrow 1$$

Apply Thm 3, $R = \mathbb{Z}F_1$.

Coherent ring.

* (Waldhausen) → goes up to 3-mfd

Mayer-Vietoris for homology sp's



$\Gamma = A *_C B$ — amalgamated free product

$$K_1\mathbb{Z}\Gamma = C(\mathbb{Z}C, \mathbb{Z}A, \mathbb{Z}B) \oplus X$$

From amalg. free product

$$K_1\mathbb{Z}C \longrightarrow \begin{matrix} K_1\mathbb{Z}A \\ \oplus \\ K_1\mathbb{Z}B \end{matrix} \longrightarrow X \longrightarrow K_0\mathbb{Z}C \longrightarrow \begin{matrix} K_0\mathbb{Z}C \\ \oplus \\ K_0\mathbb{Z}B \end{matrix} \longrightarrow$$

<u>Walhausen</u> Springer LN vol 342 ±5. category of gps ∋

Let $\Gamma = A \underset{C}{*} B$ nilpt type

$K_1\mathbb{Z}\Gamma = X \oplus \mathcal{C}(\mathbb{Z}C; \mathbb{Z}\overline{A}, \mathbb{Z}\overline{B})$

(∀ gp is constructed by $\underset{C}{*}$ and HNN constr.

1) [$\overline{A} = A - C$ (set)

There are 2 natural actions of C on $\overline{A}$ (multipl. on each sides)

$\mathbb{Z}\overline{A}$ (formal sum) is a C-bimodule.

a nil-object is $(P, Q; f, g)$, P and Q are f.g. free $\mathbb{Z}C$-module

$f: P \to Q \underset{\mathbb{Z}C}{\otimes} \mathbb{Z}\overline{B}$ C-module map.

$g: Q \to P \underset{\mathbb{Z}C}{\otimes} \mathbb{Z}\overline{A}$ 〃

(Require some nilpt conditions on f, g. To do that need filtration

∃ filtrations $0 = P_n \subset P_{n-1} \subset P_{n-2} \subset \cdots \subset P_1 = P$

$0 = Q_n \subset Q_{n-1} \subset Q_{n-2} \subset \cdots \subset Q_1 = Q$

such that $f(P_i) \subseteq Q_{i+1} \underset{\mathbb{Z}C}{\otimes} \mathbb{Z}\overline{B}$

$g(Q_i) \subseteq P_{i+1} \underset{\mathbb{Z}C}{\otimes} \mathbb{Z}\overline{A}$

In the category of $(P, Q; f, g)$, exact sequence

take iso-class. factor out by $P = Q + R$ if $0 \to (R-) \to (P-) \to$

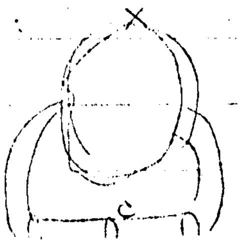
This is $\mathcal{C}(\mathbb{Z}C; \mathbb{Z}\overline{A}, \mathbb{Z}\overline{B})$

2) X fits M-V sequence

$$K_1\mathbb{Z}C \longrightarrow \begin{matrix} K_1\mathbb{Z}A \\ \oplus \\ K_1\mathbb{Z}B \end{matrix} \longrightarrow X \longrightarrow K_0\mathbb{Z}C \longrightarrow \begin{matrix} K_0\mathbb{Z}A \\ \oplus \\ K_0\mathbb{Z}B \end{matrix}$$

* $\mathcal{C} \otimes \mathbb{Q} = 0$?

* HNN :



$\pi_1(C \times I) \xrightarrow[\text{injects.}]{} \pi_1 X$

~~Conjecture~~ (Novikov)

Does $\Gamma \simeq \pi_1(M)$ asph. determine tangent bdle ?

$B\Gamma \longrightarrow BO(n)$ $B\Gamma$ & M has same hty type

$1 \to \mathbb{Z}^3 \longrightarrow \Gamma \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1$

$\Gamma \hookrightarrow GL_3(\mathbb{Z}) \hookrightarrow GL_3(\mathbb{R})$. determines a tangent bdle (in this case t

~~conjecture (Novikov)~~

Let $f: N^n \longrightarrow M^n$ he preserves pontryagin classes?

$$f^*(p_i(M)) = p_i(N) \in H^{4i}(N;\mathbb{R})$$

No, in general

$L_i(M) \in H^{4i}(M;\mathbb{R})$ (L_i is a polyn. of p_i's and vice versa with rational coeff.

$$p_0 = L_0 = 1 \in H^0$$
$$L_1 = \frac{1}{3}p_1$$
$$L_2 = \frac{1}{45}(7p_2 - p_1^2).$$

So preserv. of p_i is the same as preserv. of L_i. (Not t

But when $n=4r$, L_n is a hty invariant :

$$f^*(L_r(M)) = L_r(N) \quad \text{if} \quad f: N \longrightarrow M \text{ hty eqv.}$$

$L_r(M) \in H^{4r}(M:\mathbb{R}) = \mathbb{R}$

$\langle L_r(M), [M] \rangle \in \mathbb{R}.$

$$H^{2r}(M:\mathbb{R}) \otimes H^{2r}(M:\mathbb{R}) \xrightarrow{\cup} H^{4r}(M:\mathbb{R}) = \mathbb{R}$$

non-sing. sym. bil. form signature

<u>conjecture</u> (Novikov)

Given $\forall$ gp Γ, $\forall$ cohomology class $\alpha \in H^i(\Gamma; \mathbb{R})$ and
a map $M^n \xrightarrow[\varphi]{} B\Gamma$ (ie homo of ~~fundamental gp~~ $\pi_1 M \to \Gamma$)
then $L_\ell(M) \cup \varphi^*(\alpha)$ is a hty invariant provided $4\ell + i = di$

ie, given $f: N^n \to M^n$ a hty equiv.

$$f^*\left(L_\ell(M) \cup \varphi^*(\alpha)\right) = L_\ell(N) \cup f^*\varphi^*(\alpha).$$

<u>Thm</u> A solution due to <u>Miscenko</u>: redundant ?
Novikov's conjecture is true for ($\forall$ f.g) discrete subgp of $GL_n(\mathbb{C})$ (n=
<u>Pf</u> uses infinite dimensional bundle.. (Lusztig's thesis)

<u>Coro</u> If Γ is a discrete subgp of $GL_n(\mathbb{C})$ and $\Gamma = \pi_1(M)$ where M
a <u>closed</u> asph mfd, then up to stable homeomorphism, the
are only finite number of equivalence classes of closed as
mfds with Γ as π_1.

[π_1(closed —) is f.g.]

<u>def</u> N & M are <u>stably homeo</u> — $N \times \mathbb{R}^n$ is homeo to $M \times \mathbb{R}^n$ fo
sufficiently large n. [n = 3 enough ?]

ie, $N \times \mathbb{R}^5 \cong M \times \mathbb{R}^5 \Rightarrow N \times \mathbb{R}^3 \cong M \times \mathbb{R}^3$.

$\mathbb{R}^n \longrightarrow \mathbb{R}^n$ $\quad A \in O(n)$ is called a rigid motion

$x \longmapsto Ax + v$

$$\mathrm{Rigid}(n) = \mathbb{R}^n \rtimes O(n).$$

Γ is crystallographic if Γ is a discrete cocompact (= uniform) subgroup of $\mathrm{Rigid}(n)$.

cocompact — $\mathrm{Rigid}(n)/\Gamma$ is compact.

Thm A torsion-free f.g. virtually abelian group Γ is crystallog

Pf. Let $\mathbb{Z}^n < \Gamma$ with $(\Gamma;\mathbb{Z}^n) < \infty$. Since $\bigcap_{\gamma} \gamma^{-1}\mathbb{Z}^n\gamma$ is normal w finite index, we may assume $\mathbb{Z}^n \lhd \Gamma$. Then

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

is exact and G is finite. This gives a map

$$G \longrightarrow \mathrm{Aut}\,\mathbb{Z}^n = GL_n(\mathbb{Z}) \subset GL_n(\mathbb{R})$$

$\uparrow$

(by conjugation $z^g = \gamma^{-1} z \gamma$ for $\gamma \mapsto g$.
Well-defined since $\mathbb{Z}^n$ is abelian.)

This gives a representation$^{\alpha}$ of G on $GL_n(\mathbb{R})$

$$\alpha : G \longrightarrow \underset{\mathrm{Aut}(\mathbb{R}^n)}{\underset{\parallel}{GL_n(\mathbb{R})}} \qquad \text{(may not effective).}$$

Form $\mathbb{R}^n \rtimes_{\alpha} G$ and embed Γ into $\mathbb{R}^n \rtimes G$ by

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{R}^n \arrow[r] & \mathbb{R}^n \rtimes G \arrow[r] & G \arrow[r] & 1 \arrow[r, leftrightarrow] & 0 \in H^2(G;\mathbb{R}^n \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[u, hook, "i"] & \Gamma \arrow[r] \arrow[u, dashed, "f"'] & G \arrow[r] \arrow[u, "id"'] & 1 \arrow[r, leftrightarrow] & \theta \in H^2(G;\mathbb{Z}^n \arrow[u, "i_*"']
\end{tikzcd}$$

The existence of f corresponds to $i_*(\theta) = 0$. But since $\mathbb{R}$ is a fie

(divisible) and $|G| < \omega$, $H^2(G;\mathbb{R}^n)=0$. so that $i_*(\theta)=0$. :

Thus f exists. Then f is mono.

There exists an inner product on $\mathbb{R}^n$ such that G preserv

it. i.e., G acts on $\mathbb{R}^n$ as isometry.

(Given any inner product on $\mathbb{R}^n$, $\langle\,;\,\rangle$, define

$$[x,y] = \frac{1}{|G|}\sum_{g\in G}\langle gx, gy\rangle \quad \text{average}.$$

Get a representation $G \longrightarrow O(n)$.

$$\begin{tikzcd}
 & & \mathrm{Rigid}(n) \arrow[d, equal] & & \\
 & & \mathbb{R}^n \rtimes O(n) & & \\
1 \arrow[r] & R^n \arrow[r] \arrow[ur, "\text{1-1}"] & \mathbb{R}^n \rtimes G \arrow[r] \arrow[u, "\leftarrow \text{need not be mono}"'] & G \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[u] & \Gamma \arrow[r] \arrow[u, "f"'] & G \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

$\Gamma \hookrightarrow R^n \rtimes O(n)$ discrete, cocompact

Biebebach Theorems

B-Thm 1 (converse of above Thm)

a crystallographic group Γ is f.g. virtually abelian

Hard part will be:

$$[\Gamma : \Gamma \cap \text{Translation gp}] < \infty$$

B-Thm 2 If Γ, Γ' are 2 crystallographic groups which are isomorphic $f : \Gamma \xrightarrow{\cong} \Gamma'$ then f is the restriction of conjugation by an affine motion. [Rigidity Theorem]

$Rigid(n) \dot{<} Affine(n) = \mathbb{R}^n \rtimes ($

i.e, $\exists\ \alpha \in Affine(n)$ such that $f(\gamma) = \alpha^{-1}\gamma\alpha$ for all $\gamma \in \Gamma$.

✓ Pf of B-Thm 2 (assuming B-1) Want to find $\alpha(x) = \overset{M}{\cancel{A}}x + v$, $\overset{M}{\cancel{A}} \in GL_n(R)$, $v \in R^n$.

let $\Gamma \cap \mathbb{R}^n = A \simeq \mathbb{Z}^n$

$\Gamma' \cap \mathbb{R}^n = A' \simeq \mathbb{Z}^n$

Since $f : \Gamma \xrightarrow{\cong} \Gamma'$, $f|A : \underset{\mathbb{Z}^n}{\overset{A}{\wr\wr}} \xrightarrow{\cong} \underset{\mathbb{Z}^n}{\overset{A'}{\wr\wr}}$ so f determines an elt of $GL_n(\mathbb{Z}^n)$

and hence an elt of $GL_n(\mathbb{R}) \ni M$, $M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

let $\Gamma \xrightarrow{M^{-1}} \Gamma \xrightarrow{i} Affine(n) \xrightarrow{M} Affine(n)$, with h the composite $\Gamma \to Affine(n)$

h is id. on first coordinate

$g = j \circ f$.

$$\begin{tikzcd}
 & 1 \arrow[r] & \mathbb{Z}^n \arrow[r] & \Gamma' \arrow[r] \arrow[ddl, "j"] & G' \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[ur, "\bar{f}"] \arrow[d, dashed] & \Gamma \arrow[r] \arrow[ur, "f"] \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] & G \arrow[r] \arrow[d, dashed] & 1 & \\
1 \arrow[r] & \mathbb{R}^n \arrow[r] & Affine(n) \arrow[r] & GL_n(\mathbb{R}) \arrow[r] & 1 &
\end{tikzcd}$$

Define $\varphi : G \longrightarrow \mathbb{R}^n$ by

$$\varphi(\bar{\gamma}) = h^{-1}(\gamma)\, g(\gamma)$$

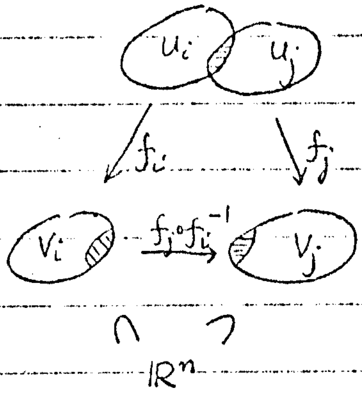
Then φ is a crossed homo $[\varphi(ab) = \varphi(a) + a\varphi(b)]$

Want to find $v \in \mathbb{R}^n$ so that

$$\alpha(x) = Mx + v.$$

$[\varphi] \in H^1(G; \mathbb{R}^n) = 0$. So φ is a principal crossed homo so $\exists\ v \in \mathbb{R}^n$ such that $\varphi(\bar{\gamma}) = \bar{\gamma}v - v$.

A closed mfd M^n is Riemannian flat if $\exists$ open cover $\{U$
together with homeo $f_i : U_i \to V_i \subset \mathbb{R}^n$ such that each $f_j \circ f_i^{-1}$
restriction of a rigid motion.



If M^n is flat riem. mfd, then M is aspherical, $\tilde{M} = \mathbb{R}^n$ and
$\pi_1 M$ is a discrete cocompact torsion-free subgroup of Rigid(n)
i.e. $\pi_1 M$ is a torsion-free crystallographic group.

Conversely, let Γ be a torsion-free crystallographic grou
Then we can construct a riem. flat mfd M with $\pi_1 M = \Gamma$.
Since Rigid(n) acts on $\mathbb{R}^n$, so does ~~$\mathbb{R}^n$~~ Γ, and $\mathbb{R}^n/\Gamma = M$

Γ : cocompact $\Rightarrow$ $\mathbb{R}^n/\Gamma$ is compact
torsion free, discrete $\Rightarrow$ mfd
chart from universal cover $\Rightarrow$ flat.

A ~~diffeomorphism~~ differen~~table~~tiable map $f : M \circlearrowleft$ is expanding if $|df(x)| > |x|$ for

(Expanding map does not preserve angle)

Thm (Epstein and Shub) Roughly, any compact flat Riem mfd
supports an expanding endo.

M supports $f: M \circlearrowleft$ —

$$\begin{tikzcd}
 & & \mathrm{Rigid}(n) \arrow[d, phantom, "\cup"] & & \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d, "\text{multpl by } s"'] & \pi_1 M \arrow[r] \arrow[d, "f_\#"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] & \pi_1 M \arrow[r] & G \arrow[r] & 1
\end{tikzcd} \qquad (*)$$

for $s \equiv 1 \mod |G|$.

Thm (Franks) If M^n supports an expanding endo, then M is asphericai and $\pi_1 M$ has polynomial growth.

A f.g. group Γ has polynomial growth: For generators γ_1, γ_2, ..., γ_m of Γ, define a function $\sigma: \mathbb{Z} \to \mathbb{Z}$ by

$\sigma(n)$ = # of group elts of Γ expressible by words in $\gamma_1 \cdots \gamma_m$ of length n or less.

Then Γ has polynomial growth if $\sigma(n)$ is dominated by a polynomial function in n.

Question Does Γ have polynomial growth, then Γ is virtually nilpotent (solved).

✓ Pf of Epstein-Schub.

Let $s \equiv 1 \mod |G|$. Filling $f_\#$ in (*) is

$$\begin{tikzcd}
\theta \in H^2(G;\mathbb{Z}^n) \arrow[d, mapsto] \arrow[d, shift left=6, "s_* = \text{multpl. by } s"] \\
\theta \in H^2(G;\mathbb{Z}^n)
\end{tikzcd}$$

$s\theta = \theta$ [ord $H^2(G;\mathbb{Z}^n)$

So we get $\varphi = f_\#$

$\varphi(\pi_1 M) \subsetneq \pi_1 M \leq \mathrm{Rigid}(n)$

$\exists\ a \in \text{Affine}(n)$ $\quad a(x) = Ax + v \quad A \in GL_n(\mathbb{R}),\ v \in \mathbb{R}^n$

such that $\ a\gamma a^{-1} = \varphi(\gamma)$. for $\ \forall \gamma \in \pi_1 M$.

<u>claim</u> $A = \lambda I$.

Now

$$\begin{tikzcd}\mathbb{R}^n/\pi_1 M \arrow[r] \arrow[rr, bend right, "f"'] & \mathbb{R}^n/\varphi(\pi_1 M) \arrow[r, "p"] & \mathbb{R}^n/\pi_1 M\end{tikzcd}$$

covering map.

We will prove

(Auslander-Kuranishi) $\forall$ finite group is a holonomy of <u>some</u> torsion-free crystallographic gp. (<u>Charlap Anna</u>

(Bieberbach) $\quad 1 \to R \to F \to G \to 1 \quad$ free resol. of G.

$\Rightarrow\ 0 \to R/[R,R] \to F/[R,R] \to G \to 1 \qquad R/[R,R] \cong \mathbb{Z}^n,\ F/[R,R]$ torsion-fr crystallogr

For a fixed integer n, $\exists$ only a finite # of $\cong$ classes of crystallographic groups in Rigid(n).

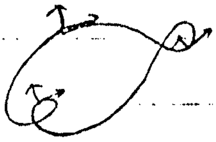
[Given G, $\quad G \to \text{Aut}\,\mathbb{Z}^n$ is finite. Now work in $H^2(G;$

<u>Thm</u> If G is $\forall$ finite group, then $\exists$ a flat mfd with holonomy G i.e., $\exists$ a torsion-free group Γ and an extension

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

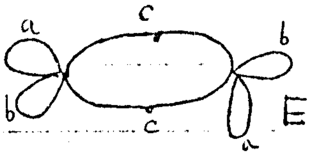
such that the holonomy representation is faithful $\left[\ G \xrightarrow{\varphi} \overset{GL_n(\mathbb{Z}}{\overset{\parallel}{\text{Aut}\,\mathbb{Z}}} \text{ is 1-1 } \right]$

(holonomy in differential geometry



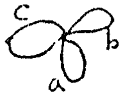
__lemma__ a subgroup S of a free group F is free with

$$rk(S)-1 = (F:S)\,(rk(F)-1).$$



$E = (F:S)$-sheeted covering.

$\downarrow p$



$X = \bigvee_{rk(F)} S^1$

$$\begin{tikzcd} \pi_1 E = S \arrow[d, hook, "\text{1-1}"'] \\ \pi_1 X = F \end{tikzcd}$$

$$\left.\begin{array}{l} H_0X = \mathbb{Z} \\ H_1X = (\pi_1 X)^{ab} = \overbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}^{rk(F)} \\ H_2X = 0 \end{array}\right) \quad \therefore \chi(X) = 1 - rk(F).$$

$$\left.\begin{array}{l} H_0E = \mathbb{Z} \\ H_1E = \cancel{\pi}(\pi_1 E)^{ab} = S^{ab} = \overbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}^{rk(S)} \\ H_2E = 0 \end{array}\right| \quad \therefore \chi(E) = 1 - rk(S)$$

Since p is $(F:S)$-sheeted, $rk(S)-1 = (F:S)\cdot(rk(F)-1)$

$$\left[\begin{array}{l} \text{If } F = \langle a,b,c\rangle, \text{ then } \quad S = \langle a, b, cac^{-1}, cbc^{-1}, c^2\rangle \\ \qquad 5-1 = 2\cdot(3-1). \end{array}\right]$$

__Pf of Thm__ let

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\varphi} G \longrightarrow 1$$

be a free presentation. Then

$$\begin{tikzcd} 1 \arrow[r] & R/[R,R] \arrow[r] \arrow[d, equal] & F/[R,R] \arrow[r] & G \arrow[r] & 1 \\ & \mathbb{Z}^n & & & \end{tikzcd}$$

We may assume that G is cyclic of prime order. Since G is abelian, $[R,R] \subset [F,F] \subset R \subset F$

$$\left(\begin{array}{l} [x,y] \to [\varphi x, \varphi y] = 0 \in G. \\ \therefore [x,y] \in \operatorname{Ker} \varphi = R. \end{array} \right.$$

Since $R/[R,R]$ is torsion free, so is $[F,F]/[R,R]$ $(\Leftarrow R \supset [F,F]$,

Now from the exact sequence

$$1 \longrightarrow [F,F]/[R,R] \longrightarrow F/[R,R] \longrightarrow F/[F,F] \longrightarrow 1$$

the middle term is torsion-free.

Now we want to show the holonomy repr is faithful. Assume the repr. is not faithful. Then $F/[R,R]$ is abelian.

($\exists\, g \in G$ such that $[r]^g = [r]$ for all $r \in R$. Since G is cycl of prime order, the action of G on $R/[R,R]$ is trivial. $\therefore [F,R] \subseteq$ [
But $[F,F] \subseteq [F,R]$ $\quad ([f,g] = [fg, e])$.

$$\therefore [F,F] \subset [R,R]$$

Thus, $[F,F] = [R,R]$. Then

$$1 \longrightarrow R^{ab} \longrightarrow F^{ab} \longrightarrow G \longrightarrow 1$$

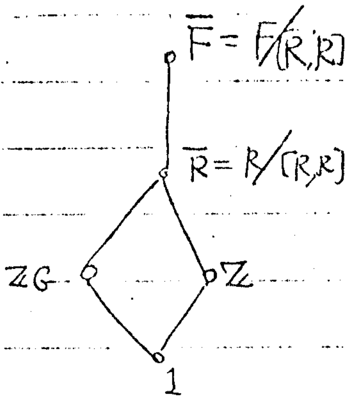
with $\operatorname{rk} R > \operatorname{rk} F$, which is impossible. $\Box$

* open problems #1

** When $G = T_2$

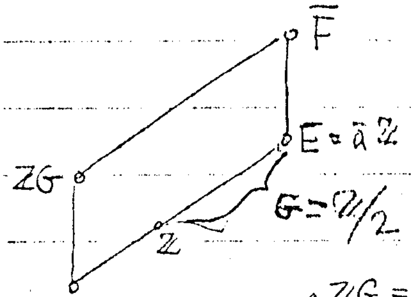
① $1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \rightarrow 1$

Let $M = \mathbb{R}^n/\Gamma$ [illegible] $\widetilde{M}$ [illegible] $\mathbb{Z}^n$



$\bar{F}$ splits over $\mathbb{Z}G$ ($\Leftarrow$ G finite & $H^2(G, \overset{\mathbb{Z}G}{\text{proj}}) = 0$)

$1 \to \bar{R} \to \bar{F} \to G \to 1$



$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}G \arrow[r] & \bar{F} \arrow[r] & E \arrow[r] \arrow[l, bend left] & 1 \end{tikzcd}$$

and E is generated by $\bar{a}$ and is ∞-cyclic

$\mathbb{Z}G$ = free ab. gp of rk 2 with basis $1, a$
σ, σ^a

$\therefore \bar{F} = F/R' = \overset{\mathbb{Z}G}{\overset{\|}{A}} \rtimes \underset{E}{\underset{\|}{T}}$

T operates on A by $t: \begin{matrix} 1 \mapsto a \\ a \mapsto 1 \end{matrix}$

$T^2 = (t^2)$ operates trivially on A

$$\begin{tikzcd} 1 \arrow[r] & A \times T^2 \arrow[r, phantom, "\subset"] \arrow[d, equal] & A \rtimes T \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, equal] & 1 \\ & \bar{R} & \bar{F} & T/T_2 & \end{tikzcd}$$

$\mathbb{Z}(T_2)$ torsion free

$\mathbb{Z} \oplus \mathbb{Z} \rtimes \mathbb{Z}$

$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus$

Algebraic Vanishing Criteria

Let $\Gamma \xrightarrow{\varphi} F \longrightarrow 1$ with $F =$ finite group.

Suppose $x \in Wh\,\Gamma$ or $\widetilde{K}_0(\mathbb{Z}\Gamma)$. Then

$x = 0$ iff $\sigma_S^*(x) = 0$ for all $S < F$ hyper-elementary subgroups

where σ_S is the transfer ~~induced~~ of $\sigma_S : \underset{\overset{\parallel}{\varphi^{-1}S}}{\Gamma_S} \hookrightarrow \Gamma$

S is <u>hyper-elementary</u> if $\exists$ exact seq. $1 \longrightarrow \underset{\text{cyclic.}}{C} \longrightarrow S \longrightarrow \underset{\text{p-group}}{P} \longrightarrow 1$

<u>eg.</u> $S = A_5$ alternating gp.

✓ [$\Rightarrow$ This splits]

$S \simeq C \times P$

(Swan) Let F be a finite group

$G(F) =$ Grothendieck construction on $\mathbb{Z}F$-lattices

($\mathbb{Z}F$-module $\ni$ f.g free as $\mathbb{Z}$

commutative <u>ring</u> associated to F

$[N_0] + [N_1] = [N_0 \oplus N_1]$ $(\Leftarrow\ 0 \to N_0 \longrightarrow N_0 \oplus N_1 \longrightarrow N_1 \longrightarrow 0)$

$[N_0] \cdot [N_1] \overset{\text{df}}{=} [N_0 \underset{\mathbb{Z}}{\otimes} N_1]$ with F-action on $N_0 \underset{\mathbb{Z}}{\otimes} N_1$ $(a \otimes b)g = ag \otimes bg$.

$1 = [\mathbb{Z}]$ where $\mathbb{Z}$ is trivial F-module.

If N_0, N_1 are F-lattices, then so is $N_0 \underset{\mathbb{Z}}{\otimes} N_1$.

Clearly "$\cdot$" gives a ring structure on $G(F)$

<u>claim</u> $K_i(\mathbb{Z}\Gamma)$ is a $G(F)$-module $(i = 0, 1)$

<u>For K_0</u> $K_0\mathbb{Z}\Gamma \times G(F) \longrightarrow K_0\mathbb{Z}\Gamma$.

$[P] \cdot [N] \overset{\text{def}}{=} [P \underset{\mathbb{Z}}{\otimes} N]$ with $(p \otimes x)\gamma = p\gamma \otimes x\varphi(\gamma)$.

$P \underset{\mathbb{Z}}{\otimes} N$: f.g prj ? Need to check only for free.

$$\mathbb{Z}\Gamma \underset{\mathbb{Z}}{\otimes} N \underset{\Gamma}{\overset{\cong}{\longleftrightarrow}} \mathbb{Z}\Gamma \underset{\mathbb{Z}}{\otimes} N_0, \quad N_0 = N \text{ with trivial } F\text{-action}$$

$$\gamma \otimes x \longrightarrow \gamma \otimes x\gamma^{-1}$$

For K_1,

$[(P,\alpha)] \in K_1\mathbb{Z}\Gamma$. $\quad P =$ f.g. free Γ-module

$\alpha : P \circlearrowleft$ Γ-auto.

$$[(P,\alpha)]\cdot[N] = [P\otimes N, \alpha\otimes \mathrm{id}].$$

Thm 1 (Frobenius Reciprocity Thm)

If $x \in K_i\mathbb{Z}\Gamma$ $(i=0,1)$ and $\gamma \in G(S)$ where $\Gamma \xrightarrow{\varphi} F =$ finite, $S \underset{\sigma}{\subsetneq} F$, $\Gamma_S = \varphi^{-1}S \xhookrightarrow{\sigma_S} \Gamma$. Then

$$\sigma_*(\gamma\,\sigma^*(x)) = \sigma_*(\gamma)\,x.$$

$S \overset{\sigma}{\subsetneq} F$ set

$$\begin{tikzcd}
{[F\times Y, Z]} \arrow[r, "\sigma^*"] \arrow[d] & {[S\times Y, Z]} \arrow[d] \\
{[F, Z^Y]} \arrow[r] & {[S, Z^Y]}
\end{tikzcd}$$

$$\begin{tikzcd}
K_i\mathbb{Z}\Gamma_S \arrow[d, "G(S)\ni\gamma"'] & K_i\mathbb{Z}\Gamma \arrow[l, "\sigma^*"'] \arrow[d, "\sigma_*(\gamma)\in G(F)"] \\
K_i\mathbb{Z}\Gamma_S \arrow[r, "\sigma_*"'] & K_i\mathbb{Z}\Gamma
\end{tikzcd}$$

$$S \xhookrightarrow{\sigma} F$$

$$\begin{tikzcd}
H^n(X) \arrow[d, "\cap\gamma"] & H^n(Y) \arrow[l, "f^*"'] \arrow[d, "\cap f_*(\gamma)\in H_m(Y)"] \\
H_{m-n}(X) \arrow[r, "f_*"'] & H_{m-n}(Y)
\end{tikzcd}$$

$$X \xrightarrow{f} Y$$

Thm 2 (Swan)

$$G(F) \ni 1 = \gamma_1 + \gamma_2 + \cdots + \gamma_n$$

? ✓ where γ_i is induced from $G(S_i)$, $\quad \gamma_i = \overset{\sigma_*(s_i)}{(\sigma_i)_*(1)}$, $\quad \sigma_i : S_i \hookrightarrow F$

$S_i =$ hyperelementary subgroup of F

Let $G(F,\mathbb{C})$ = set of finite dim'l complex vector space with F-act

$\mathbb{C}^X$ = the commutative ring of all functions $X \to \mathbb{C}$ without topology.

$\hat{F}$ = set of conjugacy classes of elements in F.

If $[\alpha] \in G(F,\mathbb{C})$, then α induces a representation $F \to GL_n(\mathbb{C})$.

$$F \xrightarrow{\alpha} GL_n(\mathbb{C}) \xrightarrow{\text{trace}} \mathbb{C}$$

Since $\operatorname{tr}(A^{-1}BA) = \operatorname{tr} B$, the above induces $\hat{F} \to \mathbb{C}$. Get

$$G(F,\mathbb{C}) \longrightarrow \mathbb{C}^{\hat{F}}.$$

For sets,

$$\begin{tikzcd} Y \arrow[r, hook, "\sigma"'] & X \end{tikzcd}$$

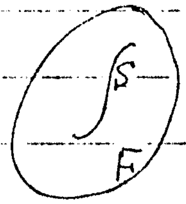
$$\Downarrow$$

$$\begin{tikzcd} \mathbb{C}^Y & \mathbb{C}^X \arrow[l, "\sigma^*"'] \end{tikzcd} \qquad \text{restriction}$$

For abelian groups,

$$\begin{tikzcd} S \arrow[r, hook, "\sigma"'] & F \end{tikzcd}$$

$$\Downarrow$$



$\begin{tikzcd} \mathbb{C}^S & \mathbb{C}^F \arrow[l, "\sigma^*"] \end{tikzcd}$ is defined by

$$\sigma^*(f)(x) = \begin{cases} 0, & x \notin S \\ [F:S]\, f(x), & x \in S \end{cases}$$

If F is abelian,

$$G(F,\mathbb{C}) \otimes \mathbb{C} \simeq \mathbb{C}^F \quad \text{algebra over } \mathbb{C}.$$

$$\begin{array}{c} \parallel \\ \mathbb{C}^{S_1} \oplus \cdots \oplus \mathbb{C}^{S_n} \end{array} \qquad S_i \text{ cyclic subgp of }$$

$$1 = r_1 + r_2 + \cdots + r_n \longleftrightarrow 1 = 1 + 1 + \cdots + 1.$$

Pf. of algebraic vanishing Criteria.

(i) $x \in \widetilde{K}_0\mathbb{Z}\Gamma$. Let $\bar{x} \in K_0\mathbb{Z}\Gamma$ such that $\bar{x} \mapsto x$.

$$\bar{x} = 1\bar{x} = r_1\bar{x} + r_2\bar{x} + \cdots + \gamma_n\bar{x}$$

Show that each $r_i\bar{x} \mapsto 0 \in \widetilde{K}_0\mathbb{Z}\Gamma$. $[\Rightarrow \bar{x} \mapsto 0$ so that $x=0]$

For $r_i = r$, (work only with r_i) let $S \underset{\sigma}{\hookrightarrow} F$, S = hyperelementa

$$\gamma = \sigma_*(s), \quad s \in G(S).$$

$$\sigma_*(s)\,\bar{x} = \sigma_*(s\sigma^*(\bar{x}))$$

$\sigma^*(x) = 0$ in $\widetilde{K}_0\mathbb{Z}\Gamma_S$ by hypothesis $\therefore \sigma^*(\bar{x}) = [P]$, P is free $\mathbb{Z}\Gamma_S$-module.

$\therefore s\sigma^*(\bar{x})$ is represented by a free Γ_S-module

Since
(induction map σ_* preserves "free") $\sigma_*(s\sigma^*(\bar{x}))$ is repr. by free.

$$\therefore r\bar{x} = \sigma_*(s)\cdot\bar{x}$$
$$= \sigma_*(s\sigma^*(\bar{x})) = 0.$$

(ii) Must show $\pm\Gamma$ is invariant under transfer, induction, $G(F)$ action.

Lemma $G(F)(\pm\Gamma) \subset \pm\Gamma$.

let $a = [M] \in G(F)$, M: F-module.

let $(\gamma) \in \pm\Gamma \subset K_1\mathbb{Z}\Gamma$. 1×1 matrix. Then γ induces an iso

$$\begin{tikzcd} \mathbb{Z}\Gamma \arrow[r, "\gamma"] & \mathbb{Z}\Gamma \\ x \arrow[r, mapsto] & \gamma x \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}\Gamma \otimes M \arrow[r, "\cong"] & \mathbb{Z}\Gamma \otimes M_0 \\ \gamma \otimes m & \gamma \otimes m\varphi(\gamma)^{-1} \end{tikzcd}$$

$$\begin{tikzcd} \alpha[(\pm\gamma)] : \mathbb{Z}\Gamma \otimes M_0 \arrow[r] & \mathbb{Z}\Gamma \otimes M_0 \\ \gamma\otimes m \arrow[r, mapsto] & \gamma x \otimes m\varphi(x)^{-1} \end{tikzcd}$$

is the composite of $x \otimes m \mapsto \gamma x \otimes m$

$x \otimes m \mapsto x \otimes mg$.

Theorem If $\Gamma < \mathrm{Rigid}(n)$ is discrete cocompact, then
$[\Gamma : \Gamma \cap \mathrm{Transl}(n)] < \infty$

Lemma $\boxed{\text{The identity component } T \text{ of } \overline{\varphi(\Gamma)}}$ is abelian, whe
$\varphi : \mathrm{Rigid}(n) \xrightarrow{\mathrm{proj}} O(n)$

Pf of Thm using lemma

(i) Let $\boxed{F = \varphi(\Gamma) \cap T}$ Then $[\varphi(\Gamma) : F] < \infty$ and F is dens

∵ $\overline{\varphi(\Gamma)} \subset O(n)$ so $\overline{\varphi(\Gamma)}$ is compact (Lie group). Since identity compone
of a compact Lie gp is open, $[\overline{\varphi(\Gamma)} : T] < \infty$.

$\therefore [\varphi(\Gamma) : F] = [\overline{\varphi(\Gamma)} \cap \varphi(\Gamma) : T \cap \varphi(\Gamma)] = [\overline{\varphi(\Gamma)} : T] < \infty$.

$\overline{F}_{\overline{\varphi(\Gamma)}} = \overline{T \cap \varphi(\Gamma)} \underset{\uparrow}{=} \overline{T \cap \overline{\varphi(\Gamma)}} = \overline{T}_{\overline{\varphi(\Gamma)}}$ so F is dense in T. ✓
T is open in $\overline{\varphi(\Gamma)}$

(ii) T is torus. (p.26 Bredon)

∵ T is connected abelian (⇐ lemma) Lie group, so it is $\cong T^k \times$
Since T is component, it is closed (in this case also open) $< \overline{\varphi(\Gamma)} < O$
and hence compact. ✓

(iii) If $T = \{1\}$, then we are done.

∵ Suppose $T = \{1\}$. Then $\varphi(F$ $F = \{1\}$ and $\varphi(\Gamma)$ is finite from (
Then $[\Gamma : \Gamma \cap \mathrm{Transl}(n)] = [\Gamma : \Gamma \cap \varphi^{-1}(0)] \leq |\varphi(\Gamma)| < \infty$ ✓

Def

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{R}^n \arrow[r] & O(n) \ltimes \mathbb{R}^n \arrow[r, "\varphi"] & O(n) \arrow[r] \arrow[l, bend right, dashed] & 1 \\
 & v \arrow[r, mapsto] & (I, v) \quad (A, w) \arrow[r, mapsto] & A & \\
 & \cup & \cup & \cup & \\
1 \arrow[r] & \boxed{\Delta \equiv \Gamma \cap \varphi^{-1}(I)} \arrow[r] & \boxed{\Gamma^* \equiv \Gamma \cap \varphi^{-1}(F)} \arrow[r] & F \arrow[r] & 1 \\
 & = \Gamma \cap \mathrm{Transl}(n) & = \Gamma \cap \varphi^{-1}(T) & &
\end{tikzcd}$$

exact & commutes. not split in general

algebraic → Let Need not semi-direct product.

$$1 \to A \to B \to C \to 1 \quad (\text{exact})$$

If A is <u>abelian</u>, then C acts on A by <u>conjugation</u>
right.

i.e, $\exists\ C \xrightarrow[\text{group homo.}]{} \operatorname{Aut} A$ left by (conjugation)

∵ Define $c \cdot a = bab^{-1}$, where $b \to c$.

(Well-def'd) If $b \to 0 \in C$. Then $b \in A$ (by exactness)

$bab^{-1} = a$ for all a. (A is abelian)

(action) $(c_1 c_2) \cdot a = c_1 c_2\, a (c_1 c_2)^{-1} = c_1 (c_2 a c_2^{-1}) c_1^{-1} = c_1 \cdot (c_2 \cdot a)$.

In the exact sequence

$$1 \longrightarrow \mathbb{R}^n \longrightarrow \underset{}{\overset{O(n) \ltimes \mathbb{R}^n \,/\!/}{\mathrm{Rigid}(n)}} \longrightarrow O(n) \longrightarrow 1.$$

$O(n)$ acts on $\mathbb{R}^n$ naturally. i.e,

$$A \cdot v = (A, 0) \cdot (I, v) = (I, A(v)) = A(v) \quad Av.$$

$$\overset{/\!/}{} (A,0)(I,v)(A^{-1},0) \overset{/\!/}{}$$

So, we may consider $\mathbb{R}^n \ni v$ as <u>point in $\mathbb{R}^n$</u> or <u>$(I, v) \in$ Rigid</u>.

(iv) Let $\boxed{W = \operatorname{Fix}(T, \mathbb{R}^n) = (\mathbb{R}^n)^T}$

If $W = \mathbb{R}^n$, Then $T = \{1\}$, we are done

∵ T acts effectively on $\mathbb{R}^n$ since so does $O(n)$. ✓

* Same is true for $1 \to \Delta \longrightarrow \Gamma^* \xrightarrow{\varphi} F$

$A \cdot v = (A, w) \cdot (I, v)$ ((A, 0) may not be in Γ)

$= (A, w)(I, v)(A^{-1}, -A^{-1}(w))$

$= (I, A(v))$

Show : <u>$W = \mathbb{R}^n$</u>

<u>Claim 1</u> $\Delta \leq W$. , $(I, \delta) \in \Gamma$

Let $\delta \in \Delta$. Then let $(A, w) \in \Gamma$.

$F(\delta) \subset \Delta$ $[A \in F \Rightarrow A \cdot \delta = (I, A(\delta)) = (A, w)(I, \delta)(A^{-1}, -A^{-1}(w)) \in \Gamma$

$T(\delta)$ connected. [consider as orbit, T is connected]

$F(\delta)$ is dense in $T(\delta)$. by (i).

Since $\Delta \subset \Gamma$, it is discrete. $F(\delta) =$ discrete, dense in $T(\delta)$ ⊂ ~~pt~~ connected — usual me

Since T is a torus, $\exists \,(\overset{\text{irrational pt.}}{B \subset T})$ such that $\{B^k \mid k \in \mathbb{Z}\}$ is dense

<u>claim 2</u> $\exists \,(\underset{\text{dense}}{A \in F}) \subset T$ such that $(\mathbb{R}^n)^A = \underset{\underset{(\mathbb{R}^n)^B}{\parallel}}{(\mathbb{R}^n)^T} = W$

$\therefore$ Let $\mathbb{R}^n = W \oplus W^\perp$ stable under $T \leftarrow B$

Let $B = \begin{pmatrix} I & 0 \\ 0 & \mathcal{B} \end{pmatrix}$. For any $A \in F$, $(\mathbb{R}^n)^A \supset (\mathbb{R}^n)^T = W$
$\quad\quad A \in T$

so $A = \begin{pmatrix} I & 0 \\ 0 & \mathcal{A} \end{pmatrix}$. Since $\mathcal{B}$ does not have eigen-value 1 (W is ma
set which is fixed under B) if we take $A \in F$ close to B, th
$\mathcal{A}$ does not have eigen-value 1. ✓

<u>claim 3</u> Pick a $\gamma = (A, a) \in \Gamma^* = \varphi^{-1}(F) \cap \Gamma$ with $\varphi(\gamma) = A$ in c
Then $\exists$ splitting ~~of~~ ψ of φ

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{R}^n \arrow[r] & O(n) \ltimes \mathbb{R}^n \arrow[r, "\varphi"] & O(n) \arrow[r] \arrow[l, dashed, bend left, "\psi"] & 1 \end{tikzcd}$$

such that translation part of γ is in W.

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{R}^n \arrow[r] \arrow[d, equal] & O(n) \times \mathbb{R}^n \arrow[r, shift left, "\varphi"] \arrow[d, "f"] & O(n) \arrow[r] \arrow[l, shift left] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{R}^n \arrow[r] & O(n) \ltimes \mathbb{R}^n \arrow[r, shift left, "\varphi"] & O(n) \arrow[r] \arrow[l, shift left, dashed, "\psi"] & 1 \end{tikzcd}$$

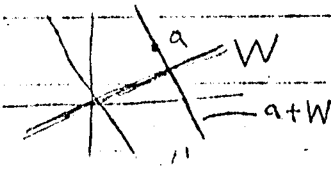
$(C, v) \mapsto (C, v + (C-I)b)$

$(C, (C-I)b) \longleftarrow\!\shortmid \; C$ (New coordinate)

Want to find $x = b$ such that
$$f(A.a)\,\phi = (A,\, a + (A-I)x), \quad a + (A-I)x \in W.$$

$(A-I)\,W^\perp = W^\perp$ [Let $x \in W^\perp$, $y \in W$. Then
$$\langle (A-I)x, y\rangle = \langle Ax - x, y\rangle = \langle Ax, y\rangle = \langle Ax, Ay\rangle = \langle x, y\rangle$$

So $a + (A-I)\,\mathbb{R}^n = a + W^\perp$, which meets W.



<u>Claim 4</u> If $\alpha \in \Gamma^*$, $\alpha(x) = Bx + b \Rightarrow b \in W$.

$\because$ Since $\mathbb{R}^n \rightarrowtail^{\chi} \mathrm{Rigid}(n) \longrightarrow O(n)$ is independent of splitting, we may suppose $\mathrm{Rigid}(n) = O(n) \ltimes \mathbb{R}^n$ with new coordinate $\rightarrow$

$a \in W$, $\gamma(x) = Ax + a$.

$\gamma\alpha\gamma^{-1}\alpha^{-1} \in \Delta$ $\qquad \gamma\alpha(x) = ABx + Ab + a$

$$\alpha^{-1}(x) = B^{-1}x - B^{-1}b.$$

$$\gamma^{-1}\alpha^{-1}(x) = A^{-1}B^{-1}x - A^{-1}B^{-1}b - a. \ (A \text{ fixes } a)$$

$$\therefore \gamma\alpha\gamma^{-1}\alpha^{-1}(x) = [A,B]x - [A,B]b - ABa + Ab + a$$

But $[A,B] = 1$ since T is abelian & A, B fixes a. So,

$$= x - b - a + Ab + a$$

$$= Ix + \underline{(I-A)(-b)}.$$

Thus, translation part of $[\gamma,\alpha]$ $\in \underset{\Downarrow}{W^{\perp}} \cap W$.

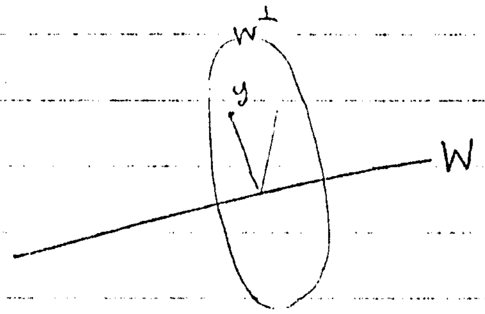
$$'' \ \chi^{-1}(Ix + (I-A)(-b)) = (I-A)(-b) \in \mathbb{R}^n$$

$\therefore (I-A)(-b) = 0 \quad \Rightarrow \quad -b \in W$.

$W \rtimes T$ — T acts on W trivially

$$\begin{tikzcd}
 & & W \rtimes T \arrow[d, equal] & & \\
0 \arrow[r] & W \arrow[r] & W \times T \arrow[r] & T \arrow[r] & 1 \\
 & \Delta \arrow[u, phantom, "\cup"] \arrow[r] & \Gamma^* \arrow[u, phantom, "\cup"] \arrow[r] & F \arrow[u, phantom, "\cup"] \arrow[r] & 1 \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

Since $(\Gamma : \Gamma^*) < \infty$ and Γ is cocompact, so $\mathbb{R}^n / \Gamma^*$ cpt.



suppose $\underset{W^{\perp}}{\overset{}{y}} = B\tilde{x} + b$ for some $Bx + b$ and $\tilde{x} \in \mathbb{R}^n$.

Let $\tilde{x} = \underset{W}{x} + \underset{W^{\perp}}{x^{\perp}}$

Then $y = Bx + Bx^{\perp} + b = x + b + Bx^{\perp}$

$\therefore y = Bx^{\perp}$ & $x = -b$

$\|y\| = \|x^{\perp}\|$. In otherword, if $\underset{W^{\perp}}{y} \sim \tilde{x}$, then $\|y\| = \|W^{\perp}$ cpnt

So if $W^{\perp} \neq 0$, $\mathbb{R}^n / \Gamma^*$ is not cpt
$\cup$ unbounded set.

$\cdot(\mathbb{R}^n)T - 1/(- \mathbb{R}^n$

<u>step 1</u> $\exists$ nbd U of $1 \in T$ such that : $\forall \overset{A}{\alpha}, \overset{B}{\gamma} \in U \subseteq O(n)$

Let $\underset{B_0}{\gamma_0} = \gamma$, $\underset{B_{n+1}}{\gamma_{n+1}} = [\alpha, \underset{B_n}{\gamma_n}]$. Then the sequence $\underset{B_n \to I}{\{\gamma_n\} \to 1}$

Recall : $\forall M \in \text{matrix}_{n\times n}(\mathbb{R})$, $\|M\| = \max\limits_{|v|=1} Mv$. Then

$\|MN\| \le \|M\|\,\|N\|$, $\|M+N\| \le \|M\| + \|N\|$, $\|MA\| = \|M\|\,\|A\| = \|M\|$ if $A \in$

We have $\exp : \text{Mat}_{n\times n}(\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ diffeo in suff. small nbd of

$$\begin{tikzcd} \text{Mat}_{n\times n}(\mathbb{R}) \arrow[r] & GL(n,\mathbb{R}) \\ 0 \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & I \arrow[u, phantom, "\cup"] \\ \text{show sym.} \to \mathfrak{o}(n) \arrow[r] & O(n) \end{tikzcd}$$

$$M \longrightarrow \exp(M) = I + M + \frac{M^2}{2!} + \cdots$$

Suppose $|A - I| < \frac{1}{4}$ & $|B - I| < \frac{1}{4}$. Let $\exp^{-1}A = a$, $\exp^{-1}B = \mathcal{B}$.

$$A = I + a + a_1 \quad\Big|\quad A^{-1} = I - a + a_2$$
$$B = I + \mathcal{B} + \mathcal{B}_1 \quad\Big|\quad B^{-1} = I - \mathcal{B} + \mathcal{B}_2$$

Then $\underset{|a_2| \le |a|^2}{|a_1| \le |a|^2}$ & $\underset{|\mathcal{B}_2| \le |\mathcal{B}|^2}{|\mathcal{B}_1| \le |\mathcal{B}|^2}$ (from $\underline{|A-1| < \frac{1}{4}}$ & $\underline{|B-1| < \frac{1}{4}}$)

Now show :

$$\boxed{|ABA^{-1}B^{-1} - I| < \frac{1}{4}|\mathcal{B}|}\ |B - I|$$

$$ABA^{-1} = I + A\mathcal{B}A^{-1} + A\mathcal{B}_1A^{-1}$$

$$\therefore (ABA^{-1})B^{-1} = (I + A\mathcal{B}A^{-1} + A\mathcal{B}_1A^{-1})\,(I - \mathcal{B} + \mathcal{B}_2)$$

$$= I + (A\mathcal{B}A^{-1} - \mathcal{B}) + \overset{R}{\underbrace{(A\mathcal{B}_1A^{-1} - A\mathcal{B}A^{-1}\mathcal{B} - \cdots)}_{6 \text{ terms}}} \text{ has } \mathcal{B}_1 \text{ or } \mathcal{B}^2,\ \mathcal{B}_2$$

ⓐ $|R| \le 6\,|\mathcal{B}|^2$.

ⓑ $(A\mathcal{B})A^{-1} = (\mathcal{B} + a\mathcal{B} + a_1\mathcal{B})\,(1 - a + a_2)$

$$= \mathcal{B} + \overset{S}{\underbrace{(\mathcal{B}a + \cdots}_{8 \text{ terms}}} \text{ has less than } |a\mathcal{B}|$$

$$|A\mathcal{B}A^{-1} - \mathcal{B}| \le 8\,|a\mathcal{B}|$$

$$\therefore |ABA^{-1}B^{-1} - I| \le |A\mathcal{B}A^{-1} - \mathcal{B}| + |R| \le 6|\mathcal{B}|^2 + 8|a\mathcal{B}| = \boxed{6|\mathcal{B}| + 8|a|}$$

$$\le \frac{1}{4}|\mathcal{B}|.\ |B - I|$$

$|a| < c\,|A - 1|$
$|\mathcal{B}| < c\,|B - 1|$?

$\frac{1}{56}$

So $|B_1 - I| < \frac{1}{4}|\mathcal{B}|\ \frac{1}{4}|B - 1| < \left(\frac{1}{4}\right)^2$ $\left(< \frac{1}{4} \right)$

$|B_2 - I| < \frac{1}{4}|B_1 - 1| < \left(\frac{1}{4}\right)^3$

⋮

$|B_n - I| < \left(\frac{1}{4}\right)^{n+1}$ $B_n \to I$

14.4

$|B - 1| = |\mathcal{B} + \mathcal{B}_1| \ge |\mathcal{B}| - |\mathcal{B}_1| \ge |\mathcal{B}| - |\mathcal{B}|^2$

$\ge \frac{1}{2}|\mathcal{B}|$ if $|\mathcal{B}| < \frac{1}{2}$)

exp is conti !

Step 2 $\exists$ integer n in the above step so that $\gamma_n = 1$

assume $\alpha, \gamma \in \varphi(\Gamma)$. [enough to show $\exists$ dense subset of T which is abelian]

Rigid (n) = Trans $(n) \supset O(n)$

Let $\hat{\alpha}, \hat{\gamma} \in \Gamma$ be such that $\varphi(\hat{\alpha}) = \alpha$ & $\varphi(\hat{\gamma}) = \gamma$.

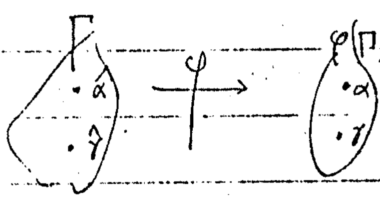
$$\hat{\alpha}(x) = \alpha(x) + a.$$

Define $\hat{\gamma}_0 = \gamma$, $\hat{\gamma}_i = [\hat{\alpha}, \hat{\gamma}_{i-1}]$ in Γ. ~~then~~ and let $\gamma_i = \varphi(\hat{\gamma}_i)$

Then $\gamma_i = [\alpha, \gamma_{i-1}]$. ~~$\hat{\gamma}_n(x) =$~~ α Let $\hat{\gamma}_i(x) = \gamma_i(x) + b_i$

Know (from step 1) : $\underline{\{\gamma_n\} \to I}$.

Want to show $\gamma_n = I$ for some n.



① Show $\hat{\gamma}_i \to 1$. $\boxed{b_i \to 0}$

$$\hat{\gamma}_n(x) = [\alpha, \gamma_{n-1}](x) = \alpha\gamma_{n-1}\alpha^{-1}\gamma_{n-1}^{-1}(x)$$

$$= \gamma_n(x) - \gamma_n(b_{n-1}) + \alpha(b_{n-1}) - \cancel{\alpha\gamma_{n-1}\alpha^{-1}(a) + a}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ " a since $a \in W$.

$$= \gamma_n(x) + \left(\alpha(b_{n-1}) - \gamma_n(b_{n-1})\right)$$

But $\|\alpha(b_{n-1}) - \gamma_n(b_{n-1})\| = \|(\alpha - \gamma_n)(b_{n-1})\| = \|(\alpha - \alpha\gamma_{n-1}\alpha^{-1}\gamma_{n-1}^{-1})(b_{n-1})\|$

$\leq \|\alpha\| \|(1 - \gamma_{n-1}\alpha^{-1}\gamma_{n-1}^{-1})(b_{n-1})\| \leq \|\alpha\| \|\gamma_{n-1}(1-\alpha^{-1})\gamma_{n-1}^{-1}\| \|b_{n-1}\| \leq \|\alpha\| \cdot \|1-\alpha^{-1}\| \cdot \|b_{n-1}\|$

$\underset{\substack{\uparrow \\ \alpha \in O(n)}}{=} \|(1-\alpha^{-1})\alpha\| \; |b_{n-1}| = \|\alpha - 1\| \; |b_{n-1}| < \frac{1}{4}|b_{n-1}\|$.

So $\hat{\gamma}_n(x) = \gamma_n(x) + b_n$ where $\gamma_n \to 1$ and $b_n \to 0$.

$\therefore$ $\hat{\gamma}_n \to 1$.

② Since Γ is discrete, $\hat{\gamma}_n = 1$ for som n.

③ $\therefore \gamma_n = 1$.

step 3 $\exists$ nbd U of $I \in O(n)$ such that if $A, B \in U$ and if $[A,[A,B]]=$
then $[A,B]=I$.

Consider $O(n) \subset O_n(\mathbb{C})$. Let

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

be eigen spaces of A. Suffices to show $BV_i = V_i$. But

$$C^n = BV_1 \oplus BV_2 \oplus \cdots \oplus BV_s$$

are eigen spaces of BAB^{-1} $[\,BAB^{-1}(BV_i) = B(AV_i) = BV_i\,]$ and A commutes with BAB^{-1} by $[A,[A,B]]=I$.

$$\left(\begin{array}{l} ABA^{-1}B^{-1}(V_i) = V_i \Rightarrow BA^{-1}B^{-1}(V_i) = V_i \Rightarrow V_i = \text{eigen space of } BA \\ \therefore A(BA^{-1}B^{-1}) = (BA^{-1}B^{-1})A \\ \therefore A(BAB^{-1}) = A(BA^{-1}B^{-1})^{-1} = (BA^{-1}B^{-1})^{-1}A = (BAB^{-1})A \end{array} \right.$$

So A should preserve eigen spaces of BAB^{-1} i.e, BV_i.

$$\therefore A(BV_i) = BV_i$$

BV_i is a sum of eigen vectors of A if B is close enough to I

$$\therefore BV_i = V_i$$

$$\left(\begin{array}{l} BV_i = \sum_j (BV_i) \cap V_j \\ \text{If } \|B-I\| < 1, \text{ then } B \text{ cannot map a unit vector to an orthogonal} \\ \text{unit vector, } BV_i \cap V_j = 0 \\ \qquad \therefore BV_i = V_i \end{array} \right.$$

QED.

§§ Wh(torsion-free crystallograph gp) = 0

(1) **Lemma** (~~Farrell~~ Farrell – Hsiang) Let Γ be a torsion-free crystallographic group with holonomy group G, and let $x \in Wh\,\Gamma$ or $x \in \tilde{K}_0(\mathbb{Z}\Gamma)$. Then there exists N_x such that $\forall s > N_x$ and $s \equiv 1 \pmod{|G|}$ then the transfer of x to $Wh\,\Gamma_{G_s}$ (resp. $\tilde{K}_0\mathbb{Z}\Gamma_{G_s}$) vanishes. $\sigma^*(x) = 0$

$\sigma : \Gamma_{G_s} \hookrightarrow \Gamma$

$$\begin{tikzcd}
 & & & 1 & \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[u] \arrow[d, dashed, bend left, "\psi"] & 1 \\
1 \arrow[r] & (s\mathbb{Z})^n \arrow[r] & \underset{\varphi^{-1}(G_s) = \Gamma_{G_s}}{\Gamma} \arrow[r, "\varphi"] & \underset{\psi(G) = G_s}{\Gamma_s} \arrow[r] \arrow[u] & 1 \\
 & & & (\mathbb{Z}_s)^n \arrow[u] & \\
 & & & 1 \arrow[u] &
\end{tikzcd}$$

vertical splits uniquely up to ~~two~~ inner auto of Γ

$H^2(G, (\mathbb{Z}_s)^n) = 0$ ~~($\Rightarrow$ splits)~~

~~$H^1(G, (\mathbb{Z}_s)^n) = 0$ ($\Rightarrow$ unique)~~

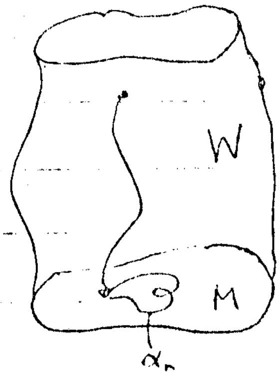
(2) **Lemma** (Ferry, Annals of Math. 77) Hilbert Q-mfd.

Given closed Riemannian ~~metric~~ mfd M, there exists $\varepsilon > 0$ such th (any ε h-cobordism (W, M) with base M is a product) i.e. $\tau(W, M) = 0 \in Wh(\pi_1 M)$.

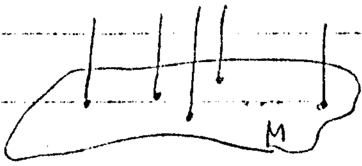
Def Pick a riem. metric on M^n. (W, N) is ε h-cobordism if $\exists$ a deformation retraction $h_t : W \times I \to W$ such th $h_0 = id$, $h_1 =$ retraction, and the following family curves

$$\alpha_p(t) = h_1 h_t(p), \quad p \in W$$

All have arc length $< \varepsilon$.



Let M^n be a closed riem. mfd. Pick finite points $a_1, a_2, \cdots a_n$



The geometric group on these points $S = \{a_1, a_2, \cdots, a_n\}$ is the free abelian group $G(S)$ generated by S.

$$\alpha : G(S) \longrightarrow G(S) \text{ auto.}$$

✓ is called an ε-auto if $\underset{\alpha^{-1}(a_i)}{\alpha(a_i)} \in G(S \cap D_\varepsilon(a_i))$ ↙ check both α

α is ε-blocked if $\exists$ partition of S, $S = S_1 \cup \cdots \cup S_t$ (disjoi where diameter $(S_i) < \varepsilon$ and $\alpha(G(S_i)) \subset G(S_i)$

Connell - Hollingsworth Conjecture Trac. AMS (1969)

Given M^n and $\varepsilon > 0$, then $\exists\, \delta > 0$ such that any δ-auto any geometric group over M^n can be expressed as a product (n+1) ε-blocked auto.

Thm (Quinn, Ends of Maps 1978)

If $R = \mathbb{Z}\Gamma$ such that $Wh(\Gamma \times \mathbb{Z}^n) = 0$. [illegible] Then the generalized conjecture (replaced geometric group by geometric R-mod is true for R and M^n.

In particular, if $\Gamma = \{1\}$ then $Wh(\mathbb{Z}) = 0$ so that C-H conj is true.

(Originated from Kirby's work on annulus conjecture)

(3) Lemma (Epstein-Schub) Let Γ be torsion-free crystallographic gp

holonomy G. $[1 \to R^n \to \Gamma \to G \to 1]$. Let $M^n = \mathbb{R}^n/\Gamma$

If $s \equiv 1 \pmod{|G|}$, $\exists$ an expanding immersion $f_s : M^n \to M^n$

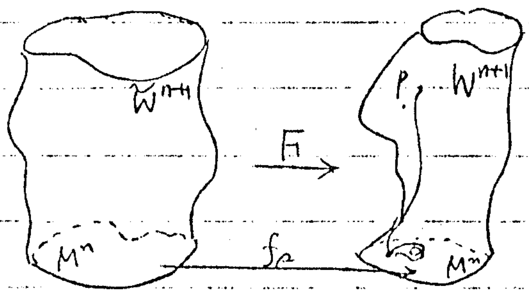
(i) $|df_s(x)| = s|x|$

(ii) $(f_s)_\#(\Gamma) = \Gamma_{G_s}$ $(\Gamma = \pi_1 M)$ $(\to f_s$ is s-sheeted

proof of (1) by (1), (2) and (3). 《Ferry & Epstein-Schub $\Rightarrow$ Farrell

✓ $\forall$ element $\in Wh\,\Gamma$ comes from an h-cobordism with base M, $\pi_1 M$

so let $\tau(W, M) = x$.



Let $\alpha_p = h_1 h_t(p)$ and let

$$\|\alpha_x\| = \max_p \{|\alpha_p|\}$$ — arc length.

Let ε be a number given by (2). Let N_x be defined by

$$N_x = \frac{2 \cdot \|\alpha_x\|}{\varepsilon} \quad \left(\Rightarrow \frac{\|\alpha_x\|}{N_x} < \varepsilon\right)$$

Let $s > N_x$ and $s \equiv 1 \pmod{\|G\|}$

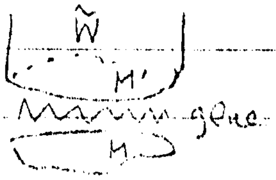
Let

$$\tilde{W} \longrightarrow W$$

be corresponding covering to $\Gamma_{G_s} \hookrightarrow \Gamma$.

claim Base space of $\tilde{W}$ is again M & $F|M = f_s$

$$\begin{tikzcd} \text{base of } W = M' \arrow[d, dashed, "G", "\text{diffeo.}"'] \arrow[dr, "F|M'"] & \\ M \arrow[r, "f_s"'] & M \end{tikzcd}$$



Get a covering $\hat{\tilde{W}}$ with
base M.

$\forall$ subgp of Bieberbach gp is also Bieberbach.

(4) <u>algebraic Lemma</u> | If Γ is a torsion-free ✓ crystallographic group. Then eith

1° $\Gamma = \Gamma' \rtimes T$

2° $\Gamma = B \underset{D}{*} C$, where $[B;D] = 2 = [C,D]$

3° $\exists$ an infinite sequence of numbers s such tha

$s \equiv 1 \pmod{|G|}$ and if S is a hyperelementary subgroup of Γ_s which projects onto G, then S is isomorphic to G_s. See (p.62

e.g. $1 \to \mathbb{Z}^3 \to \Gamma \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1$ is of type 2

(p 36)

<u>proof of Thm using (1) & (4)</u>

"$Wh\,\Gamma$ and $\tilde{K}_0 \mathbb{Z}\Gamma$ vanishes if Γ is torsion-free crystallogr

By induction (lexicographic) on $rk\,\Gamma$ ($= rk\,A$ — maximal ab. subgp

and $|G|$, $1 \to \mathbb{Z}^n \to \Gamma \to G \to 1$.

assume Thm is true for all Γ' where either

$$rk\,\Gamma' < rk\,\Gamma$$

or

$$rk\,\Gamma' = rk\,\Gamma \quad \text{and} \quad |\text{holonomy gp of } \Gamma'| < |G|.$$

when $|G| = 1$, $\Gamma = \mathbb{Z}^n$. we know $Wh(\mathbb{Z}^n) = 0$. So assume $|G| > 1$,

and prove $Wh\,\Gamma = 0 = \tilde{K}_0 \mathbb{Z}\Gamma$.

✓ <u>Case 1°</u> ① Since $\mathbb{Z}\Gamma$ is Nötherian, <u>regular</u> (p.11) ~~Z[free abelian gp] is reg~~

Proc. Sympos. in Pure Math. Vol 17

⑬ $Wh\,\Gamma' \longrightarrow Wh\,\Gamma \longrightarrow K_0(\mathbb{Z}\Gamma')$ exact

Since $Wh\,\Gamma' = 0$, ~~&~~ $\tilde{K}_0 \mathbb{Z}\Gamma' = 0$ by induction hypo, get $Wh\,\Gamma = 0$.

✓ ② Recall Serre's Thm: If R is regular, $\tilde{K}_0 R \longrightarrow \tilde{K}_0 R[x,x^{-1}] \longrightarrow$

$$\begin{tikzcd} \tilde{K}_0 R \arrow[r] \arrow[d, equal] & \tilde{K}_0 R_\alpha[x,x^{-1}] \arrow[r] \arrow[d, equal] & 0 \\ \tilde{K}_0 \mathbb{Z}\Gamma' \arrow[r] & \tilde{K}_0 \mathbb{Z}\Gamma = \tilde{K}_0 \mathbb{Z}(\Gamma' \rtimes T) \arrow[r] & 0 \end{tikzcd}$$

∴

?

(P. 19)
(P.37 (Th

so that $K_0 \mathbb{Z}\Gamma = 0$.

alg K

Case 2° ① By Waldhausen, (cf p 39)

$$\Gamma = B \underset{D}{*} C \Longrightarrow \begin{matrix} Wh\, B \\ \oplus \\ Wh\, C \end{matrix} \longrightarrow Wh\,\Gamma \longrightarrow \tilde{K}_0 \mathbb{Z} D$$

By Induction hypo, $Wh\, B = Wh\, C = \tilde{K}_0 \mathbb{Z} D = 0$ $\therefore\ Wh\,\Gamma = 0.$

② $\tilde{K}_0 \mathbb{Z}\Gamma \xrightarrow{1-1} Wh(\Gamma\times T)$

enough to show $Wh(\Gamma\times T) = 0$ $\mathbb{Z}[G\times T] = \mathbb{Z}G[x, x^{-1}$

$$\Gamma\times T = B\times T \underset{D\times T}{*} C\times T.$$

$$\therefore \begin{matrix} Wh(B\times T) \\ \oplus \\ Wh(C\times T) \end{matrix} \longrightarrow Wh\,\Gamma\times T \longrightarrow \tilde{K}_0 \mathbb{Z}(D\times T)$$

But $Wh(B\times T) = Wh\, B \oplus \tilde{K}_0 \mathbb{Z} B = 0$ by induction hypo.

Similarly

$Wh(C\times T) = 0.$

Moreover, from Serre's Thm

$$\underset{\overset{\shortparallel}{0}}{\tilde{K}_0 \mathbb{Z} D} \longrightarrow \tilde{K}_0 \mathbb{Z}(D\times T) \longrightarrow 0$$

$\tilde{K}_0 \mathbb{Z}(D\times T) = 0$ $\therefore\ \tilde{K}_0 \mathbb{Z}\Gamma = 0.$

Case 3° Let $x \in Wh\,\Gamma$ or $\tilde{K}\mathbb{Z}\Gamma$.

Pick $s > N_x$ so that 3° of algebraic lemma holds.

To show $i^* x = 0$ for all inclusion $i : \Gamma_S \hookrightarrow \Gamma$ where S is

a hyperelementary subgp of Γ_s

Either $S = G_x$ or $|\text{holo}\ \Gamma_S| < |G|$.

$S = G_x \Rightarrow$ $i^* x = 0$ by Lemma (1)

$|\text{holo}\ \Gamma_S| < |G| \Longrightarrow Wh\ T_S = 0$ by hypo.

QED of Th

$$\begin{tikzcd}(\widetilde{W}, M^n) & (W, M^n) \arrow[l, "\text{lift}"'] \\ \tilde{h}_t & h_t \arrow[l, mapsto] \\ \gamma_q & \alpha_p\end{tikzcd} \qquad \left(\begin{tikzcd}\widetilde{W}\times I \arrow[r, "\tilde{h}_t"] \arrow[d, "F\times id"] & \widetilde{W} \arrow[d, "F"] \\ W\times I \arrow[r, "h_t"'] & W\end{tikzcd}\right)$$

claim : $|\gamma_q| < \varepsilon$

$$\left|\begin{array}{l} f_r(\gamma_q) = \alpha_{F(q)} \\ |\gamma_q| = \frac{1}{r}|\alpha_{F(q)}| < \varepsilon \text{ by choice of } N_x. \end{array}\right.$$

So this is ε-h-cobordism.

By (2) Ferry's lemma, $\tau(\widetilde{W}, M) = 0$

$\| \leftarrow$ Milnor's 'Whitehead torsion'

$\sigma^*(x)$, where $\sigma : \Gamma_{G_r} \hookrightarrow \Gamma$

✓ For $\widetilde{K}_0\mathbb{Z}\Gamma$,

$$\begin{tikzcd}Wh(\Gamma\times T) \arrow[rrr, "trf"] & & & Wh(\Gamma_{G_r}\times rT) \arrow[ddd] \\ & \bar{x} \arrow[r] & 0 & \\ & & Wh(\Gamma_{G_r}\times T) \arrow[ur] & \\ \widetilde{K}_0\mathbb{Z}\Gamma \arrow[uuu, "1", "\text{1-1 by B-H-S}"'] & x \arrow[uu, mapsto] \arrow[rr, "\sigma^*"] & & \widetilde{K}_0(\mathbb{Z}\Gamma_{G_r}) \arrow[ul, "1\text{-}1"']\end{tikzcd}$$

claim: If $r \geq N_{\bar{x}}$, then $\sigma^* x = 0$ where $\sigma : \Gamma_{G_r} \hookrightarrow \Gamma$.

$$\because \begin{tikzcd}\widetilde{K}_0\mathbb{Z}\Gamma_{G_r} \arrow[r] \arrow[rrr, bend right, "id"'] & Wh(\Gamma_{G_r}\times T) \arrow[r] & Wh(\Gamma_{G_r}\times rT) \arrow[r] & K_0\mathbb{Z}\Gamma_{G_r}\end{tikzcd}$$

pf. of algebraic Lemma (4) p. 65

Need following three facts.

(1) If $s \equiv 1 \pmod{|G|}$ and $A_s{}^G \neq 0$, then $A^G \neq 0$.

∵ $0 \longrightarrow A \xrightarrow{s} A \longrightarrow A_s \longrightarrow 0$

$$\begin{tikzcd}
0 \arrow[r] & H^0(G,A) \arrow[r] & H^0(G,A) \arrow[r] & H^0(G,A_s) \arrow[r] \arrow[dr] & H^1(G,A) \arrow[r, "\cong", "s"'] & (H^1(G,A)) \arrow[r] \arrow[d, dash] & H^1(G,A_s) \arrow[r] & {} \\
 & & & & 0 \arrow[u] & \text{torsion} & 0 \arrow[u, equal] &
\end{tikzcd}$$

Get $A^G \longrightarrow A_s{}^G \longrightarrow 0$ $\qquad |G|\, H^1(G,A) = 0$

$H^1(G,A_s) = \underset{0}{\underset{\|}{H^1(G,\mathbb{Z}) \otimes A_s}} \oplus \underset{0}{\underset{\|}{H^2(G,\mathbb{Z}) * A_s}} = 0$ See Gruenberg (36)-

$\left. \begin{matrix} |G| \cdot H^1(G,\mathbb{Z}) = 0 \\ (|G|, s) = 1 \end{matrix} \right)$

$\mathrm{coker}\,(\mathrm{Hom}_G(\mathbb{Z}G, A) \longrightarrow \mathrm{Hom}(\mathfrak{g}, A)) \xrightarrow{s_*} \mathrm{Coker}\,(\mathrm{Hom}(\mathbb{Z}G; A) \longrightarrow \mathrm{Hom}(\mathfrak{g}, A))$

(2) ∃ infinite sequence of prime p such that

$(p, |G|) = 1$ & $(p-1, |G|) = 1$ or 2. $\quad |G|\, H^1(G,A) = 0$ & $(|G|, s) =$

∵ Dirichlet's thm ∃ infinitely many p's. ⇒ $p = m|G| - 1$;

$p = m|G| - 1$

$p-1 = m|G| - 2$

(3) $F < G$ with $[G:F] = 1$ or 2. $A^F \neq 0 \Longrightarrow$.

Then either $\Gamma = \Gamma' \rtimes T$ or $\Gamma = B \ast_D C$ with $[B:D] = 2 = [C:D$

∗ Torsion free subgp of cryst is cryst.

∵
$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] & \Gamma \arrow[r, "\varphi"] & G \arrow[r] & 1 \\
1 \arrow[r] & A \arrow[r] & \Gamma_0 \arrow[r, "\varphi"] & F \arrow[r] \arrow[u, phantom, "\cup"] & 1
\end{tikzcd}$$

Let $\Gamma_0 = \varphi^{-1} F$. ~~Then $\Gamma_0 = \Gamma_0' \rtimes T$.~~

$\mathrm{Hom}(\Gamma_0, T) = H^1(\Gamma_0, T)$ since action of Γ_0 on T is trivial.

Show this is not trivial gp.

Using Lyndon-Serre-Hochshield spectral sequence, ($BT \to BT$

$1 \longrightarrow A \longrightarrow \Gamma_0 \longrightarrow F \longrightarrow 1$

$$E_2^{pq} = H^p(F, H^q(A,T))$$

$$\Downarrow$$

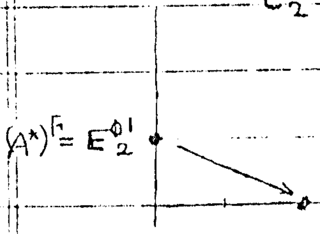
$$H^{p+q}(\Gamma_0, T)$$

If $p > 0$, E_2^{pq} is finite. $\quad E_2^{0q} = (H^q(A,T))^F$

$$E_2^{01} = (H^1(A,T))^F = (A^*)^F \neq 0 \quad (\Leftarrow A^F \neq 0)$$

$$\Downarrow$$

$(A^*)^F$ is infinite. ($A \cong \mathbb{Z}$ free)



$$\mathrm{Hom}(\Gamma_0, T) \cong H^1(\Gamma_0, T) = \sum_{p+q=1} E_2^{pq} = E_2^{01} \oplus \underset{0}{\overset{}{E_2^{10}}} \neq 0$$

So;

$\exists\ \Gamma_0 \longrightarrow T$ non-trivial homo. cyclic

So far, $H^1(\Gamma_0, T)$ is infinite group.

when $[G:F] = 1$, then Γ_0

$\Gamma \twoheadrightarrow T$ since $\exists$ non t

In this case $\mathrm{Ker} \rightarrow \Gamma \xrightarrow{} T$

$\therefore \Gamma \cong \Gamma$

If ~~$\mathscr{G}$~~ $[G,F] = 2$, then $1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow T_2 \longrightarrow 1 \quad ((\Gamma : \Gamma_0) = (G ; F) = 2)$

There are two actions of Γ on T.

(Γ, T^+) trivial action of Γ on T.

(Γ, T^-) non-trivial action of Γ using T_2 action on T since Γ-

Claim: one of $H^1(\Gamma, T^\pm)$ is infinite. [not trivial!]

If $H^1(\Gamma, T^-) \neq 0$ $\qquad$ % $H^1(\Gamma, T) \neq 0 \Rightarrow$ as in the case G

$$\varphi \in H^1(\Gamma, T^-) \Longrightarrow \varphi : \Gamma \to T \text{ is crossed homo.}$$

$$\varphi(a b) = \varphi(a) + a\,\varphi(b).$$

$\Gamma \xrightarrow{\hat{\varphi}} T \rtimes T_2 = \{ax + t \mid a \in T_2,\ t \in T\}$ infinite dihedral group.

$\gamma \longmapsto (\varphi(\gamma), \bar{\gamma}) = \bar{\gamma}x + \varphi(\gamma)$

is a group homo.

? $\hat{\varphi}$ not onto $\Rightarrow$ $\mathrm{Im}\,\hat{\varphi} = T \Rightarrow \boxed{\mathrm{Ker}\,\hat{\varphi} \rightarrowtail \Gamma \twoheadrightarrow T}$

✓ Suppose $\hat{\varphi}$ is onto. Since $T \rtimes T_2 = T_2 * T_2$

refl. at 0

refl. at 1.

generated by transl. $0 \mapsto 2$

refl. at 0

$$\Gamma \xrightarrow{\hat{\varphi}} T_2 * T_2 \longrightarrow \overset{1}{\mathscr{Q}}$$

Let $B = \hat{\varphi}^{-1}(T_2)$, $C = \hat{\varphi}^{-1}(T_2)$; $D = \hat{\varphi}^{-1}(1)$.

pf of claim
~~Now~~ use spectral sequence of $1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow T_2 \longrightarrow 1$

$$E_2^{pq} = H^p(T_2, H^q(\Gamma_0, T^{\pm}))$$
$$\Downarrow$$
$$H^{p+q}(\Gamma, T^{\pm})$$

free abel. gp

$E_2^{01} = H^0(T_2, H^1(\Gamma_0, T^{\pm})) = (H^1(\Gamma_0, T^{\pm}))^{T_2} = (H^1(\Gamma_0, T))^{T_2}$ free ab. gp of rk

One of $\mathrm{Hom}_{T_2}(\Gamma, T^-)$ & $\mathrm{Hom}_{T_2}(\Gamma_0, T)$ is $\neq 0$.

pf of (3

pf of algebraic Lemma.
Since $(p, |G|) = 1$ & $(p-1, |G|) = 1$ or 2. Fix u so that $p^u \equiv 1$ mo

$s = p^{uv}$ $(v = 1, 2, \cdots)$

These s satisfy conditions of s in 3° of (4) alg. Lemma.

Choose $S \subset \Gamma_s$

$$\begin{tikzcd} 1 \arrow[r] & A_s \arrow[r] & \Gamma_s \arrow[r, "\varphi"] & G \arrow[r] & 1 \\ & & S \arrow[u, phantom, "\cup" description] & & \end{tikzcd}$$

$A_s = A/sA$, S = hyperelementary subgp of Γ_s. $\ni$ $\varphi(S) = G$.

Want $\varphi|_S$ is iso. To show $\mathrm{Ker}(\varphi|S) = A_s \cap S = 1$;

Since S is hyper elementary, ($\rightarrow$ always split)

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] & S \arrow[r] & Q \arrow[r] & 1 \end{tikzcd}$$

, $|Q| = q$?

cyclic

q-group

$(|K|, |Q|) = 1$. $\therefore$ $S = K$

Case 1. $q | s$

$S < \Gamma_s$ and $A_s \longrightarrow \Gamma_s \xrightarrow{\varphi} G$

$A_s \rtimes G$

$Q < A_s$ and K maps onto G

$K \cap A_s < (A_s)^Q$

If $K \cap A_s \neq 0$, then this should be case (1) $\Rightarrow$ $Q \triangleleft S$

$a \in Q < A_s$

$\Rightarrow$ $S = K \rtimes Q = K \times Q$ $\Rightarrow Q < A_s^G$

$a(ba^{-1}b^{-1}) \in A_s \cap K$

normal

$\therefore$ $aba^{-1}b^{-1} = 1$

$\Rightarrow Q = 0$.

case 2. $\underline{(q, s) = 1}$

φ maps Q into G isomorphically.

$$A_s \cap S = A_s \cap K$$

claim : for $^{\forall}g \in G$, $g(A_s \wedge K)g^{-1} = A_s \wedge K$ $\big| \because$

$A_s \wedge K$ is a cyclic gp of order p^n

$$\begin{tikzcd} F \arrow[r] \arrow[d, phantom, "\shortparallel"] & G \arrow[r] & \mathrm{Aut}(A_s \wedge K) \arrow[d, phantom, "\shortparallel"] \\ \mathrm{Ker} & & \mathbb{Z}_2 \end{tikzcd}$$

since order of $A_s \wedge K$ is $p^n - p^{n-1}$
$= p^{n-1}(p-$

$\therefore [G \cdot F] = 1$ or 2

$$(A_s)^F \neq 0 \Rightarrow A^F \neq 0$$

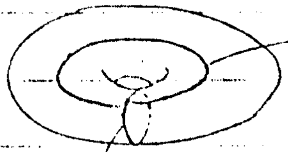
$\therefore n = 0.$

about Lens spaces (Rolfsen)

let V_1, V_2 be solid tori.

$h: \partial V_2 \to \partial V_1$ (homeo)

$M_3 \equiv V_1 \underset{h}{\cup} V_2 = V_1 \amalg V_2 / x \sim h(x)$ for $x \in \partial V_2$.



K-longitude. (generator of $H_1(V) \simeq \pi_1(V) \simeq \mathbb{Z}$
intersects some meridian of V trsv.
a single pt.

J: meridian homologically trivial in V
(homotopically)
bounds a disk in V
For some framing $h: S^1 \times D^2 \to V$, $J = h(1 \times \partial D^2)$.

longitude $= h(S^1 \times 1)$

$h_0 = id$
$h_t =$ homeo

Any two meridians of V are equiv. by an ambient isotopy o
longitude are equiv. by a homeo.
But there are infinitely many ambient isotopy classes of lo

$\partial V \xrightarrow[\cong]{h} \partial V$ can be extended to $V \xrightarrow{\cong} V \iff f$ takes merd. to m

cut $V \hookrightarrow S^3$, $X = \overline{(S^3 - V)} \hookrightarrow S^3$ V, X are mfds with $\partial =$

mfd.

meridian.

$H_i(X:\mathbb{Z}) = \mathbb{Z}, \mathbb{Z}, 0, 0, \cdots$

$H_i(V:\mathbb{Z}) = \mathbb{Z}, \mathbb{Z}, 0, 0, \cdots$

longitude

$\exists$ framing $h: S^1 \times D^2 \to V \ni h(S^1 \times 1) = 0 \in H_1(X)$.

longitude

preferred framing.

$M^3 = V_1 \underset{h}{\cup} V_2$ is connected, orientable 3-mfd

$V_1 \underset{h}{\cup} V_2 \cong V_1 \underset{k}{\cup} V_2 \iff h(m_2)$ cse $k(m_2)$

$h, k: \partial V_2 \to \partial V_1$, m_2: meridian of V_2.

Let l_1, m_1 ; l_2, m_2 be longitude & meridian of V_1 ; V_2.

$$h_*(m_2) = p l_1 + q m_1 \qquad (p, q) = 1$$

Lens space of type $L(p,q)$

$M^3 =$ lens space $\Leftrightarrow$ $M^3 = V_1 \underset{h}{\cup} V_2$.

$L(1,q) \cong S^3$

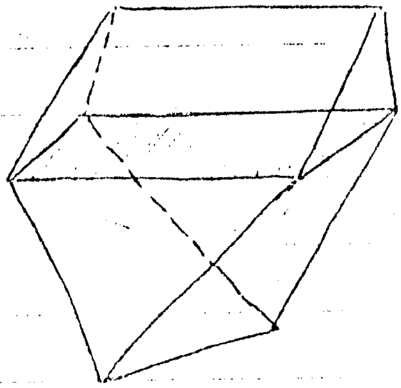
$L(0,1) \cong S^2 \times S^1$

$L(2,1) \cong \mathbb{R}P^3$

$L(p,q) \cong L(p,-q) \cong L(-p,q) \cong L(-p,-q) \cong L(p,q+kp)$.

$L(p,q)$, $0 < q < p$ exhausts all Lens sp. except S^3, $S^2 \times S^1$.

$\pi_1(L(p,q)) = \mathbb{Z}/p$.



Solid Torus + Solid torus /

(boundary identification by $l \to m'$ & $m \to \ell$

is S^3.

<u>Example</u> $Wh(\mathbb{Z}^2 \rtimes \underset{\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}}{\mathbb{Z}_3}) = 0$

$Wh(\mathbb{Z}^2 \times \mathbb{Z}_3) = 0$?
<u>not</u> know

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$ ∴ ~~A^3~~ satisfies $A^3 = I$.

and A has min. polynomial x^2+x+1.

A becomes an orthogon rotation when a new a on $\mathbb{R}^2$ is chosen;
i.e, $\exists B \in GL_2(\mathbb{R}) \ni$
~~st~~ $B^{-1}AB \in O_2(\mathbb{R})$.
~~this B corresponds to a cho~~
of new axis on $\mathbb{R}^2$

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] & \Gamma \arrow[r] & G \arrow[r] & 1 \\ & & A \rtimes G \arrow[u, phantom, "\cap"] & & \\ & & \mathrm{Rigid}(n) \arrow[u, phantom, "\cap"] & & \end{tikzcd}$$

(If Γ has no torsion, $\Gamma \subset \mathrm{Rigid}(n)$ is mono.

Let $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}_3$.

<u>$\mathbb{R}^2/\Gamma = S^2$</u>.

$$\mathbb{R}^2/\mathbb{Z}^2 = T^2$$

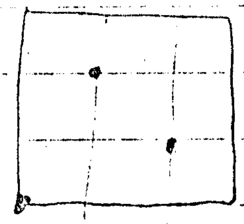
From

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} \quad m, n \in \mathbb{Z}$$

$$\begin{cases} -y = x+m \\ x-y = y+n \end{cases} \qquad \begin{aligned} 3x &= n-2m \\ y &= -\tfrac{1}{3}(n-2m)+m = \tfrac{1}{3}(n+m) \end{aligned}$$

Fixed pts

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$



By calculation of euler #, get $\mathbb{R}^2/\Gamma = S^2$.

~~Another way~~ ;

$\Gamma \twoheadrightarrow \mathbb{Z}_3^{~~*~~}$ acts freely on S^3 so that

$$S^3 \longrightarrow S^3/\mathbb{Z}_3 = L^3 \text{ (Lens space)}$$

Γ ~~[illegible]~~ acts on $\mathbb{R}^2$ and S^3.

So Γ acts diagonally on $\mathbb{R}^2 \times S^3$

$$\begin{tikzcd}
 & \left\{\begin{array}{l} L \text{ at three fixed pts} \\ S^3 \text{ otherwise.} \end{array}\right. \arrow[d] & \\
T^2 \arrow[r] & \mathbb{R}^2\times S^3/\Gamma \arrow[r] \arrow[d] & S^3/\Gamma = L \\
\text{torus} \arrow[u, dash] & S^2 &
\end{tikzcd}$$

Expanding map $\mathbb{R}^2 \times S^3$

$f_2(x, a) = (2x, a)$

$$\overset{\overset{A}{\parallel}}{\mathbb{Z}^2} \rtimes \overset{\overset{G}{\parallel}}{\mathbb{Z}_3}$$

$\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}_3 \longrightarrow \Gamma = \Gamma' \rtimes T$ (impossible) $\quad \Gamma \neq B \underset{D}{*} C$

$\exists$ infinite sequence of positive numbers $s \equiv 1 \pmod 3$

Let $S \leq \Gamma_s$ be a hyperelementary subgp.

$$\begin{tikzcd} A_s \arrow[r] & \Gamma_s \arrow[r] & G \\ & S \arrow[u, phantom, "\cup" rotate=90] & \end{tikzcd}$$

Either $S \subseteq A_s$ or $S =$ conjugate of G_s. ($S \to 1 \in G$ or S is

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi"] & \Gamma_s \arrow[r] & 1 \\ \Gamma_S \equiv \varphi^{-1} S \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & S \arrow[u, phantom, "\cup" rotate=90] & \end{tikzcd}$$

For $x \in Wh(\Gamma)$, want $\sigma^*(x) \in Wh(\Gamma_S)$

If $S \subseteq A_s$, then $\Gamma_S = \mathbb{Z}^2$. Know already. $\underset{Wh(\Gamma_S) \ni \sigma^*(x)}{\overset{\parallel}{Wh(\mathbb{Z}^2)}} = 0$. ✓

If $S = G_s$ $\quad \sigma^*(x) \in Wh(\Gamma_{G_s})$

$\Gamma \leq$ Rigid (2)

$\boxed{\mathbb{R}^2/\Gamma = S^2}$ $\quad \because \mathbb{R}^2/A = T^2$

$T^2/G = \mathbb{R}^2/\Gamma$.

$\mathbb{R}^2/\Gamma$ is mfd. $\quad \chi(\mathbb{R}^2/\Gamma) = 2$. ✓

$\mathbb{C} > G = \mathbb{Z}_3$ acts $S^3 \subseteq \mathbb{C}^2$ (free) $\Big\rangle$ get $S^3/\Gamma =$ Lens sp.
$\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}_3$ "

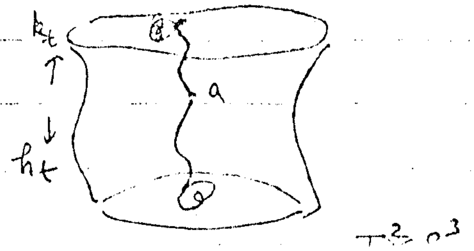
$$\begin{tikzcd} \Gamma \arrow[d] \\ \mathbb{R}^3 \times S^2 \arrow[d] \\ (\mathbb{R}^2 \underset{\Gamma}{\times} S^3 \end{tikzcd}$$

Γ acts freely on $\mathbb{R}^2 \underset{\Gamma}{\times} S^3 = T^2 \underset{G}{\times} S^3$.

$\pi_1(\mathbb{R}^2 \underset{\Gamma}{\times} S^3) = \Gamma$.

Let W^6 be an h-cobordism with one end $\quad \mathbb{R}^2 \underset{\Gamma}{\times} S^3 = T^2 \underset{G}{\times}$

$\to \quad x = \tau(W)$.



$\alpha_a = h_1 k_t(a)$

$\gamma_a = h_1(k_t(a))$ — def. retract of another end.

$Wh(\pi_1 S^3)=0$ & $Wh(\pi_1 L)=0$.

$$\begin{tikzcd} T^2 \times_G S^3 \arrow[d, "p"'] \\ T^2/G = S^2 \end{tikzcd}$$

Generalize ε h-cobordism by (h_t, k_t) is ε-h-cobordism if $|p\alpha_a| < \varepsilon$ & $|p\gamma_a| < \varepsilon$.

(Quinn)
Thm $\exists \varepsilon > 0$ such that each ε h-cobordism with base $T^2 \times_G S^3$ has 0 whitehead torsion.

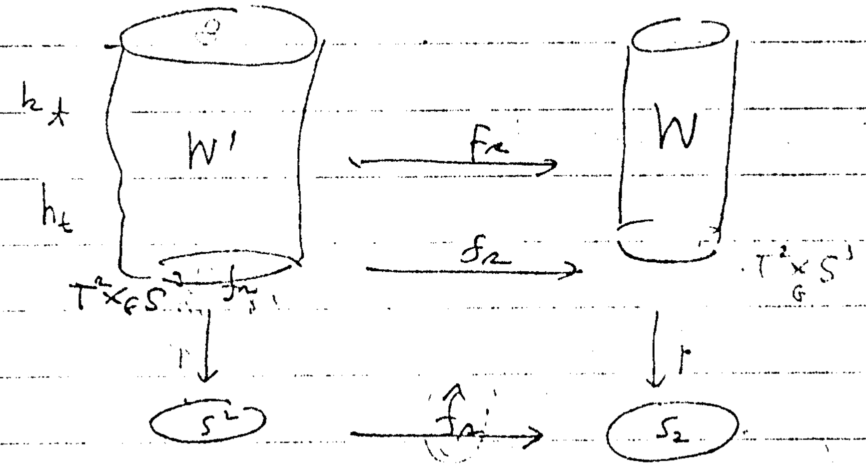
Find f_2 a smooth map (in fact, covering proj.)

$$f_2 : T^2 \times_G S^3 \circlearrowleft$$

$(f_2)_\#(\Gamma) = \Gamma_{\sigma_2}$

$f_2(y.z) = (2y, z)$

Choose a covering proj $W' \xrightarrow{F_2} W$ ~~st.~~ corresponding $\exists\, \Gamma_{\sigma_2}$



$\tau(W) = \sigma(x)$

$p(\alpha_a), p(\tilde{\gamma}_a) \xrightarrow{\hat{f}} p(\alpha_b), p(\gamma_b)$

$\|p(\gamma_b)\| = 2\|p(\gamma_a)\|$

QED modu
Quinn's Thm

M = cpt mfd with ∂

$\mathcal{S}(M)$ = homotopy structure set

objects (N, f) where $N \xrightarrow[\text{h.e.}]{f} M$ & $f|\partial N : \partial N \approx \partial M$.

$(N_1, f_1) \sim (N_2, f_2)$ equivalent $\longleftrightarrow$

$$\begin{tikzcd} N_1 \arrow[d, "F"', "\text{homeo.}"] \arrow[dr, "f_1"] & \\ N_2 \arrow[r, "f_2"'] & M \end{tikzcd}$$

* $\mathcal{S}(M)$ is an abelian group.

0-elt : $M \xrightarrow{id} M$

Thm ✓ If M^n is flat mfd, $n \neq 3, 4$ and |holonomy group of M^n| is

then $|\mathcal{S}(M^n \times D^m)| = 1$ [In fact $n+m > 4$ is enough.] — insta of wh

Corr $m = 0$: If N is htp equivalent to M^n, flat mfd $n \neq 3,4$ then $N \approx M$

? Thm (tentative) Let M^n be any mfd $n > 4$, $\pi_1 M = \Gamma$. Let

$\varphi : \Gamma = \pi_1 M \longrightarrow F \longrightarrow 1$
finite group

Let $[f] \in \mathcal{S}(M)$ such that:

If $\sigma^*[f] = 0$ for all $\sigma : \Gamma_S \hookrightarrow \Gamma$ where S ranges over all the hyper elementary subgroup of F. ($\Gamma_S = \varphi^{-1} S$), then $[f] = 0$.

$\sigma^*[f]$ is defined by $[\hat{f}]$:

$$\begin{tikzcd} \tilde{N} \arrow[r, dashed, "\hat{f}"] \arrow[d, dashed, "\text{pullback}"'] & M_S \arrow[d, "\text{covering corr. to } \Gamma_S"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Sketch of Pf of Thm. Induction on dim M and order of holonomy gr

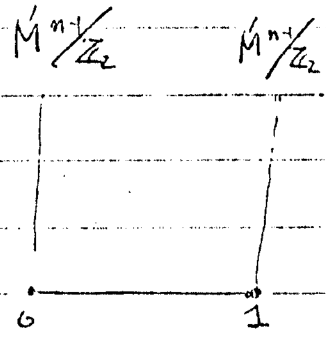
✓ (case i) $\Gamma = \Gamma' \rtimes T$. Then M is a Seifert fibration

$M'^{n-1} \longrightarrow M \longrightarrow S^1$

✓

$$\underset{0}{\overset{\parallel}{\mathcal{S}(M_1'^{n-1}\times D^{m+1})}} \longrightarrow \mathcal{S}(M'^n\times D^m) \longrightarrow \underset{0}{\overset{\parallel}{\mathcal{S}(M'^{n-1}\times D^m)}}$$

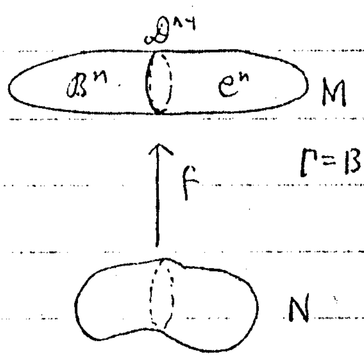
Difficulty in even dim'l case : Seifert fibering is
~~(case iii)~~



inside (0,1), fiber is M'^{n-1}

(case iii) Use above tentative Thm & analogue of Ferry's

(case ii)



$\Gamma = B \underset{D}{*} C$

Can f be pulled back?

Obstruction comes from WhΓ

So, for $|G|$ even, To take care of case $\Gamma = B\underset{D}{*}C$, we need the following algebraic fact:

<u>proposition</u> Let Γ be a crystallographic group. $1 \to A \to \Gamma \to G \to$

Then either,

(i) $\Gamma = \Gamma' \sqsupset T$

(ii) $\Gamma \dashrightarrow\!\!\!\!\to \hat{\Gamma}^{\wedge} \neq 1$ crystallographic ~~such that~~ $1 \to \hat{A} \to \hat{\Gamma} \to \hat{G} \to$ such that $\exists$ infinite sequence of positive integers $s \equiv 1$ (mod $|\hat{G}$ so that any hyper elementary subgp S of $\hat{\Gamma}_s$ which projects onto is isomorphic to $\hat{G}$

(iii) G is elementary abelian 2-groups $[G = \oplus \mathbb{Z}_2)$ and either

ⓐ $\Gamma = A \lhd T_2$ or ⓑ $\Gamma \twoheadrightarrow \bar{\Gamma}$ crystallograph $1 \to \mathbb{Z} \oplus \mathbb{Z} \to \bar{\Gamma} \to \mathbb{Z}_2 \oplus$

with holonomy representations one of

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \pm I \quad \text{and} \quad I \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Pf) assume $\Gamma \neq \Gamma' \lhd T$. Then In Algebraic Lemma, either (ii)

(iii) Suppose $\Gamma = B \underset{D}{*} C$. Then

$$\Gamma \dashrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1.$$

This implies: $\exists$ an ∞-cyclic subgroup S of A Which is invarian

under action of Φ. (A/S is again torsion-free)

Let $\Gamma/S = \Gamma_1'$ abelian by finite.

$$\begin{tikzcd} \Gamma_1' \arrow[d, dashed, "\text{may not be crystallograph.}"] \\ \mathrm{Rigid}(n-1) \end{tikzcd}$$

Let the image of this map in Rigid $(n-1)$ be Γ_1. Then we get

$$\Gamma \longrightarrow \Gamma_1 \longrightarrow 1, \qquad rk\,\Gamma_1 = rk\,\Gamma - 1.$$

Continue this process. $\Gamma_1 \to \mathbb{Z}_2 * \mathbb{Z}_2 \to 1 \rightsquigarrow \Gamma_1 \to \Gamma_2 \to 1$. etc.

$$\begin{tikzcd} \Gamma \arrow[r, two heads] \arrow[rrrr, bend right, "\varphi_i"'] & \Gamma_1 \arrow[r, two heads] & \Gamma_2 \arrow[r, two heads] & \cdots \arrow[r, two heads] & \Gamma_i \arrow[r, two heads] & \cdots \arrow[r, two heads] & \Gamma_n = 1 \end{tikzcd}$$

Let $K_i = \ker \varphi_i$. Then

$$1 < K_1 < K_2 < \cdots < K_n \lneq \Gamma.$$

Each $K_i \lhd \Gamma$. Put $A_i = K_i \cap A$. Then

$$0 < A_1 < A_2 < \cdots < A_n = A, \qquad rk\,(A_{i+1}/A_i) = 1$$

Let

$$\mathfrak{a}_i = \{ x \in A \mid s x \in A_i \text{ for some } s > 0 \}$$

$$0 < \mathfrak{a}_1 \subset \mathfrak{a}_2 < \cdots < \mathfrak{a}_n = A.$$

$\mathfrak{a}_i$ is normal in Γ and $\mathfrak{a}_{i+1}/\mathfrak{a}_i$ are infinite cyclic.

Pick a basis for A $\quad e_1, e_2, \cdots, e_n$; $\quad e_1, \cdots e_i$ span $\mathfrak{A}_i$.

$g \in G$,

$$g \longrightarrow M \text{ — } n \times n \text{ matrix upper triangular}$$

$$M = \begin{pmatrix} \ddots & \\ 0 & \ddots \end{pmatrix} \qquad M_{ii} = \pm 1$$

$$(M^2)_{ii} = 1$$

$\therefore g^2 = 1 \in G$. (Every elt of holonomy gp has order 2)

$G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots$

~~Suppose~~ hol. $\Gamma_i \longrightarrow$ hol. of Γ_{i+1} (max. abel.)

Since $\Gamma_n = 1$, $\Gamma_{n-1} = \mathbb{Z} \underset{-1}{\sqsupset} \mathbb{Z}_2$ (There are only two 1-dim crystal. g
$\mathbb{Z}$ & dihedral.)

If $T_{n-1} = \mathbb{Z}$, then Γ would be of the form $\Gamma' \sqsupset T$.

case 1. If hol $\Gamma_i = \mathbb{Z}_2$, then ~~A~~ $\Gamma = A \underset{-1}{\sqsupset} \mathbb{Z}_2$

$\because$ hol. repr. of $G = \cancel{\mathbb{Z}_2}^{T_2}$ on A.

$$A = \left\{ \begin{array}{l} T \\ T^- \\ \mathbb{Z}(T_2) \end{array} \right.$$

If $A = T$ or $\mathbb{Z}(T_2)$, then $A^G \neq 0$. $\Rightarrow \Gamma = \Gamma' \sqsupset T$ #

So $A = T^-$.

case 2. If hol $\Gamma_i = \cancel{T}_2$ but $|\text{hol.}\,\Gamma_i| > 2$.

$$\begin{tikzcd}
& K \arrow[d] & & \hat{G}_i \arrow[d] & \\
1 \arrow[r] & B_i \arrow[r] \arrow[d, "(\text{not onto})"'] & \Gamma_i \arrow[r] \arrow[d] & G_i \arrow[r] \arrow[d, dashed] & 1 \\
1 \arrow[r] & B_{i+1} \arrow[r] & \Gamma_{i+1} \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d] & 1 \\
& & 1 & 1 &
\end{tikzcd}$$

K is ∞-cyclic. and $\oplus$ summand of B_i. (as abelian gp.)
invariant under G_i.

$M \in \hat{G}_i$ then $M = \left(\begin{array}{c|c} \pm 1 & * \\ \hline 0 & I \end{array}\right)$

Conclude $\hat{G}_i = T_2$

$$= \left\{ I, \left(\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right) \right\}$$

$$G_i = T_2 \oplus T_2 = \left\{ I, -I, \left(\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right), \left(\begin{array}{c|c} +1 & -a \\ \hline 0 & I \end{array}\right) \right.$$

Construct $\Phi: \Gamma_i \to \bar{\Gamma}$

First find $\bar{B}_i \subset B_i$ invariant under G_i

and let put $\bar{\Gamma} = \Gamma_i / \bar{B}_i$

$\ni B_i / \bar{B}_i = Z \oplus Z$

Look at B_i as S-module, $S = \left\{ I, \left(\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right) \right\} \subset \hat{G}_i$

$$B_i = \oplus \begin{matrix} T \\ T \\ Z(T_2) \end{matrix}$$

$$\pm I, \quad I\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Big/ \quad \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

Connel - Hollingsworth. (p. 63)
Trans AMS (1969)

X = metric space

$S = \{p_1, p_2, \cdots p_n\}$ in X (may be the same) Form the free abel

$G(S)$ ~~generated by~~ with these pts basis.

open problems

prob 1 Does every flat Riemannian mfd bound ?

This reduces to "$\forall$ Rieman, mfd which has ~~homology~~ holonomy Π of order 2^n"
Odd sheeted covering does not change the Stiefel-Whitney numbers.
so, holomomy of $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$. $M = \boxed{T^n/G.}$ P53. M.

Prob.2 Suppose Γ acts freely, properly discontinuously on $\mathbb{R}^n \times$
with compact quotient. Then is Γ virtually torsion free

(Wall believes this is true. Cf. Groups acting freely on S
are periodic, dihedral etc.

prob.3 Generalize Biebebach Thm to Affine motions.
i.e, Let $\Gamma \subset \text{Affine} = GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ be discrete subgroup wi
$\mathbb{R}^n/\Gamma$ compact. Then is Γ virtually abelian ?

Milnor's Conjecture on Prob. 3.
$\Gamma < \text{Affine}(n)$ discrete with $\mathbb{R}^n/\Gamma$ cpt $\Leftrightarrow$ Γ is virtually poly

(Hard part) → virt. ab.
Using Johnson's result, ——— get ($\Leftarrow$) without compactness

§ Quinn's Thm

set of ε-autos $\longrightarrow Wh(\pi_1 X)$

ie, $\exists\, \varepsilon > 0$ such that any ε-auto on a geometric group on
determines an elt of $Wh(\pi_1 X)$

Given an ε-auto $f;\ G(S) \circlearrowleft$, $S = \{p_1, p_2, \cdots p_m\}$.

$C_s = \text{Carrier } f(p_s) = \{ p_i \mid p_i$ has non-zero coeff. when $f(p_s)$ is
expressed in terms of $p_i \}$

Since f is ε-auto, $C_s \subseteq S \cap D_\varepsilon(p_s)$.

$f : G(S) \circlearrowleft$ is called a <u>weak ε-auto</u> if $C_s \subset D_\varepsilon$ [diam C_s
almost "stalk to stalk"

$*$ = base pt

P_i = a path from $*$ to p_i

Q_j = " C_j

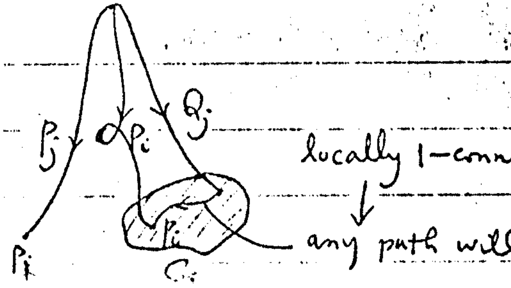
$$f(p_j) = \sum_i a_{ij}\, p_i \qquad (a_{ij}) \in GL_n(\mathbb{Z})$$

Then define

$$\hat{f} = \left(a_{ij} \boxed{P_i^{-1} Q_j} \right) \in Wh(\pi_1 \cancel{X})$$

$$P_i^{-1} Q_j \in \pi_1(X)$$

if $a_{ij} \neq 0$, then $p_i \in C_j$



<u>ε-basis</u> for a geometric group $G(S)$ is a basis $\{x_1, \cdots$ for $G(S)$, $S=\{p_1, \cdots, p_n\}$ such that

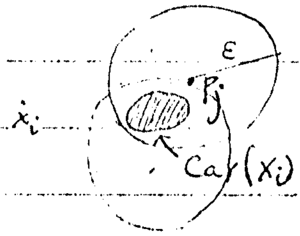
$T: p_i \to x_i$ $\quad$ $Car\, T(p_i)$

$$x_j = \sum a_{ij} p_i \,, \qquad p_j = \sum A_{ij} x_i$$

$$Car(x_j) = \{p_i \mid a_{ij} \neq 0\}$$

$\ni$ ① $\operatorname{diam} Car(x_i) < \varepsilon$

② If $A_{ij} \neq 0$, then $Car(x_i) \subset D_\varepsilon(p_j)$.



$x_1 = 2p_1 - 3p_3$

$$\begin{cases} p_1 = x_1 + 2x_2 - x_3 \\ p_2 = x_1 - x_3 \\ p_3 = \quad x_2 \end{cases}$$

then $\{p_1, p_3\} = Car(x_1)$

① $\operatorname{diam}\{p_1, p_3\} < \varepsilon$

② $\{p_1, p_3\} \subset \{$ ~~p_1, p_3~~ $\} D_\varepsilon(p_1) \cap D$

* If we define

$$T: G(S) \circlearrowleft$$

by $T(p_i) = x_i$, then T is a weak ε-auto. (actually stronger than that).

<u>Lemma 1</u> a) If T is an ε-auto of $G(S)$, then $\{T(p_i)\}$ is a $(\overset{\uparrow}{4}\varepsilon$ for $G(S)$.

2

b) If $\{x_1, \cdots x_n\}$ is an ε-basis for $G(S)$, then $\exists$ permutation $\sigma: \{1, 2, \cdots, n\} \circlearrowleft$ such that $T: G(S) \circlearrowleft$ defin by $T(p_i) = x_{\sigma(i)}$ is an ε-auto.

pf) (a) $Car(T(p_i)) \subset D_\varepsilon(p_i)$, ① ✓ $\Longrightarrow \operatorname{diam}(x_i) < 2\varepsilon$

Let $T(p_i) = x_i$. and let $\quad p_j = \sum A_{ij} \overset{T(p_i)}{\overset{\shortparallel}{\boxed{x_i}}}$. Then $T^{-1}(p_j) = \sum A_{ij} p_i$

(apply T^{-1})

$Car.(T^{-1}(p_j)) \subset D_\varepsilon(p_j)$ ―― by defn of ε-auto.

If $A_{ij} \neq 0$, then $p_i \in Car(T^{-1}(p_j))$

$$T(p_i) = x_i \in D_\varepsilon(p_i) \qquad p_i \in D_\varepsilon(p_j)$$

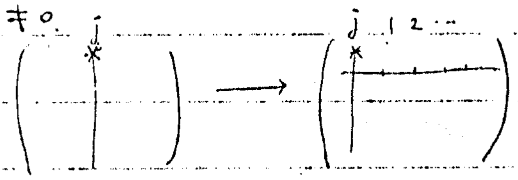
$p_i \in Car(T^{-1}(p_j)) \subset D_\varepsilon(p_j)$

$= Car(x_i) \subset D_{2\varepsilon}(p_j)$

(b) $x_j = \Sigma a_{ij} p_i$, $a = (a_{ij}) \in GL_n(\mathbb{Z})$

claim: $\exists$ a permutation of the column of changing a to ne
matrix b so that $b_{ii} \neq 0$ for all i.

det $a \neq 0$.



Then $p_i \in Car.(x_i) \subset D_\varepsilon(p_i)$ i.e, $Car(T(p_i)) \subset D_\varepsilon(p_i)$

Next, let $T^{-1}(p_j) = \sum_i A_{ij} p_i$

if $p_i \in Car(T^{-1}(p_j))$, then $A_{ij} \neq 0$. Since ~~Ei~~ (x_i) is ε-basis,

$Car(x_i) \subset D_\varepsilon(p_j) \Rightarrow p_i \in D_\varepsilon(p_j)$

$\overset{\in}{p_i} \Uparrow a_{ii} \neq 0$ $Car(T^{-1}(p_j)) \subset D_\varepsilon(p_j)$

✓ Pf

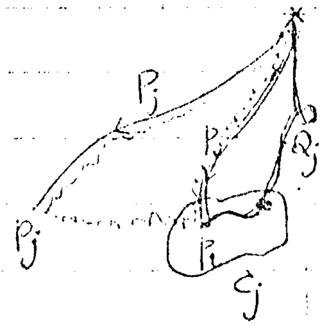
<u>Lemma 2</u> If $\overset{\text{condition ?}}{\varepsilon} = \varepsilon(X)$ small enough (depending only on X), then

$\exists$ a map $\{\varepsilon\text{-auto of geometric gps over } X\} \longrightarrow Wh\,\pi_1(X)$

$f \longmapsto \hat{f}$

such that $\widehat{fg} = \hat{f} + \hat{g}$.

Pf.



choose a path P_j from * to p_j

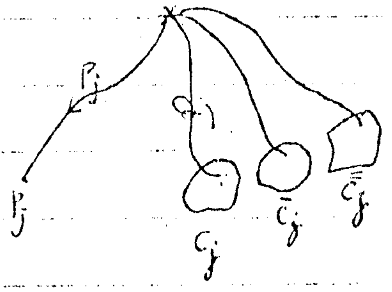
Q_j from * to $Car(p_j) \equiv c_j$

Suppose $f(p_j) = \sum_i a_{ij} p_i$ $(a_{ij}) \in GL_n(\mathbb{Z})$

Define $\hat{f} = (a_{ij} P_i^{-1} Q_j)$ $(P_i^{-1} Q_j \in \pi_1(X)$.
matrix in $\mathbb{Z}(\pi_1 X)$

To prove $\widehat{fg} = \hat{f}\,\hat{g}$, let

$\mathrm{Car}\, f p_i = C_i$, $\mathrm{Car}\, g p_i = \bar{C}_i$, $\mathrm{Car}\, gf p_i = \bar{\bar{C}}_i$



$f(p_j) = \sum a_{ij} p_i$ $\quad f \longmapsto \hat{f} = (a_{ij} p_i^{-1} Q_j)$

$g(p_j) = \sum b_{ij} p_i$ $\quad g \longmapsto \hat{g} = (b_{ij} p_i^{-1} \bar{Q}_j)$

$gf(p_j) = \sum c_{ij} p_i$ $\quad gf \longmapsto \widehat{gf} = (c_{ij} p_i^{-1} \bar{\bar{Q}}_j)$

since $C_i, \bar{C}_i, \bar{\bar{C}}_i \subset D_\varepsilon(p_i)$

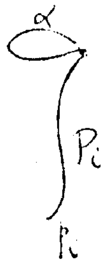
✓

$$(\hat{g}\hat{f})_{ij} = \sum_k b_{ik} p_i^{-1} Q_k\, a_{kj} p_k^{-1} p_j \quad \longleftarrow \quad p_k = Q_k \ ?$$

$p_k \in C_k$?

$$= \Big(\sum_k b_{ik} a_{kj}\Big) p_i^{-1} p_j$$

$$= c_{ij}\, p_i^{-1} p_j = \widehat{gf}_{ij}.$$



$$(\hat{f})_{ij} = a_{ij}\, \alpha^{-1} p_i^{-1} Q_j.$$

$$a_{ij}\, p_i^{-1} Q_j' = a_{ij} (p_i^{-1} Q_j\, \beta)$$

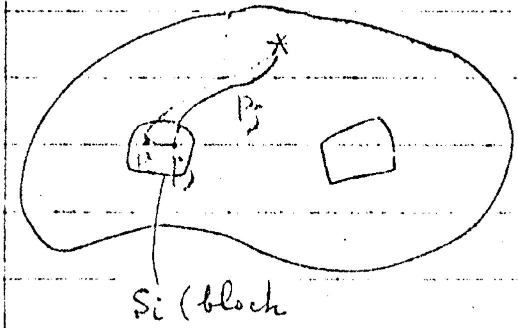
If we used Q_j' instead of Q_j, we g

$$\begin{pmatrix} a_i & a_{ij} p_i^{-1} Q_j \cdot \beta & \\ & (\quad) \cdot \beta & \\ & (\quad) \cdot \beta & \end{pmatrix} \equiv \begin{pmatrix} a_{ij} p_i^{-1} Q_j \end{pmatrix}$$

in which $\pi = h(2\pi) / \pm\pi$

condition ?

✓ **Lemma 3** If f is blocked ε-auto, then $\hat{f} = 0$.



$S = \cup S_i$

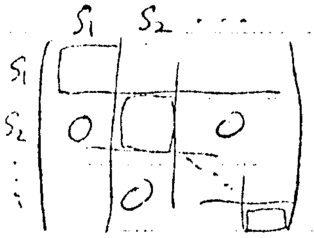
In defining the matrix $\hat{f}$, pick P_i paths from $*$ to p_i so

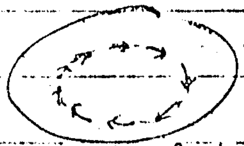
✓ if p_i, p_j are in the same partition set then $P_i = P_j$ (htp)

Then $(\hat{f}_{ij})$ is a blocked matrix.

$$f(p_j) = \sum_i a_{ij} p_i$$

(a_{ij}) =

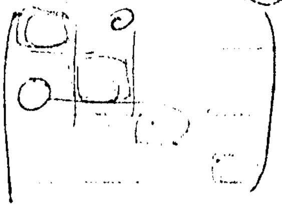




not blocked.
but $0 \in$ Wh.

$$\hat{f}_{ij} = \sum a_{ij} \boxed{P_i^{-1} P_j} = 1 \in \pi_1 X$$

$\hat{f}$ =



is a blocked with $\mathbb{Z}$-entries

✓
$$\begin{tikzcd} K_1(\mathbb{Z}) \arrow[r] & K_1(\mathbb{Z}\pi_1) \arrow[r, "\text{not onto}"] & Wh\, \pi_1 \\ \pm(1) \arrow[urr, bend right, "\pm 1 \ \pm \pi_1"', "0" near end] & & \end{tikzcd}$$

(Connell - Hollingsworth)

Conjecture Given X and $\varepsilon > 0$, $\exists \delta > 0$: $\forall \delta$-auto over X is the
P. 63 product of $(n+1)$ blocked ε-auto where $n = \dim X$.

Cor $\exists\, \delta = \delta(X)$ such that if f is an δ-auto, then $\hat{f} = 0$.

pf. let $\varepsilon_i = \left(\varepsilon(X) \text{ in Lemma } i\right)$ $i = 2, 3$.

$$\varepsilon = \min\left(\frac{\varepsilon_2}{n+1}, \varepsilon_3\right).$$

For this ε, $\exists\, \delta$ (by conj) such that [~~If take $\delta < \varepsilon_2$~~]

$f : \delta$-auto $\Rightarrow f = f_1 \cdot f_2 \cdots f_{n+1}$, each f_i is blocked ε-auto.

Then

$$\hat{f} = \widehat{f_1 \cdots f_{n+1}} = \widehat{f_1 \cdots f_n} + \hat{f}_{n+1} = \hat{f}_1 + \cdots + \hat{f}_{n+1} = 0$$

defined since ~~$\delta < \varepsilon_2$~~

f_i : ε-auto

$f_1 \cdots f_n, f_{n+1}$: $\underset{\varepsilon_2}{\overset{\wedge}{n\varepsilon}}$-auto

$\varepsilon < \varepsilon_3$.

similarly $\delta < \varepsilon_2$

(Ferry) cf. p.62 let $X \overset{\leftarrow}{\subset} K$ be a pair of finite CW-complexes
deformation retract.

Then $\exists\, \varepsilon > 0$
depending on X (not on K) such that $K \overset{\varepsilon}{\searrow} X \Rightarrow \tau(K,X$

$(K \overset{\varepsilon}{\searrow} X)$ means: let $h_t : K \circlearrowleft$ such that $h_0 = id$
$h_1 : K \to X$-retraction

Define $\alpha_p(t) = h_1 h_t(p)$

$\|\alpha_p\| < \varepsilon$ for any ε. (cf. p62)

✓ How to get h_t (h-cobordism) from a htpy equivalence $X \rightleftarrows K$?

For (Coro ⇒ Ferry), we need following Lemma.

Lemma (Trading Cells)

✓ $\forall \varepsilon > 0, \exists\, \delta > 0$: If $K \overset{\delta}{\searrow} X$, there exists $L \overset{\varepsilon}{\searrow} X$ such that

ε, δ ? $K \overset{she}{\simeq} L$ and $L - X$ consists only of cells in 2 consecutive dim

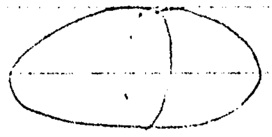
and cells are all $\varepsilon/10$ diam, and 2-cells are attached trivially.

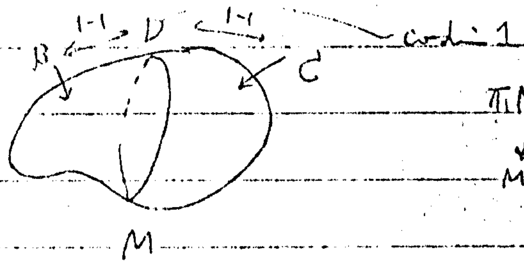
(assuming $e_1 \to X$ via e_1)
i.e. $h_1(e_1) = e_1$

We traded $e^i \longrightarrow E^2$
$e^1 \longrightarrow E^1$

(Remark on Conjecture I (p. 1)))

Conjecture If N & M are closed asph. mfd with $\simeq \pi_1$'s. Then $M^n \cong N$.





closed asph.

$\pi_1 M = \pi_1 N \ni \boxed{\Gamma}$

$\Downarrow$

$M^n = \mathbb{R}^n / \Gamma$ cryst

$\Gamma = \Gamma' \sqsupset T$

When $[B : D] = 2 = [C : D]$

$\Downarrow$ $\boxed{\Gamma = B \underset{D}{*} C \quad [B:D] = 2 = [C:D]}$

$\boxed{UNil(D; C, B) \text{ obstr.}}$

$Bx \equiv 1 \ (\Gamma)$, —

$D = 1$, $B = C = T_2$ (ex. of realization of obstr.)

Find $\Gamma = C \underset{D}{*} B$ with C, D, B, Γ fundamental gps of asph.

Then this is a counter example.

$UNil(1; T_2, T_2)$

$\Gamma = (B \underset{D}{*} C)$ ($[B : D] = 2 =$)

Question $\exists$ ~~x, y ∈~~ $x \in \mathbb{Z}C$, $y \in \mathbb{Z}B$ so that

$$\begin{tikzcd} x \in \mathbb{Z}C \arrow[d] & y \in \mathbb{Z}B \arrow[d] \\ a \in \mathbb{Z}T_2 & b \in \mathbb{Z}T_2 \end{tikzcd}$$

Under the canonical map

$$\Gamma = \underset{x \quad y}{B \underset{D}{*} C} \longrightarrow \underset{a \quad b}{T_2 \underset{1}{*} T_2} \longrightarrow 1$$

$\exists \begin{pmatrix} x - \bar{x} & -1 \\ 1 & y - \bar{y} \end{pmatrix}$ is invertible over $\mathbb{Z}\Gamma$

$-: \mathbb{Z}G \circlearrowleft$, $g \longmapsto g^{-1}$

$\sum n_g g \longmapsto \sum n_g g^{-1}$ anti-auto.

Where B, C, D = torsion free groups. ($\Rightarrow$ Γ becomes π_1(asph.)).

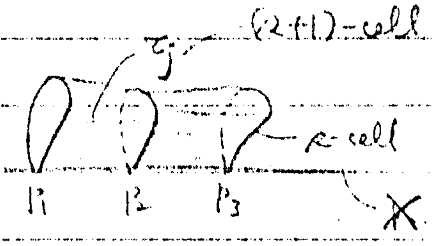
Cappell 1974, Topology (Nice paper)

Corr. ⇒ Ferry
lemma

Suppose $X \overset{he}{\rightleftarrows} K$ be given. (finite CW complexes)

Suppose $K \xrightarrow{\delta} X$ Then

$L =$



$G(\{p_i\})$ geometric gp

$$\begin{array}{ccc} C_{r+1} & \longrightarrow & C_r \\ \tau_j & \longmapsto & \partial(\tau_j) = a_j \end{array}$$

Then $\{a_j\}$ becomes a ε-basis of $G(\{p_i\})$

$$p_j = \Sigma A_{ij} a_i$$

So $f : p_j \to a_j$ is an ε-auto ($\varepsilon = \delta$ in Corr.), $\therefore \hat{f} = 0$

$$\begin{array}{r} \| \\ \tau(K,X) = \tau(L,X) \end{array} \checkmark$$

$X^n \subseteq \mathbb{R}^{2n+2}$ embed.

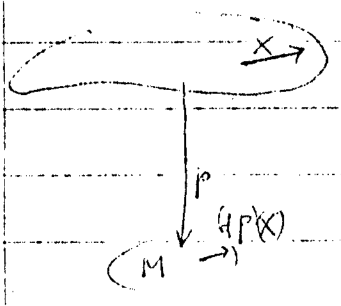
By adjusting metric on $\mathbb{R}^{2n+2}$, we can embedd $X^n \subset D^{2n+1}$, with Lipschiz condition on both sides

So we can assume $X = $ Disk.

* *

<u>Problems</u> (Pontrjagin class) $\pi_1 M = \Gamma$

Suppose M^n is a closed ~~asph.~~ mfd with universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$, Is it always possible to pick the diffeo $f : \mathbb{R}^n \to \tilde{M}$ to be expanding. i.e, $\exists\ \varepsilon > 0$ such that $|df(X)| > \varepsilon$ for each tangent vector X to $\mathbb{R}^n$ such that ~~the le~~ ~~of~~ $|X| = 1$ where the metric $|\,|$ on $\tilde{M}$ is induced from a Riema metric on M^n.



Define $|X| = |dp(X)|$.

This does not depend on $\nexists$ metrics on M

Exmple (True) M: non-positive curvature valued mfd.

Take exponential map.

◎ Homotopy invariance of pontrjagin class.

<u>Result</u> If N mfd with $\pi_1 N = \Gamma$.

$$\begin{tikzcd} N' \arrow[r, "f"'] & N \arrow[dr, "\varphi"] & \\ & & B\Gamma = M \end{tikzcd}$$

hty equivalence

(Novikov Conjecture) ~~By Novikov~~.

$L(N) \cup \varphi^*[M^n]$ is hty invariant.

i.e, $\langle L(N) \cup \varphi^*(M^n), [N] \rangle \in \mathbb{Q}$ then (L-genus is homeo-invaria by Novikov.

$$\|$$

$\langle L(N') \cup f^*\varphi^*(M), [N'] \rangle$

$G(S) \circlearrowleft f$ auto.

f gives rise to a set function $\underline{f}$: [illegible] S

by $\underline{f}(p) = \text{car.}\, f(p)$ and $\underline{f}(T) = \bigcup_{p\in T} \underline{f}(p)$ for $T \subseteq S$.

$E : G(S) \circlearrowleft$ is elementary (auto) if the assoc. matrix is either

1) diagonal with ± 1.

or 2) $I + aE_{ij}$ (elementary matrix in the sense of def. of K_1)

$H : G(S) \circlearrowleft$ is a deformation if $H = E_1 E_2 \cdots E_n$, E_i = elementary

(Even if $E_1 \cdots E_n = F_1 \cdots F_m$, we consider them differently unle

diam. of the deformation is ~~max~~

$$\text{diam}(E_1 E_2 \cdots E_n) = \max_{p\in S} \text{diam}\, \underline{E_1}(\cdots (\underline{E_{n-1}}(\underline{E_n}(p))\cdots)$$

e.g. $p_1 \ p_2 \ p_3 \ \cdots \ p_{n-1} \ p_n$

$E_i = E_{i-1,i} = \begin{pmatrix} 1 & & & \\ & \ddots & (1) & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ $i-1, i$ Then $\underset{2}{\underline{E_1}}(\cdots \underline{E_n}(p_n)\cdots) = S$

$E_{n-1} \cdots E_n(p_n)) \neq S$ $\{p_{n-1}, p_n\}$

$E_1 \cdots E_n$ is an $\boxed{\varepsilon\text{-deformation}}$ if $\text{diam}(\overset{H}{E_1 \cdots E_n}) < \varepsilon$ & $\text{diam}(\overset{H^{-1}}{E_n \cdots E_1}) <$

p. 73

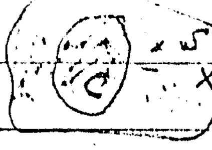
Thm (Quinn) Given $\varepsilon > 0$ and X, there $\exists\, \delta > 0$ such that $\forall$ δ-auto $f : G(S) \circlearrowleft$ is "stably" ε-deformation

$f : G(S) \circlearrowleft$ stably ε-auto — $\exists\, G_0$ geometric gp such that

$f \oplus id : G(S) \oplus G_0 \circlearrowleft$ is $E_1 \cdots E_n$ & this is is ε-defo

For (Quinn's) $\Rightarrow$ $\overset{\text{Conjecture}}{\text{(Ferry's)}}$ Take $\varepsilon = \delta$ and, and $\widehat{E_1 \cdots E_n} = \hat{E_1}$
is garanteed by ε-deformation (No upper bound of n)

The pf is enough only to show $\underline{X = \text{Disk}}$.



Def. Let C be a subset of X $\overset{\text{finite}}{S} \subseteq X$. Then
$f : G(S) \circlearrowleft$ — homo.

is an $\underline{\varepsilon\text{-auto}\ \boxed{\text{over } C}}$ if $\exists\ g : G(S) \circlearrowleft$ (group $\underline{\text{homo}}$) such th

$\uparrow$ not auto

$$pfgi = id \quad \& \quad pgfi = id_{G(S\cap C)}$$

where $G(S\cap C) \xhookrightarrow{i} G(S) \xrightarrow[\text{proj}]{p} G(S\cap C)$ $\underline{\text{dim} f, \text{dim} g < \varepsilon}$

$*$ $(f) = \left(\begin{array}{c|c} A & \\ \hline & \end{array}\right)$ If A is invertible, $\leftrightarrows$ then f is ε-auto over C.
sufficient

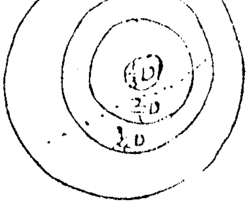
($\text{diam} f = \sup\limits_{\alpha} \text{diam}\{$

The pt is :

$$g \circ f = \left(\begin{array}{c|c} I & \\ \hline & \end{array}\right) \quad \& \quad f \circ g = \left(\begin{array}{c|c} I & \\ \hline & \end{array}\right)$$

auto (over $\frac{3}{4}D^n$)

Lemma Given n and $\varepsilon > 0$, $\exists\ \delta > 0$ such that any δ-auto
geometric gp $G(S)$ (based on D^n) over $\frac{3}{4}D^n$ is, after an
ε-$\overset{\text{deformation}}{\text{auto}}$ $\wedge$, blocked wrt $\underset{\underset{S_1}{\cup}}{\frac{2}{3}D}$ and $\underset{\underset{S_2}{\cup}}{D - \frac{1}{3}D}$.
and stabilization.



i.e, $\exists$ a geom gp $G_0 = G(S')$
such that
i) where $S' \subseteq \frac{2}{3}D^n$ and an
ε-deform. $H = E_1 \cdots E_m$ on $G(S) \oplus G(S'$
such that
$$H \circ (f \oplus id) = f_1 \oplus f_2$$

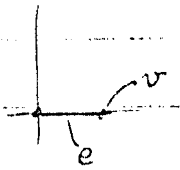
where $G(S \cup S') = G(S_1) \oplus G(S_2)$,
$S_1 \subseteq \frac{2}{3}D$, $S_2 \subseteq D - \frac{1}{3}D$.
and H leaves everything outside $\frac{2}{3}D$ fixed.

§4

Pf. of Thm ($n = 2$. i.e. on D^2)

Call the integral lattice pts "vertices" on $\mathbb{R}^2$ v

line segments (horizontal or vertical) of length ~~the~~ connecting
lattice pts. "edges" e



This gives a cell structure on $\mathbb{R}^2$

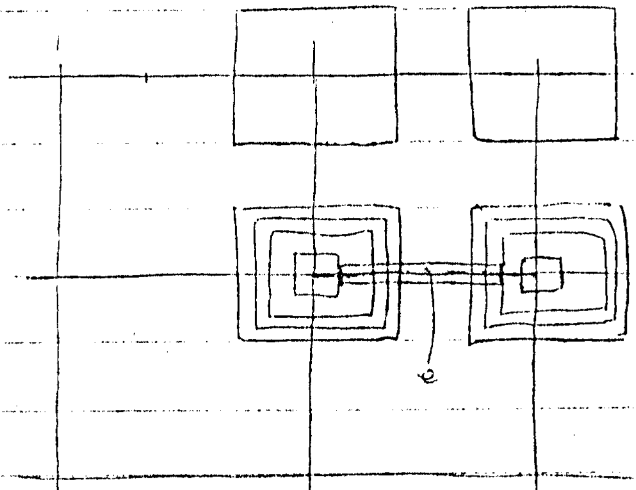
Now, we give "dual cell str. of this"

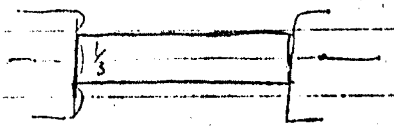
D_v = the square centered at v with side length $2/3$.

D'_v, D''_v, D'''_v, for other squares centered at v with side length $\frac{3}{4}\left(\frac{2}{3}\right)$, $\frac{2}{3}\left(\frac{2}{3}\right)$, $\frac{1}{3}\left(\frac{2}{3}\right)$.

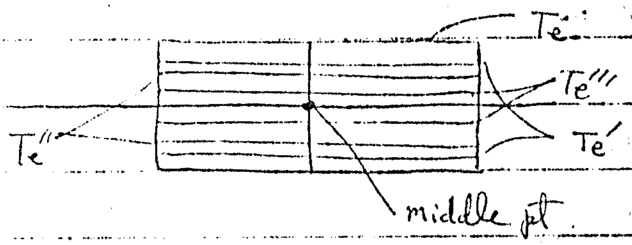
For each edge e, associate a rectangle T_e ~~contai~~ connecting D'''_{v_0} to D'''_{v_1} where v_i are the end pts of e and width of T_e is

$$\frac{1}{3} \text{ of } \frac{1}{3}\left(\frac{2}{3}\right) = \frac{2}{27}$$

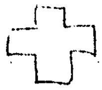




also associate rectangles T_e', T_e'', T_e''' with width $\frac{3}{4}\left(\frac{2}{27}\right)$, $\frac{2}{3}\left(\frac{2}{27}\right)$, $\frac{1}{3}\left(\frac{2}{27}\right)$.



Dual 2-cell



Let $\mathcal{D}_s$ be the square with vertices $(0,0)$, $(s,0)$, $(0,s)$ an
(s,s) where s is an integer.

Pick s so large that $s\varepsilon > 1$

It is enough to find $\delta > 0$ such that any δ-auto (base
on $\mathcal{D}_s$) is stably 1-deformation.

$$\mathcal{D}_s \longrightarrow D$$
$$x \longmapsto \frac{x}{s}$$

Then 1-auto becomes $\frac{1}{s}$-auto $\left(\frac{1}{s} < \varepsilon\right)$.

Idea of proof: For any $f: G(S) \circlearrowleft$ δ-auto, $\overset{f \oplus id}{H \circ f} = f_1 \oplus f_2 \oplus f_3$
so that $G(S) = G(S_1) \oplus G(S_2) \oplus G(S_3)$ and $S_1 \subseteq \bigcup_{v \in \mathcal{D}_s} D_v''$,
$S_2 \subseteq \bigcup_{e \in \mathcal{D}_s} Te''$, $S_3 \subseteq \mathcal{D}_s - \left(\bigcup D_v''' \cup \bigcup Te'''\right)$

If $\delta < \frac{1}{100}$ (sufficiently small) then each $G(S_1 \cap D_v'')$ is left invariant. But $Wh(1) = 0$. $f_{1,v}$ is a 1-deformation.

Suppress "stabilizing"

Given $\delta_1 > 0$, if δ is small enough then for each $v \in \mathcal{D}_s$,
$pfi: G(S \cap D_v) \hookrightarrow G(S) \xrightarrow{f} G(S) \longrightarrow G(S \cap D_v)$ is a δ_1-auto
(based on D_v) over D_v'.

Given $\varepsilon_1 > 0$, $\exists\, \delta_1 > 0$ (by the main Lemma) such that:
if pfi is δ_1-auto over D_v', then $\exists$ ε_1-deformation H_v such th

$$H_v \circ pfi = f_{v,1} \oplus f_{v,2}$$

where

$$S \cap D_v = S_{v,1} \cup S_{v,2}$$

$$S_{v,1} \subset D_v'' \quad \text{and} \quad S_{v,2} \subset D_v - D_v'''$$

and H_v fixing everything outside D_v''.

ε_1-deformation.

Let $H_1 = \prod_{v \in \mathcal{D}_2} (H_v)$. Then

$(H_1) f = f_1 \oplus f_2$, $S = S_1 \cup S_2$,

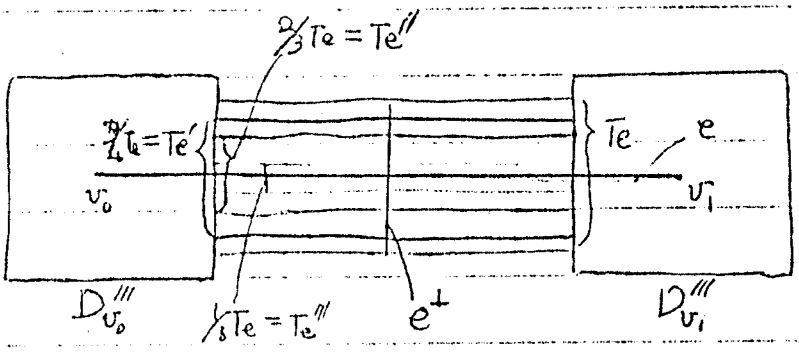
$S_1 \subset \bigcup_v S_{v,1}$, $S_2 \subset$ ~~$\bigcup_v S_{v,2}$~~ $S - S_1$

ε_1-deformation δ-auto

(since each H_v is ε_1-deform. & H_v fixes outside D''_v.)

f_2 is $(\delta + \varepsilon_1)$-auto

$G(S_2 \cap Te) \xhookrightarrow{i} G(S_2) \xrightarrow{f_2} G(S_2) \xrightarrow{p} G(S_2 \cap Te)$ is δ_2-auto of Te o



$$\begin{array}{ccccccc} Te''' & \subset & Te'' & \subset & Te' & \subset & Te \\ \| & & \| & & \| & & \\ \frac{1}{3} Te & & \frac{2}{3} Te & & \frac{3}{4} Te & & \end{array}$$

Let $g : Te \longrightarrow e^\perp$ projection

$\mathcal{S}_e = g(S_2 \cap Te) \subseteq D' = e^\perp$.

Form ~~[illegible]~~ $G(\mathcal{S}_e)$.

Since $p f_2 i : G(S_2 \cap Te) \to G(S_2 \cap Te)$ is δ_2-auto of Te over Te',

$\exists\ \varepsilon_2$ [illegible]

$Te' \mid p f_2 i = g_{1,e} \oplus g_{2,e}$

where $S_{1,e} \subset Te''$, $S_{2,e} \subset Te - Te'''$

$S_e = \bigcup S_{i,e} = S_2 \cap Te$

$f_2 = \prod g_e$ (ε-deformation)

ε_1-deformation.

Let $H_1 = \prod_{v \in \mathcal{D}_2} (H_v)$. Then

$$\boxed{H_1} f = f_1 \oplus f_2 , \quad S = S_1 \cup S_2 ,$$

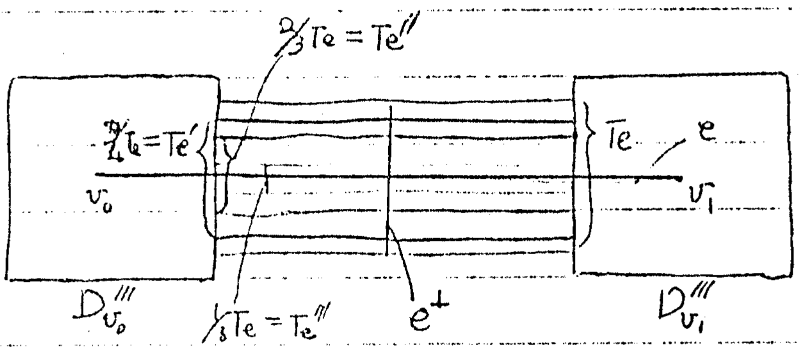
$$S_1 \subset \bigcup_v S_{v,1} , \quad S_2 \subset \overset{S - S_1}{\cancel{\bigcup_v S_{v,2}}}$$

ε_1-deformation $\quad \delta$-auto

(since each H_v is ε_1-deform. & H_v fixes outside D''_v.)

f_2 is $(\delta + \varepsilon_1)$-auto

$$G(S_2 \cap Te) \overset{i}{\hookrightarrow} G(S_2) \xrightarrow{f_2} G(S_2) \xrightarrow{p} G(S_2 \cap Te)$$ is δ_2-auto of Te



$$\begin{array}{ccccccc} Te''' & \subset & Te'' & \subset & Te' & \subset & Te \\ \| & & \| & & \| & & \\ \frac{1}{3} Te & & \frac{2}{3} Te & & \frac{3}{4} Te & & \end{array}$$

Let $g : Te \longrightarrow e^{\perp}$ projection

$$S_e = g(S_2 \cap Te) \subseteq D' = e^{\perp}.$$

Form ~~[illegible]~~ $G(S_e)$.

Since $p f_2 i : G(S_2 \cap Te) \longrightarrow G(S_2 \cap Te)$ is δ_2-auto of Te, [illegible]

[illegible]

$$Te' \cap p f_2 i = g_{1,e} \oplus g_{2,e}$$

where $S_{1,e} \subset Te''$, $S_{2,e} \subset Te - Te'''$

$$[illegible] = S_2 \cap Te$$

$$g_2 = \prod g_{2,e} \quad ([illegible]\text{-deformation})$$

$H = H_2 \circ H_1$

$$H \circ f = H_2 \circ H_1 \circ f = \varphi_1 \oplus \varphi_2 \oplus \varphi_3$$

$$\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = S$$

$$\mathcal{S}_1 \subset \cup D_v'' , \quad \mathcal{S}_2 \subset \cup T_e' , \quad \mathcal{S}_3 \subset \text{cplmt of } \cup D_v''' \cup \cup T_e'''$$

and $\varphi_i : G(\mathcal{S}_i) \circlearrowleft$

Let

$$H_3 = \varphi_1 \oplus \varphi_2 \oplus \varphi_3 .$$

QED.

Lemma (Kirby's trick) 8.5 Quinn

Given $\varepsilon > 0$ and n, $\exists\, \delta > 0$ such that any δ-auto of $\frac{2}{3}D^n$ is "almost" stably extendable to an ε-auto of D^n.

That is, $\exists$ finite set $S' \subset \frac{2}{3}D^n$

$g : G(S') \circlearrowleft$ ε-auto

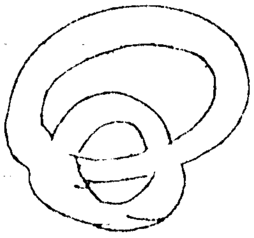
such that (i) $S \cap \frac{1}{2}D^n = S' \cap \frac{1}{2}D^n$

(ii) $f = g$ when restricted to $G(S \cap \frac{1}{2}D^n)$

Pf. $T^n - pt$ can be immersed in $\mathbb{R}^n$. Let

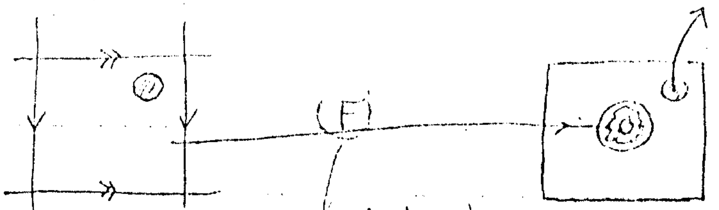
$$F : T^n - pt \longrightarrow \mathbb{R}^n$$

be immersion (C^ω local diffeo).



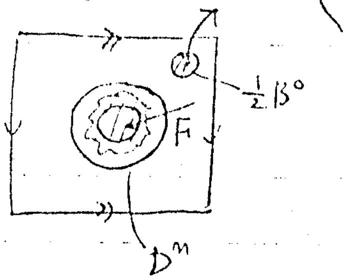
immersed punctured torus

$T^n = \mathbb{R}^n / 2\mathbb{Z}^n$ ~~[illegible]~~



$T^n - \frac{1}{2}B^\circ$ $\quad (T^n - \frac{1}{2}B^\circ) \longrightarrow \frac{2}{3}D^n \subset D^n$

We may assume $\left\langle \begin{array}{l} \operatorname{im} F \subset \frac{2}{3}D^n \\ F = id \text{ on } \frac{1}{2}D^n. \end{array}\right.$



comes from F immersion.

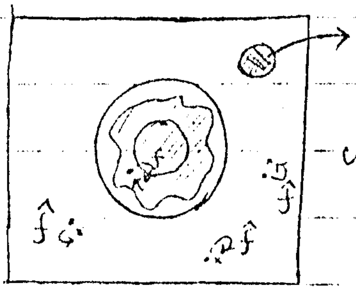
(*) Let $\varepsilon > 0$ be given. $\exists\, \delta > 0$ such that : $\forall x \in T^n - \frac{1}{2}B^\circ$, $|y - F(x)| < \delta \Rightarrow \exists ! \; y' \in Ball_{(\frac{\varepsilon}{2})}(x)$ such that $F(y') = y$. (use local

should be changed.

Let $G(S)$ and $f: G(S) \circlearrowleft$ be given geometric group and δ-auto ~~over~~ of D^n over $\frac{2}{3}D^n$. Let

$\textcircled{\hat{S}}$ $\hat{S} = F^{-1}(S)$ finite subset of $T^n - \frac{1}{2}B^0$

$\textcircled{\hat{f}}$ Define $\hat{f}: G(\hat{S}) \circlearrowleft$ "$\frac{\varepsilon}{2}$-auto" as follows:



using the property (*).

In this way, f lifts to a $\frac{\varepsilon}{2}$-auto (on $G(\hat{S})$) over $T^n - (\frac{1}{2}$
[F_1 is not local differ on $\partial(\frac{1}{2}B)$, so $\hat{f}$ cannot be $\frac{\varepsilon}{2}$-auto
all $G(\hat{S})$] Note that $\hat{f} = f$ on $G(S \cap \frac{1}{2}D)$

$\textcircled{\bar{f}}$ claim $\exists\, \bar{f}: G(\hat{S}) \circlearrowleft$ ε-auto such that $\bar{f} = \hat{f} = f$ on $\frac{1}{2}$

$\therefore$ $\hat{f}|G(\hat{S} \cap T^2 - B^0)$ is 1-1 group homo into $G(\hat{S})$
since $T^n - B^0$ is away $(\frac{1}{2} > \varepsilon \frac{\varepsilon}{2})$ from the fringe $\partial(\frac{1}{2}B)$.
$\hat{f}(G(\hat{S} \cap T^n - B^0))$ is a direct summand of $G(\hat{S})$ and contains
$G(\hat{S} \cap T - \frac{3}{2}B)$, so complementary gp is free, part of geom

$$\bar{f} \circlearrowleft G(\hat{S}) = \underset{\circlearrowleft \hat{f}}{\hat{f}(G(\hat{S} \cap T^n - B^0))} \oplus \underset{\circlearrowleft \mathrm{id}}{G(S')}$$

?

$\textcircled{\hat{\hat{S}}}$ Let $\hat{\hat{S}} = p^{-1}(\hat{S})$, $p: \mathbb{R}^n \to T^n$ projection of universal cov
infinite set.

$\textcircled{\bar{\bar{f}}}$ Define $\bar{\bar{f}}: G(\hat{\hat{S}}) \circlearrowleft$ ε-auto lifting of $\bar{f}$ to $\mathbb{R}^n$
Note: $G(\hat{\hat{S}})$ is free f.g. $\mathbb{Z}T^n$-module (T^n = free abelian gp of r
also $\bar{f}$ is a $\mathbb{Z}T^n$-iso

Pick a $\mathbb{Z}T^n$-basis $\{p_1, p_2, \cdots, p_m\}$. Then $\bar{\bar{f}}$ is a m
matrix with $\mathbb{Z}T^n$-entries.

$\bar{\bar{f}} \in GL(\mathbb{Z}T^n)$. Since $Wh(T^n)=0$ (by Bass-Heller-Swan), exist D=diagonal, $E_1, \ldots, E_r$=elementary such that

$$\bar{\bar{f}}^{-1} = D E_1 E_2 \cdots E_r.$$

$$\begin{pmatrix} 1 & 2\alpha+3\beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

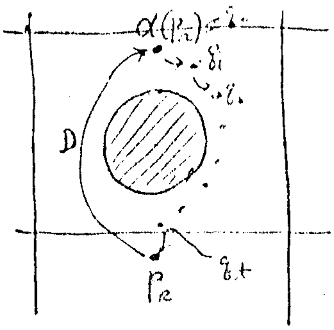
i.e., $D E_1 \cdots E_r \bar{\bar{f}} = id$ on $G(\hat{S})$. Let

$\bar{\bar{\bar{f}}}$

$$\bar{\bar{\bar{f}}}^{-1} = Q \bar{\bar{f}}^{-1} = (QD) E_1 \cdots E_r$$

where if

$$D = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \; \alpha \end{pmatrix}$$



Pick an ε-chain $\{g_0=\alpha(p_r), \ldots, p_r\}$ ~~[illegible]~~ with all translat

S'

Let $Q : G(\cancel{S'})$ ~~[illegible]~~ $G(S') \circlearrowleft$

given by

$$Q = \begin{array}{c|c} & \begin{matrix} g_0 & g_1 & & & p_r \end{matrix} \\ \begin{matrix} g_0 \\ g_1 \\ \\ \\ g_t \end{matrix} & \begin{pmatrix} 0 & 0 & & & \alpha^{-1} \\ 1 & 0 & & & \\ & 1 & & & \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix} \end{array}$$

$\alpha(g_t)=p_r$
$=\alpha^{-1} g_0$

using the basis
$\{g_0 \, g_1 \cdots g_t\}$

Then Q is ε-auto.

$\therefore \bar{\bar{\bar{f}}}$ is also ε-auto.

Now
$$QD = \left(\begin{array}{c|c} D & \\ \hline & Q \end{array}\right) = \left(\begin{array}{c|c} 1 \ddots 1 \, \alpha & \\ \hline & \begin{matrix} 0 & & \alpha^{-1} \\ 1 & \ddots & 0 \end{matrix} \end{array}\right) \longrightarrow \left(\begin{array}{c|c} 1 \ddots 1 \, \alpha & \\ \hline & 1 \ddots \alpha^{-1} \end{array}\right) = \begin{pmatrix} D & \\ & D^{-1} \end{pmatrix}$$

~~= product of~~

$$= \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1} & I \end{pmatrix} \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$ product of matrices

Let $\hat{\hat{S}} = \hat{S} \cup S'$. Then $\bar{\bar{\bar{f}}} : G(\hat{\hat{S}}) \circlearrowleft$ ε-auto.

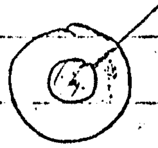
$\bar{\bar{\bar{f}}}^{-1}$ = ([illegible]) $E_1 E_2 \cdots E_r$.

Let's denote $\overset{\equiv}{f}^{-1}$ again by $E_1 E_2 \cdots E_n$

Let $H : \mathbb{R}^n \longrightarrow (D^n)^0$ be a homeo such that $H|\frac{1}{2}D^n = id$

$\textcircled{f_*}$ $(f_*) : G(H(\hat{\hat{S}})) \circlearrowleft$ $\quad \overset{\equiv}{f}$ pushed into $G(H(\hat{\hat{S}}))$

Note that $f_* = f$ on $\frac{1}{2}D^n$.



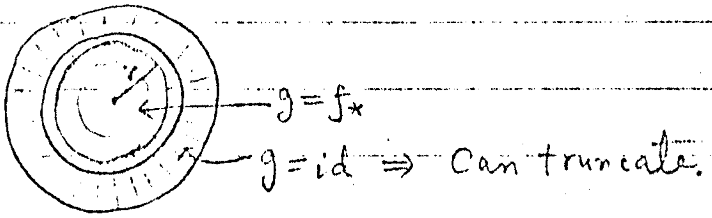
Let

$$\hat{E}_i(p) = \begin{cases} E_{i_*}(p) & \text{if } p \in D^n - rD^n. \\ p & \text{if } p \in rD^n. \end{cases}$$

$\textcircled{g}$ Then $g \equiv \hat{E}_1 \hat{E}_2 \cdots \hat{E}_n (f_*)^{\overset{\equiv}{f}}$ is the desired map.

$$= \begin{cases} \hat{E}_1 E_2 \cdots E_n f_* = id \text{ on } (D^n - rD^n) \\ id f_* = f_* \text{ on } rD^n. \end{cases}$$

So we can truncate $H(\hat{\hat{S}})$ by $\frac{2}{3}D^n$, which is finite



Pf. of main Lemma

Start with $f : G(S) \circlearrowleft \xrightarrow{\text{last lemma}} \exists \hat{S} \subset \frac{2}{3}D^n$, $\exists g : G(\hat{S}) \circlearrowleft$ r-auto su

$g|\frac{1}{2}D = f|\frac{1}{2}D$.

Then $\begin{pmatrix} I & -g \\ 0 & I \end{pmatrix}$ is a r-deformation

$\therefore \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ is $6r$-deformation

$$\begin{pmatrix} I & & \\ & I & \\ & & f \end{pmatrix} \xrightarrow[S'US'US]{} \begin{pmatrix} g & & \\ & g^{-1} & \\ & & f \end{pmatrix}$$

$S' = \hat{S} \cup \hat{S}$ — slightly moved away.

$S \cup S' = S \cup \hat{S} \cup \hat{S}$

$f \oplus g^{-1} \oplus g : G(S \cup \hat{S} \cup S^{-1}) \circlearrowleft$

$\exists\ H_1$: small $\overset{6\gamma}{\text{deformation}}$ $\ni$ $H_1 \circ (f \oplus id \oplus id) = f \oplus g^{-1} \oplus g$

$$H_1 = \begin{pmatrix} 1 & & \\ & 1 & -g^{-1} \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & g & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & -g^{-1} \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} S & \hat{S} & \hat{S} \end{array} \\ \begin{pmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{pmatrix} \end{array} \begin{array}{c} S \\ \hat{S} \\ \hat{S} \end{array} = \Big($$

$f \oplus g^{-1} \big| \tfrac{1}{2} D = f \oplus f^{-1} \big| \tfrac{1}{2} D$ since $f = g$ on $\tfrac{1}{2} D$. — auto on whole

$\exists\ H_2$: $\cdots\cdots \to H_2 \circ ((f \oplus g^{-1}) \oplus g) = f_1 \oplus f_2$ — not auto on but on som

$f_1 = H_2 (f \oplus g^{-1} \big| \tfrac{1}{3} D^n \oplus g)$ with $S_1 = S \big| \tfrac{1}{3} D^n \cup S \big| \tfrac{1}{3} D^n \cup \hat{S} \subset \tfrac{2}{3} D^n$

$(id \big| \tfrac{1}{3} D^n) \oplus g$

$f_2 = H_2 (f \oplus g^{-1} \big| D - \tfrac{1}{3} D)$

Let $\varphi = f \big| \tfrac{1}{2} D$. [really,

$$\begin{tikzcd} G(\hat{S}) \arrow[r, "\varphi"] \arrow[dr, "proj"'] & G(\hat{S}) \\ & G(\hat{S} \cap \tfrac{1}{2} D) \arrow[u, dashed, "f \big| \tfrac{1}{2} D"'] \end{tikzcd}$$

$$H_2^{-1} = \begin{pmatrix} 1 & -\varphi & 0 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ \varphi & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -\varphi & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \ddagger \\ & 1 & \\ & & 1 \end{pmatrix} , \qquad = \Big(\varphi$$

$$(H_2)^{-1} \tfrac{1}{3} D = \begin{bmatrix} f & & \\ & g^{-1} & \\ & & 1 \end{bmatrix} \Big| \tfrac{1}{3} D^n.$$

$$\begin{tikzcd}
 & S \arrow[d, "f"'] & \cup\ \hat{S} \arrow[d, "\mathrm{id}"'] & \cup\ \hat{S} \arrow[d, "\mathrm{id}"'] \\
f\oplus\mathrm{id}\oplus\mathrm{id} & S \arrow[d, "\mathrm{id}"'] & \hat{S} \arrow[d, "g^{-1}"'] & \hat{S} \arrow[d, "g"'] \\
H_1=\mathrm{id}\oplus g^{-1}\oplus g & S \arrow[d, "\overset{\varphi'}{\boxed{\varphi^{-1}}}"'] & \hat{S} \arrow[d, "\varphi"'] & \hat{S} \arrow[d, "\mathrm{id}"'] \\
H_2=\varphi^{-1}\oplus\varphi\oplus\mathrm{id} & S & \hat{S} & \hat{S}
\end{tikzcd}$$

$\varphi = f \big| f'(\frac{1}{2}D)$

$\varphi' = f' \big| f(\frac{1}{2}D), \quad ff'=f$

$= g' \big| g(\frac{1}{2}D) \quad g'=g^{-1}$

$S \supset f(\frac{1}{2}D) = g(\frac{1}{2}D) \subset \hat{S}$

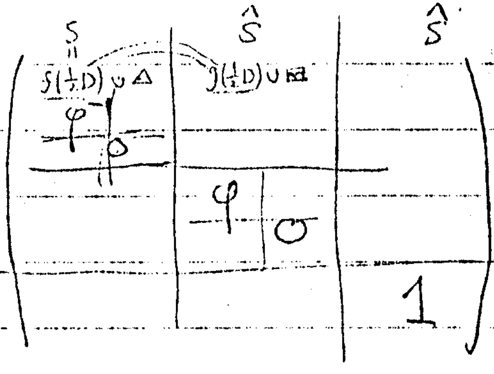
$$f_1 = \boxed{\begin{array}{ccc} \text{on } \frac{1}{3}D & \frac{1}{3}D & D \\ \hline \mathrm{id} & \mathrm{id} & g \end{array}}$$

$$f_2 = \boxed{\begin{array}{ccc} \text{on } D-\frac{1}{3}D & D-\frac{1}{3}D & \\ & & \end{array}}$$

$\overbrace{\text{id.} \quad \text{id}}$

on $\frac{1}{2}D$, so can be flocked.

$H_2 =$



$S = f(\frac{1}{2}D) \cup \triangle$

$\hat{S} = g(\frac{1}{2}D) \cup \square$

$\hat{S}$

can be repr. by
product of 6 ele
matrices

Th 1 If Γ is torsion free v. abelian f.g. group, then $Wh(\Gamma)=$
(published)

Th 2 If M^n is a closed mfd top'l $(n \neq 3, 4)$, then M has a flat ri
structure iff (i) $\pi_1 M$ is v. abelian
(ii) $\pi_i M = 0$, $i > 0$ (partially

Th 3 (Extension of 1) If Γ is v. poly $\mathbb{Z}$ ~~f.g.~~ torsion-free, then

Th 4 (〃 2) If M^n is an infra-nil-mfd $(n \neq 3, 4)$ and
a closed ~~mfd~~ asph. mfd with $\pi_1 N = \pi_1 M$, then $N \cong M$

G : conn. s.c. nilpt Lie gp. ($\Rightarrow$ G is non cpt)
$G \rtimes K$ — cpt Lie gp.
$\overset{\vee}{\Gamma}$ = discrete cocpt torsion free
$\Gamma \backslash G \rtimes K / K$ infra-nil mfd.

Conjecture (Generalization of 4)
If M^n, N^n are closed asph mfds $n \neq 3, 4$ with $\pi_1 N = \pi_1 M$ and $\pi_1 N$
poly-$\mathbb{Z}$, then $M \cong M$ (homeo).
(when index is odd, it

$\Gamma = \pi_1$ (asph)

If $\mathcal{S}(B\Gamma) = 0$, then we can calculate $\mathcal{S}(M)$ for $\pi_1 M = \Gamma$.

M = orientable mfd. closed.

$\mathcal{S}_{Top}(M)$ } assume s.h.e. $N \xrightarrow{f} M \ni \tau(f) = 0 \in$

$\mathcal{S}_0(M)$ smooth

Wall-Sullivan Exact seq. $(n \geq 5)$

simple

$$\begin{tikzcd}
L^s_{n+1}(\mathbb{Z}\pi_1 M) \arrow[r, "\varphi"] \arrow[dr] & \mathcal{S}^s_{Top}(M^n) \arrow[r, "\phi"] & {[M^n, G/Top]} \arrow[r, "\sigma"] & L^s_n(\mathbb{Z}\pi_1 M) \\
& \mathcal{S}_0(M_n) \arrow[r] & {[M^n, G/O]} \arrow[ur] &
\end{tikzcd}$$

: exact seq

$\alpha \in [X, G/O]$

α is the equiv. class of stable orthogonal vector bundle v

together with a specific fiber hty trivialization

$$\begin{tikzcd}
\mathbb{R}^n \arrow[d] & \\
E \arrow[r, "f"] \arrow[d] & X \times \mathbb{R}^n \arrow[dl] \\
X &
\end{tikzcd}$$

f: proper hty equiv. — $f^{-1}(cpt)$ is cpt

$$\begin{tikzcd}
& G/O \arrow[d] \\
& BG \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] \arrow[uur, bend left, dashed, no head] & BG
\end{tikzcd}$$

$KO(X) =$

equiv. iff

$$\begin{tikzcd}
\mathbb{R}^n \arrow[dd] & & \\
& \mathbb{R}^n \arrow[dd] & X \times \mathbb{R}^n \arrow[dd] \\
E \arrow[urr] \arrow[dr, dashed] \arrow[dd] & & \\
& E' \arrow[uur] \arrow[dd] & X \\
X & & \\
& X &
\end{tikzcd}$$

Vector bdle iso.

Defn. of ψ (In the case $\mathcal{S}_0(M)$)

Given $N^n \xrightarrow[f]{} M^n$, to define $\psi(f)$

Let $g : N \leftarrow M$ be hty inverse of f.

$g^*(\tau(N) - \tau(M)$

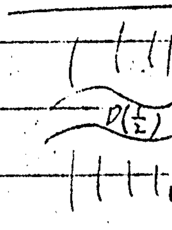
$$\begin{tikzcd} & N^n \times D^{n+2} \arrow[r, phantom, "\supset"] \arrow[dl, equal, "\text{by s-cobor thm}"'] & M \text{ embedding} \\ D(\nu_g) & \mathbb{D}(\nu_g) \arrow[u, phantom, "\cup"] \arrow[ur, "g"'] & \end{tikzcd}$$

disk bundle of ν_g

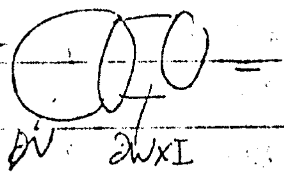
$$\begin{tikzcd} \nu_g \arrow[d] \\ M \end{tikzcd}$$ normal bundle of g.

$N \times D^{n+2} - D(\nu_g)^0$

$$\begin{tikzcd} \arrow[d, leftrightsquigarrow] \\ D(\nu_g) \overset{\text{s-cobor thm}}{=} N^n \times D^{n+2} \end{tikzcd}$$



like



$$\begin{tikzcd} D(\nu_g) = N \times D^{n+2} \arrow[r] \arrow[d] & D^{n+2} \\ M. & \end{tikzcd}$$

get proper degree 1 map

$$\begin{tikzcd} E(\nu_g) \arrow[r] \arrow[d] & R^{n+2} \times M \\ M. & \end{tikzcd}$$

make f transverse to $M \times 0$

Given $\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & M \times R^{k} \\ M & \end{tikzcd}$, get $\begin{tikzcd} N^n \subset E \arrow[r, "f"] \arrow[d] & M \times R^k \arrow[d, phantom, "\cup"] \\ M & M \times 0 \end{tikzcd}$

~~make~~ get $N^n \subset E \xrightarrow{f} M^n \times 0$ deg 1

$\exists$ a O-bdle η over M and a specific bdle $\cong$ of $f^*(\eta)$ to the stable ~~bdle~~ normal bdle of $N^n \subseteq R$

$L_0(R)$ — ring with involution ($R = \mathbb{Z}\Gamma$, with $\gamma \to \gamma^{-1}$)

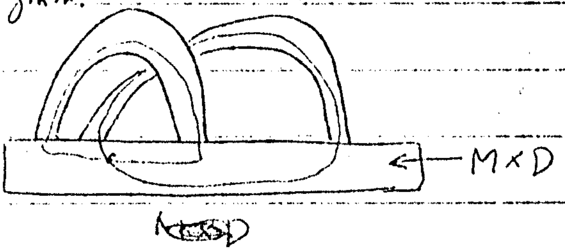
symm. invertible matrices of R

$S_0 \sim S_1$ if $\exists A \ni AS_0A^* = S_1$

$L_2(\mathbb{Z}(1)) = L_2(\mathbb{Z})$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ skew symm.

$M = [0, 1]$



$$\begin{tikzcd} N \arrow[d] \\ M \end{tikzcd}$$

W^2 cobordism $N \xrightarrow{f} M$

top.

$\pi_0(\mathrm{Homeo}^+(M^n))$

$\shortparallel$

$\mathcal{S}_{Top}(M$

Let $M = S^n$. By Smale $\mathcal{S}_{Top}(S^n) = 0$

only (Top'l!)

$\therefore \pi_n(G/Top) = L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \pmod 4 \\ \mathbb{Z}_2 & 2 \quad " \\ 0 & \text{otherwise} \end{cases}$

If $\pi_1 M = 0$, the seq becomes short exact seq (e

$0 \to \mathcal{S}(M) \to [M^n, G/Top] \to L_n(\mathbb{Z}) \to 0$

characteristic Variety Thm (Sullivan).

For $\forall$ fin cx X.

(i) $[X, G/Top] \otimes \mathbb{Q} = \overset{(4)}{\underset{i>1}{\bigoplus}} H^i(X; \mathbb{Q})$

(ii) $[X, G/Top] \otimes \mathbb{Z}[\frac{1}{2}] = KO(X)_{\otimes \mathbb{Z}[\frac{1}{2}]}$ stable o.r. vector bdles over

(iii) [illegible]

Let M^n be a closed mfd $(n \geq 5)$, then there is a long sequence

$$\longrightarrow \mathcal{S}(M\times D^1,\partial) \xrightarrow{} [M\times D^1, G/Top] \xrightarrow{\partial} L_{n+1}(\pi_1 M) \longrightarrow \mathcal{S}_{Top}(M) \longrightarrow$$

$$\longrightarrow [M, G/Top] \longrightarrow L_n(\pi_1 M)$$

✓ $[M\times D^k, \partial, G/Top]$ = f.g. abelian gp. ($\mathcal{H}^0(\ )$)

$L_k(\pi_1 M)$ is not f.g. in general.

e.g. $L_{(2)}(T_2 * T_2)$ not f.g. $= T_2 \oplus T_2 \oplus \cdots \oplus$ another stuff. (infinite)

\ infinite dihedral gp

so $\mathcal{S}_{Top}(M)$ is not f.g. eg $\mathcal{S}_{Top}(p^n \# p^n)$.

$$\underset{n=9}{L_{10}}(\pi_1\, p^9 \# p^9) \longrightarrow \mathcal{S}_{Top}(p^7 \# p^7) \longrightarrow \text{f.g.}$$

not f.g.   not f.g   Cappell Top. J.

maybe

$L(\pi) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\frac{1}{2}]$ is f.g. over $\mathbb{Z}[\frac{1}{2}]$?

In L-gp, $L(\pi, \omega)$ $\pi \xrightarrow{\omega} \mathbb{Z}$ orientation

※ so, in $L(\pi)$ we are using trivial orientation.

Thm If Γ is a virtually abelian torsion-free f.g. group (

$B\Gamma$ is orientable flat mfd of dim n. Let $\pi_1 M^n = \Gamma$. Then

$\mathcal{S}(M)$ are calculable up to an extension. i.e,

$\exists$ natural homomorphisms (f_λ): $[M^m \times D^\lambda, M\times S^{\lambda-1}; G/Top] \longrightarrow [B\Gamma \times D^t, B$

defined when $m+\lambda > n$. and t such that $m+\lambda = n+t$.

such that the following is exact

$$0 \longrightarrow \text{coker } f_{\lambda+1} \longrightarrow \mathcal{S}(M\times D^\lambda, M\times S^{\lambda-1}) \longrightarrow \ker(f_\lambda) \longrightarrow 0.$$

$$\overset{BO}{}\qquad\qquad \text{extraordinary coh.}$$

$$[X, A ; G/Top] = \mathcal{H}^0(X, A)$$

coh — surgery map — homl

$$\begin{tikzcd}
{[M^m, G/Top]} \arrow[r, "\sigma"] \arrow[d, equal] & L_m(\pi_1 M) \\
\mathcal{H}^0(M^m) \arrow[d, "\cong", "P.D."'] & \\
\mathcal{H}_{-m}(M^m) \arrow[r, "\boxed{f_*}"] \arrow[uur, dashed, "\sigma_m"] & \mathcal{H}_{-m}(B\Gamma) \arrow[uu, dashed, "\text{Misenko, Wall}"']
\end{tikzcd}$$

cf — Riemann-Ro

$\mathcal{H}(B\Gamma \times D^t, B\Gamma \times S^{t-1}) = 0$

~~$f_*(\pi_1(\ldots))=0$~~

Also, for $B\Gamma$,

$$\begin{tikzcd}
{[B\Gamma, G/Top]} \arrow[r, "\cong"] \arrow[d, "\cong"] & L_n(\pi_1 M) \\
\mathcal{H}_n(B\Gamma) \arrow[ur, dashed, "\sigma_n"] &
\end{tikzcd}$$

~~$\mathcal{H}_n(B\Gamma$~~

$\sigma_n ; \sigma_{n+1}, \sigma_{n+2}, -$ are all iso

If $\pi_1 M = \Gamma$, then $\exists$ natural map $f : M \to B\Gamma$. get $\boxed{f_*}$

When tensoring $\otimes \mathbb{Q}$ to $\mathcal{H}_{-m}(M^m) \xrightarrow{f_*} \mathcal{H}_{-m}(B\Gamma)$, this es computing $H_*(M, \mathbb{Q})$, $H_*(\Gamma, \mathbb{Q})$ and ker and coker of f_*.

<u>Char. Variety Thm.</u> (Sullivan)

$A \subset X$

(i) $[X, A ; G/Top] \otimes \mathbb{Q} = \overset{\cdots}{\underset{i}{\oplus}} H^i(X, A ; \mathbb{Q})$

(ii) $\otimes \mathbb{Z}[\frac{1}{2}] = KO(X, A) \otimes \mathbb{Z}[\frac{1}{2}]$

(iii) $\otimes \mathbb{Z}_{(2)} = \overset{\cdots}{\underset{i}{\oplus}} H^{4i}(X, A ; \mathbb{Z}_{(2)}) \oplus \overset{\cdots}{\underset{0}{\oplus}} H^{4i+2}(X, A : \mathbb{Z}/2)$.

Pf of (i).

To defining $G/Top \xrightarrow{\mathscr{S}} \prod_{i=1}^{\infty} K(\mathbb{Q}, 4i)$

is the same as defining

$$\mathscr{S} \in \prod_{i=1}^{\infty} H^{4i}(G/Top, \mathbb{Q})$$

$$\begin{tikzcd}
\left(\bigoplus_{i=1}^{\infty} \Omega_{4i}(X)\right) \otimes_{\Omega_{4*}(pt)} \mathbb{Q} \arrow[r, phantom, "\simeq"] \arrow[dr, "\mathscr{S}"'] & \bigoplus_i H_{4i}(X, \mathbb{Q}) \arrow[d, "\mathscr{S}"] \\
 & \mathbb{Q}
\end{tikzcd}$$

$\Omega_{4*}(pt)$ acts on $\mathbb{Q}$

$M^{4n} \xrightarrow[I]{} I(M) \in \mathbb{Z} \subset \mathbb{Q}$
index.

$\therefore$ $\mathbb{Q}$ is $\Omega_{4*}(pt)$-module

So want to define

$$\Omega_{4i}(G/Top) \xrightarrow{\mathscr{S}} \mathbb{Q} \qquad [\text{put } X = G/Top]$$

$$\overset{\Psi}{M \xrightarrow{p} G/Top}$$

fiber hty equiv.

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & M^{4i} \times \mathbb{R}^n \\
M^{4i}
\end{tikzcd}$$

$N^{4i} \longrightarrow M^{4i}$

$$\mathscr{S}(E) = \langle L(E_{\cdot}) \cdot L(M), [M] \rangle$$

Actually $G/Top \to \prod K(\mathbb{Q}, 4i)$ is hE.

$$\pi_i(G/Top) \otimes \mathbb{Q} = \begin{cases} 0 & \\ \mathbb{Q} & i \equiv 0 \ (4) \end{cases}$$

$$\begin{tikzcd}
E \arrow[d] \\
S^{4i}
\end{tikzcd}$$

$P_i(E) \neq 0$ by Bott Periodicity Thm.

Easy to show $\mathscr{S}$ is iso.

10/25/78:

We are interested in the following

Thm: If Γ is a torsion-free crystallographic group then Wh

We have a s.e.s.

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 ;$$

$\mathbb{Z}^n$ is the maximal abelian subgroup of Γ, and G is fin

For $s > 1$, $s \in \mathbb{Z}$, define $\Gamma_s = \Gamma/(s\mathbb{Z})^n$, a finite group.

There is then a s.e.s

$$1 \longrightarrow (\mathbb{Z}_s)^n \longrightarrow \Gamma_s \xrightarrow{\varphi} G \longrightarrow 1 .$$

(*) If $s \equiv 1 \mod |G|$ then φ splits uniquely up to inn automorphism.

For $H^2(G;(\mathbb{Z}_s)^n)$ is a $\mathbb{Z}_s$-module, but its exponent divides $|G$
so $H^2(G;(\mathbb{Z}_s)^n) = 0$. Thus the sequence splits.
Similarly $H^1(G;(\mathbb{Z}_s)^n) = 0$ so every crossed homom is principal (see p. 52). Let ψ_1, ψ_2 be two splittings, and define $\eta : G \longrightarrow (\mathbb{Z}_s)^n$ by $\eta(g) = \psi_1(g)\psi_2(g)^{-1}$.
Then η is a crossed homomorphism (writing multiplicatively, with the action of G by conjugation written as exponentiation, $\eta(gh) = \eta(h)^g \cdot \eta(g)$).
Therefore ~~(writing additively)~~ $\exists\ t \in (\mathbb{Z}_2)^n$ s.t.
$\eta(g) = t - gtg^{-1}$ ~~or~~ (additively), i.e. $\eta(g) = t\,"(gtg^{-1})^{-1}" =$
$t\,\psi_2(g)\,t^{-1}\psi_2(g)^{-1}$. But $\eta(g) = \psi_1(g)\psi_2(g)^{-1}$, so
$\psi_1(g) = t\psi_2(g)t^{-1}$. This proves (*).

Let $G_s = \psi(G)$ for a splitting ψ. $G_s \cong G$, and G_s unique up to inner automorphism.

Lemma: If $x \in Wh\,\Gamma$ or $x \in \tilde{K}_0 \mathbb{Z}\Gamma$ then $\exists\, N_x \in \mathbb{Z}$ s.t. $\forall\, s > N_x$, $s \equiv 1 \mod |G| \Rightarrow \sigma^* x = 0$, where σ = inclusion: $\Gamma_{G_s} \subset$ where $\Gamma_{G_s} = \varphi^{-1}(G_s)$, where φ is projection in the short exact sequence $1 \to (s\mathbb{Z})^n \to \Gamma \xrightarrow{\varphi} \Gamma_s \to 1$. (prf

Note Γ_{G_s} is defined only up to inner auto, but it is easy to see that if $\sigma^* x = 0$ for one choice of Γ_{G_s} then $\sigma^* x = 0$ $\forall$ conjugate choices.

The proof of the lemma will be based on the Epstein-Shub Thm (p. 53) & the following.

Let W^{n+1} be an h-cobordism with base M^n. Suppose W ($\therefore$ M) has a riem. metric on it. (W,M) is an <u>ε-h-cobordism</u> if the following condition (*) is satisfied: for a deformation retraction h_t of W onto M, where $h_0 = 1_W$ and $h_1(W) = M$, define $\gamma_x(t) = h_t(x)$ $\forall\, x \in W$. Let $\alpha_x = h_1 \circ \gamma_x$, a loop in M. Then we demand that

(*) There is a choice of h_t s.t. the family of curves $\{\alpha_x : x \in W\}$ (where $\alpha_x(t) = h_1(h_t(x))$) all have arclength $< \varepsilon$.

<u>Thm</u> (Steve Ferry): Given M, $\exists$ $\varepsilon > 0$ s.t. any ε h-cobordism W with base space M is a product, ie $\tau(W,M) = 0 \in Wh(\pi_1$

<u>Rmks</u>: This theorem was conjectured by Connell and (pron. /kʌ by Far Hollingsworth sometime during the 60's. They reduced to an algebraic result which they could not prove but which was recently proved by Quinn. Ferry proof is different.

The references are

- Connell & Hollingsworth, Trans. AMS 1969
- Ferry, Annals 1977? (paper is concerned with mfds mod on the Hilbert cube)

The original conjecture & Quinn's result are stated b
Let M^n be a closed riem. mfd, $S=\{a_1,\dots,a_m\}\subset M$ a finite set of points.

Def: The geometric group on $\{a_1,\dots,a_n\}$, denoted G_S, or
The free abelian group generated by S.

Rmk: Define a sheaf $\mathcal{S}$ over M to have stalk $\mathbb{Z}$ over each a_i and 0 elsewhere. Then $G_S = \Gamma(M;\mathcal{S}) =$ group of global sections of $\mathcal{S}$. It may be fruitful to reformulate Quinn's work geometrically, & this view might be useful in such a reformulation.

Def: $\alpha: G_S \xrightarrow{\sim} G_S$ is an ε-automorphism if $\alpha\langle a_i\rangle$ and $\alpha^{-1}\langle a$ contained in $G_{S\cap B_\varepsilon(a_i)}$ where $B_\varepsilon(a_i)$ is the ε-d about a_i.

Def: $\alpha: G_S \xrightarrow{\sim} G_S$ is ε-blocked if $\exists$ a partition of $S=S_1+S_2+$ where diam $S_i<\varepsilon$, s.t. $\alpha(G_{S_i})\subset G_{S_i}$.

In matrix terms, α ($\therefore \alpha^{-1}$) has matrix $= \begin{pmatrix} \square & & & \\ & \square & & \\ & & \square & \\ & & & \ddots\ \square \end{pmatrix}$, where each block corresponds to $\alpha|_{G_{S_i}}$, diam $S_i<\varepsilon$.

Conj (Conner-Hollingsworth): Given M^n, $\varepsilon>0$ then $\exists\ \delta>0$ s.t. any δ-automorphism of any geometric group over M can be expressed as a product of $n+1$ ε-blocked automorphisms.

Rmk: This conjecture $\Rightarrow$ Ferry's Thm.

Quinn's recent result is slightly more general, using arbitrary coeffs. rather than just $\mathbb{Z}$.

One can clearly define a <u>geometric R-module</u> jus
as one defines a geometric group.

<u>Thm</u> (Quinn): If $R = \mathbb{Z}\Gamma$ s.t. $Wh(\Gamma \times T^n) = 0$, where T^n denotes the free abelian group of rank n, then the "generalised conjecture" (i.e, with "free abelian group" replaced by "free R-module") is true for R, M

Quinn's Thm $\Rightarrow$ C.& H. Conjecture by letting $\Gamma = \{1\}$ and noting that $Wh(T^n) = 0$ by B.H.S.

<u>10/27/78</u>: We leave the Connell-Hollingsworth-Quinn material & return to the thm, p. 77.

We will use:

<u>Thm</u> (Ferry): Given a closed riemannian manifold M^n, $\exists\ \varepsilon > 0$ s.t. any ε-h-cobordism with base M^n is a product.

<u>Thm</u> (Epstein-Shub): Given $s \equiv 1 \bmod |G|$, $\exists$ an expanding immersion $f_s : M^n \longrightarrow M^n$ where $M = \mathbb{R}^n/\Gamma$, s.t.

(i) $\| df_s(X) \| = s \| X \|$, and

(ii) $(f_s)_\#(\underset{\underset{\Gamma}{\|}}{\pi_1 M}) = \Gamma_{G_s}$ (see top p. 78, or below).

<u>Prf</u>: A more careful version of the proof, pp. 53-55.

<u>Lemma</u>: Let Γ be a torsion-free crystallographic group with holonomy group G (so $\exists$ a short exact sequence

$$1 \to \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

$\mathbb{Z}^n$: maximal free abelian $\subset \Gamma$, $|G| < \infty$).

Define Γ_s, where $s \equiv 1 \bmod |G|$, by the sequence

$$1 \longrightarrow (s\mathbb{Z})^n \longrightarrow \Gamma \xrightarrow{\varphi} \Gamma_s \longrightarrow 1,$$

so we also get a sequence

$$1 \to (\mathbb{Z}_s)^n \longrightarrow \Gamma_s \longrightarrow G \longrightarrow 1$$. (cf pp. 77-78).

This splits uniquely, up to inner automorphism; let G_s be the image of G in Γ_s, determined up to inner automorphism.

Let $\Gamma_{G_s} = \varphi^{-1}(G_s) \subset \Gamma$.

Let $x \in Wh\,\Gamma$ (resp. $\tilde{K}_0 \mathbb{Z}\Gamma$).
Then $\exists\, N_x \in \mathbb{Z}$ s.t. $\forall s \in \mathbb{Z}$ with $s > N_x$, $s \equiv 1 \mod |G|$, the transfer of x to $Wh\,\Gamma_{G_s}$ (resp. $\tilde{K}_0 \mathbb{Z}\Gamma_{G_s}$) vanishes.

Prf: Case 1: $x \in Wh\,\Gamma$.
Let $M = \mathbb{R}^n/\Gamma$ and let ε be given by Ferry's Thm.
$x \in Wh\,\Gamma$ defines an h-cobordism W^{n+1} with base M s.t. $\tau(W, M) = x$.
$\exists$ deformation retract h_t of W onto M with $h_0 = id_W$, $h_1(W) = M$. $\forall\, p \in W$ let α_p be the curve $\alpha_p(t) = h_1(h_t(p$
Write $|\alpha_p|$ = length of the curve α_p.
Let $\|\alpha_x\| = \max_{p \in W} \{|\alpha_p|\}$. Define N_x by $N_x > \frac{\|\alpha_x\|}{\varepsilon}$.
Let $s > N_x$, $s \equiv 1 \mod |G|$.
Let $\tilde{W}'$ be the covering space of W corresponding to $\Gamma_{G_s} \subset \Gamma = \pi_1(W)$; $\tilde{W}'$ is a new h-cobordism, and there is a covering map $\tilde{W}' \xrightarrow{F} W$.
Let M' be the base of $\tilde{W}$. $F|_{M'} : M' \longrightarrow M$ is the covering projection corresponding to Γ_{G_s}; but so is f_s (see below). So $\exists$ $G : M' \xrightarrow[\text{diffeo}]{\sim} M$. Let $\tilde{W} := \tilde{W}' \cup_G M$, so $\tilde{W}$ is an h-cobordism with base M.

[f_s is an immersion of a cpt n-mfd to a cpt n-mfd. Such a map is locally 1-1; compactness $\Rightarrow$ inverse image of each point is finite; by invariance of domain, the map is onto; $\therefore$ a covering map.)

$$\begin{tikzcd}
\tilde{W} \arrow[r, "F"] \arrow[d, "\tilde{h}_1"'] & W \arrow[d, "h_1"] \\
M \arrow[r, "f_s"] & M
\end{tikzcd}$$

Lift h_t to a deformation retract $\tilde{h}_t$ of $\tilde{w}$ onto M.
Let $\{\beta q\}_{q\in\tilde{w}}$ be the family $\{\tilde{h}_1(\tilde{h}_t(q))$

Now $f_s(\beta_q) = \alpha_{F(q)}$.

$\therefore\ s|\beta_q| = |\alpha_{F(q)}|$, so

$$|\beta_q| = \frac{1}{s}|\alpha_{F(q)}| \leq \frac{1}{s}\|\alpha_x\| < \frac{\|\alpha_x\|}{N_x} \quad (\text{since } s > N_x)$$

$$< \varepsilon \quad (\text{def'n } N_x).$$

Since $|\beta_q| < \varepsilon$ for arbitrary q, $\tilde{W}$ is an ε-h-cobordism, so by Ferry's Thm $\tau(\tilde{w}, M) = 0$.
But $\tau(\tilde{w}, M) = \sigma^* x$ where $\sigma: \Gamma_{G_s} \hookrightarrow \Gamma$ is incl'n
(See, e.g., Milnor's survey article on Whitehead torsion, Bull. AMS, ≈1966).
This proves case (1).

Case 2: $x \in \tilde{K}_0\mathbb{Z}\Gamma$.
By BHS (Thm #2, p.12 or p.14) we have an embedding $\tilde{K}_0\mathbb{Z}\Gamma \hookrightarrow Wh(\Gamma\times T)$, where T is infinite cyclic.
Push x to $\bar{x} \in Wh(\Gamma \times T)$.
By Case 1, $\exists\ N_{\bar{x}} \in \mathbb{Z}$ s.t. $\sigma^*\bar{x} = 0$ where $\sigma: \Gamma_{G_s}\times sT \hookrightarrow \Gamma\times T$ for $s > N_{\bar{x}}$. ($\Gamma_{G_s}\times sT = (\Gamma\times T)_{G_s}$)
We have the commutative diagram (since $\Gamma_{G_s}\times sT \subset \Gamma_{G_s}\times T \subset \Gamma$

$$\begin{tikzcd}
Wh(\Gamma_{G_s}\times sT) & & & Wh(\Gamma\times T) \arrow[lll, "\sigma^*"'] \\
0 & & \bar{x} \in \arrow[ll] \arrow[lld] & \\
Wh(\Gamma_{G_s}\times T) \arrow[uu, "\beta"] & & x \arrow[u] \arrow[ld, "\sigma^*"'] & \tilde{K}_0\mathbb{Z}\Gamma \arrow[uu, hook] \\
& \tilde{K}_0\mathbb{Z}\Gamma_{G_s} \arrow[lu, hook, "\alpha"] & &
\end{tikzcd}$$

Where α is 1-1, the transfer map β is not necessari so, but $\beta\circ\alpha$ is 1-1. (Comes from studying the proj. map $Wh(\Gamma_{G_s}\times sT) \longrightarrow \tilde{K}_0\mathbb{Z}\Gamma_{G_s}$, & seeing this is a right

[illegible] .

inverse of the map $\tilde{K}_0\mathbb{Z}\Gamma_{G_s} \to Wh(\Gamma_{G_s} \times_s T)$. ?)

$\therefore\ \sigma^* x = 0$. //lemma

Prop: A subgroup of a torsion-free crystallographic group is torsion-free crystallographic. (cf Rink, p. 93)

Prf: We have $1 \to \mathbb{Z}^n \to \Gamma \to G \to 1$.
Let $\Gamma' < \Gamma$. Let $\mathbb{Z}^m = \mathbb{Z}^n \cap \Gamma'$. Then we get a s.e.s. $1 \to \mathbb{Z}^m \to \Gamma' \to G \to 1$.
$\therefore\ \Gamma'$ is virtually abelian; by the Thm on bot. p. 49, Γ' is (torsion-free) crystallographic. //Prop

11/1/78: Recall: Γ a crystallographic group, so we have a s.e.s. $1 \to A \to \Gamma \to G \to 1$,
A max'l abelian subgp of Γ, $|G| < \infty$. Define $\Gamma_s = \Gamma/sA$, where $s \in \mathbb{Z}$, so we have a s.e.s.

$$1 \to sA \to \Gamma \xrightarrow{\varphi} \Gamma_s \to 1.$$

Finally there is a s.e.s.

$$1 \to A_s \to \Gamma_s \to G \to 1,$$

where $A_s = A/sA$.
This last seq. splits uniquely up to inner auto; let $G_s = \text{im}\ G < \Gamma_s$.
Let $\Gamma_{G_s} = \varphi^{-1}(G_s) \subset \Gamma$.

Def: The rank of Γ := the rank of the free abelian group A.

To prove the theorem, top p. 77, we need 2 lemmata

Lemma: Given $x \in Wh\,\Gamma$ (or $\tilde{K}_0\mathbb{Z}\Gamma$), $\exists\ N_x \in \mathbb{Z}$ s.t. $\forall\ s > N_x$, $s \in \mathbb{Z}$, with $s \equiv 1 \bmod |G|$, $\sigma^* x = 0$ where σ = incl'n $\Gamma_{G_s} \hookrightarrow \Gamma$. //proved above

Algebraic Lemma: If Γ is a crystallographic group then either

(1) $\Gamma = \Gamma' \rtimes T$ where T is infinite cyclic, Γ' is crystallographic of rank $=$ rk $\Gamma - 1$;

(2) $\Gamma = B \underset{D}{*} C$ where $[B : D] = 2 = [C : D]$;

or (3) $\exists$ an infinite sequence of positive integers s, all $s \equiv 1$ mod $|G|$, s.t. if H is a hyperelementary subgroup of Γ_s which projects onto G then $S \cong G_s$ (i.e. if S projects onto G then it projects isomorphically onto G). (Note that this iso is an inner auto.)

Rmk: Γ need not be torsion-free in this lemma.

Assuming this second lemma for now, we prove the

Thm: If Γ is a torsion-free crystallographic group, then $\mathrm{Wh}\,\Gamma = 0 = \tilde{K}_0 \mathbb{Z}\Gamma$.

Prf: Induction on rank Γ and $|G|$.

To start the induction, note that the theorem is true for rank $= 0$ $(\Gamma = \{1\})$ and, given rank n, for $|G| = \{1\}$ $(\Gamma = T^n$; $\mathrm{Wh}\, T^n = 0 = \tilde{K}_0 \mathbb{Z} T^n$ by BHS).

choose Γ, and

So assume that the theorem is true for all Γ' where either (i) rank Γ' < rank Γ, or (ii) rank $\Gamma' =$ rank Γ and $|G'| < |G|$, where $G' =$ holonomy group of

~~[illegible]~~ In partic, assume $|G| > 1$.

Claim A: $\mathbb{Z}\Gamma$ is regular.

($\because$) $\Gamma = \pi_1 M^n$, M^n aspherical, so Γ is of finite cohomological dimension. $\mathbb{Z}A$ is Noetherian, & Γ is a finite extension of A, so $\mathbb{Z}\Gamma$ is Noeth. //A

We use the Algebraic Lemma to split the proof into 3 cases.

(1) $\Gamma = \Gamma' \rtimes T$. We use the extended BHS Thms, pp 43-44. Since $\mathbb{Z}\Gamma$ is regular, the nilpotent groups vanish, and there is an exact sequence $\mathrm{Wh}\,\Gamma' \longrightarrow \mathrm{Wh}\,\Gamma \longrightarrow \tilde{K}_0\mathbb{Z}\Gamma'$. Since rank $\Gamma' <$ rank
↳ easy
$\mathrm{Wh}\,\Gamma' = 0 = \tilde{K}_0\mathbb{Z}\Gamma'$, so $\mathrm{Wh}\,\Gamma = 0$.

For $\tilde{K}_0\mathbb{Z}\Gamma$, consider Serre's Thm (p. 24): if R is regular then $\tilde{K}_0R \overset{epi}{\twoheadrightarrow} \tilde{K}_0R[x,x^{-1}]$. The corresponding theorem is true for twisted Laurent series rings as well: $\tilde{K}_0R \overset{epi}{\twoheadrightarrow} \tilde{K}_0R_\alpha[x,x^{-1}]$.
In particular, there is an epi $\tilde{K}\mathbb{Z}\Gamma' \twoheadrightarrow \tilde{K}_0\mathbb{Z}\Gamma$.
But $\tilde{K}_0\mathbb{Z}\Gamma' = 0$ by induction, so $\tilde{K}_0\mathbb{Z}\Gamma = 0$.

(2) $\Gamma = B \underset{D}{*} C$. We use Waldhausen's Thm, pp. 46-7.
Again the "nilpotent" construction vanishes by regularity, so there is an exact sequence

$$\rightarrow \begin{matrix}\mathrm{Wh}\,B\\ \oplus\\ \mathrm{Wh}\,C\end{matrix} \longrightarrow \mathrm{Wh}\,\Gamma \longrightarrow \tilde{K}_0\mathbb{Z}D \longrightarrow$$

B, C, D are crystallographic groups of smaller rank, again $\mathrm{Wh}\,\Gamma = 0$.
Now $\Gamma = B \underset{D}{*} C \Rightarrow \Gamma \times T = (B \times T) \underset{(D\times T)}{*} (C \times T)$.
(Can see geometrically using Van Kampen's Thm & crossing
a circle.)
By BHS, $\tilde{K}_0\mathbb{Z}\Gamma \hookrightarrow \mathrm{Wh}(\Gamma \times T)$, so we show $\mathrm{Wh}(\Gamma\times$
By Waldhausen $\Rightarrow$

$$\rightarrow \begin{matrix}\mathrm{Wh}(B\times T)\\ \oplus\\ \mathrm{Wh}(C\times T)\end{matrix} \longrightarrow \mathrm{Wh}(\Gamma \times T) \longrightarrow \tilde{K}_0\mathbb{Z}(D \times T) \rightarrow \text{ is exact.}$$

$\mathrm{Wh}(B \times T) = \mathrm{Wh}(C \times T) = 0$, since BHS $\Rightarrow \mathrm{Wh}(B \times T) =$
$\mathrm{Wh}\,B \oplus \tilde{K}_0\mathbb{Z}B$ (regularity here again) $= 0 = \mathrm{Wh}(C\times$

For $K_0\mathbb{Z}(D\times T)$: Serre's Thm $\Rightarrow K_0\mathbb{Z}D \twoheadrightarrow K_0\mathbb{Z}(D\times T)$.

But $\operatorname{rk} D < \operatorname{rk} \Gamma \Rightarrow \tilde{K}_0\mathbb{Z}D = 0$.

$\therefore \operatorname{Wh}(\Gamma\times T) = 0, \Rightarrow \tilde{K}_0\mathbb{Z}\Gamma = 0$.

(3) Let $x \in \operatorname{Wh}(\Gamma)$ (or $\tilde{K}_0\mathbb{Z}\Gamma$).

Take N_x as in ~~Alg.~~ p. 83 Lemma, & choose $s > N_x$ to be one of the s specified by the algebraic lemma, case (3).

Consider $1 \to sA \to \Gamma \xrightarrow{\varphi} \Gamma_s \to 1$.

We show $i^*x = 0$ $\forall$ inclusion maps $i : \Gamma_H \to \Gamma$ where H is a hyperelementary subgroup of Γ_s.

~~We have the s.e.s. $1 \to sA \to \Gamma_H \xrightarrow{\varphi} H \to 1$, so~~

~~H is the holonomy of Γ_H~~ the image of H in G (under $\Gamma_s \twoheadrightarrow G$) is the holonomy of Γ_H (since we have a s.e.s. $1 \to A \to \Gamma_H \to (\operatorname{im} H \subset G) \to 1$)

so by our choice of s (alg. lemma), either $H = G_s$ or $|\text{holonomy } \Gamma_H| < |G|$.

In this second case, by the induction hypothesis $\operatorname{Wh}\Gamma_H = \tilde{K}_0\mathbb{Z}\Gamma_H = 0$, so of course $i^*x = 0$. In the first case $\Gamma_H = \Gamma_{G_s}$, so by the Lemma, bot. p. 83 $i^*x = 0$.

We now apply the algebraic vanishing criterion, p. 6 with $F = \Gamma_s$, to see that $x = 0$.

But x was an arbitrary elt. of $\operatorname{Wh}\Gamma$ or $\tilde{K}_0\mathbb{Z}\Gamma$ so $\operatorname{Wh}\Gamma = 0 = \tilde{K}_0\mathbb{Z}\Gamma$. //Thm, p. 84.

Rmk: The group Γ of p. 42, with a s.e.s. $1 \to \mathbb{Z}^3 \to \Gamma \to \mathbb{Z}_2\oplus\mathbb{Z}_2 \to 1$ is crystallographic torsion free, so $\operatorname{Wh}\Gamma = 0$. Farrell believes he's got a proof that for any torsion-free virtual poly-$\mathbb{Z}$ group Γ, $\operatorname{Wh}\Gamma = 0$.

Proof of algebraic lemma: We have the seq. $1 \to A \to \Gamma \to G \to 1$.

Lemma 1: If $s \equiv 1 \mod |G|$ and $A_s^G \neq 0$ (i.e. G has a nontrivial fixed point in A_s) then $A^G \neq 0$.

Pf: We use group cohomology. $A^G = H^0(G;A)$.

Consider the s.e.s.

$$0 \to A \xrightarrow{\times s} A \xrightarrow{\varphi} A_s \to 0 .$$

This induces a cohomology l.e.s.

$$0 \to H^0(G;A) \to H^0(G;A) \xrightarrow{\varphi_*} H^0(G;A_s) \to H^1(G;A) \xrightarrow{(\times s)_*} H^1(G;A) \to H^1(G;A_s) = 0 .$$

Since $(\times s)_*$ is onto and $H^1(G;A)$ is finite, $(\times s)_*$ must be a iso, so φ_* is onto. $\therefore$ if $A_s^G \neq 0$, $A^G \neq 0$ as well. //Lem

Lemma 2: $\exists$ an infinite sequence of primes p s.t. $(p, |G$ and $(p-1, |G|) = 1$ or 2.

Rmk: We will need only one such p.

Pf: We use Dirichlet's Thm: in any arithmetic sequen there are infinitely many primes. So consider $\{m|G| - 1 : m \in \mathbb{Z}\}$. Let $p = m|G| - 1$ be prime for some m; then $(p, |G|) = 1$, & $p - 1 = m|G| - 2 \Rightarrow (p-1, |G|)$ divides 2, so $(p-1, |G|) = 1$ or 2. //Lemma 2

11/3/75: Lemma 3: If F is a subgroup of G s.t. $A^F \neq 0$ an $[G:F] = 1$ or 2, then either

(1) $\Gamma = \Gamma' \rtimes T$ or

(2) $\Gamma = B \underset{D}{*} C$ where $[B:D] = 2 = [C:D]$.

Pf. We will use group cohomology. Notation: H^i(group; coeff indicates arbitrary coeffs, while H^i(group, coeffs) mean trivial coeffs.

Prf: Let $\Gamma_0 = \varphi^{-1}(F) \subset \Gamma$.

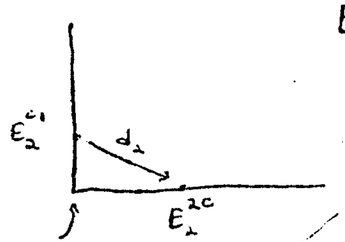
We first show $\Gamma_0 = \Gamma_0' \rtimes T$. To prove this, it suffices to show $\exists$ a nontrivial homom $\Gamma_0 \to T$.

Now $\mathrm{Hom}(\Gamma_0, T) = H^1(\Gamma_0, T)$ (comm $\Rightarrow$ trivial coeffs). To compute this group we write the Lyndon-Hochschild spectral sequence for $1 \to A \to \Gamma_0 \to F \to 1$.

(Rmk: This sequence is related to that of the ? or same as? fibration $BA \to B\Gamma_0 \to BF$)

$$E_2^{pq} = H^p(F; H^q(A, T))$$
$$\Downarrow$$
$$H^{p+q}(\Gamma_0, T).$$

F is finite, so if $p > 0$, E_2^{pq} must be finite.



only possible infinite groups; on q-axis

$$E_2^{0q} = H^0(F; H^q(A, T))$$
$$= (H^q(A,T))^F \text{ (fixed points). In particular}$$
$$E_2^{01} = (H^1(A,T))^F.$$

But $H^1(A,T) = \mathrm{Hom}(A,T) = A^*$; so $E_2^{01} = (A^*)^F$. It is easy to see that $A^F \neq 0 \Rightarrow (A^*)^F \neq 0$.

Thus $E_2^{01} \neq 0$. Since E_2^{01} is a subset of the free abelian group A^*, E_2^{01} is free abelian nonzero, in particular E_2^{01} is infinite.

The only take $d_k \neq 0$ at E_k^{01} is when $k = 2$.

But $\mathrm{im}\, d_2 \subset E_2^{20}$ which is finite, so $\ker d_2$ is infinite (in fact, again a nonzero free abelian group).

Thus $H^1(\Gamma_0, T)$ is a nonzero free abelian group, & in particular $\Gamma_0 = \Gamma_0' \rtimes T$.

Now if $F = G$, $\Gamma_0 = \Gamma$ and we're done. In fact, if $H^1(\Gamma, T$ 0 we're done, as above (Case (1) holds).

assuming $[G:F]=2$

In general, we must consider the s.e.s.

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \xrightarrow{\pi} T_2 \longrightarrow 1 .$$

Let T^+ be T with the trivial Γ-action, & let T^- be T with the action defined as follows: if $T = \{nx : n \in \mathbb{Z}\}$, & $T_2 = \{\pm 1\}$, acting on T by mult'n, then for $\gamma \in \Gamma$, $\gamma(nx) := (\pi\gamma).(nx)$.

Claim: One of $H^1(\Gamma; T^\pm)$ is infinite and contains elts of order $\neq 2$.

($\because$) We apply the spectral seq. argument to the s.e.s. on the top of this page.

$$E_2^{pq} = H^p(T_2; H^q(\Gamma_0; T^\pm))$$
$$\Downarrow$$
$$H^{p+q}(\Gamma; T^\pm).$$

As before, it suffices to prove E_2^{01} is free abelian & nonzero.

$$E_2^{01} = H^0(T_2; H^1(\Gamma_0; T^\pm))$$
$$= H^1(\Gamma_0; T^\pm)^{T_2}.$$

Now $\pi(\Gamma_0) = 1 \Rightarrow \Gamma_0$ acts trivially on both T^+ and T^-, so $H^1(\Gamma_0; T^\pm) = H^1(\Gamma_0, T)$ and $E_2^{01} = H^1(\Gamma_0, T)^{T_2}$. (But the T^2-action depends on whether we have $T^\pm$.) $H^1(\Gamma_0, T)$ is free abelian (proved on p. 88) so it suffices to show $E_2^{01} \neq 0$.

Any rep'n of T_2 on a free abelian group can be decomposed into a sum of T^+'s, T^-'s, and $\mathbb{Z}(T_2)$'s; only the T^- rep'n has no (nontrivial) fixed point.

The T_2-action on $H^1(\Gamma_0; T^-)$ is the negative

of the action on $H^1(\Gamma_0; T^+) = H^1(\Gamma_0, T)$ so even if one of these rep'ns consists of all T^-'s, the other has a fixed point. This proves the claim. //claim

Now if $H^1(\Gamma; T^+) \neq 0$ we've got case (1), as mentioned above. If $H^1(\Gamma; T^+) = 0$ then $H^1(\Gamma; T^-) \neq 0$, & is in fact free abelian.

Now $H^1(\Gamma; T^-) = \{\text{crossed homoms}\} / \{\text{principal crossed ho}$

So $\varphi \in H^1(\Gamma; T^-)$ is a map $\varphi : \Gamma \longrightarrow T$ s.t.

$\varphi(a \cdot b) = \varphi(a) + a \cdot \varphi(b)$ (where $a \cdot \varphi(b)$ is the action of $a \in \Gamma$ on T).

Given φ, construct $\hat{\varphi} : \Gamma \longrightarrow T \rtimes T_2$ as follows:

Let $ax + t \in T \rtimes T_2$ where $a \in T_2$, $t \in T$, x an indeterminate, and we multiply be composition of polynomials. $(a = \pm 1)$.

Then $\hat{\varphi}(\gamma) = \bar{\gamma} x + \varphi(\gamma)$ where $\bar{\gamma} = \pi \gamma \in T_2$.

It is easy to see that $\hat{\varphi}$ is a homomorphism.

If $\hat{\varphi}$ is of order 2, then im $\hat{\varphi}$ could be $\cong T_2$; so choose φ not of order 2, in which case im $\hat{\varphi}$ i an infinite subgroup, which must be $\cong T$ or $T \rtimes T_2$.

If im $\hat{\varphi} \cong T$ then we're in case (1) again. So suppose im $\hat{\varphi} \cong T \rtimes T_2$; assume wlog that $\hat{\varphi} : \Gamma \longrightarrow T \rtimes$ is epi.

<u>Claim</u>: $T \rtimes T_2 \cong T_2 * T_2$.

(∵) Both act as subgroups of Rigid (1) on $\mathbb{R}$.

$T_2 * T_2 = \langle a : a^2 = 1 \rangle * \langle b : b^2 = 1 \rangle$ acts as follows:

a is reflection in 0.

b " " " 1.

This is an effective action.

$T \rtimes T_2 = \langle x \rangle \rtimes \langle c : c^2 = 1 \rangle$ acts as follows:

x is translation by 2,
c is reflection in 0.

This is also an effective action.

Now it is clear that $T \rtimes T_2 \cong T_2 * T_2$ by $c \mapsto a$, $x \mapsto ba$ or, equivalently, $a \mapsto c$, $b \mapsto (x, c)$. //cl

So we have $\hat{\varphi} : \Gamma \xrightarrow{epi} T_2 * T_2$.

Let $B = \hat{\varphi}^{-1}(T_2)_1$ (1st copy)
$C = \hat{\varphi}^{-1}(T_2)_2$ (2nd copy)
$D = \ker \hat{\varphi}$.

Then $\Gamma = B \underset{D}{*} C$. //Lemma 3

Proof of algebraic lemma

Spose (1) & (2) don't occur; must show (3) does.
Let p be a prime s.t. $(p, |G|) = 1$ & $(p-1, |G|) = 1$ or 2, as in Lemma 2, p. 87. By picking an appropriate u, we can get $p^u \equiv 1 \mod |G|$. Fix p, u; let
~~[illegible] be the infinite sequence of~~
s be the infinite sequence p^{uv}, $v = 1, 2, 3, \ldots$;
all $s \equiv 1 \mod |G|$.

Consider s.e.s.'s $1 \to A_s \to \Gamma_s \xrightarrow{\varphi} G \to 1$ where
$A_s = A/sA$.

Let H be a hyperelementary subgroup of Γ_s, and suppose $\varphi(H) = G$. We must show $H \cong G_s$; i.e. $A_s \cap H$

H hyperelem $\Rightarrow \exists$ s.e.s.

$$1 \to K \to H \to Q \to 1$$

where K is cyclic, Q a q-group (q prime).

We can always assume $(|K|, |Q|) = 1$ so $H = K \rtimes Q$.

Case i : $q | s$

Case ii : $q \nmid s$

Suppose case (i) holds. $(s, |G|) = 1$ so, since $q|s$, we must have $\varphi(Q) = 1$; thus $Q \subset A_s$.

Thus $G = \varphi(K)$.

Claim A : $A_s \cap K \subset (A_s)^G$.

(∵) G is a quotient of the cyclic group K, so G lea

K fixed. $/\!/_A$

If $K \cap A_s \neq 0$ then $(A_s)^G \neq 0$ & so by lemma 1 (p. 87) $A^G \neq$

now lemma 3 $\Rightarrow$ Case (1) or (2) holds.

So $K \cap A_s = 0$.

Spose $a \in Q, b \in K$; then $aba^{-1}b^{-1} \in A_s$ since $a \in A_s$ & A_s is normal $\Rightarrow ba^{-1}b^{-1} \in A_s$. But also $aba^{-1}b^{-1} \in K$ since $K \lhd H$; $\therefore$ $[a,b] = 1$, & so $H = K \times Q$, i.e. K commutes with Q. $\therefore$ $Q \subset (A_s)^G$ and, again, $Q \neq 1 \Rightarrow$ $(A_s)^G \neq 1 \Rightarrow A^G \neq 1 \Rightarrow$ Case (1) or (2) holds $*$, so $Q = 1$;

$\therefore$ $H = K$ and $A_s \cap H = 1$.

Suppose case (ii) holds. $(s, q) = 1 \Rightarrow \ker(\varphi/_Q) = Q \cap A_s =$

So $A_s \cap H = A_s \cap K$. (Prf: over).

Claim B : $\forall g \in G$, $g(A_s \cap K)g^{-1} = A_s \cap K$.

(∵) A_s is abelian, so $A_s \cap K$ is normalised by A_s; also, $A_s \cap K = A_s \cap H \lhd H$; since H and A_s generate Γ_s, the claim follows. $/\!/_B$

Now $A_s \cap K$ is a cyclic group of order p^n for some $n \in \mathbb{N}$

We have $\psi : G \longrightarrow \mathrm{Aut}(A_s \cap K)$.

If $n \neq 0$, $|\mathrm{Aut}(A_s \cap K)| = p^n - p^{n-1} = p^{n-1}(p-1)$.

Since $(p-1, |G|) = 1$ or 2, $(p, |G|)=1$, we have $|\psi(G)| \leq 2$ (i.e. $\psi(G) \cong \mathbb{Z}_2$ or $\psi(G)=1$).

Let $F = \ker \psi$; $[G:F] = 1$ or 2.

$(A_s)^F \neq 0$ $((A_s)^F = A_s \cap K)$ so $A^F \neq 0$, so again Case 1 or 2 must hold: $\times$.

This argument worked only for $n>0$; so we must have $n=0$, & $K \cap A_s = 1$. //Algebraic lemma.

This finally finishes the proof of the theorem on p. 84.

Rmk (cf prop, p. 83): A subgroup of a crystallographic group need not b crystallographic. Ex: Let $\Gamma = \mathbb{Z}\oplus\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by $\alpha(-1) = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$. Then $\mathbb{Z}\oplus 0 \rtimes_\alpha \mathbb{Z}_2 \subset \Gamma$; but $\mathbb{Z}\oplus 0 \rtimes_\alpha \mathbb{Z}_2 = \mathbb{Z}\times\mathbb{Z}_2$ which is not crystallographic since the holonomy rep'n is not faithful.

Digression: Let $\Gamma \subset$ Rigid (n), $v_0 \in \mathbb{R}^n$.

Let $c = \{x \in \mathbb{R}^n : (x, v_0) \leq d(x, \gamma v_0) \ \forall \gamma \in \Gamma\}$

c is the crystal containing v_0.

(c seems to be convex — this should us rigidity.) Note that ~~[illegible]~~ translates of c tesselate $\mathbb{R}^n$. Conversely, given such a tesselation, let $\Gamma \subset$ Rigid (n) be the sgrp gen by reflections in the sides of one crystal c. This (somehow) gives a splitting $\Gamma = \mathbb{Z}^n \rtimes G$.

11/6/78: We wish to show that $\mathrm{Wh}\,\Gamma = 0$ if Γ is virtually poly-$\mathbb{Z}$ and torsion free.

The new ideas we will need will come from studying the following

Prop: Let $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}_3$ where $\mathbb{Z}_3$ acts on $\mathbb{Z}^2$ by $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. ($\Gamma$ is crystallographic, but not torsion free)

Then $\mathrm{Wh}\,\Gamma = 0$.

Rmk: This depends on the fact that $\mathrm{Wh}\,\mathbb{Z}_3 = 0$.

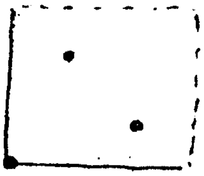
Prf: We first analyze the action of Γ.

Claim A: $\mathbb{R}^2/\Gamma = S^2$.

($\because$) Let $A = \mathbb{Z}^2 < \Gamma$.

$\mathbb{R}^2/A = T^2$. T^2 is a branched cover over $\mathbb{R}^2/\Gamma$ with three branch points, since $\mathbb{R}^2/\Gamma = T^2/\mathbb{Z}_3$ & this $\mathbb{Z}_3$-action has 3 fixed points.

These are (in $[0,1)\times[0,1)$) the points (x,y) s.t. $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} n \\ m \end{pmatrix}$ where $n, m \in \mathbb{Z}$; the solution is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$



Since the action about each fixed point is rotation by 120° (essentially), $\mathbb{R}^2/\Gamma$ is a 2-mfd. By an easy euler characteristic argument, $\chi(\mathbb{R}^2/\Gamma) = 2$, so $\mathbb{R}^2/\Gamma = S^2$. //A

Now $\mathbb{Z}_3$ acts freely on S^3, as mult'n by $e^{2\pi i/3}$, and $S^3/\mathbb{Z}_3$ is a lens space L^3.

Since $\Gamma \xrightarrow{epi} \mathbb{Z}_3$, Γ acts on S^3.

Γ acts freely on $\mathbb{R}^2 \times S^3$ (Cartan construction).

Factor out by this action to get $M = \mathbb{R}^2 \times_\Gamma S^3$;

$\pi_1(M) = \Gamma$.

We have fibrations

$T^2 \longrightarrow \mathbb{R}^2 \times_\Gamma S^3 \longrightarrow S^3/\Gamma = L^3$ (an actual fibre bundl

and $\mathbb{R}^2 \times_\Gamma S^3 \xrightarrow{p} \mathbb{R}^2/\Gamma = S^2$. This second fibration is no

a bundle because there are 3 exceptional fibres: over these points the inverse image is L; elsewhere it is S^3.

Now Epstein-Shub & Bieberbach Rigidity thms do not require torsion-free, so we could use these to find an expanding endo, but we proceed differently (Since we can f_s down explic

We will consider self-immersions of M which are fibre-preserving (for the fibration p), isometries along the fibre and expanding in the base direction.

In fact, $\forall$ $s \in \mathbb{Z}$, define

$f_s : \mathbb{R}^2 \times S^3 \longrightarrow \mathbb{R}^2 \times S^3$ by

$$f_s(x, a) = (sx, a).$$

Then f_s induces a map $f_s : M \longrightarrow M$ with the desired properties.

(Rmk: This f_s is not differentiable at the 3 exception points, but is expanding away from these points. (Here we mean induced map on S^2.) See below.)

11/8/78 : Now we wish to show that $Wh(\Gamma) = 0$, where $\Gamma = \mathbb{Z}^2 \rtimes_{\left(\begin{smallmatrix}0 & -1\\ 1 & -1\end{smallmatrix}\right)} \mathbb{Z}_3$, ~~with~~ by using the algebraic vanishin

conditions.

Rmk: If $Wh\,\Gamma = 0$ then the same argument will show $Wh(\Gamma \times T) = 0$, so $\tilde{K}_0 \mathbb{Z}\Gamma \subset Wh(\Gamma \times T)$

is also zero.

Notation: $A = \mathbb{Z}^2$, $G = \Gamma/A = \mathbb{Z}_3$.

Γ is crystallographic, so by the Algebraic Lemma p. 84, one of 3 cases must occur.

Now it is easy to see that (1), (2) can't occur, si (essentially) the G-action "mixes up" the $\mathbb{Z}^2$-fact too much. Thus there is a sequence of positive num s, all $s \equiv 1 \mod |G|^{c3}$, s.t. if $H \subset \Gamma_s$ is hyperelem and H projects onto G in the extension $1 \to A_s \to \Gamma_s \to G$ then $H \cong G$.

Since $G = \mathbb{Z}_3$, the image of an arbitrary hyperelem sgp of Γ_s in G is either 1 or G. Thus, if H is hyperelem then either $H \subset A_s$ or else H projec onto G (i.e. $H = G_s$).

We will (as before) apply the algebraic vanishing condi (p. 63), to $\Gamma \xrightarrow{\varphi} \Gamma_s \longrightarrow 1$.

Let $x \in Wh\, \Gamma$. $x = 0$ if $\sigma^* x = 0 \in Wh\, \Gamma_H$ $\forall$ hyperel $H \subset \Gamma_s$, where $\Gamma_H = \varphi^{-1}(H)$.

If $H \subset A_s$ then $\Gamma_H = \mathbb{Z}^2$; by BHS, $Wh\, \Gamma_H = 0$, so $\sigma^* x = 0$.

So we need analyze only $\sigma^* x \in Wh\, \Gamma_{G_s}$. Here th geometry (pp. 94-5) appears.

Recall $\Gamma \subset Rigid(2)$; $\mathbb{R}^2/A = T^2$, & $\mathbb{R}^2/\Gamma = T^2/G$, & an easy arg. $\Longrightarrow T^2/G = S^2$. G acts freely on S^3 by $\times e^{2\pi i/3}$; $S^3/G = L^3$, lens space.

Γ acts on S^3 by $\Gamma \xrightarrow{\varphi} G$ so Γ acts on $\mathbb{R}^2$

This action is free and so $M = \mathbb{R}^2 \times_\Gamma S^3 = T^2 \times_G S^3$

a 5-mfd with $\pi_1(M) = \Gamma$.

Let W^6 be an h-cobordism with base M, s.t. $\tau(W,M) = x \in Wh\,\Gamma$.

Consider M the "bottom" of W. There are two deformati retractions, h_t of W onto M, k_t of W onto its "top"
~~(Unfortunately we know nothing about the top.)~~

∀ $a \in W$, define curves $\alpha_a(t)$, $\beta_a(t)$ by

$$\alpha_a(t) = h_1 h_t(a), \quad \beta_a(t) = h_1 k_t(a).$$

(We project $k_t(a)$ into M since we know nothing abo the "top" of W.)

Our old definition of an ε-h-cobordism & Ferry's Th ~~were not applicable to this situation~~ will not give us our desired result, so we extend them.

Recall we have a fibr'n $p: T^2 \times_G S^3 \longrightarrow S^2$ (top p. 95

Def: W is an ε-h-cobordism if $|p\alpha_a| < \varepsilon$, $|p\beta_a| < \varepsilon$ ∀ $a \in W$.

Unfortunately as p is not differentiable, this defin doesn't quite mean what we want it to. In fact since there are only finitely many singular poi we could ignore the problem, but for the sake of generalizing later, we will reinterpret the defn as

Def′: W is an ε-h-cobordism if each $p\alpha_a$, $p\beta_a$ lies in an ε-ball on S^2 (where we measure ε-balls w.r.t. an arbitrary fixed metric).

Thm. $\exists$ $\varepsilon > 0$ s.t. each ε-h-cobordism with base $M = T^2 \times_G S^3$ has zero Whitehead torsion.

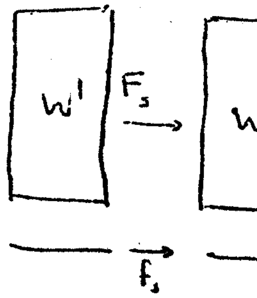
Rmk: This theorem is not in general true (i.e. for other crystallographic groups Γ for which we might mimic this construction). It depends on the fact that for the singular fibres L, $Wh(\pi_1 L) = Wh(\mathbb{Z}_3^{\#}) = 0$.

Assuming the theorem for now, we continue with our example.

Our given W, of course, need not be an ε-h-cobor

We consider $f_s : \mathbb{R}^2 \times_\Gamma S^3 \longrightarrow \mathbb{R}^2 \times_\Gamma S^3$, smooth maps (in fact covering proj'ns) s.t. $(f_s)_{\#}(\Gamma) = \Gamma_{G_s}$; namel $f_s[y,z] = [sy, z]$.

Let $W' \xrightarrow{F_s} W$ be the covering space corresponding to $\Gamma_{G_s} \subset \Gamma \cong \pi_1(W)$.



As on p. 81, W' has $T^2 \times_G S^3 = M$ as base space.

Lift the deformations k_t, h_t to $\tilde{k}_t, \tilde{h}_t$ of W' onto its top & bottom (resp) to get curves $\tilde{\alpha}_a, \tilde{\beta}_a$ in $\forall a \in W'$. It's easy to see that

$$f_s(\tilde{\alpha}_a) = \alpha_b, \quad f_s(\tilde{\beta}_a) = \beta_b \quad \text{where } b = F_s(a).$$

As mentioned on p. 95, f_s is not expanding in the fib direction, but the induced map $\hat{f}_s$ on S^2 is expanding:

(see caveat, next page)

$$\begin{tikzcd} T^2 \times_G S^3 \arrow[r, "f_s"] \arrow[d, "p"'] & T^2 \times_G S^3 \arrow[d, "p"] \\ S^2 \arrow[r, "\hat{f}_s"] & S^2 \end{tikzcd}$$

$\hat{f}_s(p\tilde{\alpha}_a) = p\alpha_b$, $\hat{f}_s(p\tilde{\alpha}_a) = p\alpha_b$, $b = F_s(a)$.

$\hat{f}_s$ is technically not an expanding endo (3 ba points) but it is expanding enough to insure that for sufficiently large s, W' is an ε-h-cobor

Thus by the Thm, top p. 96, W' is trivial.

So $0 = \tau(W', M) = \sigma^* x \in Wh(\Gamma_{a_s})$.

This proves the proposition. //Prop, p. 94.

It remains to prove the Theorem; here we only indicate a quick sketch:

If p were a fibration (bundle), and $Wh(\pi_1(\text{fibre}))$ were nice (here the fibres would all be S^3 & $\pi_1(S^3) = 0$) then the theorem holds.

For singular fibrations, put little discs aroun the sing. fibres & apply the thm to the nice fibration away from the singular fibres. Near the singular fibres we have 3 h-cobordisms. The base is a bundle over L^3 so $\pi_1(\text{base}) = \mathbb{Z}_3$, but $Wh(\mathbb{Z}_3) = 0$, so for purely algebraic reasons these small h-cobordisms are trivial. //Thm

We are now interested in the following

Thm: If $M^n \underset{h.e.}{\simeq} N^n$ where N is flat then $M \cong$

for $n \neq 3, 4$.

We first need a new

Algebraic Lemma: Let Γ be a crystallographic group, with short exact sequence $1 \to A \to \Gamma \to G \to 1$

Then either

(1) $\Gamma = \Gamma' \rtimes T$

(2) ∃ epimorphism $\Gamma \longrightarrow \hat{\Gamma}$ where $\hat{\Gamma}$ is a nont crystallographic group which satisfies condition (3 the previous algebraic lemma, p. 84, or

(3) G is an elementary abelian 2-group (i.e., the direct sum of $\mathbb{Z}_2$'s) and either

(a) $\Gamma = A \rtimes T_2$ with holonomy representati given by multiplication by -1, or

(b) Γ maps epimorphically onto a crystallogra group $\bar{\Gamma}$ where, in the extension

$$1 \longrightarrow \bar{A} \longrightarrow \bar{\Gamma} \longrightarrow \bar{G} \longrightarrow 1,$$

$\bar{A} \cong \mathbb{Z} \oplus \mathbb{Z}$, $\bar{G} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the holonomy representation is either

$$\left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\right\} \quad \text{or} \quad \left\{\pm I, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}.$$

11/10/78:

Rmks: we need this lemma because we wish to do surgery, & can't handle the case $B \ast_D C$. We consider the ideas behind the thm, p. 99, more carefully.

Recall the abelian group $\mathcal{S}(M) = \{$equivalence classes of h.e.'s $f: N \longrightarrow M$ with $f|_{\partial N}: \partial N \longrightarrow \partial M$ a homeo$\}$, where the equiv rel'n is: $(f: N \longrightarrow M) \sim (g: S \longrightarrow M)$ iff ∃ homeo $F: N \xrightarrow{\approx} S$ s.t.

$$\begin{tikzcd} N \arrow[rd, "f"] \arrow[d, "F"'] & \\ S \arrow[r, "g"'] & M \end{tikzcd}$$

homotopy commutes, and

$$\begin{tikzcd} \partial N \arrow[rd, "f|_{\partial N}"] \arrow[d, "F|_{\partial N}"'] & \\ \partial S \arrow[r, "g|_{\partial S}"'] & \partial M \end{tikzcd}$$

commutes exactly. The zero element of $\mathcal{S}(M)$ is $[1_M]$. (cf. pp 30 ff).

Thm (Farrell-Hsiang): M^n flat, $n \neq 3, 4$, and holonomy of M odd

$|\mathcal{S}(M \times D^m)| = 1 \quad \forall\, m$.

Rmk: $n \neq 3, 4$ important only because we want $n+m>4$.

Cor: $N^n \underset{h.e.}{\simeq} M^n$, $n \neq 3, 4$, M flat $\Rightarrow$ $N \cong M$ (homeo.)

(∵) Let $m=0$ / cor.

The following conjecture ~~[illegible]~~ (being studied by a student of Hsi
would enable us to get rid of the holonomy conditi
It is an analogue of the Algebraic Vanishing Criterion
(Frobenius Induction).

Conj: Let $\Gamma = \pi_1 M^n$, $n>4$, & let $\varphi: \Gamma \to F$ be an epimorph
where F is a finite group. Let $[f] \in \mathcal{S}(M)$. If $\sigma^*[f] =$
for all $\sigma: \Gamma_S \hookrightarrow \Gamma$, where S ranges over the hyperel
tary subgroups of F, then $[f]=0$.

Here $\sigma^*[f]$ is defined as follows. $\Gamma_S \subset \Gamma$ is a subg
of finite index, so there is a compact covering spa

$$\begin{tikzcd} M_s \arrow[d, "p"'] \\ M \end{tikzcd}$$

corresponding to Γ_S. Let $\tilde N = f^* M_s$ in the diagra
$\tilde N$ is a covering space of N.
Then $\tilde f: \tilde N \to M_s$ is a homotopy
equivalence, and $\sigma^*[f] := [\tilde f] \in \mathcal{S}(M_s)$.

$$\begin{tikzcd} \tilde N \arrow[r, "\tilde f"] \arrow[d, "p'"'] & M_s \arrow[d, "p"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Using the conjecture we prove the theorem on the bottom of

Pf (Farrell-Hsiang): Induction on n and on $|G|$, $G=$ holon
$|G|$ odd $\Rightarrow$ (2) of the "old" algebraic Lemma (p. 84) doesn
occur. If (3) holds, use the conjecture above together
with an analogue of Ferry's result (or a result
Ferry-Chapman). If (1) holds, use an analogue of BHS
In fact if $\Gamma = \Gamma' \rtimes T$ then $\exists$ fibration $(M')^{n-1} \to M^n \to S$
Farrell-Hsiang have proved earlier that then there
an ex. seq. $\mathcal{S}(M' \times D^{n+1}) \longrightarrow \mathcal{S}(M \times D^n) \longrightarrow \mathcal{S}(M' \times D^n)$
[illegible] by induction hypoth., $\mathcal{S}(M' \times D^k) = 0$ $\forall k$. // Farrell H

Of course Farrell-Hsiang were able to prove their theorem without using the conjecture, but they had to use the full machinery of surgery theory.

Suppose $|G|$ is even. Then there is a Seifert fibering $M \to [0,1]$ which is a fibration (= bundle in Farrell's terminology) over $(0,1)$ and has exceptional fibres over $\{0,1\}$. (The fibre over $\{0,1\}$ is $N/$a $\mathbb{Z}_2$-action, where $N =$ fibre over $(0,1)$).

We have $M = \mathcal{B} \underset{\mathcal{D}}{\cup} \mathcal{E}$ where $\mathcal{D} = N$, $\mathcal{B} =$ inv. image of $[0,\frac{1}{2}]$, $\mathcal{E} =$ inv. image of $[\frac{1}{2},1]$.

Then $\pi_1 M = \Gamma = B \underset{D}{*} C$ where $B = \pi_1(\mathcal{B})$, $C = \pi_1(\mathcal{E})$, $D = \pi_1(\mathcal{D}$

Suppose $N \xrightarrow{f} M$ is a h.e. We want to model the decomposition of M, as above, in N (which we ca certainly do if $N \cong M$). We start by finding an analogue to the codim 1 submfd $\mathcal{D}$. The obstruction group to doing this is the unitary nilpotent group $UNil(D; B, C)$, & it is very hard to analy

cf. pp. 115-116

The purpose of the more general (new) algebraic lemma (pp. 9 is to avoid such problems

Pf of Alg. Lemma: Suppose (1) doesn't hold. Then either (2) or (3) of the previous algebraic lemma (p 84) mu hold. If old (3) holds then let $\hat{\Gamma} = \Gamma$; then new (2) holds.

Suppose old (2) holds. $\Gamma = B \underset{D}{*} C$, $[B:D] = 2 = [C:D]$

then there is a map $\Gamma \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1$.

Claim A: $\exists$ an infinite cyclic subgroup $S \subset A$ s.t. S is invariant under the action of G (so $S \lhd \Gamma$), an

s.t. A/S has no torsion.

(∴) Take the proof on pp 89-90 and run it backwards //A

Let $\Gamma_1' = \Gamma/S$. Γ_1' needn't be crystallographic, but it is an extension of an abelian group by a finite group ($1 \to A/S \to \Gamma_1' \to G \to 1$). As shown on pp. 49-50, howev

there is a map $\Gamma_1' \xrightarrow{\alpha} \text{Rigid}(n)$ with finite kernel; define $\Gamma_1 := \text{im}\,\alpha$. Thus Γ maps onto the crystallographic group Γ_1, and $\text{rank}\,\Gamma_1 = \text{rank}\,\Gamma - 1$.

Continue the process. $\Gamma_1 \not\cong \hat{\Gamma}_1 \rtimes T$ or else $\Gamma \cong \hat{\Gamma} \rtimes T$;

if Γ_1 satisfies old (3) then Γ satisfies new (2) (with $\hat{\Gamma} = \Gamma_1$); so again assume there is an epi $\Gamma_1 \to \mathbb{Z}_2 * \mathbb{Z}_2$

and continue as above.

If at any point we reach a Γ_r satisfying old (3), we're done; otherwise we get a sequence of epimorphisms

$$\Gamma = \Gamma_0 \xrightarrow{\alpha_1} \Gamma_1 \xrightarrow{\alpha_2} \Gamma_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \Gamma_n \longrightarrow 1.$$

Let $\varphi_i = \alpha_i \circ \alpha_{i-1} \circ \cdots \circ \alpha_1 : \Gamma \longrightarrow \Gamma_i$.

Let $K_i = \ker \varphi_i$. This defines a seq. of subgrou

$$1 \subset K_1 \subset K_2 \subset \cdots \subset K_n = \Gamma,$$ and each

$K_i \lhd \Gamma$.

Then $0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = A$ is a filtration of

where $A_i = K_i \cap A$, each $A_i \lhd \Gamma$.

A_{i+1}/A_i may have torsion, but we know it must

be of rank 1.

Let $a_i = \{x \in A : sx \in A_i$ for some $s > 0\}$.
Then we get a filtration

$$0 \subset a_1 \subset a_2 \subset \cdots \subset a_n = A,$$

where each $a_i \lhd \Gamma$ and a_{i+1}/a_i is infinite cyclic.
Choose a basis $(e_1, \dots, e_n)$ of A s.t. $\{e_1, \dots, e_i\}$ spans a_i.
Each $g \in G$ acts on A, and so determines a matrix M wrt the basis $(e_1, \dots, e_n)$; $a_i \lhd \Gamma \Rightarrow M$ is upper triangular.

$M \in GL(n, \mathbb{Z}) \Rightarrow$ det $M = \pm 1$, so $\prod(m_{ii}) = \pm 1$, so each m_{ii} (diagonal entries) $= \pm 1$. Thus all the diagonal entries of M^2 are all $+1$. But M^2 has finite order; it is an easy exercise to show that these conditions $\Rightarrow M^2 = I$.
But G acts faithfully on A, so $g^2 = 1$. Thus every element of G has order 2, i.e. G is an elementary abelian 2-group.

<u>11/13/78</u>. Now we must show that there is an epi $\Gamma \to \overline{\Gamma} \to 1$ where $1 \to \mathbb{Z} \oplus \mathbb{Z} \to \overline{\Gamma} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1$, or else $\Gamma = A \rtimes_{(-1)} T_2$.
Consider again the sequence

$$\Gamma = \Gamma_0 \xrightarrow{\alpha_1} \Gamma_1 \xrightarrow{\alpha_2} \Gamma_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \Gamma_n = 1.$$

Since α_i takes max. abelian sgp to max abelian sgp, the induced map in holonomy sends the holonomy of Γ_i <u>onto</u> the holonomy of Γ_{i+1}.
Therefore $\Gamma_{n-1} \cong \mathbb{Z} \rtimes \mathbb{Z}_2$; for there are only two 1-dimensional crystallographic groups, & if $\Gamma_{n-1} = \mathbb{Z}$ then $\Gamma \cong \Gamma' \rtimes \mathbb{Z}$, so Γ_{n-1} must be $\mathbb{Z} \rtimes \mathbb{Z}_2$.

Subcase (a): If hol $\Gamma_i = \mathbb{Z}_2$, then $\Gamma = A \rtimes_{-1} \mathbb{Z}_2$.

(∵) $G = \text{hol}\, \Gamma_i = T_2$. Consider the holonomy representation of T_2 on A.

$A = \oplus$ indecomposables, each iso. to T, T^- or $\mathbb{Z}T_2$.

If T or $\mathbb{Z}T_2$ occurs then $A^G \neq 0$; therefore $\Gamma = \Gamma' \rtimes T$ (p. 88), which we've already ruled out, so $A = \oplus T^-$, and so the holonomy representation is $\times(-1)$.

Subcase (b): Suppose hol $\Gamma_{i+1} = \mathbb{Z}_2$ but $|\text{hol}\, \Gamma_i| > 2$.

Let B_i be maximal abelian in Γ_i and $G_i = \text{hol}\, \Gamma_i$. If $\beta_i = \alpha_i|_B$ and γ_i is the induced map on holonomy we have the diagram

$$\begin{tikzcd}
1 \arrow[r] & B_i \arrow[r] \arrow[d, "\beta_i"] & \Gamma_i \arrow[r] \arrow[d, "\alpha_i"] & G_i \arrow[r] \arrow[d, "\gamma_i"] & 1 \\
1 \arrow[r] & B_{i+1} \arrow[r] & \Gamma_{i+1} \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

Let $K = \ker \beta_i$. K is infinite cyclic and a direct summand (as abelian groups) of B_i, & K is invariant under G_i. Pick a basis of B_i s.t. G_i acts as $\left[\begin{array}{c|c} \pm 1 & * \\ \hline 0 & * \end{array}\right]$. Let $\hat{G}_i = \ker \gamma_i$. If $M \in \hat{G}_i$, the corresponding matrix is $M = \left[\begin{array}{c|c} \pm 1 & * \\ \hline 0 & I \end{array}\right]$.

Claim A: $\hat{G}_i = T_2$.

(∵) If $M = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & I \end{array}\right]$ then $M = I$, since M is of finite order. If $M \neq 1$, then $M = \left[\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right]$. Suppose $M' \neq 1$ also, $M' = \left[\begin{array}{c|c} -1 & b \\ \hline c & I \end{array}\right]$.

Then $M(M')^{-1} = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & I \end{array}\right] = I \Rightarrow M = M'$.

Thus $\hat{G}_i = T_2$. //A

In fact $\hat{G}_i = \left\{ I, \left[\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right] \right\}$.

$\therefore$ $G_i = T_2 \oplus T_2$ (elem. 2-group of order 4).

Now $I \in G_i$ and $\left[\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right] \in G_i$.

Claim B : $-I \in G_i$.

($\because$) The two elements in G_i mapping to -1 in T_2 are of the form $\left[\begin{array}{c|c} -1 & * \\ \hline 0 & -I \end{array}\right]$ or $\left[\begin{array}{c|c} 1 & * \\ \hline 0 & -I \end{array}\right]$.

Now $\left[\begin{array}{c|c} 1 & * \\ \hline 0 & -I \end{array}\right]\left[\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{c|c} -1 & * \\ \hline 0 & -I \end{array}\right]$, so $\exists$ element of form $\left[\begin{array}{c|c} -1 & * \\ \hline 0 & -I \end{array}\right]$ in G_i, mapping to $-1 \in T_2$.

Again, since this mx has finite order, it must be $-I$

so $-I \in G_i$. //B

Thus $G_i = \left\{ \pm I, \left(\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right), \left(\begin{array}{c|c} 1 & -a \\ \hline 0 & -I \end{array}\right) \right\}$

Now we construct $\overline{\Gamma}$ to be a quotient of Γ_i ($\therefore$ of Γ). We do this by finding $\overline{B}_i \subset B_i$, invariant under the action of G_i, s.t. $B_i/\overline{B_i} = \mathbb{Z} \oplus \mathbb{Z}$, and we will set $\overline{\Gamma} = \Gamma_i/\overline{B}_i$.

This is done as follows:

$\pm I$ leave any subspace invariant

Consider then the subgroup $S = \left\{ I, \left(\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right) \right\} < G_i$; if a subgroup $\overline{B}_i \subset B_i$ is S-invariant then it is G_i-invariant.

B_i is an $S \cong T_2$-module, so B_i is a direct sum of T's, T^-'s and $\mathbb{Z}(T_2)$'s.

If a $\mathbb{Z}T_2$ occurs, let $\bar{B}_i = \oplus$ all other factors, so $B_i/\bar{B}_i = \mathbb{Z}T_2$ (if there is more than one $\mathbb{Z}T_2$-facto all but one go into $\bar{B}_i$). If $\mathbb{Z}T_2$ doesn't occur, we *can't* have $B_i = \oplus T^-$, for in this case S would act by -1, but we see that the nontrivial elt of S has $+1$-entries as w

So there is at least one T and at least one T

let $\bar{B}_i = \oplus$ everything else.

We can see that we get one of the 2 desire representations on $\mathbb{Z} \oplus \mathbb{Z} = B_i/\bar{B}_i$. //Alg. lemma, pp. 99-10

We leave this material for now and return to geometric groups (pp 79-80).

Ref: Connell-Hollingsworth, Trans. AMS 1969.

Recall the

Def: Let X be a "nice" compact metric space, e.g. a finite simplicial complex or compact manifold. Let $S = \{p_1, \dots, p_n\} \subset X$. Let $G(S)$ be the free abelian group with these points as basis; $G(S)$ is the *geometric group* on S.

Rmk: We could also construct the free R-module with basis S.

Def: Let $f: G(S) \xrightarrow{\sim} G(S)$ be an automorphism. Denote by $B_\varepsilon(p_t)$ the ε-ball about p_t.

f is an <u>ε-automorphism</u> if $\forall\ t = 1, \dots, n$, $f(p_t) \in G(S \cap B_\varepsilon(p_t))$ and $f^{-1}(p_t) \in G(S \cap B_\varepsilon(p_t$

We will define a map, for small enough ε, from $\{\varepsilon\text{-autos. over } \underline{\text{all}} \text{ geo. groups}\} \xrightarrow{\varphi} Wh(\pi_1 X)$. It turns out the for ε small enough, $\varphi = 0$ (even though n can increase w/o bound).

First we construct φ.

<u>Def</u>: Let f be an automorphism of $G(S)$. The <u>carrier</u> of $f(p_t)$, denoted car $f(p_t)$ or C_t, is $\{p_i \in S : \text{if we write } f(p_t) = \hat{\sum_{j=1}} a_j p_j,\ a_i \neq 0\} \subset S$.

If f is an ε-auto, $C_t \subset S \cap B_\varepsilon(p_t)$.

We would like to define φ for <u>weak</u> ε-autos., i.e. autos. s.t. $C_t \subset$ some ε-disc $\forall t$ (but not necessarily $B_\varepsilon(p_t)$). Unfortunately the properties of such maps aren't quite nice enough. Instead, we make the following

11/15/78: <u>Def</u>: Let $S = \{p_1, \dots, p_n\}$. An <u>ε-basis</u> for $G(S)$ is basis $\{x_1, \dots, x_n\}$ s.t.

(1) if $x_j = \sum_{i=1}^n a_{ij} p_i$ and $\text{Car}(x_j) = \{p_i : a_{ij} \neq 0$ then $\text{diam Car}(x_j) < \varepsilon$, and

(2) if $p_j = \sum_{i=1}^n A_{ij} x_i$ and $A_{ij} \neq 0$ then $\text{Car}(x_i)$ $B_\varepsilon(p_j)$. ((2) is like an "inverse" to (1)).

Given an ε-basis $\{x_1, \dots, x_n\}$, define

$T: G(S) \xrightarrow{\sim} G(S)$ by

$T(p_i) = x_i$.

T is not an ε-auto, but is more than just a weak ε-auto. (Note (1) $\Rightarrow T$ is an ε-auto.)

Lemma 1: (a) If T is an ε-automorphism of $G(S)$, $S = \{p_1, \dots, p_n\}$, then $\{Tp_i : i = 1, \dots, n\}$ is a 2ε-basis for $G(S)$.

(b) If $\{x_1, \dots, x_n\}$ is an ε-basis for $G(S)$, then $\exists\ \sigma \in \Sigma_n$ (symmetric group) s.t. the map $T: G(S) \longrightarrow G(S)$ defined by $Tp_i = x_{\sigma(i)}$ is an ε-automorphism.

Prf: (a) $\operatorname{Car}(Tp_i) \subset B_\varepsilon(p_i)$, so diam $\operatorname{Car} T(p_i) <$

For the second condition, suppose $p_j = \sum A_{ij} Tp_i$

so $T^{-1} p_j = \sum A_{ij} p_i$. By def'n of ε-auto,

$\operatorname{Car}(T^{-1} p_j) \subset B_\varepsilon(p_j)$, so if $A_{ij} \neq 0$, $p_i \in B_\varepsilon(p_j)$.

Now $\operatorname{Car}(Tp_i) \subset B_\varepsilon(p_i) \subset B_{2\varepsilon}(p_j)$.

(b) Write $x_j = \sum a_{ij} p_i$. The matrix $a = (a_{ij})$

$GL_n(\mathbb{Z})$, so $\det a \neq 0$.

We claim that $\exists$ a permutation of the columns of a changing a to a new matrix b, with $b_{ii} \neq 0\ \forall i$.

($\because$) $a = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, $\det a \neq 0$. Expand along the first row; some entry $\overset{\neq 0}{a_{1j}}$, with

corresponding minor $\det A_{ij} \neq 0$.
Permute the j^{th} column to the first place to get a matrix

$$a' = \left|\begin{array}{c|c} a_{ij} & * \\ \hline * & A_{ij}\end{array}\right| .$$

$\det A_{ij} \neq 0$ so continue by induction. This proves the claim.

Assume now for ease of notation that $a_{ii} \neq 0$ $\forall$ & we show that the map $T : p_i \longmapsto x_i$ is an ε-auto. $a_{ii} \neq 0 \Rightarrow p_i \in \mathrm{Car}(x_i)$.

But diam $\mathrm{Car}(x_i) < \varepsilon \Rightarrow \mathrm{Car}(x_i) \subset B_\varepsilon(p_i)$.

Finally we must show $\mathrm{Car}(T^{-1}p_j) \subset B_\varepsilon(p_j)$.

Write $p_j = \Sigma A_{ij} x_i$, so $T^{-1}p_j = \Sigma A_{ij} T^{-1}x_i = \Sigma A_{ij} p_i$

Suppose $p_i \in \mathrm{Car}(T^{-1}p_j)$; then $A_{ij} \neq 0$, and so $\mathrm{Car}(x_i) \subset B_\varepsilon(p_j)$. But $p_i \in \mathrm{Car}(x_i)$ since $a_{ii} \neq 0$. Thus $\mathrm{Car}(T^{-1}p_j) \subset B_\varepsilon(p_j)$. // Lemma 1.

Lemma 2: $\exists$ $\varepsilon > 0$ depending only on X s.t. there is a map {ε-automorphisms of geometric groups over X}
Wh $\pi_1 X$.

Denote the image of f by $\hat{f}$, and choose ε small enough that if $\hat{f}$ and $\hat{g}$ are defined, so is $\widehat{fg}$ (fg is a 2ε-automorphism). Then $\widehat{fg} = \hat{f}\,\hat{g}$.
(Or, additively, $\hat{f} + \hat{g}$.)

Rmk: Lemmata 1 & 2 give a map $\{\varepsilon\text{-bases}\} \to \mathrm{Wh}\,\pi_1 X$
(Two permutations have image differing by a

permutation matrix.

Pf: Pick a basepoint $* \in X$. Let $f : G(S) \xrightarrow{\sim} G(S)$ be an ε-auto. Choose paths $P_1, \dots, P_n$ from $*$ to $p_1, \dots, p_n$ resp. Let $C_i = \mathrm{Can}(fp_i)$ and let $Q_1, \dots, Q_n$ be paths from $*$ to $C_1, \dots, C_n$ resp.
(Note that we could choose $Q_i = P_i$ if we wish)
Define a matrix $(a_{ij}) \in GL_n(\mathbb{Z})$ by $fp_j = \sum a_{ij} p_i$
Define a matrix $\hat{f}$ over $\mathbb{Z}\pi_1 X$ by $\hat{f}_{ij} = a_{ij} P_i^{-1} Q_j$
Here if $a_{ij} = 0$, $\hat{f}_{ij} = 0$; if $a_{ij} \neq 0$ then $p_i \in C_j \subset B_\varepsilon(p_j)$. Join the endpoint of Q_j (which in C_j) to p_i by a small path; then $P_i^{-1} Q_j$ is a path based at $*$, whose homotopy class depends only on P_i, Q_j for small enough ε.
We must show $\hat{f} \in GL(\mathbb{Z}\pi_1 X)$. To do this, we
show $\widehat{gf} = \hat{g}\hat{f}$. ($g, f$: $\varepsilon/2$-autos).
Let $C_i = \mathrm{Can}(fp_i)$, $\bar{C}_i = \mathrm{Can}(gp_i)$, $\bar{\bar{C}}_i = \mathrm{Can}(gfp_i)$.
Now $C_i, \bar{C}_i, \bar{\bar{C}}_i$ are all in $B_\varepsilon(p_i)$, so we can use the same path p_i from $*$ to $C_i, \bar{C}_i, \bar{\bar{C}}_i$.
Write $fp_j = \sum a_{ij} p_i$, $gp_j = \sum b_{ij} p_i$, so

$$gf p_j = \sum c_{ij} p_i \text{ where } c = b \cdot a.$$

Now

$$(\hat{g}\hat{f})_{ij} = \sum_{k=1}^{n} b_{ik} P_i^{-1} P_k \cdot a_{kj} P_k^{-1} P_j$$

$$= \sum_{k=1}^{n} b_{ik} a_{kj} P_i^{-1} P_j$$

$$= c_{ij} P_i^{-1} P_j$$

$$= (\widehat{gf})_{ij}$$

Now in particular $\hat{f}^{-1}\hat{f} = I$ so $\hat{f}$ is invertible. Finally we show the map $f \mapsto \hat{f} \in Wh\,\pi_1 X$ is indep of choices.

Suppose we pick another path $*$ to p_1, say. This is equivalent (up to homotopy) to choosing $\alpha \in \pi_1($ & letting the new path be $P_1 \cdot \alpha$ (read right to left)

Now the new $\hat{f}_{1j} = a_{ij}\,\alpha^{-1} P_1^{-1} Q_j$. Thus we've multiplied the whole first row by $\alpha^{-1} \in \pi_1 X$. But in $Wh(\pi_1 X)$ we factor out by matrices corresponding to group elements, so $f \mapsto \hat{f}$ is well-defined. //Lemma

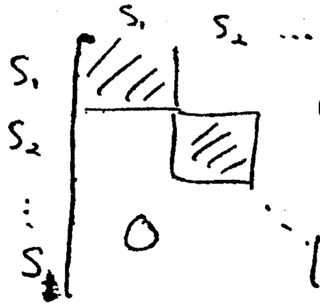
Rmk: The $\pm\pi_1 X$ in the definition of $Wh(\pi_1 X)$ co in with the permutation changing an ε-basis to an ε-auto.

1/17/78: Def: Let $S = \{p_1, \dots, p_n\}$. f is a blocked ε-automorphi of $G(S)$ if $\exists$ a partition $S = S_1 \dot\cup S_2 \dot\cup \dots \dot\cup S_t$ s.t. diam $S_i < \varepsilon$ $\forall i$ & $f(G(S_i)) \subset G(S_i)$, where we think of $G(S_i) \subset \overset{t}{\underset{i=1}{\oplus}} G(S_i) = G(S)$ in the obvious way.

Lemma 3: If f is a blocked ε-automorphism, then $\hat{f} = 0 \in Wh\,\pi_1 X$.

Prf: Draw a path P_i to each $p_i \in S$ s.t., if $p_i, p_j \in$ some S_k, then $P_i = P_j$ (i.e. "essentially equal," differing only by a small path in an ε-ball). Pick $Q_i = P_i$ (as in lemma 2).

Now the matrix $(\hat f_{ij})$ is a blocked diagonal matrix, since $a = (a_{ij})$, defined by $f p_j = \sum_{i=1}^{n} a_{ij} p_i$, is such a matrix, with each block corresponding to one of the S_j.



Now $\hat f_{ij} = a_{ij} P_i^{-1} P_j$; whenever $a_{ij} \neq 0$, $P_i = P_j$, so $\hat f_{ij} = a_{ij} \cdot 1$ $(1 \in \pi_1 X)$. ∴ $\hat f \in \text{im } \underset{\overset{\|}{\pm 1}}{K_1(\mathbb{Z})} \longrightarrow K_1(\mathbb{Z}\pi_1 X)$.

But this ± 1 goes to $0 \in Wh\, \pi_1 X$; so $\hat f = 0$. //

Conj (Connell - Hollingsworth): Given X and $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. any δ-automorphism over X is the product of $n+1$ blocked ε-automorphisms, where $n = \dim X$.

This conjecture was finally proved by Quinn.

Assuming the conj. for now, we have its

Cor: $\exists\, \varepsilon$, depending only on X, s.t. if f is an ε-automorphism then $\hat f = 0$.

Prf. $\exists\, \varepsilon_1 > 0$ s.t. $\hat f$ is defined for ε_1-autos. Let $\varepsilon_2 = \varepsilon_1/(n+1)$. Now the composite of $n+1$ ε_2-autos. is an ε_1-auto, & we have $\widehat{f_1 \circ \cdots \circ f_{n+1}} = \hat f_1 + \cdots + \hat f_{n+1}$ for ε_2-autos. f_i.

Now the conjecture $\Rightarrow \exists\ \varepsilon$ (the "δ" of the statement on previous page) s.t. any ε-auto is the product of $n+1$ blocked ε_2-autos. If f is an ε-auto, write $f = f_1 \circ \cdots \circ f_{n+1}$, where each f_i is ε_2-blocked; then $\hat{f} = \hat{f}_1 + \cdots + \hat{f}_{n+1} = 0 \in Wh\,\pi_1 X$. //

Def: Let $X \subset K$ be a pair of finite CW complexes (or simplicial complexes). K *ε-deformation retracts* o X, written $K \overset{\varepsilon}{\searrow} X$, if there is a 1-parameter family of continuous maps $K \longrightarrow K$, $0 \leq t \leq 1$, s. $h_0 = 1_K$ and h_1 is a retraction onto X, satisfying the following condition: $\forall\ p \in K$ define a path α_p in X by $\alpha_p(t) = h_1 h_t(p)$; then $\operatorname{diam} \alpha_p < \varepsilon\ \ \forall\ p \in K$. We say $\{h_t\}_{t\in I}$ is an *ε-deformation retraction*.

We use the Cor. to prove Ferry's Thm:

~~Thm. Let $X \subset K$ be a pair of finite CW (or simplic~~
~~complexes s.t. $K \overset{\varepsilon}{\searrow} X$.~~

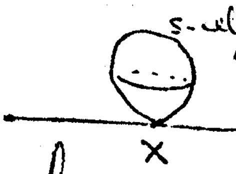
Thm: Let X be a CW (or simplicial) complex. Then $\exists\ \varepsilon > 0$, depending only on X, s.t. if $K \overset{\varepsilon}{\searrow} X$ th $\tau(K, X) = 0$.

Lemma: X as above; n fixed positive integer $\geq \dim X$; $\forall\ \varepsilon > 0\ \exists\ \delta > 0$, depending on ε and X, s.t. if $K \overset{\varepsilon}{\searrow} X$ and $\dim K \leq n$ the $\exists\ L$ s.t. $L \overset{\delta}{\searrow} X$ and

(i) $\tau(K, X) = \tau(L, X)$

(ii) $L - X$ contains only cells of dimensions s and $s+1$

Further, we can assume that the s-cells are attached to X at single points ("trivially attached") and that each cell of $L - X$ has diameter $< \varepsilon/10$.



Rmk: This lemma is (essentially) straight out of Whitehead's original paper on Whitehead torsion.

11/20/78:

Digression: Recall our original conjecture 1 (p. 2): If M, N are closed aspherical manifolds with $\pi_1 M \cong \pi_1 N$ then $M \cong N$.

One way of trying to get a counterex. is as follows:

Suppose $M = \mathcal{B} \underset{\mathcal{D}}{\cup} \mathcal{C}$, $\operatorname{codim}_M \mathcal{D} = 1$, and $f: N \to M$ is a h.e. Cappell has considered the question of when can we "model" the splitting of M in N?

Let $B = \pi_1(\mathcal{B})$, $D = \pi_1(\mathcal{D})$, $C = \pi_1(\mathcal{C})$ and suppose $D \to B$, $D \to C$ are 1-1. If $[B:D] = [C:D] = 2$ then there is an obstruction group $UNil(D; C, B)$, the "unitary nilpotent group", to mimicking the splitting in N.

Ex: If $D = 1$, $B = C = T_2$, then $UNil(D; C, B) \neq 0$.

(*) Conj If B, C, D are torsion free then $UNil(D; C, B) = 0$.

Suppose this is not true. Let $\pi_1 M = \Gamma$, M aspherical, & $\Gamma = C \ast_D B$. If $UNil(D; C, B) \neq 0$ then we get an immediate counterex. to Conj 1.

~~(Obtain (apparently this conj (*) has nothing to do with the counterex.)~~

Note if $\Gamma = B \ast_D C$, ~~and $UNil(D; C, B) \neq 0$~~ there is

an epi $\longrightarrow T_2 * T_2$; so one way to show $UNil(D; C, B) \neq 0$ is to show that the associated map $UNil(D; C, B) \xrightarrow{\varphi_*} UNil(1; T_2, T_2) \neq 0$ is onto, where B, C, D are torsion free.

This question comes down to the following:

Let $T_2 * T_2 = \langle a, b : a^2 = b^2 = 1 \rangle$.

If G is any group, define the map $\mathbb{Z}G \to \mathbb{Z}G$, written $x \longmapsto \bar{x}$, by the antiauto. $g \longmapsto g^{-1}$ (i.e. $\sum n_g g \longmapsto \sum n_g g^{-1}$).

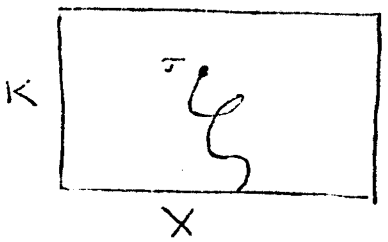
<u>Qu</u>: Are there $x \in \mathbb{Z}C$, $y \in \mathbb{Z}B$ s.t. $\varphi x = a \in \mathbb{Z}T_2$, $\varphi y = b \in \mathbb{Z}T_2$ (a, b the generators of the T_2 factors as above) s.t. $\begin{pmatrix} x - \bar{x} & -1 \\ 1 & y - \bar{y} \end{pmatrix}$ is invertible over $\mathbb{Z}$

<u>Ref</u>: Cappell, <u>Topology</u> 1974

We now return to the lemma, pp. 114-115.

<u>Prf sketch</u>: We won't bother to keep ε's & δ's straight. We suppose $K - X$ has cells of dim'n 0, 1, 2, $s+1$. We will <u>trade</u> each cell of dim i ($i < s$ for a cell of dim $i+2$.

Subdivide X & K first so that all cells have diamet less than $\varepsilon/10$. Now we begin cell trading.

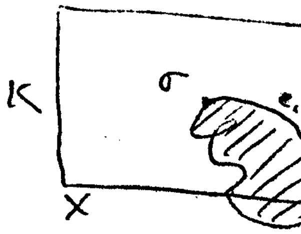


Let σ be a 0-cell in K; we want to trade in σ for a 2-cell τ. [To keep ε's & δ's straight, we trade in <u>all</u> 0-cells at once (in one step), then <u>all</u> 1-cells, &c. Since $\dim K \leq n$ we can control the metric criteria this way.]

Consider the path $h_t(\sigma)$, which is not necessarily a cell.

Let τ be the 2-cell , with $\partial\tau = e_0 \cup e_1$.

Glue τ to K by $e_0 \xrightarrow{\alpha} h_t(\sigma)$ to get $\tau \underset{\alpha}{\cup} K$. Form $\tau \underset{\alpha}{\cup} K/e_1$ by sending e_1 to a point.



Now $X \subset \tau \underset{\alpha}{\cup} K/e_1$, im. of $\sigma \in X$, & $Wh(\tau \underset{\alpha}{\cup} K/e_1, X) = Wh(K, X)$.

[For $Wh(\tau \underset{\alpha}{\cup} K, X) = Wh(K, X)$, and $Wh(X \cup e_1, X) =$ $\therefore Wh(\tau \underset{\alpha}{\cup} K, X \cup e_1) = Wh(K, X)$. Now if we collapse e_1 to a point, we haven't changed the difference cell structure (i.e. $\tau \underset{\alpha}{\cup} K - X \cup e_1$).]

Of course we must do this without messing up th metric control — make sure the htpy ext'n thm & cellular approx. thm work with ε-control. The ε-control cellular approx. thm is used to insure that $h_t(\sigma) \subset$ 1-skeleton, so $K \underset{\alpha}{\cup} \tau/e_1$ is still a complex.

Now continue inductively. //Lemma

<u>Pf of Thm</u> (p 114). We have $L \overset{\varepsilon}{\simeq} X$, $\tau(L, X) = \tau(K, X)$. Look at the def'n of τ and consider the matrix defining this element of $Wh \pi_1 X$. Since the cells of L are so small, this element is $\hat{f}$ for some ε-auto. f, constructed as follows.

Let $p_1, \dots, p_n$ be the points of X at which the 0-cells are attached; let $S = \{p_1, \dots, p_n\}$.

We get a geometric group $G(S)$ on X.

Write $L - X = \bigcup \sigma_i^s \cup \bigcup \tau_j^{s+1}$.

Each τ_j may be incident to several σ_i's. $\{\tau_j\}$ is a basis for C_{s+1} (cell complex). Let $a_j = \partial\tau_j \in C_s$; the $\{a_j\}$ are basis elements of C_s. (cf. pp identify C_s with $G(S)$ [by $\sigma_i \longleftrightarrow p_i$], so we have defined a new basis $\{a_j\}$ of $G(S)$. Since diam $\tau_j < \varepsilon/10$, we see that a_j = lin. comb. of p_i which are close together (with $3\varepsilon/10$, in fact). Write $p_j = \Sigma A_{ij} a_i$. If $A_{ij} \neq 0$, consider the corresponding σ_i; deform σ_i to X; the cell can't move far (since the map is an ε-def. retraction). So $A_{ij} \neq 0 \Rightarrow a_i$ is close to p_j.

[I don't think I understand this argument] (over)

Thus the $\{a_j\}$ are an ε-basis of $G(S)$; so we get an ε-auto f (see Lemma 1, p. 109) whose associated Whitehea torsion elt. $\hat{f}$ is precisely $\tau(L, X)$.
But for small ε, $\hat{f} = 0$ (Cor of Connell-Hollingsworth, p. 103).
This proves Ferry's Theorem. //

So we must verify the Connell-Hollingsworth conjecture, which we do by Quinn's methods. (See handout, from a seminar given by Quinn, Summer 1978)

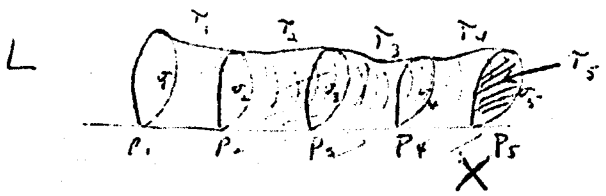
We will verify C-H for n-cubes, which will suffice to prove it in general. For a simplicial complex X^n embeds in $\mathbb{R}^{2n+1}$; the induced metric may

Basic idea (I think): Suppose

$$(*)\begin{cases} a_1 = p_1 - p_2 \\ a_2 = p_2 - p_3 \\ a_3 = p_3 - p_4 \\ a_4 = p_4 - p_5 \\ a_5 = p_5 \end{cases} \quad \text{so} \quad p_1 = \sum_{i=1}^{5} a_i$$

where $d(p_i, p_{i+1}) < \varepsilon$ but $d(p_i, p_j) > \varepsilon$ if $|i-j| > 1$.

This would give a picture as follows $(s=1)$:



Now $L \searrow X$ but not by an ε-def. retraction, since σ_1 has to be pulled all the way to τ_5 to be collapsed.

Thus $(*)$ above can't happen.

is different from the original ~~[illegible]~~ metric, but there is a "Lipschitz bound" between them (compactness). By shrinking the embedding, can assume $X \subset D^{2n+2}$. Now prove the statement "$\forall \varepsilon \exists \delta \ldots$" in D^{2n+2}, which will imply it for the original metric also.

11/22/75 (Notes taken from Chris Stark)

Quinn's work

Let 2^S = the power set of S.
If $f: G(S) \longrightarrow G(S)$ is an auto., Quinn defines a map $\underline{f}: 2^S \longrightarrow 2^S$ by

$$\{p_1, \ldots, p_k\} \longmapsto \bigcup_{j=1}^{k} \operatorname{car} f(p_j).$$

Def: A homomorphism $E: G(S) \longrightarrow G(S)$ is an *elementary automorphism* is an automorphism which either

(i) sends each $p_i \in S$ to $\pm p_i$, or
(ii) fixes all but one element of S, and sends that one point p_i to $p_i \pm p_j$ for some $p_j \in S$, $p_j \neq p_i$.

In matrix form, E is either $\begin{bmatrix} \pm 1 & & 0 \\ & \pm 1 & \\ 0 & & \ddots \ \pm 1 \end{bmatrix}$ (diagonal in $GL_n(\mathbb{Z})$)

~~or~~  or $I \pm E_{kl}$ where E_{kl} is the $n \times n$ matrix with 1 in the (k,l)-position, zeroes elsewhere.

Def. A homomorphism $H: G(S) \longrightarrow G(S)$ is a *deformation* if H is a sequence $E_1, E_2, \ldots, E_q$ of elementary automorphisms; q may be arbitrarily large.

Rmk: Because our interest is in seqs. of elem. aut
rather than simply individual autos., we could
allowed arbitrary elementary matrices $\underline{I} + a\,\mathcal{E}_{ij}$
in the definition. At a later date, Farrell mentioned
this more general def'n of elementary auto.;
doesn't matter which one we use.

Def: The diameter of the deformation H is
$$\max_{p\in S} \operatorname{diam}(\underline{E}_1(\underline{E}_2(\cdots(\underline{E}_q(\{p\}))\cdots))),$$
$\underline{E}_i$ defined as on p. 119.

Ex: If $S = \{p_0,\dots,p_n\}$, $E_i = \mathcal{E}_{i-1,i}$ for $i=1,\dots,n$, then
$$\underline{E}_1(\underline{E}_2(\cdots(\underline{E}_n(p_n))\cdots)) = S$$

Def. An ε-deformation is a deformation $H = E_1,\dots,E_q$ s.t
$\operatorname{diam} H < \varepsilon$ and $\operatorname{diam} H^{-1} < \varepsilon$, where $H^{-1} = E_q,\dots,$
(Rmk: I think H^{-1} should be $E_q^{-1},\dots,E_1^{-1}$)

Quinn proves the following modified form of the C-H conjecture. Note that the number of factors in the prod
is allowed to be unbounded.

Thm: Given $\varepsilon > 0$ and X, $\exists$ $\delta > 0$ s.t. any δ-automorphism over X is "stably" an ε-deformation $E_1,\dots,E_q$; i.e. $\exists$ a geometric group G_0 such that $H\oplus \mathrm{id} : G(S)\oplus G_0 \xrightarrow{\sim} G(S)\oplus G_0$ is an ε-deformation.

Rmks. 1. C-H and Quinn allow a point of X to appear several times in S. This doesn't seem to be truly needed, since we could "stamp our feet" & knock the multiple points apart. (These ideas come up on pp 137 ff.)

2. We are not out of the woods yet.
We would like

$$\varphi(E_1 \cdots E_q) := [\widehat{(E_1 \cdots E_q)}] \in \text{Wh}\, \pi_1 X$$

to be

$$\varphi(E_1) + \cdots + \varphi(E_q).$$

This requires the restriction of the diameter of H^{-1} in
the def'n of ε-deformation, & some care with t

Embed X in a disc I^N. We will prove Quinn's Th
on a disc, & it will follow readily for X.

Def: If $C \subset X$ and S is finite $\subset X$, then a ma
$f: G(S) \longrightarrow G(S)$ is an <u>ε-automorphism over C</u> if
there is a group homomorphism $g: G(S) \to G(S)$ s.t
$pfgi = id$ and $pgfi = id$, where

$$\left.\begin{array}{l} i: G(S \cap C) \hookrightarrow G(S) \text{ is inclusion,} \\ \text{and } p: G(S) \twoheadrightarrow G(S \cap C) \text{ is projection} \end{array}\right\} \text{canonical m}$$

;
i.e. fg and gf have matrices of the form

$$\begin{array}{c|cc} & S\cap C & S-C \\ \hline S\cap C & I & * \\ S-C & * & * \end{array} \quad .$$

[<u>and</u> f, g are ε maps]

<u>Lemma</u>: Given n and $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. any δ-autom
(based on D^n) over $\frac{3}{4}D^n$ is stably and after an ε-d
matrix blocked (over $\frac{3}{4}D^n$) w.r.t. $\frac{2}{3}D^n$ and $D^n - \frac{1}{3}D^n$.

In other words, ~~[illegible]~~ let $f: G(S) \to G(S)$ be a δ-autom
phism over $\frac{3}{4}D^n$. Then $\exists$ a geometric group $G_0 = G(S'$
where $S' \subset \frac{2}{3}D^n$, and an ε-deformation $E_1, \ldots, E_q =$
of $G(S) \oplus G_0$, s.t. $H(f \oplus id) = f_1 \oplus f_2$, where
$G(S \cup S') = G(S_1) \oplus G(S_2)$, $S_1 \subset \frac{2}{3}D^n$, $S_2 \subset D^n - \frac{1}{3}D^n$,
$f_i: G(S_i) \xrightarrow{\sim} G(S_i)$ is an automorphism, and H lea
fixed everything outside $\frac{2}{3}D^n$.

11/27/75 :

We first show how the lemma $\Rightarrow$ the theorem.

We do this only in the case $n=2$, which suffices to indicate the general procedure.

Rmks: The Thm is 8.4 in Quinn's notes; the lemma is 8.6. Farrell refers to the lemma as the "Main Lemma"

Prf: Lemma $\Rightarrow$ Thm $(n=2)$.

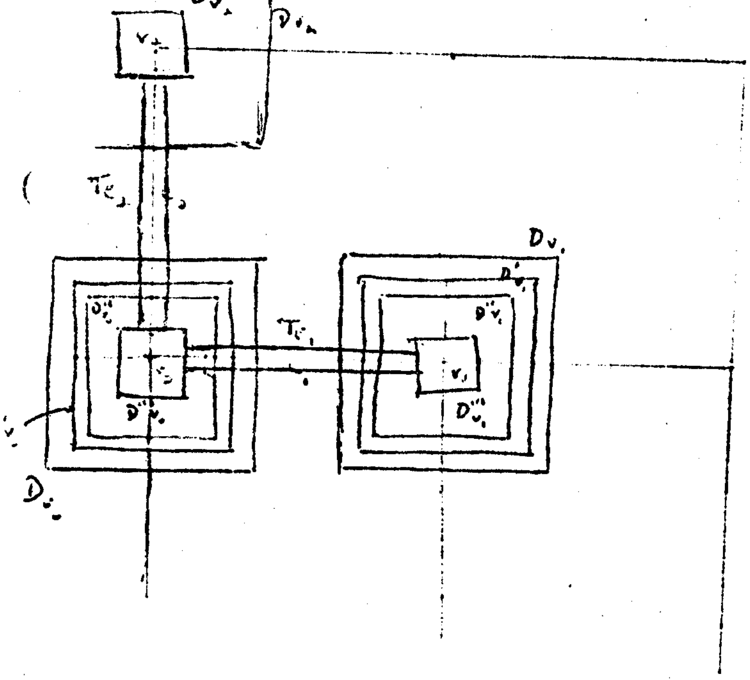
Define a cell structure on $\mathbb{R}^2$ as follows: the integral lattice points are vertices, the horiz. & vertical line segments of length 1 connecting the vertices are edges. Denote vertices by v, edges b

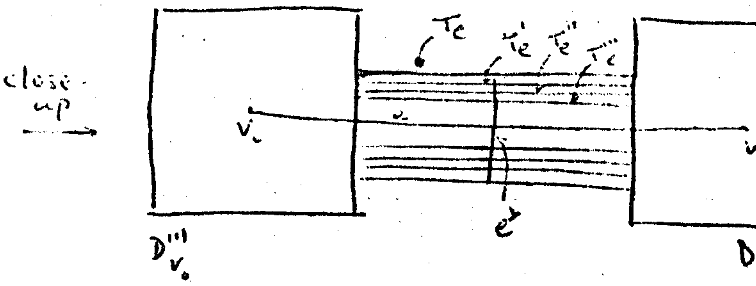
We now consider a "dual" cell structure as follows. $\forall$ vertex v, let D_v be the square in $\mathbb{R}^2$ centered at v with sides of length $\frac{2}{3}$ (so if $D_v \cap D_{v'} = \emptyset$, in fact $d(D_v, D_{v'}) = \frac{1}{3}$). Let D_v', D_v'', D_v''' b smaller squares centered at v with sides of leng $\frac{3}{4}\cdot\frac{2}{3}$, $\frac{2}{3}\cdot\frac{2}{3}$, $\frac{1}{3}\cdot\frac{2}{3}$.

$\forall$ edge e, let T_e be the rectangle with "core" e, connecting D''_{v_0} to D'''_{v_1} where v_0, v_1 are the endpoints of e, and of width $w = \frac{1}{3}$ (width of D_v''). Again, associate smaller rectangles T_e', T_e'', T_e''' in the same fashion, with widths $\frac{3}{4}w$, $\frac{2}{3}w$, $\frac{1}{3}w$.

Let $e^\perp$ be the perpendicular segment to e, at the midpoint of e, of length w.

Each component of $\bigcup_v D_v''' \cup \bigcup_e T_e'''$ is a region C, shaped like





side of D'''_v : $\frac{2}{9}$

width of T_e : $\frac{2}{27}$

width of T'''_e : $\frac{2}{81}$

Length of T_e : $\frac{25}{2}$

Clearly all the D_v's, T_e's etc. are identical up to rigid motion.

We're given $n(=2)$ and $\varepsilon>0$; we want to find a δ.

Let $\mathcal{D}_s$ be the square in $\mathbb{R}^2$ with vertices $(0,0),(s,0),(0,s),(s,s)$, $s\in\mathbb{Z}_+$.

Pick s large enough that $s\varepsilon>1$. It suffices to find $\delta>0$ s.t. any δ-auto. f (based on $\mathcal{D}_s^{\#}$) is stably a 1-deformation. [In fact, we get f to be a 3-def., which is clea
(Alternatively, we could have worked with $\mathcal{D}_1 = I^2$ by shrinking our cell structure small enough that the distance between vertices is s.)

Naive start. We're given f; we try to show $f=f_1\oplus f_2\oplus f_3$ where each $f_i : G(S_i)\xrightarrow{\simeq} G(S_i)$, ~~[illegible]~~. $S=S_1\dot\cup S_2\dot\cup S_3$, with $S_1\subset\bigcup_{v\in\mathcal{D}_s} D''_v$, $S_2\subset\bigcup_{e\in\mathcal{D}_s} T''_e$, $S_3\subset\bigcup_{c\in\mathcal{D}_s} C$ = complement of $\bigcup D'''_v \cup \bigcup T'''_e$.

Suppose we can do this.

f_1 is supported on a union of small squares which are far apart, so if δ is small enough ($\frac{1}{100}$, e.g.) no basis element can be moved from 1 square to another; therefore each square is left invariant, i.e. $f_{1,v} := f|_{G(S_1 \cap D_v'')}$ leaves $G(S_1 \cap D_v'')$ invar

But $Wh(1) = 0$; so $\hat{f}_{1,v} = \Pi$ elementary matrices, i.e. $f_{1,v}$ is a deformation of size $< \operatorname{diam} D_v'' = \frac{4\sqrt{2}}{9} < 1$. Similarly for $f_{2,e} = f|_{G(S_2 \cap T_e'')}$, $f_{3,c}$. Then f is a deformation of size < 1.

Unfortunately, the naive proof doesn't quite work; we can't get $f = f_1 \oplus f_2 \oplus f_3$. Instead, we try to find a 1-deformation H st.

$$(k) \quad Hf = f_1 \oplus f_2 \oplus f_3 \, ,$$

where again each f_i is a 1-deformation; Now f will be a 2-deformation (which is good enough; just choose s st. $s\varepsilon > 2$).

Unfortunately (k) also doesn't quite hold; we must stabilise to get

$$(2) \quad H(f \oplus id) = f_1 \oplus f_2 \oplus f_3 .$$

Again, this is enough to prove the thm.

We must therefore prove (2). For convenience's sake, however, we will assume the main lemma holds without stabilisation, and use it to show (k) instead. In fact, we will show

$$(\xi) \quad H_2 H_1 f = f_1 \oplus f_2 \oplus f_3 \qquad \text{where}$$

each H_i is a 1-deformation and the f_i are as above, therefore also 1-deformations; so f is (strictly, but this is suppressed) a 3-deformatio

Let $i: G(S\cap D_v) \hookrightarrow G(S)$ be inclusion,
$p: G(S) \longrightarrow G(S\cap D_v)$ be projection. ~~fi~~

Given $\varepsilon_1 > 0$, then ~~$\exists$~~ we can make δ small enough that $\forall v \in \mathcal{D}_s$ pfi is a ε_1-auto (based on D_v) over D_v'.
Note δ must be small enough that $f(G(S\cap D_v')) \subset G(S\cap D_v)$.
[In fact if δ_1 is small enough, can let $\delta = \varepsilon_1$]

Now, given $\varepsilon_1 > 0$, the lemma $\Rightarrow$ $\exists$ $\varepsilon_1 > 0$ (which will be the δ_1 above) s.t. if pfi is a δ_1-auto over D_v' then $\exists$ ε_1-deform'n H_v s.t. $H_v pfi = f_{v,1} \oplus f_{v,2}$ where $S\cap D_v = S_{v,1} \cup S_{v,2}$, $S_{v,1} \subset D_v''$, $S_{v,2} \subset D_v - D_v'''$.

Let $H_1 = \prod_{v\in\mathcal{D}} H_v$. Since all the D_v's are disjoint, H_1 is a 1-deformation (in fact an ε_1-deformation).
$H_1 f = f_1 \oplus f_2$ where $S = S_1 \cup S_2$, $S_1 = \cup S_{v,1}$, $S_2 = S - S_1$.
$S_1 \subset D_v''$ as desired, but S_2 is not as nice as we need; all we know is that $S_2 \subset \mathcal{D}_s - \cup D_v'''$.

11/29/75.
(**Recall**: Let $A \subset X$, S finite $\subset X$. A map $f: G(S) \longrightarrow G(S)$ is a <u>δ-auto over A</u> if $\exists$ $g: G(S) \longrightarrow G(S)$ [like an inverse] s.t. $pfgi = id_{G(S\cap A)}$, $pgfi = id_{G(S\cap A)}$, and both f and g are δ-maps, i.e. $\mathrm{Car}\, f(p_i) \subset B_\delta(p_i)$, $\mathrm{Car}\, g(p_i) \subset B_\delta(p_i)$ $\forall i$.)

Return to the proof:

H_1 an ε_1-deformation, f a δ-auto $\Rightarrow f_2$ is a $(\delta+\varepsilon_1)$-

$\forall$ edge e of $\mathcal{D}_3$, we again consider

$$\begin{tikzcd} G(S_2 \cap T_e) \arrow[r, bend left, "i"] & G(S_2) \arrow[l, bend left, "p"] \end{tikzcd}.$$

Given $\varepsilon_2>0$ If δ, ε_1 are sufficiently small, then each pf_2i a δ_2-auto of T_e over T_e'.

Now pick $\varepsilon_2 = \frac{1}{400}$, say; by lemma 2 $\exists\ \delta_2$ s.t. any δ_2-map is ε_2-deformable to a blocked auto. (Thus we first pick $\varepsilon_2 = \frac{1}{400}$; this gives δ_2, which gives bound on ε_1 and δ, ε_1 gives a bound on δ_1, whi gives another bound on δ; this determines δ).

Now we can't apply the main lemma to our collection $(T_e, T_e', T_e'', T_e''')$. Instead, we will project T_e onto

Of course all distances will be reduced.

More precisely let $q: T_e \to e^\perp$ be orthogonal projec

Let $S_e = q(S \cap T_e)$. [If $p_1 \neq p_2$ we will consider qp_1, qp_2 as distinct points.] Now apply the main lemma to $G(S_e)$ with $\varepsilon_2 = \frac{1}{400}$ to get our δ_2, a mentioned above.

Now pull the deformation given by the lemma back all of T_e. The result is still a deformation, but we lose metric control in the direction parallel to e. A priori, this could result in problems if $n>2$, si we need to build up at each stage. However it turns out that we keep just enough control in the right direction (parallel to $e^\perp$) that

the process can be continued in higher dimensions.
Then we get a deformation H_e s.t. $H_e p f i = g_{1e} \oplus g_{2e}$, where $S_2 \cap T_e = S_{1,e} \cup S_{2,e}$, $S_{1,e} \subset T_e''$ and $S_{2,e} \subset T_e - T_e'''$. H_e is a 1-deformation but in the $e^{\perp}$-direction we can say more; it is a $\frac{1}{400}$-deformation.

Let $H_2 = \prod_{e \subset \Delta_s} H_e$, a 1-deformation.

Let $H = H_2 H_1$. We can write:

$$Hf = \varphi_1 \oplus \varphi_2 \oplus \varphi_3,$$

a decomposition and a partition $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ of S, s.t. $\mathcal{S}_1 \subset \bigcup_{v \in \hat{\Delta}_s} D_v''$, $\mathcal{S}_2 \subset \bigcup_{e \subset \Delta_s} T_e''$ and $\mathcal{S}_3 \subset \bigcup_{c \subset \hat{\Delta}_s} C$ (cf. p. 124). Each of these pieces (D_v'', T_e'', C) has diameter < 1, and since H_e is a $\frac{1}{400}$-deformation in the $e^{\perp}$-direction, φ_3 really keeps each region C invariant. So $\varphi_1 \oplus \varphi_2 \oplus \varphi_3$ really is a 1-deformation as desired. //thm

<u>Rmk</u>: We fudged a little; diam $C > 1$, but is less than some $1 + \lambda$, λ small. So we get f is a $(3 + \lambda)$-deformation which is obviously good enough.

<u>Rmk</u>: Quinn keeps much more rigid metric control by continually subdividing. This is necessary since he is dealing with an arbitrary handlebody decomposition of a manifold, while we are using a specific nice decomposition of a disc D_s.

Next we turn to the proof of the Main Lemma. The main idea is the <u>Kirby trick</u> (Quinn's notes, Lemma 8.5).

Kirby trick: Given $\varepsilon > 0$ and $n \in \mathbb{Z}_+$, $\exists\ \delta > 0$ s.t. any δ-automorphism f of D^n over $\frac{2}{3}D^n$ is "almost" stably extendable to an ε-automorphism of D^n; i.e., let $G(S) \xrightarrow{f} G(S)$ be a δ-automorphism over $\frac{2}{3}D^n$; then $\exists$ finite $S_0 \subset D^n - \frac{1}{2}D^n$ and an ε-automorphism $g: G(S \cup S_0) \xrightarrow{\simeq} G(S \cup S_0)$ s.t. $g|_{G(S \cap \frac{1}{2}D^n)} = f|_{G(S \cap \frac{1}{2}D^n)}$.

13/4/75: We restate the Kirby trick slightly:

Lemma: Given $\varepsilon > 0$ and $n \in \mathbb{Z}_+$, $\exists\ \delta > 0$ s.t. any δ-automorphism f of D^n over $\frac{2}{3}D^n$ is stably almost extendable to an ε-automorphism of D^n; i.e., if $f: G(S) \longrightarrow G(S)$ is a δ-automorphism over $\frac{2}{3}D^n$ then $\exists$ finite $S' \subset D^n$ and an ε-automorphism $g: G(S') \xrightarrow{\simeq} G(S')$ s.t.

(i) $S \cap \frac{1}{2}D^n = S' \cap \frac{1}{2}D^n$

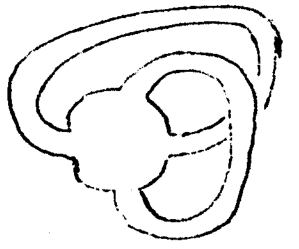
(ii) $f|_{G(S \cap \frac{1}{2}D^n)} = g|_{G(S' \cap \frac{1}{2}D^n)}$.

Further, we can assume $S' \subset \frac{2}{3}D^n$.

Rmk. S' corresponds to $S \oplus S_0$ above, but it is in fact not necessary that $S \subset S'$.

Pf. The proof is based on the "torus trick," which is in turn based on the following fact. Let $p \in T^n$. Then there is a smooth immersion $F: T^n - p \looparrowright \mathbb{R}^n$. (Equivalently, if B is a small ball about p in T^n, $F: T^n - \mathring{B} \looparrowright \mathbb{R}^n$).

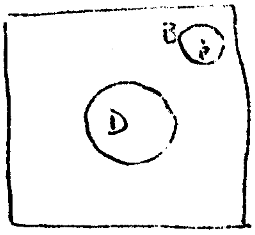
Ex. $n = 2$: the image of F looks like:



Now think of $T^n = \mathbb{R}^n / 2\mathbb{Z}^n$;

a fundamental domain $= [0, 2]^n$.

Let $D^n = D$ = the closed 1-disc about $0 \in \mathbb{R}^n$; $D \subset\subset [0,2)^n$; let B be another closed n-disc in T^n disjoint from D^n, ctr'd at $p \notin D$:



We can define F s.t.

(i) im $F \subset \frac{2}{3}\overset{\circ}{D}{}^n$, and

(ii) $F|_{\frac{1}{2}D} = id_{\frac{1}{2}D = \frac{1}{2}D^n}$, where we think of D both in T^n and in $[0,2)^n \subset \mathbb{R}^n$.

Let $F_1 = F|_{T^n - \frac{1}{2}\overset{\circ}{B}}$. Let $\hat{S}_1 = F_1^{-1}(S) \subset T^n - \frac{1}{2}\overset{\circ}{B}$.

Now we will construct a "sort of" γ-cute $\hat{f}$ of $G(\hat{S}_1)$.

We use uniform continuity & the fact that F is an immersion!

For γ sufficiently small (in particular, we'll want $\gamma < \varepsilon/2$), ~~[illegible]~~ lemma) s.t. $\exists\ \delta$ (which will be the δ of the lemma) s.t. $\forall\ x \in T^n - \frac{1}{2}\overset{\circ}{B}$, $\forall\ y$ which is δ-close to $F(x) \in D^n$, $\exists!\ y' \in T^n - p$ within γ of x s.t. $F(y') = y$.

(i.e. $\exists\ \gamma$ which is ~~[illegible]~~ small enough that F is a homeo on γ-balls.)

Now if $y \in S$ and f is a δ-cute over $\frac{2}{3}D^n$, Car $f(y)$ is within δ of y. We would like, therefore, to lift f to a γ-cute $\hat{f}$ (on $T^n - \frac{1}{2}\overset{\circ}{B}$) over $T^n - (\frac{1}{2}+\gamma)\overset{\circ}{B}$. This doesn't quite work, since ~~[illegible]~~ if $x \in \partial(\frac{1}{2}B)$, ~~[illegible]~~ $y \in$ Car $f(F(x))$, y may not be in im $F|_{T^n - \frac{1}{2}B^{\circ}}$ (i.e. the y' above may be in $\frac{1}{2}\overset{\circ}{B}$).

→ I don't think $\hat{S}_2$ is used

So define $\hat{S}_2 = F_2^{-1}(S)$; let φ = projection: $G(\hat{S}_1) \longrightarrow G(\hat{S}_1 \cap (T^n - (\frac{1}{2}+\gamma)$

Now define $\hat{f} : G(\hat{S}_1) \longrightarrow G(\hat{S}_1)$ as follows: ~~[illegible]~~

~~[illegible]~~

~~[illegible]~~

for $p \in \hat S_1$, $\hat f(p)$ is the elt $q \in G(\hat S_1)$ within γ of p s.t. $F(\mathrm{Can}\, q) =$ ~~[illegible]~~ $\mathrm{Can}\, f(F(\varphi(p)))$

<u>Claim A</u>: $\exists\ \hat S^{\text{finite}} \subset T^n -$ ~~[illegible]~~ p s.t. $|\hat S| = |\hat S_1|$ and s.t. $\hat S \cap (T^n - 2\mathring B) = \hat S_1 \cap (T^n - 2\mathring B)$, and $\exists$ a 2γ-automorphism $\bar f : G(\hat S) \xrightarrow{\sim} G(\hat S)$ s.t.

$$\bar f|_{\frac12 D} = \hat f|_{\frac12 D} = f|_{\frac12 D}$$

(where $f|_{\frac12 D}$ means $f|_{G(S \cap \frac12 D)}$, etc.).

<u>Rmk</u>: $\hat f|_{\frac12 D} = f|_{\frac12 D}$ since $F|_{\frac12 D} = \mathrm{id}$.

<u>Pf</u>: $\hat f|_{T^n - \mathring B}$ is a monomorphism $G(\hat S_1 \cap (T^n - \mathring B)) \hookrightarrow G(\hat S$

Because $\hat f$ is a γ-auto over $T^n - (\frac12 + \delta)\mathring B$, one can show that $\hat f(G(\hat S_1 \cap (T^n - \mathring B)))$ is in fact a direct summand of $G(\hat S_1)$.

Now $G(\hat S_1 \cap (T^n - \frac32 \mathring B)) \subset \hat f(\hat S_1 \cap (T^n - \mathring B)) \Rightarrow$ the complementary summand is in $\hat S_1 \cap \frac32 \mathring B$. This summand is stably free. If we were working over an arbitrary ring we would have to stabilise to make it free, but since we're working over $\mathbb Z$, the complementary summand is free.

By a dimension count, $\dim_{\mathbb Z}$(compl. summand) $= |\hat S \cap \mathring B|$.

So define $f' : G(\hat S_1) \to G(\hat S_1)$ by

$$f|_{T^n - \mathring B} = \hat f|_{T^n - \mathring B}$$

$f'|_{\mathring B} =$ arbitrary map onto complementary summand.

Now we have lost some metric control, which we must regain.

Consider the piecewise differentiable map $G : 2B \xrightarrow{\simeq} 2B$ taking B onto $\frac{1}{10}B$, s.t. $\|dG\| \leq 2$.

Now we've shrunk the "bad" part of f' into $\frac{1}{10}B$, & kept our metric control up to a factor of 2.

Let $\hat{S} = G(\hat{S}_1)$ and $\bar{f}$ the induced map on $G(\hat{S})$.

We're still not out of the woods, since we want an ε-auto of D^n, not $T^n - \frac{1}{2}\hat{E}$.

Now we have

$$\begin{tikzcd} & \mathbb{R}^n \arrow[d, "p"', "/\mathbb{Z}^n"] \\ D \arrow[ur, hook] \arrow[r, hook, bend right] & T^n \end{tikzcd}$$

$\hat{S} \subset T^n$; let $\hat{\hat{S}} = p^{-1}(\hat{S}) \subset \mathbb{R}^n$; note $|\hat{\hat{S}}| = \infty$. As before we can take $\bar{f} : G(\hat{S}) \to G(\hat{S})$ and lift it to $\bar{\bar{f}} : G(\hat{\hat{S}}) \to$

We lose no metric control here since p is a local isometry, so $\bar{\bar{f}}$ is still a 2ε-auto.

As before $\bar{\bar{f}}|_{\frac{1}{2}D} = f|_{\frac{1}{2}D}$.

We will identify $\mathbb{R}^n$ with $\mathring{D}$ by means of the specific homeo $H : \mathbb{R}^n \xrightarrow{\simeq} \mathring{D}$, given by

$$H(x) = \begin{cases} x, & x \in \frac{1}{2}D \\ s(\|x\|) \cdot x, & x \in \mathbb{R}^n - \frac{1}{2}\mathring{D}, \end{cases}$$

where $s : [\frac{1}{2}, \infty) \xrightarrow{\simeq} [\frac{1}{2}, 1)$ is a homeo (e.g. $s(t) = 1 - \frac{1}{4t}$)

Now we can identify $\hat{\hat{S}}$ with its image $H(\hat{\hat{S}}) \subset \mathring{D}$, write $f_* : G(H(\hat{\hat{S}})) \xrightarrow{\simeq} G(H(\hat{\hat{S}}))$ for the map induced by $\bar{\bar{f}}$. $f_*|_{\frac{1}{2}D} = f|_{\frac{1}{2}D}$.

12/6/71. Quick summary. Given $\varepsilon > 0$ and n, $\exists\ \delta > 0$ s.t. if $f : G(S) \to G(S)$ is a δ-auto (on D^n) over $\frac{2}{3} D^n$, S finite $\subset D^n$, then $\exists\ 2\gamma$ auto (γ can be chosen arbitrarily small, e.g. $\gamma = \frac{\varepsilon}{10}$) $\bar{\bar{f}} : G(\hat{S}) \longrightarrow G(\hat{\hat{S}})$ where $|\hat{S}| = \infty$, $\hat{S} \subset \mathbb{R}^n$, s.t. $\hat{S} \cap \frac{1}{2} D^n = S \cap \frac{1}{2} D^n$, and $\bar{\bar{f}}|_{\frac{1}{2}D^n} = f|_{\frac{1}{2}D^n}$.

- Continue w/ prf.

We will use BHS to truncate $\hat{S}$ to a finite set.

Now $G(\hat{S})$ is a free f.g. $\mathbb{Z}T^n$-module, where T^n is the free abelian group of rank n (acting as covering transformations of $\mathbb{R}^n$ over the n-torus); $G(\hat{S})$ has $\mathbb{Z}T^n$-basis $\hat{S}$. Further, $\bar{\bar{f}}$ is a T^n-module automorphism.

The same thing goes for f_* (with the T^n-module structure on $H(\hat{S}) \subset D^n$ induced from that on $\hat{S}$).
f_* is a 2δ auto ($2\delta < \varepsilon$) as desired, the problem is that $|\hat{S}| = \infty$.

Let $p_1, \dots, p_m$ be a basis for $G(\hat{S})$ as a T^n module (As mentioned on the previous page, we will think of $\hat{S}$ as the original $\hat{S} \subset \mathbb{R}^n$ and also as $H(\hat{S}) \subset D^n$.)

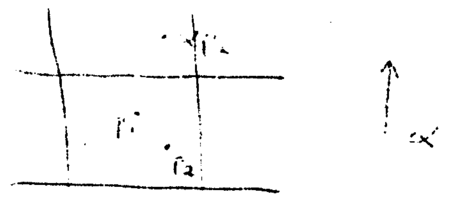
Now we can represent f_* by an $m \times m$ matrix A with entries in $\mathbb{Z}T^n$. By BHS, A is (possibly after stabilisation) a product of elementary (& diagonal) matrices. (Rmks (1) diagonal mx is nec. since in defining WhT^n we mod out by the group action. (2) stabilisation is no problem as long as we put the extra points far from $\frac{1}{2} D^n$.)

Actually, we ~~constructs~~ with $A^{-1} = DE_1 \cdots E_s$ where D is diagonal and E_i is elementary.

Rmk: If we ignore the module structure, each E_i represents a product of elem. matrices, but these are all blocked away from each other & so commute. (?)

Rmk Suppose $\alpha, \beta \in T^n$ and $E_5 = \begin{pmatrix} 1 & 5\alpha + 6\beta \\ 0 & 1 \end{pmatrix}$. Rewrite $E = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^5 \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}^6$. Thus we can assume the off-diagonal element of each E_i is $\pm$(a group elt.).

Ex Suppose α = translation up 1 unit (on $\mathbb{R}^2$).



Suppose $E = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ operating on p_1 & p_2. Then $\forall \gamma \in T^n$, $\gamma p_1 \xrightarrow{E} \gamma p_1$, but $\gamma p_2 \xrightarrow{E} \gamma p_2 + \alpha \gamma p_1$

Now since $A^{-1} = DE_1 \cdots E_s$ we have

$$DE_1 \cdots E_s f_* = id.$$

We must get rid of the D-factor; we do that (last lecture). So assume $D = I$; we now truncate the E_i's, replacing each E_i by $\hat{E}_i$ where $\hat{E}_i$ acts on the points p (i.e. $\mathbb{Z}$-basis elements) of $\hat{\hat{S}}$ by

$$\hat{E}_i(p) = \begin{cases} E_i(p), & p \notin rD^n \\ p, & p \in rD^n \end{cases} \text{ where } \tfrac{1}{2} \leq r \leq 1, \ r \text{ to be picked later.}$$

Now $\hat{E}_1 \cdots \hat{E}_s f_*$ will be the desired ε-auto. First, since $E_1 \cdots E_s f_* = id$, $\hat{E}_1 \cdots \hat{E}_s f_*$ will be identity near ∂D. Also, $\hat{E}_1 \cdots \hat{E}_s f_* |_{\frac{1}{2}D^n} = f_* |_{\frac{1}{2}D^n} = f|_{\frac{1}{2}D^n}$.

of course the points p of $H(\hat{S})$ converge to ∂D^n; let r_1 be close to 1, $1 > r_1 >> r$, and discard all the points of $\hat{S}$ $(= H(\hat{S}))$ in $D^n - r_1 D^n$. This leaves only finitely many points in the remaining set which will be the desired S'.

Thus conditions (i) & (ii) of the Lemma (p. 128) are satisfied; it remains to show $\hat{E}_i \mapsto \hat{E}_i f_*$ is an ε-auto. It is not hard to show (using e.g. $\hat{S} \subset \mathbb{R}^n$) that truncation of elem. matrices leaves them as automorph

So we must check the metric control. $\bar{\bar{f}}_*$ is a 2γ-map; the E_i's don't have to be. But diam(Cu $E_{p_i} \cup p_i$) is bounded over all $p \in \hat{S}$. Thus when we apply H, the induced E_i's (call them $(E_i)_*$) are λ_i-autos, where λ_i depends on p but $\lambda_i \to 0$ as $p \to \partial D^n$ (since H contracts distances at ∞

Now we pick r so that on the fringe, $D^n - rD^n$, each E_i is a $\frac{\gamma}{2s}$-auto, where $s = \#$ of elem. matri in the product. This gives metric control on $D^n - rD^n$; and on rD^n such control is immediate, since $\hat{E}_i = \mathrm{id}$ there.

12/18/78 Before returning to the problem of getting rid of D we digress for a moment.

Prob: Suppose M^n is a closed manifold whose universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$. Is it always possib to choose the diffeomorphism $f : \mathbb{R}^n \to \tilde{M}$ to be expandin i.e. $\exists\ \varepsilon > 0$ s.t. $\|df(X)\| > \varepsilon$ $\forall$ tangent vector X

$\mathbb{R}^n$ s.t. $\|X\| = 1$, where the metric on $\tilde{M}$ is induced from a riemannian metric on M?
(Here if $\|\cdot\|_M$ is a metric on M, the induced metric $\|\cdot\|$ on $\tilde{M}$ is defined by $\|X\| = \|Dp(X)\|_M$ where $X \in T\tilde{M}$, p = projection: $\tilde{M} \to M$.)

Ex. Such an f exists if M is a nonpositively curved manifold (let $p: \mathbb{R}^n \to M$ be the exponential map; this map is always expanding for $K_M \leq 0$).

Rmk: The solution of this problem bears on Novikov's conjecture (see pp 47-49). In fact, suppose we prove the problem for M^n where $\pi_1(M) = \Gamma$. Now let N^n be any manifold with $\pi_1(N) = \Gamma$. There is a unique (up homotopy) map $\varphi: N \longrightarrow B\Gamma = M$. One can show (given the truth of the problem-assertion for M) that $L(N) \cup \varphi^*[M]$ is a htpy invariant; i.e if $f: N' \to N$ is a h.e. then $\langle L(N) \cup \varphi^*[M], [N] \rangle = \langle L(N') \cup f^*\varphi^*[M], [N'] \rangle$.
This is a special case of Novikov's conjecture.
Note $[M] \in H_*(\Gamma; \mathbb{Q})$.

We return to Quinn's work: On p. 133 we found elementary matrices $E_1, \dots, E_s$ and a diagonal matri D over $\mathbb{Z}T^n$ s.t. $DE_1 \cdots E_s \bar{\bar{f}} = id$. By the action of T^n on $K_1\mathbb{Z}T^n$ (recall $Wh\, T^n = K_1\mathbb{Z}T^n / \pm T^n$) we
→ mod out by "trivial uni
can assume $D = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1_\alpha \end{bmatrix}$, $\alpha \in T^n$.

To get rid of D we will <u>change</u> $\bar{\bar{f}}$ & construct a <u>new</u> auto. g with all the important properties

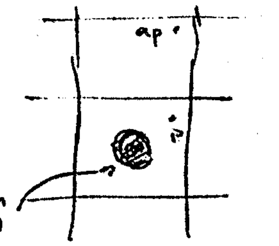
of $\bar{\bar{f}}$, except with the corresponding $D = I$.

The basic fact is that $\left[\begin{array}{c|c} D & 0 \\ \hline 0 & \pm\alpha^{-1} \end{array}\right]$ ((m+1)×(m+1)-matrix)

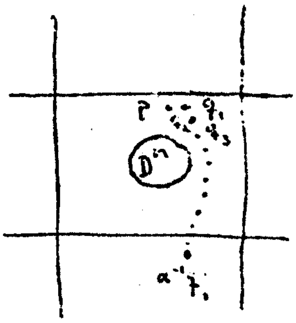
is a product of elementary matrices.

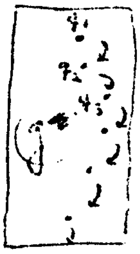
A special argument is needed for the special case $n=$ ($D^1 = [-1, 1]$) since $\partial D^1 - D^1$ is not connected; but thi case is easy. The following argument works for $n > 1$.

Suppose for simplicity α = translation up 1 unit, actin on p. We can choose the position in D where $\pm\alpha \neq 1$ appears; so we can insure that $p \in (\hat{S} \cap [0, 2]^n) - D^n$.



We would like to add a new basis element q to $\hat{S}$ with $\bar{\bar{f}}(q) = \alpha(q)$ (so $D(q) = \alpha^{-1}(q)$) but now $\bar{\bar{f}}$ is not a $\partial\Gamma$-auto. So instead we add a chain of points in D^n. In our simple example, $n=2$ & $\alpha = y$ (i.e. +1 i y-direc'n), we add a large set of new basis elts. which are very close to each other; call them $q_1, q_2, \ldots$. g is now constructed so that $g|_{\hat{S}} = f|_{\hat{S}}$, and g takes each q_i to q_{i+1} and the last q_i to $\alpha^{-1} q_1$ $\longrightarrow$





The matrix for this operation is

$$A = \begin{bmatrix} 0 & \alpha^{-1} \\ I & 0 \end{bmatrix}.$$

But by row & column operations A becomes

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \alpha^{-1} \end{bmatrix}.$$

Now $E_s \cdots E_1 g = \left|\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \; y \end{matrix} & 0 \\ \hline 0 & \begin{matrix} I & \big| \, y^{-1} \end{matrix} \end{array}\right|$ (writing $\alpha = y$)

$$\simeq \left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \; y \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \; y^{-1} \end{matrix} \end{array}\right].$$ But $\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ = product of el

matrices. In fact, if A is an arbitrary matrix,

$$⊛ \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A^{-1} & I \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}.$$

12/11/78: Thus $E_s \cdots E_1 g$ is a product of elem matrices $E'_1 \cdots E'_p$ with no D-term, as desi

so $E'^{-1}_p \cdots E'^{-1}_1 E_s \cdots E_1 g = id$;

This proves the lemma, p. 128.

We now prove the Main Lemma (p. 121): Given $n \in \mathbb{Z}$

~~[illegible]~~ $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. if S finite $\subset D^n$ and

$f: G(S) \longrightarrow G(S)$ is a δ-automorphism over $\frac{3}{4}D^n$, the

$\exists$ finite sets S', S_1, S_2 s.t. $S \cup S' = S_1 \dot{\cup} S_2$, $S_1 \subset \frac{2}{3}D^n$,

$S_2 \subset D^n - \frac{1}{3}D^n$, ~~and~~ an ε-deformation H, and autos.

$f_i : G(S_i) \xrightarrow{\sim} G(S_i)$, s.t. $H(f \oplus id) = f_1 \oplus f_2 : G(S \cup S') \xrightarrow{\sim} G(S \cup S')$

Prf: By the lemma we've just proved, $\exists$ finite $\hat{S}$ and

$\gamma (= \frac{\varepsilon}{50}$ or so$)$-auto $g : G(\hat{S}) \xrightarrow{\sim} G(\hat{S})$ s.t.

(i) $\hat{S} \subset \frac{2}{3}D^n$ (by an easy modification of the prf of lemma)

(ii) $\hat{S} \cap \frac{1}{2}D^n = S \cap \frac{1}{2}D^n$

(iii) $g|_{\frac{1}{2}D^n} = f|_{\frac{1}{2}D^n}$.

Define $S' := \hat{S} \cup \hat{S}$ (double each point or, equivalently,

move $\hat{S}$ off itself $\varepsilon/1000000$ and let $S' = \hat{S} \cup (\text{new copy of } \hat{S})$).

Consider

$$f \oplus g^{-1} \oplus g : G(S \cup \hat{S} \cup \hat{S}) \xrightarrow{\sim} G(S \cup \hat{S} \cup \hat{S});$$ in matrix form,

this is $$\begin{bmatrix} f & & \\ & g^{-1} & \\ & & g^{*} \end{bmatrix}.$$

<u>Claim A</u>. $\exists$ small deformation H_1 s.t.

$$H_1(f \oplus id \oplus id) = f \oplus g^{-1} \oplus g$$

$(\because)$ Set $$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -g^{-1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & g & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -g^{-1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since g is a γ-automorphism, H is a 3γ-deformation. $//_A$

Now $f \oplus g^{-1}|_{\frac{1}{2}D} = f \oplus f^{-1}|_{\frac{1}{2}D}$.

<u>Claim B</u>. $\exists$ small deformation H_2 s.t.

$$H_2(f \oplus g^{-1} \oplus g) = f_1 \oplus f_2,$$

where $$f_1 = H_2(f \oplus g^{-1}|_{\frac{1}{3}D}) \oplus g = id|_{\frac{1}{3}D^n} \oplus g,$$ acting on $G(S$

$$S_1 = (S \cap \tfrac{1}{3}D^n) \cup (\hat{S} \cap \tfrac{1}{3}D^n) \cup \hat{S}$$
$$= (S \cap \tfrac{1}{3}D^n) \cup (S \cap \tfrac{1}{3}D^n) \cup \hat{S} \subset \tfrac{2}{3}D^n,$$

and where

$$f_2 = H_2(f \oplus g^{-1})|_{D^n - \frac{1}{3}D^n},$$ acting on $G(S_2)$,

$$S_2 = (S \cap (D^n - \tfrac{1}{3}D^n)) \cup (\hat{S} \cap (D^n - \tfrac{1}{3}D^n)).$$

(We can split like this because $H_2(f \oplus g^{-1}) = id$ on rD (where $\frac{1}{2} > r > \frac{1}{3}$) and so in particular $H_2(f \oplus g^{-1})|_{\frac{1}{3}\partial D} = id$.)

(:) Define $\varphi : G(\hat{S}) \longrightarrow G(\hat{S})$ by

$$G(\hat{S}) \xrightarrow{\text{proj'n}} G(\hat{S} \cap \tfrac{1}{2}D^n) \xrightarrow{f|_{\frac{1}{2}D^n}} G(\hat{S}).$$

φ^{-1} is not quite defined, since f may take points of $\frac{1}{2}D$ out of $\frac{1}{2}D$. But since f is a δ-auto ove $\frac{3}{4}D$, $\exists\ \hat{f} : G(\hat{S}) \to G(\hat{S})$ s.t. $f\hat{f} = \text{id}$ on $\frac{3}{4}D^n$.

Define $\psi : G(\hat{S}) \to G(\hat{S})$ by

$$G(\hat{S}) \xrightarrow{\text{proj'n}} G(\hat{S} \cap \tfrac{1}{2}D^n) \xrightarrow{\hat{f}|_{\frac{1}{2}D^n}} G(\hat{S}).$$

Now let H_2^{-1} be defined by

$$H_2^{-1} = \begin{pmatrix} 1 & -\varphi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \psi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\varphi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Now

$H_2^{-1}(1 \oplus 1 \oplus g) = (\varphi \oplus \psi \oplus g)$; on most of $\frac{1}{2}D^n$

e.g. on rD where $\frac{1}{3} < r < \frac{1}{2}$ — $\varphi = f$, $\psi = f^{-1}$

So. $H_2(f \oplus g^{-1})|_{rD} = \text{id}$ $//_B$ // Main Lemma.

<u>Rmk</u>: f_1 of the main lemma is in fact an ε-auto; f_2 only an ε-map.

This finishes Quinn's proof of the Connell-Hollingsw conjecture.

<u>Thms & Conjectures</u> (Farrell & Hsiang): listed in decreasing order of certainty.

<u>Thm 1</u>. If Γ is a torsion-free virtually abelian finitely generated group then $\mathrm{Wh}\,\Gamma = 0$.

<u>Rmks</u>: (1) Such a Γ is crystallographic (see p. 49)
(2) This Thm is proved on p. 84 ff.

<u>Thm 2</u>. If M^n is a closed manifold, $n \neq 3, 4$, then M has a flat riemannian structure iff

(1) $\pi_1 M$ is virtually abelian and
(2) $\pi_i M = 0$, $i > 1$.

(Comments on this Thm, bottom p. 52)

<u>Thm 3</u>: If Γ is a torsion-free virtually poly-$\mathbb{Z}$ group then $\mathrm{Wh}\,\Gamma = 0$.

(Comments, p. 94)

<u>Def</u>. Let G be a nilpotent simply connected Lie group, K a compact Lie group (e.g., a finite group), and $\Gamma < G \rtimes K$ a discrete cocompact torsion-free subgroup. Then $\Gamma \backslash G \rtimes K / K$ is an <u>infranilmanifold</u>.

<u>Thm 4</u>. If M^n is an infranilmanifold, $n \neq 3, 4$, and N^n is a closed aspherical manifold with $\pi_1 N = \pi_1 M$, then M and N are homeomorphic.

<u>Conj</u>: If M^n, N^n are closed aspherical manifolds, $n \neq$ s.t. $\pi_1 M = \pi_1 N$ is virtually poly-$\mathbb{Z}$, then M and are homeomorphic.

<u>Rmk</u>. Farrell is "pretty sure" of:

<u>Thm 5</u>. If in the conjecture above, Γ is the poly-$\mathbb{Z}$ subgroup of $\pi_1 M$ and $[\pi_1 M : \Gamma]$ is odd, then $M \underset{\text{homeo}}{\cong} N$.

12/13/78: Surgery and Structures

M^n: orientable closed manifold.

Def. $\mathcal{S}_0(M)$ = set of smooth structures on M, is defined as the set of equivalence classes of simple homotopy equivalences $N^n \xrightarrow{f} M$, where $(N_1 \xrightarrow{f_1} M) \approx (N_2 \xrightarrow{f_2} M)$ if $\exists$ diffeomorphism $g: N_1 \xrightarrow{\sim} N_2$ s.t. the diagram

$$\begin{tikzcd} N_1 \arrow[dr, "f_1"] \arrow[d, "g"'] & \\ N_2 \arrow[r, phantom, "(H)"] \arrow[r, "f_2"'] & M \end{tikzcd}$$

homotopy commutes. We similarly define $\mathcal{S}_{Top}(M)$, where we require only that g be a homeomorphism.

(See pp. 30 ff.) Note here we are considering only h.e.'s f s.t. $\tau(f) = 0 \in Wh\,\pi_1 M$.)

Wall-Sullivan exact sequence

Write CAT for Top or O. For $n \geq 5$ there is an exact sequence

$$\cdots \longrightarrow L^s_{n+1}(\mathbb{Z}\pi_1 M) \xrightarrow{\varphi} \mathcal{S}_{CAT}(M^n) \xrightarrow{\psi} [M, G/CAT] \xrightarrow{\sigma} L^s_n(\mathbb{Z}\pi_1 M)$$

Rmk: If CAT = Top, this can be made into an exact sequence of abelian groups; not so if CAT = O.

We define (roughly) the terms in the sequence. G/O is defined only up to homotopy. If X is a topological space, $\alpha \in [X, G/O]$, then α is the equivalence class of a stable O-vector bundle (i.e. orthogonal vector-bundle) over X, together with a specific fibre homotopy trivialisation. Here we have the diagram at right where f is a proper homotopy equivalence (i.e. $f^{-1}(cpt)$ is cpt), which preserves fibres.

$$\begin{tikzcd} \mathbb{R}^n \arrow[d] & & \mathbb{R}^n \arrow[d] \\ E \arrow[rr, "f"] \arrow[dr] & & X \times \mathbb{R}^n \arrow[dl] \\ & X & \end{tikzcd}$$

[Actually we need not assume f is fibre-preserving; if it's a proper h.e., it can be fiddled around w
to make it fibre-preserving.
Atiyah has a paper on Thom complexes which deals with these ideas.]

If we think of vector bdles as classified by BO and spherical fibrations by BG, then G/O is the fibre of the bundle

$$\begin{tikzcd} BG \\ BO \arrow[u] \\ G/O \arrow[u] \end{tikzcd}$$

We define equivalence as follows:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & X \times \mathbb{R}^n \arrow[dl] \\ X & \end{tikzcd}$$

and

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d] & X \times \mathbb{R}^n \arrow[dl] \\ X & \end{tikzcd}$$

are <u>equivalent</u> if $\exists$ vector-bundle isomorphism $h : E \xrightarrow{\sim} E'$ s.t. $gh \simeq f$.

Now we define ψ. We will assume CAT = O; similar (but harder) work does the constructions in Top.

Suppose $[f : N^n \xrightarrow{h.e.} M^n] \in \mathcal{S}_O(M)$. We wish to define $\psi f \in [M, G/O]$. Let g be a homotopy inverse to

Homotope g to an embedding $M \underset{g}{\hookrightarrow} N \times D^{n+2}$.

Let ν_g be the associated normal bundle.
(Essentially, $\nu_g = g^* \tau N - \tau M$) Now $M \subset D(\nu_g) \subset N \times D^{n+2}$ where $D(\nu_g)$ = disc b of ν_g is embedded as a tubular nbhd of M.

Then (one shows) $N \times D^{n+2} - (D(\nu_g))^\circ$ is an s-cobordism
$\therefore$, by Smale's s-cobordism thm, a product; so
$N \times D^{n+2} - (D(\nu_g))^\circ \cong \partial D \nu_g \times I$.

Thus $N \times D^{n+2} = D\nu_g \times (\underbrace{\partial D\nu_g \times I}_{= \text{collar on } \partial D\nu_g})$

$$= D\nu_g .$$

$\therefore$ Proj'n $D\nu_g \longrightarrow D^{n+2}$ defines a proper degree 1 map $E(\nu_g) \longrightarrow \mathbb{R}^{n+2}$. This is how one defines the fibre h.

$$\begin{tikzcd} E(\nu_g) \arrow[r, "h"] \arrow[d] & \mathbb{R}^{n+2} \times M \arrow[dl] \\ M & \end{tikzcd}$$

Define $\Psi f = [E(\nu_g), h] \in [M, G/O]$

Next we define (roughly) σ and L_n.

Given

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d] & M^n \times \mathbb{R}^s \\ M^n & \end{tikzcd}$$

Make $F \pitchfork (M \times 0)$ by a small perturbation.

This defines $N^n = F^{-1}(M \times 0) \subset E$, together with a map $f : N \longrightarrow M$. $\deg f = 1$; further, there is an O-bundle η over M and a specific bundle iso. of $f^*\eta$ to the stable normal bundle of N^n (i.e. ν_g where $g : N^n \hookrightarrow \mathbb{R}^{2n+2}$).

We would like to make f a s.h.e. by surgery. The algebraic obstruction to doing so is measured by $L_n^s(\mathbb{Z}\pi, M)$. This could be given as a def'n of $L_n^s(\mathbb{Z}\pi, M)$, but in fact there is an algebraic def'n, which can be used actually to compute this group.

Roughly, let R be a ring with involution $r \mapsto \bar{r}$. An element of $L_0(R)$ is a cogradience class of (arbitrarily large) hermitian invertible matrices: here, cogradience is the equiv. rel'n $S_0 \approx S_1$ if

$\exists$ invertible A s.t. $A S_0 A^* = S$.

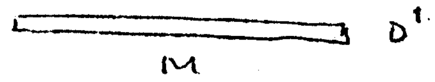
(Ex: $R = \mathbb{Z}\Gamma$ with involution $\bar{\gamma} = \gamma^{-1}$. If $\Gamma = \{1\}$, then ~~[illegible]~~ the involution is the identity, and in this case hermitian = symmetric.)

For $L_2(R)$, use skew-hermitian matrices instead.

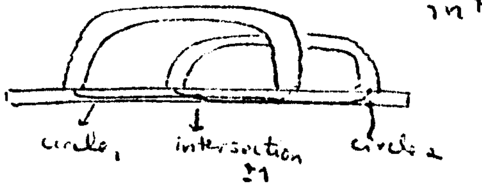
(Ex: $\Gamma = 1$. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in L_2(\mathbb{Z})$. (Actually, this element is equivalent to $1 \in L_2(\mathbb{Z})$.))

Geo. interp. of $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$: Suppose e.g. $M = [0,1]$.

(i) Thicken M to $M \times D^1$:



(ii) Add 1-handles which link according to the matrix. (i.e. extend their cores to circles; these intersect according to the matrix).



Let W^2 = the resulting cobordism (not an h-cobordism).

$\partial W = M + N$; since S is invertible, one can show $N \simeq M$; in the proof (which is essentially algebraic) a particular h.e. $f: N \xrightarrow{\simeq} M$ is produced.

Ex: $M = S^n$. In Top, $\mathcal{S}(S^n) = 0$ (Poincaré conjecture).
$\therefore$ the exact sequence $\Rightarrow \pi_n(G/Top) = L_n(\mathbb{Z})$.
(Actually to be more careful, we'd use $\mathcal{S}(D^{n+1}, \partial D^n) - ($
$L_n(\mathbb{Z})$ is known; ~~[illegible]~~ so

$$\pi_n(G/Top) = L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n \equiv 0 \ (4) \\ \mathbb{Z}_2, & n \equiv 2 \ (4) \\ 0, & n \text{ odd} \end{cases}$$

Now if $\pi_1 M = 1$, then $C = 0$ and one obtains the s.e.s.

$$0 \longrightarrow \mathcal{S}(M^n) \xrightarrow{\psi} [M, G/Top] \xrightarrow[\text{(onto)}]{\sigma} L^s_n(\mathbb{Z}) \longrightarrow 0 .$$

Further, this s.e.s. splits.

To compute the middle term (in order to get at $\mathcal{S}(M)$ use Sullivan's characteristic variety thm (cf pp 37-

<u>Sullivan's Characteristic Variety Theorem</u>:

For any finite complex X,

(1) $[X, G/Top] \otimes \mathbb{Q} \cong \bigoplus_{i=1}^{\infty} H^{4i}(X; \mathbb{Q})$

(2) $[X, G/Top] \otimes \mathbb{Z}[\frac{1}{2}] \cong KO(X) \otimes \mathbb{Z}[\frac{1}{2}]$

(3) $[X, G/Top] \otimes \mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots] \cong$
$$\bigoplus_{i=1}^{\infty} H^{4i}(X; \mathbb{Z}[\tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \dots]) \oplus \bigoplus_{i=1}^{\infty} H^{4i+2}(X; \mathbb{Z}_2)$$

<u>2/15/75</u>: Consider again the surgery long exact sequence

$$\to \mathcal{S}(M\times D^1, \partial) \to [(M\times D^1, \partial), G/Top] \to L_{n+1}(\pi, M) \xrightarrow{\varphi} \mathcal{S}_{Top}(M) \xrightarrow{\psi} [M, G/Top] \xrightarrow{\sigma} L_n(\pi, M)$$

The terms $[M, G/Top]$, $[(M\times D^1, \partial), G/Top]$ are f.g. abelian groups, since these are $0^{\underline{th}}$ terms in some extraordinary cohomology theory which has a spectral sequence converging to it with E_2 term in ordinary cohomology.

Unfortunately, $L_{n+1}(\pi, M)$ may be infinitely generated; ~~in this~~ ~~[illegible]~~ e.g. $L_3(T \ast T_2)$ is inf gen. So if $\pi_1 M^9 = T \ast T$ say, then $\mathcal{S}(M)$ is infinitely generated; e.g. $M = \mathbb{P}^9 \# \mathbb{P}^9$.

<u>Rmk</u>: $Wh(T \ast T_2) = Wh(T_2 \ast T_2) = 0$ (Waldhausen)

<u>Ref</u>: Cappell, <u>Topology</u> $\simeq 1974$

<u>Rmk</u>: The problem with $L_2(T \ast T_2)$ is 2-torsion. Farrell thinks that $L_2(T \ast T_2) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ may be f.g. over $\mathbb{Z}[\frac{1}{2}]$.

Thm: If Γ is a torsion free, finitely generated* virtually abelian group (and $B\Gamma$ is an orientable flat n-manifold) and M^m is a manifold with $\pi_1 M = \Gamma$, then $\mathcal{S}(M)$ is calculable up to an extension; i.e. $\exists$ natural homomorphisms
$f_s : [(M \times D^s, M \times S^{s-1}), G/Top] \longrightarrow [(B\Gamma \times D^t, B\Gamma \times S^{t-1}), G/Top]$
defined when $m+s > n$, where $t = m+s-n$, s.t. the sequence

$$0 \longrightarrow \operatorname{coker} f_{s+1} \longrightarrow \mathcal{S}(M \times D^s, M \times S^{s-1}) \longrightarrow \ker f_s \longrightarrow 0$$

is exact.

Idea of prf: By Sullivan's Thm, the groups with G/Top are (after tensoring) ordinary cohomology. One can think of $[(X, A), G/Top]$ as equal to $\mathcal{H}^0(X, A)$ for some cohom. theory $\mathcal{H}^*$ (like $[(X, A), BO]$). One can dualise to get a homology theory $\mathcal{H}_*$.
We have the commutative diagram (σ_m defined as $\sigma . D$

$$\begin{tikzcd}
{[M^m, G/Top]} \arrow[r, "\sigma"] \arrow[d, equal] & L_m(\pi_1 M) \\
\mathcal{H}^0(M) \arrow[d, "\cong"', "D \text{ (Poincaré duality - shown to be an iso by Ranicki)}"] & \\
\mathcal{H}_m(M) \arrow[uur, "\sigma_m"'] &
\end{tikzcd}$$

In fact in general one has a map $\mathcal{H}_*(X) \xrightarrow{\sigma_*} L_*(\pi_1 X)$ for any complex X, even though the original map σ defined only for manifolds.
Similarly we have

$$\begin{tikzcd}
{[B\Gamma, G/Top]} \arrow[r, "\sigma"] \arrow[d, "\cong"', "D"] & L_n(\pi_1 M) \\
\mathcal{H}_n(B\Gamma) \arrow[ur, "\sigma_n"'] &
\end{tikzcd}$$

But Hsiang-Farrell $\Rightarrow \mathcal{S}(B\Gamma) = 0$, so $\sigma : [B\Gamma, G/Top] \to L_n(\pi, M)$ which is iso, so σ_n is an iso. By crossing $B\Gamma$ with a disc, we see

(i) $\mathcal{S}(B\Gamma \times D^t, B\Gamma \times S^{t-1}) = 0$, and so

(ii) $\sigma_{n+t} : \mathcal{H}_{-n-t}(B\Gamma \times D^t, B\Gamma \times S^{t-1}) \xrightarrow{\sim} L_{n+t}(\pi, M)$ is an iso.

But σ is periodic of period 4, so all the σ_j are i

$\sigma_* : \mathcal{H}_*(B\Gamma) \xrightarrow{\sim} L_*(\Gamma)$.

There is a natural map $f : M \to B\Gamma$ since $\pi_1 M = \Gamma$, we get an induced map $\mathcal{H}_{-m}(M) \xrightarrow{f_*} \mathcal{H}_{-m}(B\Gamma)$.

σ_m natural $\Rightarrow$ the following diagram commutes:

$$\begin{tikzcd}
 & L_m(\pi, M) & \\
\mathcal{H}_{-m}(M) \arrow[ur, "\sigma_m"] \arrow[rr, "f_*"] & & \mathcal{H}_{-m}(B\Gamma) \arrow[ul, "\sigma_m", "\cong"']
\end{tikzcd}$$

Thus $\ker f_* = \ker \sigma_m$. We recover the maps f_s of the theorem by passing backwards by $\mathbb{Z}$ to get back to $\mathcal{H}^0$. //

<u>Rmk</u>. Tensoring with $\mathbb{Q}$ turns the extraordinary homology map above into

$f_* : H_*(M, \mathbb{Q}) \to H_*(\Gamma, \mathbb{Q})$.

using Sullivan's theorem.

So we return to the

<u>Sullivan Char'istic Variety Thm</u>: $A \subset X$, pair of complexes

(1) $[(X, A), G/Top] \otimes \mathbb{Q} = \bigoplus_{i=1}^{\infty} H^{4i}(X, A; \mathbb{Q})$

(2) $[(X, A), G/Top] \otimes \mathbb{Z}[\frac{1}{2}] = KO(X, A) \otimes \mathbb{Z}[\frac{1}{2}]$

(3) $[(X, A), G/Top] \otimes \mathbb{Z}_{(2)} = \bigoplus_{i=1}^{\infty} H^{4i}(X, A; \mathbb{Z}_{(2)}) \oplus \bigoplus_{i=1}^{\infty} H^{4i+2}(X, A; \mathbb{Z}$

Pf sketch. We show only the first statement, using methods that apply to the other two, but only with several technical modifications.

We wish to define $G/Top \xrightarrow{s} \prod_{i=1}^{\infty} K(\mathbb{Q}, 4i)$, which is the same as defining an element $s \in \prod_{i=1}^{\infty} H^{4i}(G/Top; \mathbb{Q}$

(should this be

Consider $\left(\bigoplus_{i=1}^{\infty} \Omega_{4i}(X)\right) \otimes_{\Omega_{4*}(pt)} \mathbb{Q}$.

$\Omega_{4n}(pt.) = \Omega_{4n}$ = ordinary bordism theory. $\Omega_{4n} \xrightarrow{I} \mathbb{Z}$, defined by $I(M)$ = index of M, is a well-defined map $\Omega_{4*}(pt) \to$ $\mathbb{Z} \subset \mathbb{Q}$, which defines an $\Omega_{4*}(pt)$-module structure on $\mathbb{Q}$.

Fact $\left(\bigoplus_{i=1}^{\infty} \Omega_{4i}(X)\right) \otimes_{\Omega_{4*}(pt)} \mathbb{Q} \cong \bigoplus_{i=1}^{\infty} H_{4i}(X; \mathbb{Q})$.

Now a coho. class (at least, with coeffs in $\mathbb{Q}$), is a homom $s : \bigoplus H_{4i}(X, \mathbb{Q}) \longrightarrow \mathbb{Q}$, i.e. a homom

$$s : \left(\bigoplus_i \Omega_{4i}(X)\right) \otimes_{\Omega_{4*}(pt)} \mathbb{Q} \longrightarrow \mathbb{Q}$$

We define s on each $\Omega_{4i}(G/Top)$ and then show $\otimes\,\mathbb{Q}$ induces s as desired.

An element of $\Omega_{4i}(G/Top)$ is a pair $[M^{4i}, \varphi]$ where $\varphi : M^{4i} \to G/Top$ is a map, i.e., a bdle

$$\begin{tikzcd} E \arrow[r, "\hat{\varphi}"] \arrow[d] & M^{4i} \times \mathbb{R}^n \\ M^{4i} & \end{tikzcd}$$

together with a specific fibre htpy triv'n $\hat{\varphi}$.

Take $s[M, \varphi]$ to be (something like) $\langle L(\xi) L(M), [M] \rangle \in$ index formula (this comes coming from the Hirzebruch from surgery.)

This defines a (continuous) map

$$s : G/Top \longrightarrow \prod_{i=1}^{\infty} K(\mathbb{Q}, 4i) .$$

Next we want s to be an iso on $\pi_*(G/Top)\otimes\mathbb{Q}$.

$$\pi_s(G/Top)\otimes\mathbb{Q} = \begin{cases}\mathbb{Q} & \text{if } 4|s \\ 0 & \text{otherwise.}\end{cases}$$

So since we're dealing with rational vector-spaces, we need show only that $s_*: \pi_*(G/Top)\otimes\mathbb{Q} \longrightarrow \mathbb{Q}$ is nonzero, i.e. we must find a bundle $\begin{matrix}E\\ \downarrow\\ S^{4i}\end{matrix}$ such that $p_i(E) \neq 0$.

($E\oplus\cdots\oplus E$ is fibre htpy trivial, so the extra condition is vacuous.) Such an E comes from the Bott periodicity Theorem. //