[87] M. Rees, "The ending laminations theorem direct from Teichmüller geodesics," preprint.
[88] H. Rubinstein and S. Wang, "π_1-injective surfaces in graph manifolds." *Comm. Math. Helv.* **73** (1998), 499–515.
[89] P. Scott, "Compact submanifolds of 3-manifolds," *J. London Math. Soc.* **7** (1974), 246–250.
[90] G. P. Scott, "Subgroups of surface groups are almost geometric," *J. London Math. Soc.* **17** (1978), 555–565.
[91] G.P. Scott, Correction to"Subgroups of surface groups are almost geometric," *J. London Math. Soc.* **32** (1985), 217–220.
[92] P. Scott and T. Tucker, "Some examples of exotic non-compact 3-manifolds," *Quart. J. Math. Oxford* **40**(1989), 481–499.
[93] J. Simon, "Compactification of covering spaces of compact 3-manifolds.," *Mich. Math. J.* **23** (1976), 245–256.
[94] T. Soma, "Existence of ruled wrappings in hyperbolic 3-manifolds," *Geom. Top.* **10** (2006), 1173–1184
[95] T. Soma, "3-manifold groups with the finitely generated intersection property," *Trans. A.M.S.* **331**(1992), 761–769.
[96] J. Souto, "A note on the tameness of hyperbolic 3-manifolds," *Topology* **44**(2005), 459–474.
[97] D. Sullivan, "The density at infinity of a discrete group of hyperbolic motions," *Publ. I.H.E.S.* **50**(1979), 419–450.
[98] D. Sullivan, "Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups," *Acta Mathematica* **155** (1985), 243–260.
[99] D. Sullivan, "Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups," *Acta. Math.* **153**(1984), 259–277.
[100] P. Susskind, "Kleinian groups with intersecting limit sets," *Journal d'Analyse Mathematique* **52**(1989), 26–38.
[101] W.P.Thurston, *The Geometry and Topology of Three-Manifolds*,Princeton University course notes,available at http://www.msri.org/publications/books/gt3m/ (1980).
[102] W.P. Thurston, "Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle," preprint.
[103] W.P. Thurston, "Hyperbolic Structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary," preprint.
[104] T.W. Tucker, "Non-compact 3-manifolds and the missing-boundary problem," *Topology* **13**(1974), 267–73.
[105] P. Tukia, "On the dimension of limit sets of geometrically finite Möbius groups," *Ann. Acad. Sci. Fenn.* **19** (1994), 11–24.
[106] J.H.C. Whitehead, "Some examples of exotic noncompact 3-manifolds," *Quart. J. Math. Oxford Ser.* **40** (1989), 481–499.
[107] D. Wise, "Subgroup separability of the figure 8 knot group," *Topology* **45** (2006), 421–463.

Topological Rigidity and Geometric Applications

F. T. Farrell *

Abstract

This is an informal article surveying geometric applications of the topological rigidity theorem of Lowell Jones and myself. These applications are due to various researchers including Lowell Jones, Pedro Ontaneda, M.S. Raghunathan, C.S. Aravinda, J.-F. Lafont and myself. The article is the direct outgrowth of my series of four lectures given at the Beijing International Conference and Instructional Workshop on Discrete Groups (2006); which in turn built on a series of three lectures I gave at the 22nd Annual Workshop in Geometric Topology held in Colorado Springs (2005).

Lecture 1. The best of all possible maps is sometimes not good enough

Let me start by recalling

Borel's Conjecture. *Any homotopy equivalence between closed aspherical manifolds is homotopic to a homeomorphism.*

Definition. X is aspherical $\Leftrightarrow$ $\pi_i X = 0$ for $i \neq 1$ ($\Leftrightarrow$ its universal cover $\tilde{X}$ is contractible).

Examples. Complete *non-positively* curved Riemannian n-manifolds are *aspherical.* (Cartan showed there universal covers are diffeomorphic to $\mathbb{R}^n$.)

For "most k" and "most twists" Thurston showed M^3 is aspherical. In fact has -1 curvature.

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Some low dim. examples

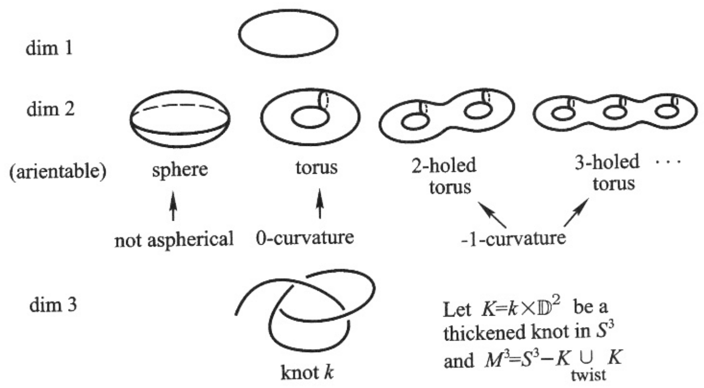


Figure 1:

Theorem (F-Jones 1993). *Let M^m and N^m be a pair of closed aspherical manifolds with $\pi_1 M = \pi_1 N$ (and $m \neq 3,4$). If M is a non-positively curved Riemannian manifold, then M and N are homeomorphic.*

Our proof (to be sketched in Lecture 2) is motley: long, indirect and non-constructive. So we wondered whether a more direct approach might work at least when *both* M and N are *non-positively curved* Riemannian manifolds. And we discovered that differential geometers were doing this in *many important cases.* But some remained open. I'll talk about one today.

Harmonic maps

Let $f : M \to N$ be a smooth map between closed Riemannian manifolds. It's *energy* $E(f) \in [0, +\infty)$ is defined as follows:

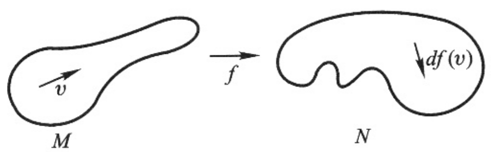


Figure 2:

$$E(f) = \int_{v\in SM} \frac{1}{2}|df(v)|^2.$$

A critical point of this functional

$$E : C^\infty(M,N) \to [0,+\infty)$$

is called a *harmonic map*. In particular a function of minimum energy inside a homotopy class is harmonic.

Tension vector field τ_f

For each $v \in T_\alpha M$ let α_v denote the (unique) geodesic with $\dot{a}_v(0) = v$.

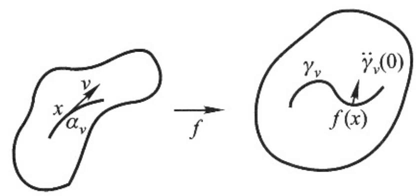


Figure 3:

Let $\gamma_v = f \circ \alpha_v$. Let $e_1, e_2, \ldots, e_m$ be an orthonormal basis for $T_x M^m$. Then $\tau_f(x) \in T_{f(x)}N$ is $\sum_{i=1}^{m} \ddot{r}e_i(0)$.

Important Fact. f is harmonic $\Leftrightarrow \tau_f \equiv 0$.

Consequently

1. Isometries are harmonic.
2. Closed geodesics are harmonic.

Also τ determines a partial flow on $C^\infty(M,N)$ by the (Heat Equation) PDE

$$\dot{f} = \tau_{f_t}; \quad f_0 = f \in C^\infty(M,N).$$

Which "should flow" f to a harmonic map $\hat{f}$ as $t \to \infty$. When $M = S^1$ this is a standard technique for flowing a closed curve α_0 to a geodesic α_∞.

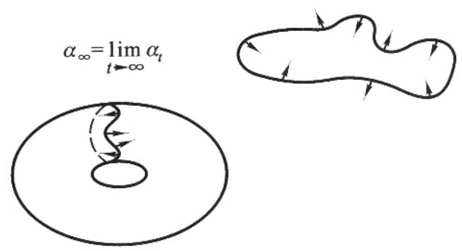


Figure 4:

Basic Theorem (Eells-Sampson (existence), Hartman-Al'ber (uniqueness)). *If M, N are closed Riemannian manifolds and N is negatively curved, then there* *exists a unique harmonic map $\hat{f}$ homotopic to $f \in C^\infty(M, N)$. In fact*

$$\hat{f} = \lim_{t \to \infty} f_t$$

where f_t solves the above PDE. (Uniqueness requires that $f_\#(\pi_1 M)$ is *not cyclic*; e.g. this is satisfied when f is a homotopy equivalence.)

Remark. Think of $\hat{f}$ as a kind of *best of all possible maps* in the homotopy class of f.

Basic Problem. Let $f : M^n \to N^n$ be a homotopy equivalence between *negatively curved* Riemannian manifolds. Is $\hat{f}$ a *diffeomorphism*, or at least a *homeomorphism*? □

Remarks.

1. This problem is due to Lawson-Yau and is a differential geometric variant of Borel's Conjecture.
2. When $n = 2$, Schoen-Yau and Sampson (1978) independently proved that $\hat{f}$ is a *diffeomorphism.*
3. When $n \geqslant 3$ and both M^n and N^n are locally symmetric (i.e. sectional curvatures are constant under parallel translation) Mostow's Strong Rigidity Theorem (1973) implies that $\hat{f}$ is a *diffeomorphism* (in fact an isometry up to scaling).

Here is a *specious argument* suggesting that $\hat{f}$ is a diffeomorphism in general.

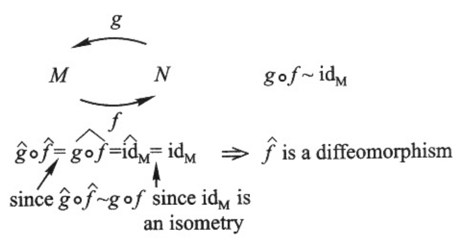


Figure 5:

The flaw in this argument is $\hat{g} \circ \hat{f}$ may not be harmonic since compositions of harmonic maps may not be harmonic.

Example. $\mathbb{R} \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}$.

$$f(t) = (t, 0) \qquad g(x, y) = x^2 - y^2 \qquad g(f(t)) = t^2.$$

In fact Jones and I were able to construct examples where $\hat{f}$ *couldn't be a diffeomorphism* because M and N were *not* diffeomorphic.

In our examples M^m is a certain hyperbolic manifold (constant -1 curvature) and

$$N^m = M^m \# \Sigma^m$$

i.e. the connected sum of M^m and an *exotic sphere* (a manifold homeomorphic but *not* diffeomorphic to S^m) Σ^m.

Let $f : M^m \to M^m \# \Sigma^m$ be the obvious homeomorphism. Then $\hat{f}$ is *not* a diffeomorphism. But it is still possible that $\hat{f}$ is a *homeomorphism.*

At this point we were stumped. But eventually (1998) Jones, Ontaneda and I proved the following result.

Theorem (F-Jones-Ontaneda). *In every dim $n \geqslant 6$, there exists a pair of closed negatively curved Riemannian manifolds M^n, N^n such that*

1. M^m and N^n are homeomorphic.
2. M^m and N^n are *not* PL homeomorphic.
3. M^m is a hyperbolic manifold (i.e. has constant -1 curvature).

Definition. Smooth manifolds M and N are PL-homeomorphic if they can be smoothly triangulated by the same simplicial complex.

Corollary (F-Jones-Ontaneda). *Let $f : M^n \to N^n$ be the homeomorphism from part* 1 of the above theorem, then $\hat{f}$ is *not* a homeomorphism.

Proof. We start by recalling the statement of the Smooth Hauptvermutung proved by Scharlemann and Siebenmann in 1973.

Smooth Hauptvermutung (Scharlemann-Siebenmann). Let $f : M^n \to N^n$ $(n \geqslant 6)$ be a smooth homeomorphism. Then M^n and N^n are PL homeomorphic.

Example. $f(x) = x^3$ is a smooth homeomorphism $\mathbb{R} \to \mathbb{R}$ which is *not* a diffeomorphism since $f^{-1}(x) = x^{1/3}$.

Proof of Corollary continued. But the harmonic map $\hat{f}$ is $C^\infty(M, N)$; i.e. $\hat{f}$ is smooth. Hence if $\hat{f}$ were a homeomorphism, then M and N would be PL homeomorphic by the Smooth Hauptvermutung contradicting property 2 of The Theorem. □

However there is the following more refined version of the Basic Problem on Yau's 1982 Problem List.

Problem 111. Let $f : M^n \to N^n$ be a *diffeomorphism* between closed negatively curved Riemannian manifolds. Is $\hat{f}$ a homeomorphism?

Our first approach to this problem (1996), although inconclusive, is simple and I think interesting. Let me briefly describe it.

Using results of Waldhausen and Igusa on the smooth pseudo-isotopy space of $S^1 \times \mathbb{D}^{n-1}$, one can construct for any closed hyperbolic manifold N^n $(n \geqslant 11)$ a self-diffeomorphism $f : N \circlearrowleft$ such that

1. $f \sim \mathrm{id}_N$.

2. But f is *not* topologically isotopic to id_N.

Now let g_0 be the given hyperbolic metric on N and g_1 be the Riemannian metric such that

$$f : (N, g_1) \to (N, g_0)$$

is an isometry. And let g_t, $t \in [0, 1]$, be a path of Riemannian metrics connecting g_0 to g_1. (This is easy to do.) Consider

$$f : (N, g_t) \to (N, g_0)$$

and let $\hat{f}^t$ be the unique harmonic map homotopic to f given by the Basic Theorem Sampson and Schoen-Yau showed that $\hat{f}^t$ varies continuously with the metrics g_t. Therefore if each harmonic map $\hat{f}^t$ were one-to-one, then $t \mapsto \hat{f}^t$ would be a topological isotopy between $\hat{f}^0$ and $\hat{f}^1$. But $\hat{f}^0 = \mathrm{id}_N$ since $\mathrm{id}_N : (N, g_0) \to (N, g_0)$ is an isometry and $\mathrm{id}_N \sim f$. Also $\hat{f}^1 = f$ since $f : (N, g_1) \to (N, g_0)$ is also an isometry.

Conclusion. *For some* $t \in (0, 1)$

$$\hat{f}^t : (N, g_t) \to (N, g_0)$$

is a harmonic map homotopic to the diffeomorphism f*; but* $\hat{f}^t$ *is not a homeomorphism.*

But unfortunately (N, g_t) may have some *positive curvature*. However there is also the following well known problem.

Problem 7.1 [Burns-Katok list 1984]. Is the space of negatively curved Riemannian metrics on a given closed manifold path connected?

Of course if you could answer this question positively for N, then this would solve Problem 111 on Yau's list. However Ontaneda and I have recently shown (2006) that the answer is No for all manifolds of dimension $\geqslant 10$. I'll discuss this result in a later lecture.

But after a few more false starts Ontaneda, Raghunathan and I solved Problem 111 in 2000.

Theorem (F-Ontaneda-Raghunanthan). *For every* $n \geqslant 6$ *there is a diffeomorphism* $f : M^n \to N^n$ *between a pair of closed negatively curved Riemannian manifolds such that* $\hat{f}$ *is not a one-to-one function (hence not a homeomorphism).*

Our proof depends on the following.

Construction. Given $n \geqslant 6$, there exists a n-dimensional closed hyperbolic manifold T and a homeomorphism $g : S \to T$ such that

1. S is a negatively curved Riemannian manifold.
2. S and T are *not* PL homeomorphic.

3. But there exists a connected 2-sheeted cover $\tilde{T} \to T$ such that $\tilde{g} : \tilde{S} \to \tilde{T}$ *is homotopic to a diffeomorphism.*

Here $\tilde{S} \to S$ is the corresponding 2-sheeted cover and $\tilde{g}$ is the lift of g to $\tilde{S}$. In particular the following diagram commutes:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{g}} & \tilde{T} \\ \downarrow & & \downarrow \\ S & \xrightarrow[g]{} & T. \end{array}$$

(Note that $\tilde{g}$ is a homeomorphism.)

Proof of Theorem. Let $M = \tilde{S}$, $N = \tilde{T}$ and $f : M \to N$ be the diffeomorphism with $f \sim \tilde{g}$ given by property 3 of the Construction. We need to show that $\hat{f}$ is *not* one-to-one. For this we first show that $\hat{f}$ is a lift of $\hat{g}$. Since $g \sim \hat{g}$ we can lift this homotopy to one between $\tilde{g}$ and a lift $\tilde{\hat{g}}$ of $\hat{g}$. Note that $\tilde{\hat{g}} \sim f$. But $\tilde{\hat{g}}$ is also clearly harmonic; hence $\hat{f} = \tilde{\hat{g}}$ by the uniqueness property in the Basic Theorem. Therefore the following diagram commutes:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\hat{f}} & \tilde{T} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\hat{g}} & T. \end{array}$$

Now because of property 2 of the Construction together with the Smooth Hauptvermutung, $\hat{g}$ is *not* a homeomorphism. But $\hat{g}$ is *onto* since it is a homotopy equivalence. Therefore $\hat{g}$ is *not* univalent. Hence $\hat{f}$ cannot be univalent by the pigeon hole principle (4 pigeons $\to$ 2 holes). □

Remarks. Some crucial ingredients behind the proof of the Smooth Hauptvermutung are Moise's Theorem that homeomorphic 3-manifolds are diffeomorphic and Sard's Theorem that regular values of a smooth map are dense in its range.

The key to the Construction is the following

Lemma. *Given an integer* $n \geqslant 6$ *and a positive real number* r*, there exists a pair of closed connected orientable hyperbolic manifolds* M^m, N^{m-1} *and a pair of cohomology classes* $\alpha \in H^1(M, \mathbb{Z}_2)$*,* $\beta \in H^2(M, \mathbb{Z}_2)$ *satisfying:*

1. *N is a codim 1 totally geodesic submanifold of M whose normal tubular neighborhood has width $\geqslant r$.*
2. *N does not depend on r (but M may).*
3. $\alpha \cup \beta \neq 0$.
4. $[N^{m-1}]$ *is dual to* α.
5. β *is co-spherical.*

The proof of this result is via an extension of earlier work of Millson and Raghunathan. M is arithmetically constructed. In fact $\pi_1 M$ is commensurable with $O(q, \mathbb{Z}\sqrt{2})$ where $q(x_0, x_1, \ldots, x_m) = -\sqrt{2}\, x_0^2 + x_1^2 + \cdots + x_m^2$. The class β is also dual to a codim 2 totally geodesic submanifold K^{m-2} of M^m with trivial normal bundle so that K^{m-2} and N^{m-1} intersect transversally and $[K \cap N]$ is dual to $\alpha \cup \beta$. Furthermore there exists a 3-dimensional totally geodesic submanifold S of M^m such that

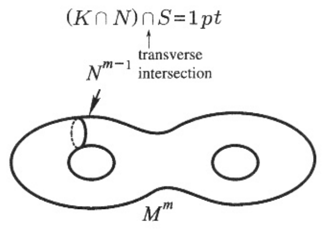


Figure 6:

Finally I indicate how to make the Construction using the Lemma. $T = M$ and S is obtained from M by cutting apart along $N^{m-1} \subset M^m$ and then regulating with a twist using a certain self-diffeomorphism $f : N \circlearrowleft$

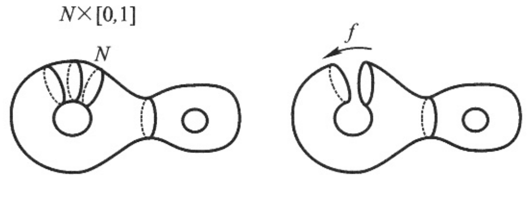


Figure 7:

Since the number of such f is equal to the cardinality of the set $[N \times [0,1], \partial; \mathrm{Top}/O]$ which is finite, if r is large enough, then S supports a negatively curved Riemannian metric.

We use α and β to determine an element

$$\gamma \in [N \times [0,1], \partial; \mathrm{Top}/O]$$

as follows:

$$\begin{array}{ccccccc}
 & \tilde{M} & \longrightarrow & S^2 \times S^1 & \longrightarrow & S^3 & & \\
 & \downarrow & & \downarrow{\scriptstyle \mathrm{id} \times Z^2} & & \downarrow{\scriptstyle \text{degree } 2} & & \\
N \times [0,1] \subseteq & M & \xrightarrow{\beta \times \tilde{\alpha}} & S^2 \times S^1 & \longrightarrow & S^3 & \xrightarrow[\text{generator}]{} & \mathrm{Top}/O \\
 & & & & & \| & & \downarrow{\scriptstyle \simeq \text{ on } \pi_3 = \mathbb{Z}_2} \\
N \times \partial\ [0,1] & & \longrightarrow & & & \text{base pt} \in S^2 \wedge S^1 & & \mathrm{Top}/\mathrm{PL} = K(\mathbb{Z}_2, 3)..
\end{array}$$

Addendum. (Assuming the Theorem's notation) There exists a number $T > 0$ such that f_t is *not* univalent for all $t \geqslant T$. Here f_t denotes the solution to the PDE $\dot{f}t = \tau_{f_t}$, $f_0 = f$.

The proof of this result uses Scharlemann's extension of the Smooth Hauptvermutung to cell like maps and the recent solution of the Poincaré Conjecture.

Final Remark. Besson, Courtois and Gallot have proposed an alternative candidate for "best of all possible maps" which they call the natural map f^* associated to a homotopy equiv. $f : M \to N$. And they have used f^* to give a new proof of Mostow's Rigidity Theorem for hyperbolic manifolds. But Marco Varisco has observed that the natural map f^* shares the same problems as the harmonic map $\hat{f}$; in particular f^* is also *not* one-to-one in the context of the above theorem.

Lecture 2. Topological rigidity

This lecture is devoted to sketching a proof of the following special case of Borel's Conjecture.

Topological Rigidity Theorem (F-Jones). *Let $f : N^m \to M^m$ be a homotopy equivalent between closed m-dimensional manifolds ($m \geqslant 5$) such that M^m supports a non-positively curved Riemannian metric. Then f is homotopic to a homeomorphism.*

Our sketch will concentrate on the important instance of this theorem where M^m is a hyperbolic manifold; i.e. has constant -1 sectional curvatures.

We start by considering the special situation where there exists an h-cobordism W^{m+1} between M^m and N^n; i.e. W^{m+1} is a compact manifold with

$$\partial W^{m+1} = M^m \coprod N^m$$

and W^{m+1} is homotopy equivalent to $M^m \times [0,1]$ and f is induced by this homotopy equivalence.

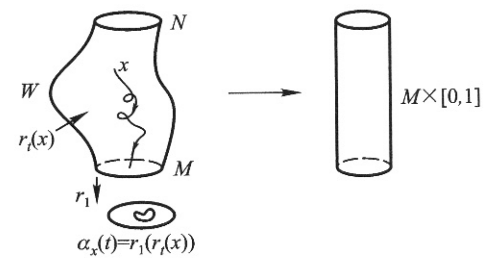


Figure 8:

If we could show that W is a cylinder over M (i.e. homeomorphic to $M \times [0,1]$) then we would be done in this important special case. To accomplish this, first note that there is a deformation retraction r_t, $t \in [0,1]$, of W^{m+1} onto M^m : $r_0 = \mathrm{id}_W$, $r_1 : W \to M$ is a retraction.

For each $x \in W$

$$\alpha_x(t) = r_1(r_t(x))$$

is a loop in M, and $\{\alpha_x \mid x \in W\}$ are called the *tracks* of r_t.

Definition. W is said to be ϵ-controlled if there exists a deformation retraction whose tracks all have diameter $\leqslant \epsilon$.

The following result due to Steve Ferry gives a metric way of recognizing when W is a cylinder.

Controlled h-cobordism Theorem (Ferry). *Given a closed Riemannian manifold M^m ($m \geqslant 5$), there exists $\epsilon = \epsilon_M > 0$ such that every ϵ_M-controlled h-cobordism with base M is a cylinder.*

Now let $\{\alpha_x \mid x \in W\}$ be the tracks of a fixed *smooth* deformation retraction r_t of W onto M. (Smoothing theory shows that W has a smooth structure.) And let

$$\beta = \text{maximum arc length of these tracks.}$$

If $\beta < \epsilon_M$, then W is a cylinder (by Ferry's Theorem) and we are done. But usually $\beta >> \epsilon_M$. (In fact if M had constant $+1$ curvature, say M was a Lens space with $\pi_1 M = \mathbb{Z}_p$ ($p \geqslant 5$), then there are h-cobordisms W which are *not* cylinders. So we must use the negative curvature condition somehow.)

Method for shrinking tracks

Consider the following picture in $\mathbb{H}^m$ (hyperbolic m-space) which is the universal cover of M^m. Here we use the Poincaré model for $\mathbb{H}^m$; i.e.

$$\mathbb{H}^m = \mathrm{Int}\ \mathbb{D}^m.$$

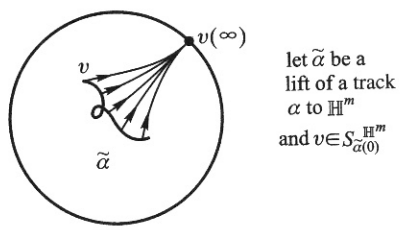


Figure 9:

Equip $\tilde{\alpha}$ with a vector field in the asymptotic manner shown. This construction is equivariant so it produces a vector field $v\alpha$ along α for each $v \in S_{\alpha(0)}M^m$.

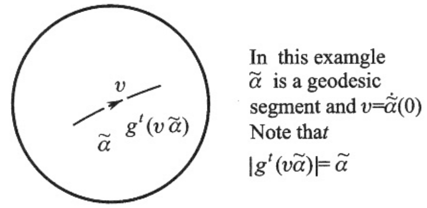


Figure 10:

And flowing $v\alpha$ forward via the geodesic flow $g^t : SM \circlearrowleft$ apparently "shrinks" $v\alpha$. But in reality it only makes it "skinny" as the following example shows.

Now there exists an h-cobordism $\mathcal{W}^{2m+1}$ (namely $\mathcal{W} = r_1^*(SM \to M)$) with base SM and a deformation retraction $R_t : \mathcal{W} \circlearrowleft$ whose tracks are $\{g^s \circ (v\alpha)\}$ for some big s. (We call $\mathcal{W}$ the *asymptotic transfer* of W.) Although $\mathcal{W}$ may not be ϵ_{SM}-controlled, its tracks are arbitrarily skinny and none are longer than β. Hence $\mathcal{W}$ is a cylinder by the following result.

Foliated Control Theorem (F-Jones 1986). *Given a closed Riemannian manifold M^m ($m \geqslant 3$) and $\beta > 0$, there exists an $\epsilon = \epsilon_{M,\beta}$ such that any (β,ϵ)-controlled h-cobordism $\mathcal{W}$ with base SM is a cylinder.*

Definition. Here (β, ϵ)-controlled means that each track is ϵ-close to a path lying in an orbit of the geodesic flow g^t and having diameter $\leqslant \beta$ in that orbit.

But we needed to show that W is a cylinder and we've only shown that $\mathcal{W}$ is a cylinder. To compare W to $\mathcal{W}$ we need another more algebraic way of thinking about the set $\mathrm{Wh}(B)$ of all h-cobordisms with a given base B. (This method traces back to JHC Whitehead and Smale.)

It turns out that $\mathrm{Wh}(B)$ is an Abelian group with the cylinder $B \times [0,1]$ as its zero element. (In fact $\mathrm{Wh}(B)$ depends only on $\pi_1 B$.) Also a continuous map $p : \mathcal{B} \to B$ induces a group homomorphism

$$p_* : \mathrm{Wh}(\mathcal{B}) \to \mathrm{Wh}(B)$$

and if p is a fiber bundle, there is a transfer homomorphism

$$p^* : \mathrm{Wh}(B) \to \mathrm{Wh}(\mathcal{B})$$

in the other direction which is topologically defined; while p_* is algebraically defined.

In fact in the case considered above

$$\mathcal{W} = p^*(W)$$

where $\mathcal{B} = SM$ and $B = M$. Furthermore Doug Anderson has determined the composition $p_* \circ p^*$.

Theorem (D. Anderson). *$p_* \circ p^*$ is multiplication by $\chi(F)$ – the Euler characteristic of the fiber F of $p : \mathcal{B} \to B$* (least when $\pi_1 B$ acts trivially on the homology of F via the holonomy representation).

In the situation at hand $p : SM \to M$, F is S^{m-1}. And hence

$$\chi(S^{m-1})W = p_* p^*(W) = p_*(\mathcal{W}) = p_*(0) = 0.$$

If $\chi(S^{m-1}) = 1$, then we'd be done. Unfortunately $\chi(S^{m-1}) = 0$ or 2. Hence in this way, we *don't* get $W = M \times [0,1]$. But we can make an alternate construction (more complicated) whose fibre is $\mathbb{D}^m$ (note $\chi(\mathbb{D}^m) = 1$) and having the same shrinking property for tracks. In this way we *do* get that W is a cylinder and that

$$\mathrm{Wh}(M) = 0.$$

But in the general case, we only have a homotopy equivalence

$$f : N \to M$$

and no h-cobordism connecting N and M. Then we form a set $\mathcal{S}(M)$ whose elements are equivalence classes of homotopy equivalences. Here

$$f_i : N_i \to M \quad (i = 0, 1)$$

are equivalent $\Leftrightarrow$ there exists a homeomorphism $F : N_0 \to N_1$

$$\begin{array}{ccc} N_0 & \xrightarrow{f_0} & M \\ {\scriptstyle F}\downarrow & \nearrow_{f_1} & \\ N_1 & & \end{array}$$

such that $f_0 \sim f_1 \circ F$.

Using surgery theory and the fact that $\mathrm{Wh}(M) = 0$, one can put an Abelian group structure on $\mathcal{S}(M)$ such that the equivalence class of a homeomorphism is the zero element, and having properties similar to those of $\mathrm{Wh}(B)$. (This is done by Kirby-Siebenmann.) In particular given a fiber bundle

$$p : \mathcal{B} \to B$$

there are group homomorphisms

$$\mathcal{S}(\mathcal{B}) \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathcal{S}(B)$$

where Ranicki's algebraic surgery theory is used to define p_*. And p^* has the following simple topological description. If $f : N \to B$ represents an element in $\mathcal{S}(B)$, then $p^*(f) \in \mathcal{S}(B)$ is represented by

$$\hat{f} : \mathcal{N} \to \mathcal{B}$$

where $\mathcal{N} \to N$ is the pullback of $\mathcal{B} \to B$ along f and $\hat{f}$ fits naturally into the diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\hat{f}} & \mathcal{B} \\ \downarrow & & \downarrow \\ N & \xrightarrow[f]{} & B. \end{array}$$

In analogy with the situation for $\mathrm{Wh}(B)$, we would like to have a "nice formula" for the composite $p_* \circ p^*$. And there is such a formula when B is aspherical provided the *integral form* of Novikov's Conjecture is true for B.

We recall that Novikov's Conjecture for an aspherical manifold B asserts that for any element $f : \mathcal{B} \to B$ in $\mathcal{S}(B)$

$$f^*(p_i(B)) = p_i(\mathcal{B})$$

where p_i are the rational Pontryagin classes in $H^{4i}(\ ,\mathbb{Q})$. Mischenko (1974) showed this conjecture is true for every non-positively curved Riemannian manifold B.

The integral form of Novikov's Conjecture is a bit stronger statement which we proceed to formulate.

The composite map

$$\mathcal{B}^n \xrightarrow[f]{} B^n \subseteq B^n \times \mathbb{R}^{n+1}$$

is homotopic to an embedding with normal bundle v whose total space can be identified with $B^n \times \mathbb{R}^{n+1}$ (using the h-cobordism Theorem). Thus v is a $\mathbb{R}^{n+1}$-bundle equipped with a fiber homotopy trivialization; i.e. it determines a homotopy class of maps

$$\mathcal{N}_f : B \to G/\mathrm{Top}$$

called the normal invariant of $f \in \mathcal{S}(B)$. The integral form of Novikov's Conjecture asserts that $\mathcal{N}_f$ is homotopic to a constant map for every $f \in \mathcal{S}(B)$. Hsiang and I (1981) showed this conjecture is true when B is a closed non-positively curved Riemannian manifold. There is consequently the following "nice formula" in this case:

$$p_* \circ p^* = \text{multiplication by index } (F^{4i})$$

where F^{4i} is a $4i$-dimensional manifold which is the fiber of $p : \mathcal{B}^{n+4i} \to B^n$. (Again $\pi_1 B$ is required to act trivially on the homology of F via the holonomy representation.) In particular when $F = \mathbb{C}P^{2i}$

$$p_* \circ p^* = \mathrm{id}$$

since $\mathrm{Index}(\mathbb{C}P^{2i}) = 1$. (This result is essentially due to Wall.)

Using this formula and a foliated analogue of the Chapman-Ferry "α-approximation Theorem" in place of the foliated form of Ferry's "controlled h-cobordism Theorem", we could proceed as in the case of $\mathrm{Wh}(M)$ to show that $\mathcal{S}(M) = 0$ *except that*

$$\mathrm{Index}(S^{4i}) = 0.$$

This is a major difficulty which took Jones and I a couple of years to get around.

The idea that works is to use *unordered pairs* of unit length tangent vectors instead of a single vector when doing the asymptotic transfer. The space F^m of such pairs has

$$\text{Index}(F^m) = 1$$

when $m = \dim M$ is odd; in fact when $\dim m = 3$, this space is $\mathbb{C}P^2$. This last fact can be seen by considering the map $sl_2(\mathbb{C}) \to F^3$ defined by $A \mapsto$ the eigenspaces of A. And noting that F^m is the space of all unordered pairs $[u, v]$ of vectors $u, v \in S^{m-1}$.

Since the details now get quite involved, this is a good place to end my sketch of the proof of the Top Rigidity Theorem. A fuller sketch can be found in my article in the ICTP lectures notes series (2002) volume "Topology of High Dimensional Manifolds".

But before leaving this topic let me make a few remarks from the sketch of proof in our Proc. NAS announcement (1989). First the space F^m is *not* a manifold when $m > 3$. But it does have a natural stratification with two strata: the top stratum $T = \{[u, v] \mid u \neq v\}$ and the bottom stratum $B = \{[u, u] \mid u \in S^{m-1}\}$. Now for convenience let us again restrict to the case where M is hyperbolic; i.e. $\tilde{M} = \mathbb{H}^m$. And let α be a path in $\mathbb{H}^m$ and $\omega = [u, v] \in F_{\alpha(0)}(\mathbb{H}^m)$. We can asymptotically equip α with two vector fields $u\alpha$ and $v\alpha$ (as above) and then define $\omega\alpha = [u\alpha, v\alpha]$. Focusing at ∞ in the two directions u, v gives two points U, V on the sphere at ∞. If $u \neq v$, then $U \neq V$ and there is a geodesic line L_ω connecting U to V. We call L_ω the *marking* of $\mathbb{H}^m$ determined by ω. Thus if $\omega \in T$, we have a new way to shrink $\omega\alpha$; namely flow $\omega\alpha$ perpendicularly towards the marking L_ω.

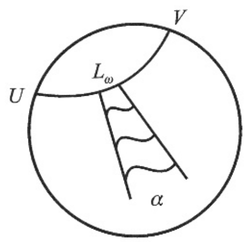


Figure 11:

Note that this is only a partial flow since it can't go past the marking. But this partial radial flow is an important new tool used in trying to homotope the transferred homotopy equivalence to a homeomorphism.

Another remark is that by combining the above techniques with ideas from Cheeger-Fukaya-Gromov collapsing theory Jones and I obtained the following extension of topological rigidity valid even for non-compact manifolds.

Topological Rigidity Theorem (General Case) (F-Jones 1998). *Let $f : N^m \to M^m$ $(m \geqslant 5)$ be a homotopy equivalence between aspherical manifolds such that*

1. *There exists a compact subset $K \subset M$ with $f : N - f^{-1}(K) \to M - K$ a homeomorphism.*
2. *Either $\Gamma = \pi_1(M)$ is a discrete subgroup of $GL_n(\mathbb{R})$ or there exists a complete A-regular Riemannian manifold $\mathcal{M}$ of non-positive curvature with $\pi_1(\mathcal{M}) = \Gamma$(e.g. $\mathcal{M}$ can be any complete pinched negatively curved manifold).*

Then there exists a bigger compact set $\mathcal{K} \supset K$ such that f is homotopic to a homeomorphism rel $N - f^{-1}(\mathcal{K})$.

(We recall that a Riemannian metric is A-regular if there is a sequence of nonnegative numbers A_i such that $|\nabla^i R| < A_i$ for all $i = 0, 1, 2, \ldots$; where R is the curvature tensor.)

Remark. The A-regularity property in condition 2 is the fundamental one; the subgroup of $GL_n(\mathbb{R})$ condition is derived from it.

There are two interesting corollaries of this General Case giving more information about closed manifolds. One of these which I'll now discuss is about (complete) affine flat manifolds; the other is discussed in a later lecture. A *complete affine flat* manifold is a manifold whose universal cover is the vector space $\mathbb{R}^n$ and whose deck transformation group is a subgroup of

$$\text{Aff}(\mathbb{R}^n) = \text{all affine motions of } \mathbb{R}^n.$$

An *affine motion* is a composition of a translation and a linear transformation.

Theorem (F-Jones 1998). *Closed (complete) affine flat manifolds are topologically rigid; i.e. any isomorphism between their fundamental groups is induced by a homeomorphism.*

In dim 3 this is due to Fried and Goldman, and in dim 4 to Abels, Margulis and Soifer. In dim $\geqslant 5$ the result does not follow directly from the Topological Rigidity Theorem (closed form) since many of these affine flat manifolds M^m *do not* support non-positively curved Riemannian metrics because of Yau's Ph.D. thesis (1971). But M^m is clearly aspherical and it is easy to show that $\pi_1(M^m)$ is a discrete subgroup of $GL_{m+1}(\mathbb{R})$. Hence this theorem follows from the Topological Rigidity Theorem (General Case) by letting K be the null set.

And the following is a very interesting open question.

Question 3. Are closed (complete) affine flat manifolds smoothly rigid (or at least PL rigid) and perhaps even affinely rigid?

Remark. One of the motivations for Borel's Conjecture was the classical result of Bieberbach asserting that closed Riemannian flat manifolds are affinely rigid.

Example. Closed does not imply complete for affine flat manifolds. In fact $S^1 \times S^{n-1}$ is closed but not complete; its universal cover is $\mathbb{R}^n - 0$ and has an infinite cyclic group of deck transformations generated by the affine motion $v \to 2v$.

Lecture 3. The Ricci flow and other applications

We now continue with our program of finding geometric applications for topological rigidity. Let's start by recalling another problem from Yau's 1982 list.

Problem 13. Let M_1 and M_2 be closed Einstein manifolds with negative curvature. Suppose $\pi_1 M_1 \simeq \pi_1 M_2$ and dim $M_1 \geqslant 3$. Is M_1 isometric to M_2 (up to scaling)?

Let me recall the definitions of Ricci curvature Ric and of an Einstein manifold.

$$\mathrm{Ric} : T_x M \to \mathbb{R}$$

is a quadratic function defined on each tangent space $T_x M$ by

$$\mathrm{Ric}(v) = \text{average of the sectional curvatures of all the 2-planes in } T_x M \text{ containing } v.$$

Here $|v| = 1$ and $\mathrm{Ric}(tv) = t^2 \mathrm{Ric}(v)$.

An Einstein manifold is a Riemannian manifold of constant Ricci curvature; i.e.

$$\mathrm{Ric}\big|_{SM} \text{ is constant.}$$

Said a little bit differently. There exists a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}(v) = \lambda |v|^2$$

for all vectors v tangent to M.

Rugang Ye (1993) proposed a method for showing that No is the answer to Problem 13. His method uses Hamilton's Ricci Flow technique for improving a Riemannian metric g_0 on a manifold N. Hamilton's technique is analogous to the harmonic map procedure (discussed yesterday) for flowing a closed curve

$$\alpha : S^1 \to M$$

to a geodesic thru the 1-parameter family of curves α_s satisfying the PDE

$$\frac{\partial \alpha_s(t)}{\alpha s} = \ddot{\alpha}_s(t), \quad \alpha_0 = \alpha.$$

Hamilton's PDE is

$$\dot{g}_t = \frac{2}{n} r g_t - \mathrm{Ric}_{g_t}$$

where r is the average scalar curvature (and $n = \dim N$).

Ye's program was the following. Recall from my first lecture that, for any $\epsilon > 0$, Jones and I had constructed a non-diffeomorphic pair M and N of homeomorphic Riemannian manifolds with M hyperbolic and the sectional curvatures of (N, g_0) pinched within ϵ of -1. Ye conjectured that there exists $\epsilon_n > 0$ such that if $\epsilon \leqslant \epsilon_n$, then the Ricci flow g_t of g_0 converges to a negatively curved Einstein metric g_∞ on N. If so then M and (N, g_∞) would negatively answer Problem 13 since M and N are *not* even diffeomorphic. But Ontaneda and I have recently shown that unfortunately Ye's conjecture is *not* true.

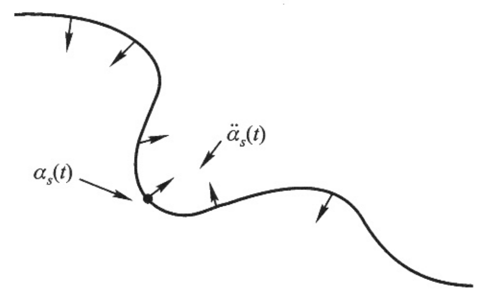


Figure 12:

Theorem. *Given $n > 10$, $\epsilon > 0$ there exists a closed n-dimensional Riemannian manifold (N, g) with all it sectional curvatures in $[-1-\epsilon, -1]$; but the Ricci flow g_t (with $g_0 = g$) does* not *converge (in C^2-topology) to a negatively curved (Einstein) metric on N.*

Addendum. Such examples exist where N supports a hyperbolic metric. And there are also such examples where N is *not* homotopically equivalent to a closed hyperbolic manifold.

To prove this Theorem, we use the Construction (F-Ontaneda-Raghunathan) also mentioned in my first lecture.

This Construction can be paraphrased as saying there exists a closed smooth manifold $\mathcal{N}^n$ equipped with both a hyperbolic metric g_0 and an ϵ-pinched close to -1 metric g_1, and a pair of order 2 cyclic subgroups

$$C_i \subseteq \mathrm{Iso}(g_i)$$

each acting freely on $\mathcal{N}^n$ and such that $\mathcal{N}/C_0$ is homeomorphic but *not* PL-homeomorphic to $\mathcal{N}/C_1$. And we would like a 1-parameter family g_t of ϵ-pinched close to -1 negatively curved Riemannian metric connecting g_0 to g_1. But we don't know how to do this. (It will be shown in Lecture 4 that this is impossible in general.) However by passing to a sufficiently large regular finite sheeted cover

$$N^n \xrightarrow[p]{} \mathcal{N}^n$$

(using our knowledge of $\mathcal{P}(\mathcal{N})$ coming from an addendum to the Top Rigidity Theorem (1989) for the construction) we are able to do this for the pullback metrics

$$h_0 = p^*(g_0), \quad h_1 = p^*(g_1).$$

Stated precisely, we construct a closed smooth manifold N^n equipped with a 1-parameter family h_s, $s \in [0, 1]$, of Riemannian metrics and a pair of finite subgroups

$$G_i \subseteq \mathrm{Iso}(h_i) \quad i = 0, 1$$

satisfying the following 5 properties:

1. sectional curvatures $h_s \subseteq [-1-\epsilon, -1]$.
2. $G_0 \simeq G_1$; in fact they are conjugate subgroups of $\mathrm{Top}(N)$.
3. $N \to N/G_i$ is a covering space.
4. N/G_0 and N/G_1 are *not* diffeomorphic.
5. h_0 has constant -1 sectional curvatures.

Now we are positioned to complete the proof of the Theorem.

Obviously if the Ricci flow does not converge to a negatively curved (Einstein) metric for one of these Riemannian metrics h_s, then we are done. So, let us assume that the Ricci flow converges to a negatively curved metric for each h_s. We now show this leads to a contradiction. Write $h_{s,t}$ for the Ricci flow starting at $h_{s,0} = h_s$, $0 \leqslant t < \infty$, and converging to the negatively curved Einstein metric j_s.

We claim that all these j_s are isometric up to scaling. This is a consequence of the following result of Ye together with the fact that the function $h_{s,t}$ is jointly continuous in s and t for $(s,t) \in [0,1] \times [0,+\infty)$.

Stability Theorem [Rugang Ye]. *Let j be a negatively curved Einstein metric, then the Ricci flow starting at any metric sufficiently close to j converges to a metric which is isometric to j up to scaling.*

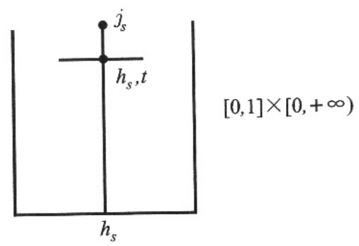


Figure 13:

In particular j_1 is isometric to j_0 (up to scaling). But $j_0 = h_0$ since Einstein metrics are left fixed under the Ricci flow and h_0 has constant -1 curvature. Consequently j_1 has constant -1 curvature also.

Another basic property of the Ricci flow is that

$$\mathrm{Iso}(h_1) \subseteq \mathrm{Iso}(h_{1,t})$$

for all $t \geqslant 0$; i.e. the metric gets *more beautiful* as you flow it. Consequently

$$G_1 \subseteq \mathrm{Iso}(h_1) \subseteq \mathrm{Iso}(j_1).$$

Therefore j_1 induces a Riemannian metric on N/G_1 and this metric has constant -1 sectional curvature as does N/G_0 (by property 5). And by property 2 above N/G_0 and N/G_1 are homeomorphic. Hence they are diffeomorphic by Mostow's Rigidity Theorem.

But this contradicts property 4 above proving the Theorem. □

The next application illuminates the difference between the sign of the eigenvalues of the curvature operator and the sign of the sectional curvatures of a perhaps different Riemannian metric on the same smooth manifold. We start with some definitions.

Let V be a finite dimensional inner product space. Then the exterior product $\bigwedge^2 V = V \wedge V$ is also an inner product space where

$$|u \wedge v| = \text{ area of the parallelogram spanned by } u \text{ and } v.$$

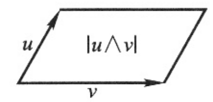


Figure 14:

The *curvature operator* on a Riemannian manifold M at $x \in M$ is a self-adjoint linear transformation

$$\mathcal{R} : \bigwedge^2 T_x M \circlearrowleft$$

and the sectional curvatures of the plane spanned by $u, v \in T_x M$ is given by

$$\frac{\mathcal{R}(u \wedge v) \cdot (u \wedge v)}{|u \wedge v|^2}.$$

Note that if $\mathcal{R}$ is negative definite, then all sectional curvatures are negative; but not necessarily vice-versa. In fact in Peter Petersen's text book "Riemannian Geometry" there is the following question.

Question. Does every closed negatively curved manifold admit a Riemannian metric whose curvature operator has no positive eigenvalues; i.e. $\mathcal{R}$ is a non-positive operator.

C.S. Aravinda and I have recently answered this question.

Theorem (F-Aravinda 2005). *The answer is No. In fact for every $\epsilon > 0$, there exists a 16-dimensional manifold (M, g) all of whose sectional curvatures lie in the interval $[-4, 1-\epsilon]$; but for every Riemannian metric h on M, $\mathcal{R}$ has a positive eigenvalue.*

Proof. The manifold M^{16} is the connected sum $N^{16} \# \Sigma^{16}$ where Σ^{16} is the unique 16-dimensional exotic sphere and N^{16} is a Cayley hyperbolic manifold of sufficiently large injectivity radius. Using a result of Boris Okun, we (2003) showed that M^{16} is *not* diffeomorphic to N^{16} although M^{16} and N^{16} are of course homeomorphic.

Remark. Examples of closed complex and quarternionic hyperbolic manifolds and exotic spheres Σ such that $N \# \Sigma$ supports a Riemannian metric of negative sectional curvature and where N and $N\#\Sigma$ are *not* diffeomorphic were constructed by Farrell-Jones (1994) and Aravinda-Farrell (2004) respectively. The Farrell-Jones examples are almost 1/4-pinched; but unlike in the Cayley hyperbolic case the exotic Aravinda-Farrell quaternionic hyperbolic examples are not.

Now Theorem follows immediately from Kevin Corlette's Superrigidity Result.

Superrigidity Theorem (Corlette). *Let M^{16} be a closed Cayley hyperbolic manifold and (N^{16}, h) be a closed Riemannian manifold whose curvature operator $\mathcal{R}$ has no positive eigenvalue. If $\pi_1(M^{16}) \simeq \pi_1(N^{16})$, then M^{16} and N^{16} are isometric up to scaling. In particular they are diffeomorphic.*

Remark. An analogous statement for a real hyperbolic manifold M (i.e. one of constant -1 curvature) is obviously false. Since its curvature operator is negative definite, if we perturb it slightly it stays negative definite; but probably gets non-constant sectional curvatures.

I will next discuss what I think is a quite interesting and virtually open problem.

Problem. Find interesting geometric conditions (beyond aspherical with isomorphic fundamental groups) that will imply smooth (or PL) rigidity but don't force isometry (up to scaling). For example: Let $f : N^n \to M^n$ be a homotopy equivalence between closed negatively curved manifolds. When is f homotopic to a diffeomorphism or at least a PL homeomorphism?

Towards addressing this Problem let me recall how Eberlein and O'Neill associate a sphere at infinity $\tilde{M}(\infty)$ to the universal cover $\tilde{M} \to M$ of a complete negatively curved manifold M. A point in $\tilde{M}(\infty)$ is an asymptopy class $[r]$ of geodesic rays in $\tilde{M}$.

Definition. Two unit speed rays $r_0(t)$, $r_1(t)$ $t \in [0, \infty)$ are asymptotic if they stay a bounded distance apart at $t \to \infty$; i.e. $\{d(r_0(t), r_1(t)) \mid t \in [0, \infty)\}$ is bounded from above.

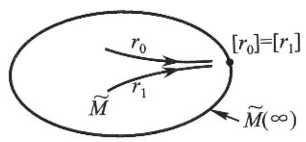


Figure 15:

They show that $\tilde{M}^n(\infty)$ is homeomorphic to S^{n-1} and in fact $\tilde{M} \cup \tilde{M}(\infty)$ is homeomorphic to $\mathbb{D}^n$. Also the deck transformation action of $\pi_1(M)$ on $\tilde{M}$ extends to $\tilde{M}(\infty)$. And Mostow associated to the homotopy equivalence $f : M \to N$ a $\pi_1 M$-equivariant homeomorphism

$$\bar{f} : \tilde{M}(\infty) \to \tilde{N}(\infty)$$

which depends (essentially) only on

$$f_\# : \pi_1 M \to \pi_1 N.$$

To do this he uses a lift $\tilde{f} : \tilde{M}$ ⟲ of f and shows that $\tilde{f}(r)$, although rarely a geodesic ray itself, is always within a finite distance of a geodesic ray s in $\tilde{N}$. And defines

$$\bar{f}([r]) = [s].$$

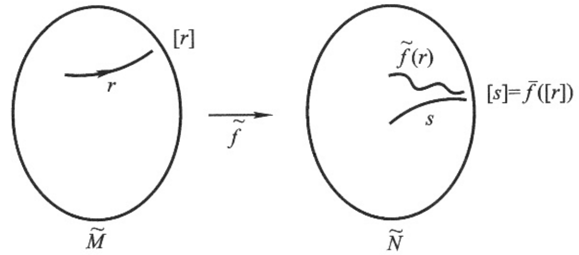


Figure 16:

And when the sectional curvatures are strictly 1/4-pinched; i.e. lie in the interval $(-4, -1]$, Hirsch and Pugh showed that $\tilde{M}(\infty)$ has a *natural* C^1-structure; in particular the $\pi_1 M$ action on $\tilde{M}(\infty)$ is also C^1.

Remark. It is "rare" that $\tilde{M}(\infty)$ has a natural C^∞ structure. In fact if the sectional curvatures are all in $\left(-\frac{9}{4}, -1\right]$ this forces those curvatures to be constant; i.e. M to be hyperbolic and $(\tilde{M} \cup \tilde{M}(\infty), \tilde{M}(\infty))$ to be the Poincaré model.

This is due to Kanai and Hamenstädt.

Theorem (F-Jones). *If the sectional curvatures of both M^n and N^n $(n \geqslant 5)$ are strictly 1/4-pinched and*

$$\bar{f} : \tilde{M}(\infty) \to \tilde{N}(\infty)$$

is a C^1-diffeomorphism, then f is homotopic to a PL-homeomorphism and to a diffeomorphism when n is odd.

Addendum. When n is even, N^n is diffeomorphic to the connected sum $M^n\#s$-copies of Σ^n, where $s = \chi(M)$, and

Unfortunately the condition "$\bar{f}$ is a C^1-diffeomorphism" is *very far* from necessary; e.g. Mostow showed that when $n = 2$ if M^2 and N^2 both have constant -1 curvature and $\bar{f}$ is a C^1-diffeomorphism, then f is homotopic to an *isometry*.

Question 1. Does $\bar{f}$ being a C^1 diffeomorphism always imply that f is homotopy to an isometry? Or at the other extreme can Σ^n in Addendum ever be an exotic sphere?

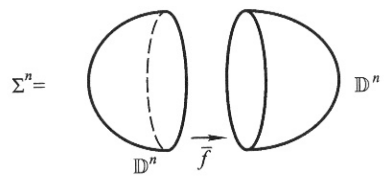


Figure 17:

Definition. When M is negatively curved, there is a function $l : \pi_1 M \longrightarrow [0, \infty)$ defined by

$$l(\alpha) = \text{length of the unique closed geodesic representing } \alpha.$$

And M, N have isomorphic marked length spectra if there exists an isomorphism $\phi : \pi_1 M \longrightarrow \pi_1 N$ such that

$$l_N \circ \phi = l_M.$$

Corollary. *If two odd dimensional and strictly $1/4$-pinched negatively curved manifolds of dim $\geqslant 5$ have isomorphic marked length spectra, then they are diffeomorphic.*

This is a consequence of the Theorem because of a result due to Hamenstädt showing under the above assumptions that $\bar{f}$ is C^1. In fact it is conjectured that isomorphic marked length spectra imply M is *isometric* to N. And Hamenstädt showed this is true if one of the manifolds has constant -1 curvature.

Question 2. Let M be a closed negatively curved Riemannian manifold and N be a smooth manifold homeomorphic to M. Does N also support a negatively curved Riemannian metric?

Remark. For complete finite volume open manifolds with pinched negative curvature, the answer is in general No (F-Jones, Proc. AMS 1993).

Recall that the inner automorphisms of a group Γ form a normal subgroup $\mathrm{Inn}(\Gamma)$ of the group of *all* automorphisms $\mathrm{Aut}(\Gamma)$ and that the factor group

$$\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$$

is called the group of all outer automorphisms of Γ.

Let $\mathcal{E}(M)$ be the function space consisting of all self-homotopy equivalences of M. It is an H-space. And an element $f \in \mathcal{E}(M)$ determines an automorphism $f_\# : \pi_1 M \circlearrowleft$ well defined only up to composition with an inner automorphism. Hence we get an H-space map

$$\mathcal{E}(M) \to \mathrm{Out}(\pi_1 M)$$

which descends to a group homomorphism

$$\pi_0 \mathcal{E}(M) \to \mathrm{Out}(\pi_1 M)$$

since $f \sim g \Rightarrow f_\# = g_\#$ in $\mathrm{Out}(\pi_1 M)$. And Hurewicz showed that this is an isomorphism when M is aspherical.

Remark. Hurewicz's result was another motivation for Borel's Conjecture.

Now consider the group homomorphism α which is the composition

$$\mathrm{Top}(M) \to \mathcal{E}(M) \to \mathrm{Out}(\pi_1 M).$$

And note that whenever M is topologically rigid, then this map is onto. However one can ask another interesting question about α.

Question 3. Assuming M is aspherical, does every finite subgroup $F \subseteq \mathrm{Out}(\pi_1 M)$ split back to $\mathrm{Top}(M)$?

F. Raymond and L. Scott (1977) gave a counterexample where M is a nilmanifold. Their example uses the fact that $\mathrm{Center}(\pi_1 M) \neq 1$.

However Question 3 is quite viable when M is a closed negatively curved manifold. It has *Yes* for the answer when M^n has constant -1 curvature (due to Mostow for $n \geqslant 3$ and to Kerckhoff when $n = 2$). Now I come to the second application to closed manifolds of the Topological Rigidity Theorem (General Case) to which I alluded in my last lecture.

Theorem (Farrell-Jones 1998). *Let M^n be a closed negatively curved manifold ($n \geqslant 5$) and $F \subseteq \mathrm{Out}(\pi_1 M)$. If $p^{-1}(F)$ is torsion-free* (where $p : \mathrm{Aut}(\pi_1 M) \to \mathrm{Out}(\pi_1 M)$ *is the canonical epimorphism*) then *F splits back to* $\mathrm{Top}(M^n)$.

Remark. Rips showed that $\mathrm{Out}(\pi_1 M)$ is a finite group when M is negatively curved and $\dim(M) \geqslant 3$.

Proof. Since Center $(\pi_1 M) = 1$, ker $p = \pi_1 M$. And the condition that $\Gamma = p^{-1}(F)$ is torsion-free allows us to construct a finite aspherical complex K with $\pi_1 K = \Gamma$.

The hard work is involved in showing that this complex can actually be taken to be a closed manifold N^n. This is a consequence of the Topological Rigidity Theorem (General Case) via Ranicki's algebraic formulation of surgery theory.

Now let $\hat{N}$ be the finite sheeted cover corresponding to $\pi_1 N \subseteq \Gamma$, then $\hat{N}$ is homotopically equivalent to M and hence homeomorphic to M by the Topological Rigidity Theorem.

Now F acts on $M = \hat{N}$ via the deck transformations of the regular covering space $\hat{N} \to N$. In this way F splits back to $\mathrm{Top}(M)$. In trying to remove the torsion-free condition from $\Gamma = p^{-1}(F)$ in the above theorem, one is led to the following Test question.

Question 4. Let $\alpha \in \mathrm{Aut}(\pi_1 M)$ be an element of order 2 and $\hat{\alpha} : \tilde{M}(\infty) \circlearrowleft$ be the natural involution given by Mostow's Construction. (Here M is assumed to be a closed negatively curved manifold.) Is the fixed point set $\tilde{M}(\infty)^{\hat{\alpha}}$

(i) an ANR
(ii) a sphere
(iii) a locally flatly embedded sphere
(iv) an unknotted sphere?

Remarks.

1. When M has constant curvature, Mostow's Rigidity Theorem shows that all these are true.
2. Lafont and I have constructed examples in the more rarefied setting of CAT(-1) (smooth) manifolds where $\tilde{M}(\infty)$ is a sphere but
   a) $\tilde{M}(\infty)^{\hat{\alpha}}$ is *not* an ANR.
   b) $\tilde{M}(\infty)^{\hat{\alpha}}$ is a knotted codimensional 2 sphere (in fact everywhere not locally flatly embedded).
   c) We also show the positive result: If $\tilde{M}(\infty)^{\hat{\alpha}}$ is a locally flatly embedded codimensional 2 sphere, then it is *unknotted*.

I now wish to formulate a final question apropos this topic.

Although Jones and I (1990) have constructed examples of closed negatively curved manifolds M where

$$\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1 M)$$

is *not* onto, one can ask the following.

Question 5. Does the union of the images

$$\mathrm{Diff}(M, \theta) \to \mathrm{Out}(\pi_1 M)$$

as θ varies over all smooth structures on M fill up $\mathrm{Out}(\pi_1 M)$? Stronger yet, does the union of the images

$$\mathrm{Iso}(M, \theta, g) \to \mathrm{Out}(\pi_1 M)$$

fill up $\mathrm{Out}(\pi_1 M)$ where g also varies over all negatively curved Riemannian metrics on (M, θ)?

Remark. We note that if the answer to this *stronger* Question 5 is Yes, then Yes is also the answer to Question 4.

Lecture 4. The space of negatively curved metrics on M^n

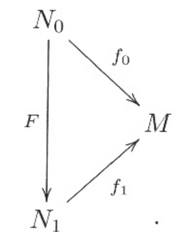
Most of the questions we've formulated above relate to the Teichmüller space $\mathcal{T}(M^m)$ of all (marked) negatively curved Riemannian metrics on a given closed smooth manifold M^m (which is assumed to admit at least one such metric). To be precise, a point in $\mathcal{T}(M^m)$ is an equivalence class of maps

$$f : N \to M$$

where N is a closed negatively Riemannian manifold (with $\mathrm{vol}(N) = 1$) and f is a diffeomorphism. Two such maps

$$f_i : N_i \to M \quad i = 0, 1$$

are equivalent iff there exists an isometry $F : N_0 \to N_1$ such that $f_1 \circ F \sim f_0$; i.e. the following diagram is homotopy commutative:



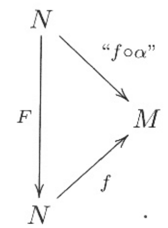
Also the maps are "close to equivalent" if there exists such a diffeomorphism F which is "close to an isometry". And $\mathcal{T}(M)$ topologized in this way is a Hausdorff but infinite dimensional space; in fact a Fréchet manifold; cf. Besse and Ebin for more details.

When M_g is a surface of genus $g \geqslant 2$ and we restrict our construction by using only constant $2(1-g)$ curvature manifolds N^2, this construction gives the classical Teichmüller space T_g which is homeomorphic to $\mathbb{R}^{6(g-1)}$. And Hamilton has shown that the Ricci flow gives a deformation retraction of $\mathcal{T}(M_g)$ onto T_g in this case. Hence $\mathcal{T}(M_g)$ is contractible.

Also notice that the image $\mathcal{I}$ of

$$\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1 M)$$

acts naturally on $\mathcal{T}(M)$. And under this action if $\alpha \in \mathcal{I}$ fixes $[f] \in \mathcal{T}(M)$ then the isometry $F \in \mathrm{Iso}(N)$ maps to $\alpha \in \mathrm{Out}(\pi_1 M)$



In this way Nielsen solved Question 6 for 2-manifolds in the case of elements $\alpha \in \mathrm{Out}(M_g)$ of prime power order.

Recall that P.A. Smith showed that any action of Z_{p^i} on $\mathbb{R}^n$ has a fixed point.

This is one motivation for trying to discover facts about the topology of $\mathcal{T}(M)$. Another is that Einstein metrics are precisely the critical points of the total scalar curvature functional

$$S : \mathcal{T}(M) \to \mathbb{R}.$$

Here

$$S(f : N \to M) = \int_P \text{sec. curvature } (P)$$

where P varies over all 2-planes tangent to N.

By a parametrized version of the techniques used to construct negatively curved manifolds which are homeomorphic but not diffeomorphic to constant -1 curvature manifolds, Ontaneda and I [37] obtained the following information about $\pi_k(\mathcal{T}(M^m))$.

Theorem. (F-Ontaneda) *For every integer $k_0 \geqslant 1$, there exists an integer $n_0 = n_0(k_0)$ such that the following holds for any closed hyperbolic manifold M^n where $n \geqslant n_0$: M^n has a finite sheeted cover N^n such that*

$$\pi_k(\mathcal{T}(N^n)) \neq 0$$

for every $1 \leqslant k \leqslant k_0$ satisfying $n + k \equiv 2 \bmod 4$ (i.e. $k \equiv 3n + 2 \bmod 4$) (i.e. for roughly every 4th integer k between 1 and k_0).

Corollary. *For each integer $k \geqslant 1$, there exists a smooth fibre bundle*

$$E \to S^k$$

each of whose fibres is equipped with a negatively curved Riemannian metric (varying continuously from fibre to fibre). But, although the fibres are diffeomorphic to a closed hyperbolic manifold, it is impossible to equip them with hyperbolic metrics varying continuously from fibre to fibre.

Of course this corollary begs me to ask the following.

Question 6. Let $E \to B$ be a fibre bundle whose fibres are diffeomorphic to a closed negatively curved manifold. Is it always possible to equip its fibres with negatively curved Riemannian metrics (varying continuously from fibre to fibre)?

But Question 6 is closely related to Problem 7.1 on the list compiled by Burns-Katok which I mentioned in my first lecture. Hence the negative solution to Problem 7.1 recently given by Ontaneda and myself in [38] clearly gives examples (with $B =$ circle) which also answers Question 6 negatively. And let me end this series of lectures by discussing in some detail the solution to Problem 7.1.

Let M^n be a closed smooth manifold ($n = \dim M^n$) and $(C_p)^\infty$ denote the infinite Abelian group which is the countable direct sum of cyclic groups C_p of prime order p. The space of negatively curved Riemannian metrics on M^n is denoted by $\mathrm{Met}^{\sec<0}(M^n)$, and give this space the C^∞-topology.

Theorem (Farrell-Ontaneda [38]). *Assume $n \geqslant 10$ and $\mathrm{Met}^{\sec<0}(M^n) \neq \emptyset$. Then*

1. $\mathrm{Met}^{\sec<0}(M^n)$ *has infinitely many path components K.*
2. *For each component K and each prime $p \neq 2$, $(C_p)^\infty \subseteq \pi_{2p-4}(K)$.* (Here we must assume that $2p-4 < \dfrac{n-10}{2}$. For example, if $p = 3$, then $(C_3)^\infty \subseteq \pi_2(K)$ provided $n > 14$.)
3. $(C_2)^\infty \subseteq \pi_1(K)$ *provided $n \geqslant 14$.*

I'll now sketch the proof of this result by specializing to the main case of showing that $\mathrm{Met}^{\sec<0}(M^n)$ is disconnected. For simplicity sake, I'll assume that M^n ($n \geqslant 10$) is orientable and consequently contains a non-trivial embedded closed

geodesic α with trivial normal bundle. Then we identify $S^1 \times 2\mathbb{D}^{n-1}$ with a tubular neighborhood of radius 2 of α by orthogonally trivializing its normal bundle. (To do this we'd need to assume the normal injectivity radius of α is at least 2. This is not a crucial assumption; any radius $2r$ will do; it just makes our notation simpler.)

Now using Waldhausen's work on the space of smooth pseudoisotopies $P^s(S^1 \times \mathbb{D}^m)$ there exists a self-diffeomorphism

$$f : S^1 \times S^{n-2} \times [1,2]$$

which is id on $S^1 \times S^{n-2} \times \{1,2\}$

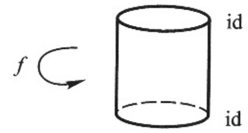


Figure 18:

but which is *not topologically* isotopic to id rel $S^1 \times S^{n-2} \times 1$.

Clearly f extends to a self-diffeomorphism ϕ of M^n by setting $\phi = \mathrm{id}$ outside

$$S^1 \times S^{n-2} \times [1,2] \subseteq S^1 \times 2\mathbb{D}^{n-1} \subseteq M^n.$$

Let g be a given Riemannian metric on M^n with negative sectional curvatures. We proceed to show that g and $\phi_*(g)$ are in different components of $\mathrm{Met}^{\sec<0}(M^n)$ thus proving our result. (Recall that $\phi_*(g)$ denotes the Riemannian metric on M^n such that $\phi : (M^n, g) \to (M^n, \phi_*(g))$ is an isometry.) If not then there exists a path g_t of metrics in $\mathrm{Met}^{\sec<0}(M)$ with $g_0 = g$ and $g_1 = \phi_*(g)$. We will show this assumption leads to a contradiction thus proving our Theorem. It is okay to assume the g_t is a smooth path. But we now make a *very strong* extra assumption about these metrics g_t and at the end of our proof indicate how to get rid of this assumption.

Extra Assumption. All the metrics g_t agree with g on $S^1 \times \mathbb{D}^{n-1} \subseteq S^1 \times 2\mathbb{D}^{n-1} \subseteq M^n$.

Let $\hat{M} \xrightarrow{P} M$ be the covering space of M corresponding to the ∞-cyclic subgroup of $\pi_1 M$ generated by $[\alpha]$. Each metric g_t induces via p a metric on $\hat{M}$ which we continue to denote by g_t.

Since α lifts to $\hat{M}$ we can enumerate the components K_i of

$$p^{-1}(S^1 \times 2\mathbb{D}^{n-1}) = \coprod_{i=0}^{\infty} K_i$$

so that

$$p : K_0 \to S^1 \times 2\mathbb{D}^{n-1}$$

is a homeomorphism and $p : K_i \to S^1 \times 2\mathbb{D}^{n-1}$ is the universal cover

$$\mathbb{R} \times 2\mathbb{D}^{n-1} \to S^1 \times 2\mathbb{D}^{n-1}$$

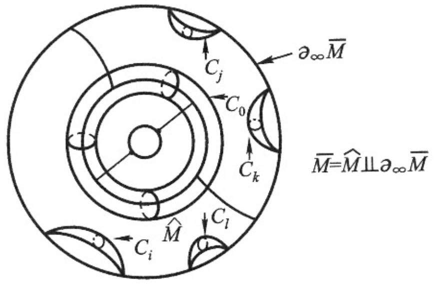


Figure 19:

for each $i > 0$. Identify k_0 with $S^1 \times 2\mathbb{D}^{n-1}$ via p and hence α with $S^1 \times 0 \subseteq K_0$. Now compactify $\hat{M}$ to $\bar{M}$ by adding at ∞ ideal points corresponding to the $g = g_0$ geodesic rays emanating perpendicularly from α. (Note we get the same compactification if we use the g_t-geodesic rays since they are quasi-geodesic rays relative to g_0.) Now define self-diffeomorphisms $f_t : \hat{M} \circlearrowleft$ as indicated in the picture

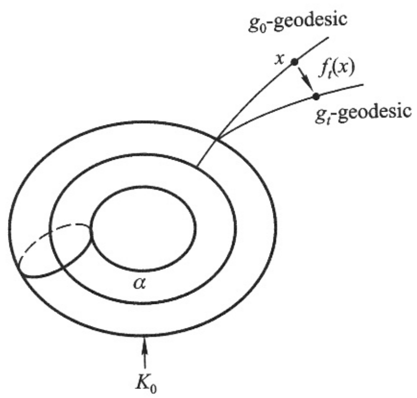


Figure 20:

These diffeomorphisms extend to *self-homeomorphisms* $\bar{f}_t : \bar{M} \circlearrowleft$. Clearly $\bar{f}_0 = \text{id}$ and each $\bar{f}_t\,|_{S^1\times\mathbb{D}^{n-1}} = \text{id}$.

Hence they restrict to a topological isotopy $\bar{f}_t : \bar{M} - S^1 \times \mathring{\mathbb{D}}^{n-1} \circlearrowleft$. (Where $\mathring{\mathbb{D}}^{n-1}$ denotes Interior $(\mathbb{D}^{n-1})$.) But we can identify $\bar{M} - S^1 \times \mathring{\mathbb{D}}^{n-1}$ with $S^1 \times S^{n-2} \times [1, +\infty]$ and then $\bar{f}_t$ is a topological isotopy rel $S^1 \times S^{n-2} \times 1$ of id to $\bar{f} \overset{Df}{=} \bar{f}_1|$. And one easily verifies the following:

1. $\bar{f}\big|_{\bar{M}-(\cup K_i)} = \text{id}$
2. $\bar{f}\big|_{K_0 - \frac{1}{2}\mathring{K}_0} = f$
3. $\bar{K}_i = \mathbb{D}^n$ if $i \neq 0$
4. $\bar{f}\big|_{\partial \bar{K}_i} = \text{id}$ if $i \neq 0$.

Now using the Alexander isotopy on each $\bar{f}|_{\bar{K}_i}$, $i \neq 0$, one gets a topological isotopy from $\bar{f}$ to f, rel $S^1 \times S^{n-2} \times 1$. Therefore id is topologically isotopic to f, rel $S^1 \times S^{n-2} \times 1$ which is our desired contradiction. □

Let me now briefly discuss how the "Extra Assumption" is removed. Using a result due to Sampson that the unique geodesic α_t in (M, g_t) homotopic to α varies smoothly with t, we can arrange the following.

Fact. There exists a smooth path $\hat{g}_t$ of complete pinched negatively curved metrics on $\hat{M}$ such that

1. $\hat{g}_0 = g_0$, $\hat{g}_1 = g_1$;
2. $\alpha_t = \alpha$ for all t;
3. the lifts of $\hat{g}_t$ to the universal cover $\tilde{M}$ are all quasi-isometric.

Hence we can find a small number $\delta > 0$ such that, for each t and each point $(\theta, v, \delta) \in S^1 \times S^{n-2} \times \delta$, there is a unique (unit speed) $\hat{g}_t$-geodesic ray γ emanating perpendicularly to α and passing through (θ, v, δ); in fact hitting this tube exactly once.

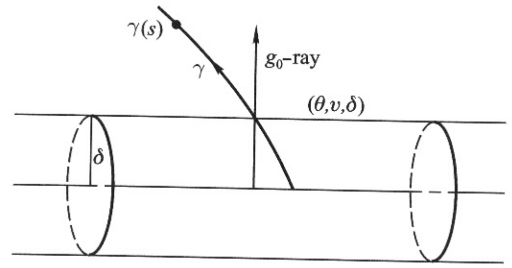


Figure 21:

Now change the definition of $f_t(\theta, v, s)$ to be $\gamma\left(\frac{\delta}{\bar{\delta}}s\right)$ where $\gamma(\bar{\delta}) = (\theta, v, \delta)$.

Final comments

One is also concerned with the quotient space $\mathcal{M}(M^n)$ of the Teichmüller space $\mathcal{T}(M^n)$ obtained by forgetting the (homotopy) markings. We call $\mathcal{M}(M^n)$ the "moduli space" of negatively curved Riemannian metrics on the given closed smooth manifold M^n. This is the classical moduli space when $n = 2$. And when $n > 2$ it is the orbit space of $\mathcal{T}(M^n)$ under the natural action of a finite group; this finite group is the subgroup $\mathcal{I}$ of $\text{Out}(\pi_1 M)$ which is the image of the natural homomorphism $\text{Diff}(M^n) \to \text{Out}(\pi_1 M^n)$ mentioned above; i.e. $\mathcal{M}(M^n) = \mathcal{T}(M^n)/\mathcal{I}$. By combining the ideas used to solve Problem 7.1 with the tapering of metrics techniques used to solve Problem 111, Ontaneda and I produced examples in [39] of closed smooth manifolds M^n where $\mathcal{M}(M^n)$ is disconnected. There are examples in all dimensions $n \geqslant 10$ and in these examples M^n supports a real hyperbolic metric.

Let me end these lectures with a final question that I think is quite interesting

Question 7. Does there exist a closed manifold M^n such that $H_k(\mathcal{M}(M^n), \mathbb{Q}) \neq 0$ for some $k > 0$?

Remark. Ontaneda and I construct in [39] examples M^n where $H_k(\mathcal{M}(M^n)) \neq 0$ for arbitrarily large k. (M^n depends on k.) But so far we have only found elements of finite order in $H_k(\mathcal{M}(M^n))$ when $k > 0$.

References

[1] H. Abels, G.A. Margulis and G.A. Soifer, Properly discontinuous groups of affine transformations with orthogonal linear part, C.R. Acad. Sci. Paris Sér I Math. 324 (1997), 253-258.

[2] S.I. Al'ber, Spaces of mappings into manifold of negative curvature, Dokl. Akad. Nauk. SSSR 178 (1968), 13-16.

[3] D.R. Anderson, The Whitehead torsion of the total space of a fiber bundle, Topology 11 (1972), 179-194.

[4] C.S. Aravinda and F.T. Farrell, Exotic negatively curved structures on Cayley hyperbolic manifolds, Jour. of Diff. Geom. 63 (2003), 41-62.

[5] C.S. Aravinda and F.T. Farrell, Exotic structures and quaternionic hyperbolic manifolds, in Proc. of the Int. Conf. on Algebraic Groups and Arithmetic, TIFR, Mumbai (2004), 507-524.

[6] C.S. Aravinda and F.T. Farrell, Nonpositivity: Curvature vs. Curvature Operator, Proc. AMS 132 (2005), 191-192.

[7] A.L. Besse, Einstein Manifolds, Ergebnisse Series vol. 10, Springer-Verlag, Berlin, 1987.

[8] G. Besson, G. Courtois and S. Gallot, Minimal entropy and Mostow's rigidity Theorems, Ergodic Theory & Dynam. Sys. 16 (1996), 623-649.

[9] K. Burns and A. Katok, Manifolds with non-positive curvature, Ergodic Theory & Dynam. Sys. 5 (1985), 307-317.

[10] T.A. Chapman and S. Ferry, Approximating homotopy equivalences by homeomorphisms, Amer. J. Math. 101 (1979), 567-582.

[11] J. Cheeger, K. Fukaya, and M. Gromov, Nilpotent structures and invariant metrics on collapsed manifolds, JAMS 5 (1992), 327-372.

[12] K. Corlette, Archimedean superrigidity and hyperbolic geometry, Ann. Math. 135 (1992), 165-182.

[13] P. Eberline and B. O'Neill, Visibility manifolds, Pacific J. Math. 46 (1973), 45-109.

[14] D.G. Ebin, The manifold of Riemannian metrics, Proc. Sympos. Pure Math. 15 (1968), 11-40.

[15] J. Eells and L. Lemaire, Another report on harmonic maps, Bull. London Math. Soc. 20 (1988), 385-524.

[16] J. Eells and J.H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964), 109-160.

[17] F.T. Farrell, The Borel Conjecture, in Topology of High-Dimensional Manifolds, Number 1, edited by F.T. Farrell, L. Goettsche, and W. Lueck, ICTP Lecture Notes, Trieste-Italy, 2002,225-298.

[18] F.T. Farrell and W.C. Hsiang, On Novikov's conjecture for non-positively curved manifolds I, Annals of Math. 113 (1981), 199-209.

[19] F.T. Farrell and L.E. Jones, K-theory and dynamics I, Annals of Math. 124 (1986), 531-569.

[20] F.T. Farrell and L.E. Jones, Topological analogue of Mostow's rigidity theorem, Jour. of the AMS 2 (1989), 257-370.

[21] F.T. Farrell and L.E. Jones, Compact negatively curved manifolds (of dim $\neq 3, 4$) are topologically rigid, Proceedings of the National Academy of Science, USA, 86 (1989), 3461-3463.

[22] F.T. Farrell and L.E. Jones, Negatively curved manifolds with exotic smooth structures, Jour. of the AMS 2 (1989), 899-908.

[23] F.T. Farrell and L.E. Jones, Smooth non-representability of $\mathrm{Out}(\pi_1 M)$, Bull. of LMS 22 (1990), 485-488.

[24] F.T. Farrell and L.E. Jones, Topological rigidity for compact non-positively curved manifolds, Proc. Sympos. Pure Math. 54 Part 3 (1993), 229-274.

[25] F.T. Farrell and L.E. Jones, Complex hyperbolic manifolds and exotic smooth structures, Inventiones Mathematicae 117 (1994), 57-74.

[26] F.T. Farrell and L.E. Jones, Exotic smoothings of hyperbolic manifolds which do not support pinched negative curvature, Proc. AMS 121 (1994), 627-630.

[27] F.T. Farrell and L.E. Jones, Smooth rigidity and C^1-conjugacy at ∞, Comm. in Anal. and Geom. 2 (1994), 563-578.

[28] F.T. Farrell and L.E. Jones, Some non-homeomorphic harmonic homotopy equivalences, Bull. LMS 28 (1996), 177-182.

[29] F.T. Farrell and L.E. Jones, Rigidity for aspherical manifolds with $\pi_1 \subset GL_m(\mathbb{R})$, Asian Jour. of Math. 2 (1998), 215-262.

[30] F.T. Farrell, L.E. Jones and P. Ontaneda, Hyperbolic manifolds with negatively curved exotic triangulations in dimensions larger than five, J. Diff. Geom. 48 (1998), 319-322.

[31] F.T. Farrell, L.E. Jones and P. Ontaneda, Examples of non-homeomorphic harmonic maps between negatively curved manifolds, Bull. London Math. Soc. 30 (1998), 295-296.

[32] F.T. Farrell and J.-F. Lafont, Finite automorphisms of negatively curved Poincaré duality groups, Geometric and Functional Analysis 14 (2004), 283-294.

[33] F.T. Farrell and J.-F. Lafont, Involutions of negatively curved groups with wild boundary behavior, Pure and Applied Math Quarterly (to appear).

[34] F.T. Farrell and P. Ontaneda, Harmonic cellular maps which are not diffeomorphisms, Inventiones Mathematicae 158 (2004), 497-513.

[35] F.T. Farrell and P. Ontaneda, Exotic structures and the limitations of certain analytic methods in geometry, Asian J. of Math. 8 (2004), 639-652.

[36] F.T. Farrell and P. Ontaneda, A caveat on the Ricci flow for pinched negatively curved manifolds, Asian J. of Math. 9 (2005), 401-406.

[37] F.T. Farrell and P. Ontaneda, The Teichmueller space of pinched negatively curved metrics on a hyperbolic manifold is not contractible, (submitted for publication).

[38] F.T. Farrell and P. Ontaneda, On the topology of the space of negatively curved metrics, (submitted for publication).
[39] F.T. Farrell and P. Ontaneda, The moduli space of negatively curved metrics on a hyperbolic manifold, (in preparation).
[40] F.T. Farrell and P. Ontaneda and M.S. Raghunathan, Non-univalent harmonic maps homotopic to diffeomorphisms, Jour. of Diff. Geom. 54 (2000), 227-253.
[41] S. Ferry, The homeomorphism group of a compact Hilbert cube manifold in an ANR, Ann. of Math. 106 (1977), 101-119.
[42] D. Fried and W.M. Goldman, Three-dimensional affine crystallographic groups, Adv. in Math. 47 (1983), 1-49.
[43] U. Hamenstädt, Regularity of time-preserving conjugacies for contact Anosov flows with C^1-Anosov splitting, Ergodic Theory & Dynam. Sys. 13 (1993), 65-72.
[44] R. Hamilton, The Ricci flow on surfaces, Contemporary Mathematics 71 (1988), 237-261.
[45] P. Hartman, On homotopic harmonic maps, Canad. J. Math. 19 (1967), 673-687.
[46] A.E. Hatcher, Concordance spaces, higher simple homotopy theory, and applications, Proc. Sympos. Pure Math. 32 (1978), 3-21.
[47] M. Hirsch and C. Pugh, Smoothness of horocycle foliations, J. Diff. Geom. 10 (1975), 225-238.
[48] K. Igusa, The stability theorem for pseudoisotopies, K-Theory 2 (1988), 1-355.
[49] M. Kanai, Geodesic flows on negatively curved manifolds with smooth stable and unstable foliations, Ergodic Th. & Dynamical Sys. 8 (1988), 215-239.
[50] S.P. Kerckhoff, The Nielsen realization problem, Ann. of Math. 117 (1983), 235-265.
[51] R.C. Kirby and L.C. Siebenmann, Foundational essays on topological manifolds, smoothings, and triangulations, Ann. of Math. Studies, Vol. 88, Princeton Univ. Press, Princeton, NJ (1977).
[52] A.S. Mischenko, Infinite dimensional representations of discrete groups and higher signatures, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 81-106.
[53] G.D. Mostow, Strong Rigidity of Locally Symmetric Spaces, Ann. of Math. Studies, Vol. 78, Princeton Univ. Press, Princeton, 1973.
[54] B. Okun, Nonzero degree tangential maps between dual symmetric spaces, Algebraic & Geometric Topology 1 (2001), 709-718.
[55] P. Petersen, Riemannian Geometry, Graduate Texts in Math., 171, Springer-Verlag, NY, 1998.
[56] A. Ranicki, Algebraic L-theory and topological manifolds, Cambridge Tracts in Mathematics, Vol. 102, Cambridge University Press, 1992.
[57] F. Raymond and L.L. Scott, Failure of Nielsen's theorem in higher dimensions, Arch. Math. (Basel) 29 (1977), 643-654.
[58] J. Sampson, Some properties and applications of harmonic mappings, Ann. Sci. Ecole Norm. Sup. 11 (1978), 211-228.
[59] M. Scharlemann, Smooth CE maps and smooth homeomorphisms, Lecture

Notes in Math., vol. 341, Springer, 1978, pp. 234-240.
[60] M. Scharlemann and L. Siebenmann, The Hauptvermutung for smooth singular homeomorphisms, Manifolds Tokyo 1973, (University of Tokyo Press, Tokyo, 1975), 85-91.
[61] R. Schoen and S.-T. Yau, On univalent harmonic maps between surfaces, Invent. Math. 44 (1978), 265-278.
[62] W. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, BAMS 6 (1982), 357-381.
[63] M. Varisco, personal communication.
[64] F. Waldhausen, Algebraic K-theory of topological spaces, I, Proc. Sympos. Pure Math. vol. 32, AMS, Providence, RI, 1978, 35-60.
[65] C.T.C. Wall, Surgery on compact manifolds, second edition edited by A.A. Ranicki, Mathematical Surveys and Monographs, 69, Amer. Math. Soc., Providence, RI, 1999.
[66] S.-T. Yau, On the fundamental group of compact manifolds of nonpositive curvature, Ann. of Math. 93 (1971), 579-585.
[67] S.-T. Yau, Seminar on differential geometry, Ann. of Math. Studies, no. 102, Princeton Univ. Press, Princeton, NJ, 1982.
[68] R. Ye, Ricci flow, Einstein metrics and space forms, Trans. AMS 338 (1993), 871-896.